Charles Griebell

Homework 20

OK to post (but not completely right and not completely finished, contains errors in the conjecture code)

```
> with(LinearAlgebra);
```

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CARE, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, CompressedSparseForm, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DARE, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, FromCompressedSparseForm, FromSplitForm, GaussianElimination, GenerateEquations, GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct, LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, ProjectionMatrix, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SplitForm, StronglyConnectedBlocks, SubMatrix, SubVector, SumBasis, SylvesterMatrix, SylvesterSolve, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip] **(1)**

```
> read `C:/Users/cgrie/Dynam Models Bio/Homeworks/HW21/DMB.txt` ;
```

First Written: Nov. 2021

This is DMB.txt, A Maple package to explore Dynamical models in Biology (both discrete and continuous)

accompanying the class Dynamical Models in Biology, Rutgers University. Taught by Dr. Z. (Doron Zeilbeger)

The most current version is available on WWW at:
http://sites.math.rutgers.edu/~zeilberg/tokhniot/DMB.txt .
Please report all bugs to: DoronZeil at gmail dot com .

For general help, and a list of the MAIN functions,
type "Help();". For specific help type "Help(procedure_name);"

For a list of the supporting functions type: Help1();
For help with any of them type: Help(ProcedureName);

For a list of the functions that give examples of Discrete-time dynamical systems (some famous),
type: HelpDDM();
For help with any of them type: Help(ProcedureName);

For a list of the functions continuous-time dynamical systems (some famous) type: HelpCDM();
For help with any of them type: Help(ProcedureName);

------------------------------ **(2)**

PROBLEM 1: Make up a different problem

(i) For the following vectors:

Determine which vectors are eigenvectors of the matrix

$$\begin{bmatrix} 1 & 6 \\ 6 & -4 \end{bmatrix}$$

Create the Eigenvectors to

```
> A:= Matrix([[1,6],[6,-4]]);
  Ev := Eigenvectors(A):
  print(`Eigenvectors of A are:`);

  Ival1:= Ev[1][1];
  Ivec1:= Column(Ev[2],1);
```

```
Ival2:= Ev[1][2];
Ivec2:= Column(Ev[2],2);
```

$$A := \begin{bmatrix} 1 & 6 \\ 6 & -4 \end{bmatrix}$$

Eigenvectors of A are:

$$Ival1 := 5$$

$$Ivec1 := \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

$$Ival2 := -8$$

$$Ivec2 := \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \qquad (3)$$

#Create some false eigenvectors to mix in with the correct ones

THE FOLLOWING EIGENVECTORS ARE:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 12 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{3}{2} \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

#IN the pop quiz we were supposed to compute the matrix-vector product of the coefficient matrix A and the eigenvector v and compute the same eigenvector v multiplied by the eigenvalue λ (a scalar) and test if

$$\begin{bmatrix} 1 & 6 \\ 6 & -4 \end{bmatrix} v = \lambda v \quad \text{is correct}$$

The answers would be

$$\begin{bmatrix} -4 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 6 \end{bmatrix} \quad (4)$$

and

$$\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \quad (5)$$

(same vector) associated with -8 eigenvalue AND

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad (6)$$

Part 2 of problem 1 (create a continuous differential equation)

A squirrel's weight increases if its caloric intake exceeds its caloric expenditures.
A fat squirrel burns more calories than an emaciated squirrel. In this case, a squirrel twice as fat burns twice as many calories **(Instantaneous rate)**

Ignoring biological specifics, let 1 calorie = 1 kilogram of fat, and assume that calories burned only depends on a squirrels weight

$$\begin{bmatrix} -4 \\ 6 \end{bmatrix}$$
$$\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad (7)$$

instantaneous rate of Calories consumed per unit time (K) is assumed to be constant

The Instantaneous RATE of calories burned with respect to weight can be modeled as:

$$\frac{d(weight)}{dt} = caloriesEaten(t) - caloriesBurned(t)$$

$$\frac{d(weight)}{dt} = K - 2 \cdot weight(t)$$

Tus, via separation of variables (Ignore weight as a function of t, because we dont want to multiply by an extra dt from chain rule),

$$\int \frac{d(weight)}{K - 2 \cdot weight} = \int dt$$

Which is integrated as follows (via u-substitution)

Let $u = K - 2 \cdot weight$

Therefore, $du = -2 \cdot d(weight)$

Therefore, rewrite integral as:

$$-\frac{1}{2}\int \frac{1}{u}\,du = \int dt \xrightarrow{After\,evaluating\,indefinite\,integrals} -\frac{1}{2}\ln|u| = t + c \xrightarrow{un\,-\,substituting} -\frac{1}{2}\ln|K - 2 \cdot weight| = t + c$$

let our initial weight of squirrel be 400 kilorgams and its daily caloric intake be 900 calories. Therefore, the value of c is obtained as follows:

$$-\frac{1}{2}\ln|900 - 400| = 0 + c$$

```
> evalf((-1/2)*ln(500));
```

$$-3.107304049 \quad \textbf{(8)}$$

#Therefore, with these initial conditions, $c = -3.107304049$

```
> squir_weight := t = (-1/2)*ln(K-2*w) + 3.107304049
```

$$squir\_weight := t = -\frac{\ln(K - 2\,w)}{2} + 3.107304049 \quad \textbf{(9)}$$

Take a random time

```
> t_rand := evalf(rand(0.01..10.1));
```

$t\_rand := (\,) \mapsto RandomTools{:-}Generate(float('range' = 0.01 ..10.1, 'method' = 'uniform'))$ **(10)**

Therefore the weight of a squirrel at random time t_rand is :

```
> r := t_rand();
  f:= evalf(subs({K = 900, t = r},squir_weight));
  print(`t=`, r);
  solve(f,w);
```

$r := 3.967188605$

$f := 3.967188605 = -0.5000000000 \ln(900. - 2.\, w) + 3.107304049$

$t=,\ 3.967188605$

449.9104463 **(11)**

#So that's finished!

Problem 2:

Part (a):

```
> Help(Orb);
```

Orb(F,x,x0,K1,K2): Inputs a transformation F in the list of variables x with initial point pt, outputs the trajectory of

of the discrete dynamical system (i.e. solutions of the difference equation): x(n)=F(x(n-1)) with x(0)=x0 from n=K1 to n=K2.

For the full trajectory (from n=0 to n=K2), use K1=0. Try:

*Orb(5/2*x*(1-x),[x], [0.5], 1000,1010);*

*Orb([(1+x+y)/(2+x+y),(6+x+y)/(2+4*x+5*y),[x,y], [2.,3.], 1000,1010);* **(12)**

```
> #Denote Lynx population at time n as L(n)
  #Denote Hare population at time n as H(n)

  #Represent the underlying transformation with the

  Lynx := 2*L +3*H;
  Hare := 3*L + H;


  disPops := Orb([Lynx,Hare],[L,H],[10.,20.],0,10);

  print(`at the start of the 10th year, there were`);
```

```
print(disPops[10][1]. `lynxes`);
print(disPops[10][2]. `hares`);
```

$$Lynx := 2\,L + 3\,H$$

$$Hare := 3\,L + H$$

$disPops := [[10., 20.], [80., 50.], [310., 290.], [1490., 1220.], [6640., 5690.], [30350., 25610.], [137530., 116660.], [625040., 529250.], [2.837830 \times 10^{6}, 2.404370 \times 10^{6}], [1.2888770 \times 10^{7}, 1.0917860 \times 10^{7}], [5.8531120 \times 10^{7}, 4.9584170 \times 10^{7}], [2.65814750 \times 10^{8}, 2.25177530 \times 10^{8}]]$

at the start of the 10th year, there were

1.2888770×10^{7} *lynxes*

1.0917860×10^{7} *hares* **(13)**

THE CONTINUOUS CASE

```
> #Another way to do this is to do the diff command


  Lynx:= diff(L(t),t) = 2*L(t) + 3*H(t);
  Hare:= diff(H(t),t) = 3*L(t) + H(t);

  #With 10 lynxes and 20 hares

  print(`given 10 lynxes and 20 hares as our initial conditions`):
  print(`the corresponding dynamical system is`);

  particular_sol := dsolve({L(0)=10,H(0)=20,Lynx,Hare});



  print(``);
  print(``);
  print(`how many rabbits and hares are there at 10 years?`);

  p_sol_10 := subs(t=10,particular_sol):
  populations:= evalf(p_sol_10);

  print(`Where H(10) corresponds to hare and L(10) corresponds to
  Lynx`);
```

$$Lynx := \frac{d}{dt} L(t) = 2\, L(t) + 3\, H(t)$$

$$Hare := \frac{d}{dt} H(t) = 3\, L(t) + H(t)$$

given 10 lynxes and 20 hares as our initial conditions

the corresponding dynamical system is

$$particular\_sol := \left\{ H(t) = \left(10 + \frac{20\sqrt{37}}{37}\right) e^{\frac{(3+\sqrt{37})\,t}{2}} + \left(10 - \frac{20\sqrt{37}}{37}\right) e^{-\frac{(-3+\sqrt{37})\,t}{2}}, \right.$$

$$L(t) = \frac{\left(10 + \frac{20\sqrt{37}}{37}\right) e^{\frac{(3+\sqrt{37})\,t}{2}} \sqrt{37}}{6} - \frac{\left(10 - \frac{20\sqrt{37}}{37}\right) e^{-\frac{(-3+\sqrt{37})\,t}{2}} \sqrt{37}}{6}$$

$$\left. + \frac{\left(10 + \frac{20\sqrt{37}}{37}\right) e^{\frac{(3+\sqrt{37})\,t}{2}}}{6} + \frac{\left(10 - \frac{20\sqrt{37}}{37}\right) e^{-\frac{(-3+\sqrt{37})\,t}{2}}}{6} \right\}$$

how many rabbits and hares are there at 10 years?

$$populations := \{H(10) = 7.021456243 \times 10^{20}, L(10) = 8.288551198 \times 10^{20}\}$$

Where H(10) corresponds to hare and L(10) corresponds to Lynx **(14)**

PROBLEM 3

Carefully read
https://sites.math.rutgers.edu/~zeilberg/Bio21/LadasSri.pdf

and confirm the claims for randomly chosen values of the parameters for conjecture 1.

$x_{n+1} = x_n(1 - b - c) + y_n(1 - \exp(-a x_n))$, n=1,2,...
$y_{n+1} = (1 - y_n) b + y_n(\exp(-a x_n))$

where $0 < b + c \le 1$, $0 < a$

underlying transformation:

$$\begin{bmatrix} 1-b-c & 1-\exp(-ax) \\ \frac{1}{y}-b & \exp(-ax) \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$

#THat is not necessary because orb already does the job, although evaluating the jacobian of that matrix is a way to evaluate stability

$$\begin{bmatrix} (1-b-c)\, x_n + \left(1-\mathrm{e}^{-ax\_}\right) y_n \\ \left(\frac{1}{y\_\_}-b\right) x_n + \mathrm{e}^{-ax\_} y_n \end{bmatrix} \tag{15}$$

```
> print(Orb);
```

proc(F, x, x0, K1, K2) **(16)**

```
    local x1, i, L, i1, i2;
    if not (type(F, list) and type(x, list) and type(x0, list) and nops(F) = nops(x) and
    nops(x) = nops(x0) and type(K1, integer) and type(K2, integer) and 0 <= K1 and K1
     < K2) then
        print(bad input); RETURN(FAIL)
    end if;
    x1 := x0;
    for i from 0 to K1 - 1 do
        x1 := [seq(subs({seq(x[i2] = x1[i2], i2 = 1 ..nops(x))}, F[i1]), i1 = 1 ..nops(F))]
    end do;
    L := [x1];
    for i from K1 to K2 do
        x1 := [seq(subs({seq(x[i2] = x1[i2], i2 = 1 ..nops(x))}, F[i1]), i1 = 1 ..nops(F))];
        L := [op(L), x1]
    end do;
    L
end proc
```

```
> Orb*
Error, `;` unexpected
```

```
> #Make some random conditions for a
  rand_param := rand(0.01..100);

  print(`a`);
  a_rand := rand_param();
  b_rand := 2:
  c_rand := 2:
  while (b_rand + c_rand) > 1 do
  b_rand := rand_param();
  c_rand := rand_param();
```

```
od:
print(`b`);
b_rand;
print(`c`);
c_rand;
print(`b+c`);
print(b_rand+c_rand);
```

$rand\_param := () \mapsto RandomTools:-Generate(float('range' = 0.01 ..100, 'method' = 'uniform'))$

$$a$$

$$a\_rand := 50.93653700$$

$$b$$

$$0.03603940400$$

$$c$$

$$0.07587722142$$

$$b+c$$

$$0.1119166254 \quad \textbf{(17)}$$

NOW FOR OUR ORB COMMAND

```
> digits := 3;
```

$$digits := 3 \quad \textbf{(18)}$$

```
> Orb([(1-b_rand-`c_rand`)*x + y*(1-evalf(exp(-`a_rand`*x))),(1-y)*
  `b_rand` + y*(evalf(exp(-`a_rand`*x)))],[x,y],[1.,1.],10,20);
```

CONJECTURE 1: Prove if $R_0 = \dfrac{a}{b+c} > 1$ then the solution

CONJECTURE 2:
Prove if

Question 5: see worked out by hand

Question 6: The stable fixed point is NOT a global attractor because Global attractors require ALL starting points to eventually tend to the stable point.

Starting at x(-9) does not lead to the stable fixed point. (view images below)

Problem 5

Find the fixed Points (and stable fixed Points)

$$x(n) = \frac{x(n-1)}{10+x(n-1)}$$

UNDERLYING TRANSFORMATION

$$f(x) = \frac{x}{10+x}$$

Then, $f(x) = x$ is a stable fixed point

We see $f(0) = \frac{0}{10+0} = 0$ is a fixed Point

$(10+x)f(x) = x$

$(10+-9)-9 = -9$

$1(-9) = -9$ is also a fixed Point

wifi: light186

Now, To determine if The fixed points are stable, and if they are subsequently global attractors

Stability: Because Recurrence equation;

$|f'(x)| < 1 \Rightarrow$ Stable x

$$f'(x) = \frac{\frac{dx}{dx}(10+x) - x\frac{d(10+x)}{dx}}{(10+x)^2} = \frac{10+x-x}{(10+x)^2} = \frac{10}{(10+x)^2}$$

Therefore, $f'(0) = \frac{10}{(10+0)^2} = \frac{1}{10}$ and $\left|\frac{1}{10}\right| < 1 \Rightarrow f'(0)$ is a stable fixed Point

$f'(-9) = \frac{10}{(1)^2}$ which has absolute value $10 > 1 \Rightarrow f'(-9)$ is not a stable fixed point

Problem 5

wifi: light1

is $x=0$ a global Attractor for

$$x(n) = \frac{x(n-1)}{10+x(n-1)} \text{ ?}$$

Pick $2 = \frac{x}{10+x} \Leftrightarrow 20+2x = x$

$\Leftrightarrow 20 = -x$

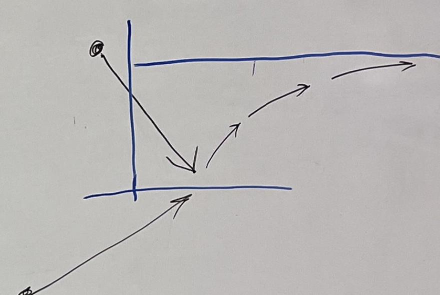
$\Rightarrow x = -20$

$$f(x) = \frac{x}{x+10}$$ find Limit via l'Hopital

$$\lim_{x\to\infty} f(x) = \frac{\lim_{x\to\infty} x'}{\lim_{x\to\infty}(x+10)'} = \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty} 1}$$

$f(x)=1$ ← will, for some values

☆ It appears that a $f(x)$ value that is greater tha[n] IS Likely Always



The Equation is phase-independent,
SO The long-term Behavior is solely dependent on the initial value of f.

Pick $-2 = \frac{x}{10+x}$

$-20 - 2x = x$

$-20 = 3x \Rightarrow x = \frac{-20}{3}$

2 possible Routes:

I Do not think $x_n = \frac{x(n-1)}{10+x(n-1)}$

where $X=0$ is a global Attractor because

values of $f(x)$ eventually tend to $f(x)=1$ as $X \to \infty$

Some

$X = \frac{0.1}{0.1+10} = \frac{0.1}{10.1}$ which is smaller than 0.1

But $1 = \frac{X}{X+10}$

$X+10 = X$

" IS Not an acceptable Initial Condition