Problem 1)

$a_n = 4a_{n-1} - 6a_{n-2} + 4a_{n-3} - a_{n-4}$, $a_0 = 0$, $a_1 = 1$
$a_2 = 8$, $a_3 = 27$

i) Find a_n for $1 \leq n \leq 8$

$a_1 = 1$
$a_2 = 8$
$a_3 = 27$
$a_4 = 4(27) - 6(8) + 4(1) - 0 = 64$
$a_5 = 4(56) - 6(27) + 4(8) - 1 = 93$
$a_6 = 4(93) - 6(56) + 4(27) - 8 = 136$
$a_7 = 4(136) - 6(93) + 4(56) - 27 = 183$
$a_8 = 4(183) - 6(136) + 4(93) - 56 = 232$

ii) explicit formula?

$a_n = n$

iii) Can you prove it?

Yes using maple we can use the expand function

So,

expand$(4(n-1) - 6(n-2) + 4(n-3) - (n-4))$

Maple then tells us that it is indeed

n

Problem 2)

$$\frac{dy}{dt} = \frac{y^3}{(t+1)}, \quad y(0)=1$$

$$\int \frac{dy}{y^3} = \int \frac{dt}{(t+1)}$$

$$-\frac{1}{2y^2} = \ln(t+1) + C$$

$$-\frac{1}{2(1)^2} = \ln(0+1) + C$$

$$-\frac{1}{2} = 0 + C$$

$$C = -1/2$$

$$-\frac{1}{2(y)^2} = \ln(t+1) - 1/2$$

$$y = \pm\sqrt{-\frac{1}{2\ln(1+t)-1}}$$

$$y = \sqrt{-\frac{1}{2\ln(1+t)-1}}$$

Problem 3)

$$y''(t) - 3y'(t) + 2y(t) = 0 \quad , \quad y(0) = 2 \ , \ y'(0) = 3$$

$$(e^{rt})'' - 3(e^{rt})' + 2(e^{rt}) = 0$$

$$r^2 e^{rt} - 3re^{rt} + 2e^{rt} = 0$$

$$e^{rt}(r^2 - 3r + 2) = 0$$

$$r = 2 \ , \ r = 1$$

$$y = K_1 e^{2t} + K_2 e^{t}$$

$$y' = 2K_1 e^{2t} + K_2 e^{t}$$

$$2 = K_1 + K_2$$

$$3 = 2K_1 + K_2$$

$$K_1 = 1 \quad K_2 = 1$$

so, $\boxed{y(t) = e^{2t} + e^{t}}$

Problem 4)

i) $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

$$\det \begin{vmatrix} 3-\lambda & -4 \\ 4 & 3-\lambda \end{vmatrix}$$

$$(3-\lambda)^2 + 16$$

$$\lambda^2 - 6\lambda + 25$$

$$\boxed{\begin{aligned} \lambda_1 &= 3+4i \\ \lambda_2 &= 3-4i \end{aligned}}$$

ii) $\lambda_1 = 3+4i$

$$\begin{vmatrix} 3-3+4i & -4 \\ 4 & 3-3+4i \end{vmatrix}$$

$$\left[\begin{array}{cc|c} 4i & -4 & 0 \\ 4 & 4i & 0 \end{array}\right]$$

$$4ix_1 - 4x_2 = 0$$
$$4x_1 + 4ix_2 = 0$$
$$ix_1 - x_2 = 0$$
$$x_1 + ix_2 = 0$$
$$x_1 = -ix_2$$

$$i(-ix_2) - x_2 = 0$$
$$x_2 - x_2 = 0$$
$$0 = 0 \quad \text{free variable.}$$

so, $V_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_1 = \quad 3-4i$

$$\left[\begin{array}{cc|c} -4i & -4 & 0 \\ 4 & -4i & 0 \end{array}\right]$$

$-4i_{x_1} - 4_{x_2} = 0$

$4_{x_1} - 4i x_2 = 0$

$i x_1 - x_2 = 0$

$x_1 - i x_2 = 0$

$1 x_1 = x_2$

$x_1 - i(1 x_1) = 0$

$0 = 0$

free

$$V_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$