> #OK to post
#Julian Herman, 22nd November, 2021, Assignment 22

```
> read `/Users/julianherman/Documents/Rutgers/Fall 2021/Dynamical Models In
    Biology/HW/DMB.txt`
```

First Written: Nov. 2021

This is DMB.txt, A Maple package to explore Dynamical models in Biology (both discrete and continuous)
accompanying the class Dynamical Models in Biology, Rutgers University. Taught by Dr. Z. (Doron Zeilbeger)

The most current version is available on WWW at:
http://sites.math.rutgers.edu/~zeilberg/tokhniot/DMB.txt .
Please report all bugs to: DoronZeil at gmail dot com .

For general help, and a list of the MAIN functions,
type "Help();". For specific help type "Help(procedure_name);"

For a list of the supporting functions type: Help1();
For help with any of them type: Help(ProcedureName);

For a list of the functions that give examples of Discrete-time dynamical systems (some famous), type: HelpDDM();
For help with any of them type: Help(ProcedureName);

For a list of the functions continuous-time dynamical systems (some famous) type: HelpCDM();
For help with any of them type: Help(ProcedureName);

------------------------------ **(1)**

> #1) I did not get anything wrong

>

> #2)

> #a)THIS IS A DISCRETE DYNAMICAL SYSTEM: USE A SYSTEM OF DIFFERENCE EQUATIONS!

> # Let x(n) = the number of lynxes at the start of year 'n'

> # Let y(n) = the number of hares at the start of year 'n'

> # x(n)=2·x(n-1) +3·y(n-1)

> # y(n)=3·x(n-1) +y(n-1)

> # x(0)=20; y(0)=10

> # x(9)=? ; y(9)=?

> Help(Orb)

Orb(F,x,x0,K1,K2): Inputs a transformation F in the list of variables x with initial point pt, outputs the trajectory of
of the discrete dynamical system (i.e. solutions of the difference equation): x(n)=F(x(n-1)) with x(0)= x0 from n=K1 to n=K2.
For the full trajectory (from n=0 to n=K2), use K1=0. Try:
*Orb(5/2*x*(1-x),[x], [0.5], 1000,1010);*
*Orb([(1+x+y)/(2+x+y),(6+x+y)/(2+4*x+5*y)],[x,y], [2.,3.], 1000,1010);* **(2)**

> $F := [2\cdot x + 3\cdot y, 3\cdot x + y]$

$$F := [2\,x + 3\,y, 3\,x + y] \quad (3)$$

> Orb(F, [x, y], [20, 10], 0, 9)

[[20, 10], [70, 70], [350, 280], [1540, 1330], [7070, 5950], [31990, 27160], [145460, 123130], [660310, 559510], [2999150, 2540440], [13619620, 11537890], [61852910, 52396750]] **(4)**

> %[-2]

[13619620, 11537890] **(5)**

> #At the start of the tenth year (which corresponds to n=9 because we started our first year at n= 0), there are 13619620 lynxes and 11537890 hares. Note: there is a bug in Orb(): it displayed 11 values instead of 10, so the 10th year is actually the second to last value!

>

> #b)

> # Let x(t) = the number of lynxes at time 't'

> # Let y(t) = the number of hares at time 't'

> # x'(t) = 2·x(t) +3·y(t)

> # y'(t) = 3·x(t) +y(t)

> # x(0)=20; y(0)=10

> # x(10)=? ; y(10)=?

> evalf(dsolve({diff(x(t), t) = 2·x(t) + 3·y(t), diff(y(t), t) = 3·x(t) + y(t), x(0) = 20, y(0) = 10}, {x(t), y(t)}))

$$\{x(t) = 16.57595949\,e^{4.541381265\,t} + 3.424040509\,e^{-1.541381265\,t}, y(t) = 14.04194430\,e^{4.541381265\,t} - 4.041944304\,e^{-1.541381265\,t}\} \quad (6)$$

> subs(t = 10, %)

$$\{x(10) = 16.57595949\,e^{45.41381265} + 3.424040509\,e^{-15.41381265}, y(10) = 14.04194430\,e^{45.41381265} - 4.041944304\,e^{-15.41381265}\} \quad (7)$$

> evalf(%)

$$\{x(10) = 8.758846449 \times 10^{20}, y(10) = 7.419856090 \times 10^{20}\} \quad (8)$$

> #After ten years, there are $8.758846449 \times 10^{20}$ lynxes **and** $7.419856090 \times 10^{20}$ hares!

>

>

> #3)

> $x(n+1) = x(n)\cdot(1-b-c) + y(n)\cdot(1-\exp(-a\cdot x(n)))$

$$x(n+1) = x(n)\,(1-b-c) + y(n)\,\left(1-e^{-a\,x(n)}\right) \quad (9)$$

> $subs(n = n-1, \%)$

$$x(n) = x(n-1)\,(1-b-c) + y(n-1)\,\left(1-e^{-a\,x(n-1)}\right) \quad (10)$$

> $y(n+1) = (1-y(n))\cdot b + y(n)\cdot\exp(-a\cdot x(n))$

$$y(n+1) = (1-y(n))\,b + y(n)\,e^{-a\,x(n)} \quad (11)$$

> $subs(n = n-1, \%)$

$$y(n) = (1-y(n-1))\,b + y(n-1)\,e^{-a\,x(n-1)} \quad (12)$$

> #Our system put into canonical form:

> $x(n) = x(n-1)\,(1-b-c) + y(n-1)\,\left(1-e^{-a\,x(n-1)}\right)$:

> $y(n) = (1-y(n-1))\,b + y(n-1)\,e^{-a\,x(n-1)}$:

>

> #Let a=2.3, b=0.25, c=0.53

> $x(n) = x(n-1)\cdot 0.22 + y(n-1)\,\left(1-e^{-2.3\cdot x(n-1)}\right)$

$$x(n) = 0.22\,x(n-1) + y(n-1)\,\left(1-e^{-2.3\,x(n-1)}\right) \quad (13)$$

> $y(n) = (1-y(n-1))\cdot 0.25 + y(n-1)\,e^{-2.3\cdot x(n-1)}$

$$y(n) = 0.25 - 0.25\,y(n-1) + y(n-1)\,e^{-2.3\,x(n-1)} \quad (14)$$

> #R_0= $\frac{a}{b+c} = \frac{2.3}{0.25+0.53} = 2.948717949 > 1$

>

> $Help(Orb)$

Orb(F,x,x0,K1,K2): Inputs a transformation F in the list of variables x with initial point pt, outputs the trajectory of

of the discrete dynamical system (i.e. solutions of the difference equation): x(n)=F(x(n-1)) with x(0) =x0 from n=K1 to n=K2.

For the full trajectory (from n=0 to n=K2), use K1=0. Try:

*Orb(5/2*x*(1-x),[x], [0.5], 1000,1010);*

*Orb([(1+x+y)/(2+x+y),(6+x+y)/(2+4*x+5*y)],[x,y], [2.,3.], 1000,1010);* **(15)**

>

> $F := \left[0.22\cdot x + y\cdot\left(1-2.718^{-2.3\,x}\right),\, 0.25 - 0.25\cdot y + y\cdot 2.718^{-2.3\cdot x}\right]$:

>

> #For R_0 > 1, the left hand side (limit as n goes to infinity of our system) is represented by the below orbit

> $\{op(Orb(F, [x,y], [0.25, 0.33], 1000, 1010))\}$

$$\{[0.1867905273, 0.4172135548]\} \quad (16)$$

> #The right hand side (the positive solutions of setting F[1] = x, F[2]=y and solving for {x,y}) is

represented below

```
> solve({0.78·x = y·(1 − 2.718^(−2.3·x)), y = 1 − x·(1 + 2.12)}, {x, y})[-1]
```

Warning, solutions may have been lost

$$\{x = 0.1867905273, y = 0.4172135548\} \quad (17)$$

> *#Conjecture 1 holds true: left hand side is equal to the right hand side:* {[0.1867905273, 0.4172135548]} = {x = 0.1867905273, y = 0.4172135548}

> *# x=0.1867905273 meaning about 18.68% infected people, y=0.4172135548 meaning about 41.72% susceptible people, and removed = 1-x-y=1-0.1867905273-0.4172135548= 0.3959959179 meaning about 39.60% removed people. This represents the unique positive endemic equilibrium!*

>

```
> r := rand(0.0 ..1.0) :
> for i from 1 to 20 do a := r( ) : b := r( ) : print(a, b, {op(Orb(F, [x, y], [a, b], 1000, 1010))}) od:
```

0.1987987412, 0.4809582675, {[0.1867905273, 0.4172135547]}
0.04152840698, 0.7844649847, {[0.1867905273, 0.4172135547]}
0.5768645319, 0.2978867746, {[0.1867905273, 0.4172135549]}
0.3297654079, 0.7800864131, {[0.1867905273, 0.4172135549]}
0.1738224444, 0.8687019171, {[0.1867905273, 0.4172135549]}
0.06198813631, 0.1269248574, {[0.1867905273, 0.4172135547]}
0.5895574564, 0.9792717505, {[0.1867905273, 0.4172135549]}
0.7519247060, 0.9801380609, {[0.1867905273, 0.4172135547]}
0.5997879218, 0.1948514864, {[0.1867905273, 0.4172135549]}
0.7811327643, 0.1675393935, {[0.1867905273, 0.4172135549]}
0.4915531458, 0.2834466080, {[0.1867905273, 0.4172135548]}
0.5183632973, 0.8118137952, {[0.1867905273, 0.4172135549]}
0.9724067291, 0.8437990268, {[0.1867905273, 0.4172135549]}
0.9556374600, 0.4440261112, {[0.1867905273, 0.4172135549]}
0.5332300168, 0.1254245114, {[0.1867905273, 0.4172135549]}
0.4700048787, 0.4132809028, {[0.1867905273, 0.4172135549]}
0.6354209980, 0.01238757408, {[0.1867905273, 0.4172135547]}
0.7134669132, 0.002123326392, {[0.1867905273, 0.4172135547]}
0.8530681252, 0.7827858269, {[0.1867905273, 0.4172135549]}
0.8502406271, 0.5948554975, {[0.1867905273, 0.4172135549]} **(18)**

> *#The endemic equilibrium is true for all of the above initial conditions... it is most likely globally stable!*

>

> *#Lets try for R_0 <= 1*
 # a=0.5, b=0.3, c=0.4
$$\#R\_0 = \frac{a}{b+c} = \frac{0.5}{0.3+0.4} = 0.7142857143 \leq 1$$

```
>
> F := [x·0.3 + y(1 − 2.718^(−0.5 x)), (1 − y)·0.3 + y·2.718^(−0.5·x)]:
>
> #For R_0 <= 1, the left hand side (limit as n goes to infinity of our system) is represented by the below orbit
> {op(Orb(F, [x, y], [0.2, 0.33], 1000, 1010))}
```

$$\{[1.428571428 \times 10^{-10}, 0.9999999996]\} \tag{19}$$

```
> #The right hand side (the positive solutions of setting F[1] = x, F[2]=y and solving for {x,y}) is represented below
```

$$> \ solve\left(\left\{0.7\cdot x = y\cdot\left(1 - 2.718^{-0.5\cdot x}\right), y = 1 - x\cdot\left(1 + \frac{0.4}{0.3}\right)\right\}, \{x, y\}\right)[1]$$

Warning, solutions may have been lost

$$\{x = 0., y = 1.\} \tag{20}$$

```
> #For R_0 <= 1, the solution approaches the infection-free state, where the population is composed of 0 % infected people and 100 % susceptible!
>
>
>
> #4)
> x(n+1)=x(n)·(1-b) + (1-x(n))·(1-exp(-a·x(n-1)))
```

$$x(n+1) = x(n)(1-b) + (1 - x(n))\left(1 - e^{-a\,x(n-1)}\right) \tag{21}$$

```
> subs(n = n − 1, %)
```

$$x(n) = x(n-1)(1-b) + (1 - x(n-1))\left(1 - e^{-a\,x(n-2)}\right) \tag{22}$$

```
> #where 0<b<1, 0<a, and 0<x(0), x(1)<1
> #Let: a = 2.5, b=0.5
```

$$> \ \#R\_0 = \frac{a}{b} = \frac{2.5}{0.5} = 5 > 1$$

$$> \ x(n) = x(n-1)\cdot(0.5) + (1 - x(n-1))\left(1 - e^{-2.5\cdot x(n-2)}\right)$$

$$x(n) = 0.5\,x(n-1) + (1 - x(n-1))\left(1 - e^{-2.5\,x(n-2)}\right) \tag{23}$$

```
> F := 0.5·z[1] + (1 − z[1])·(1 − 2.718^(−2.5·z[2])) :
> Digits := 6 :
> #The below corresponds to the left hand side (limit as n goes to infinity of the second order difference equation above):
> {op(Orbk(2, z, F, [0.23, 0.78], 1000, 1010))}
```

$$\{0.610107\} \tag{24}$$

```
> #The below corresponds to the right hand side (the positive solution to setting the difference equation equal to z and replacing all x[n's] with z because a fixed point occurs when x[n]=x[n-1]=x[n-2]=z ):
```

$\# z = z\cdot(1-b) + (1-z)\left(1 - 2.718^{-az}\right)$

$\# z - z + bz = (1-z)\left(1 - 2.718^{-az}\right)$

$\# bz = (1-z)\left(1 - 2.718^{-az}\right)$ #This is the equation we solve for z

> $solve\left(\left\{0.5\, z = (1 - z)\cdot\left(1 - 2.718^{-2.5\, z}\right)\right\}, \{z\}\right)[2]$

Warning, solutions may have been lost

$$\{z = 0.610107\} \quad (25)$$

> #The LHS = RHS, therefore, the conjecture is verified!

>

> #Let's try for R_0 = $\frac{a}{b} = \frac{0.17}{0.83} = 0.204819 < 1$!

> $x(n) = x(n-1)\ (1-b) + (1 - x(n-1))\ \left(1 - e^{-a\, x(n-2)}\right)$:

> $F := z[1]\cdot(1 - .83) + (1 - z[1])\cdot\left(1 - 2.718^{-0.17\cdot z[2]}\right)$:

> #LHS:

> $\{op(Orbk(2, z, F, [0.13, 0.38], 1000, 1010))\}$

$$\{1.80448 \times 10^{-768}, 1.06146 \times 10^{-767}, 6.24388 \times 10^{-767}, 3.67287 \times 10^{-766}, 2.16051 \times 10^{-765}, 1.27089 \times 10^{-764}, 7.47585 \times 10^{-764}, 4.39756 \times 10^{-763}, 2.58680 \times 10^{-762}, 1.52165 \times 10^{-761}, 8.95086 \times 10^{-761}\} \quad (26)$$

> #RHS:

> $solve\left(\left\{0.83\cdot z = (1 - z)\cdot\left(1 - 2.718^{-0.17\, z}\right)\right\}, \{z\}\right)[2]$

Warning, solutions may have been lost

$$\{z = 0.\} \quad (27)$$

> #LHS converges to 0 = RHS =0!

>

>

>

> #5)

> $x(n) = \frac{x(n-1)}{10 + x(n-1)}$

$$x(n) = \frac{x(n-1)}{10 + x(n-1)} \quad (28)$$

> #To find the fixed points: set the underlying transformation(in terms of z) equal to z and solve for z.

> #Underlying transformation: F(z)= $\frac{z}{10 + z}$

> # $\frac{z}{10+z} = z \quad \rightarrow \quad 10\, z + z^2 = z \quad \rightarrow \quad 9\, z + z^2 = 0 \quad \rightarrow \quad z(9 + z) = 0 \quad \rightarrow \quad z = 0, z = -9$

> #Fixed points: x(n)=0, x(n)=-9

> #To check if the fixed points are stable: evaluate |F'(z)| < 1

> #F'(z)= $\frac{10 + z - z}{(10+z)^2} = \frac{10}{(10+z)^2}$

> #$|F'(z = 0)| < 1$?

> #|0.1| < 1? YES, therefore, x=0 is a STABLE fixed point!

> #$|F'(z = -9)| < 1$?

> #|10| < 1? NO, therefore, x=-9 is an UNSTABLE fixed point!

> *#These results are confirmed below:*

> $FP\left(\left[\frac{z}{10+z}\right], [z]\right)$

$$\{[-9], [0]\} \quad \textbf{(29)}$$

> $SFP\left(\left[\frac{z}{10+z}\right], [z]\right)$

$$\{[0.]\} \quad \textbf{(30)}$$