Ok to post

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1) $x'(t) = 3x(t) - y(t)$

$y'(t) = 2x(t)$ $\quad x(0)=2$, $y(0)=3$

(i) $x''(t) = 3x'(t) - y'(t)$

$\Rightarrow x''(t) = 3x'(t) - 2x(t)$

$x''(t) - 3x'(t) + 2x(t) = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$\lambda = 2, 1$

$x(t) = c_1 e^{2t} + c_2 e^{t}$

$x'(0) = 3x(0) - y(0)$

$$x'(0) = 6 - 3 = 3$$

$$x(0) = c_1 + c_2 = 2$$

$$x'(0) = 2c_1 + c_2 = 3$$

$$c_1, c_2 = 1$$

$$\boxed{x(t) = e^{2t} + e^{t}}$$

$$y(t) = 3e^{2t} + 3e^{t} - 2e^{2t} - e^{t}$$

$$= e^{2t} + 2e^{t}$$

(ii) $$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$(3-\lambda)(-\lambda) + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1, 2$$

$\lambda = 1$

$$\begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2v_1 - v_2 = 0$$

$$v_2 = 2v_1$$

$$v_a = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda = 2$

$$\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = v_2$$

$$v_b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(t) = A\begin{bmatrix}1\\2\end{bmatrix}e^{t} + B\begin{bmatrix}1\\1\end{bmatrix}e^{2t}$$

$$x(0) = \begin{bmatrix}A+B\\2A+B\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$$

$$A+B=2 \quad \Rightarrow \quad B = 2-A$$

$$2A+B=3 \quad \Rightarrow \quad 2A+2-A=3$$

$$A=1, \quad B=1$$

$$x(t) = e^{t} + e^{2t}$$

$$y(t) = 2e^{t} + e^{2t}$$

(iii) see maple code

2) $x'(t) = x(t) + 9y(t)$

$y'(t) = 3x(t)$

$x(0) = 4$
$y(0) = 0$

(i) $x''(t) = x'(t) + 9y'(t)$

$\Rightarrow x''(t) - x'(t) - 27x(t) = 0$

$\lambda^2 - \lambda - 27 = 0$

$\frac{1 \pm \sqrt{1 + 4(27)}}{2} = \frac{1}{2} \pm \frac{\sqrt{109}}{2}$

$x(t) = c_1 e^{\frac{1}{2} + \frac{\sqrt{109}}{2} t} + c_2 e^{\frac{1}{2} - \frac{\sqrt{109}}{2} t}$

$c_1 + c_2 = 4$

$\frac{\left(-\frac{1}{2} + \frac{\sqrt{109}}{2}\right)}{9} c_1 + \frac{\left(-\frac{1}{2} - \frac{\sqrt{109}}{2}\right)}{9} c_2 = 0$

Using maple:

$$c_1 = 2 + \frac{2\sqrt{109}}{109}, \quad c_2 = 2 - \frac{2\sqrt{109}}{109}$$

$$x(t) = \left(2 + \frac{2\sqrt{109}}{109}\right) e^{\frac{1}{2}+\frac{\sqrt{109}}{2}t} + \left(2 - \frac{2\sqrt{109}}{109}\right) e^{\frac{1}{2}-\frac{\sqrt{109}}{2}t}$$

$$y(t) = \frac{x'(t) - x(t)}{9}$$

(Using maple)

$$y(t) = 12\,\frac{\left(e^{\left(\frac{1+\sqrt{109}}{2}\right)t} - e^{\left(\frac{1+\sqrt{109}}{2}\right)t}\right)\sqrt{109}}{109}$$

(ii) $x'(t) = x(t) + 9y(t)$

$y'(t) = 3x(t)$

$x(0) = 4$
$y(0) = 0$

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$(1-\lambda)(-\lambda) - 27 = 0$$

$$\lambda^2 - \lambda - 27 = 0$$

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{109}}{2}$$

$$\lambda = \frac{1}{2} + \frac{\sqrt{109}}{2}$$

Using maple $\Rightarrow$ $v_a = \begin{bmatrix} 9 \\ -\frac{1}{2} + \frac{\sqrt{109}}{2} \\ 1 \end{bmatrix}$

$$\lambda = \tfrac{1}{2} - \frac{\sqrt{109}}{2}$$

Using maple $\Rightarrow$ $v_b = \begin{bmatrix} a \\ -\frac{1}{2} - \frac{\sqrt{109}}{2} \\ 1 \end{bmatrix}$

$$x(t) = A\begin{bmatrix} a \\ -\frac{1}{2} + \frac{\sqrt{109}}{2} \\ 1 \end{bmatrix} e^{\left(\frac{1}{2} + \frac{\sqrt{109}}{2}\right)t} + B\begin{bmatrix} a \\ -\frac{1}{2} - \frac{\sqrt{109}}{2} \\ 1 \end{bmatrix} e^{\left(\frac{1}{2} - \frac{\sqrt{109}}{2}\right)t}$$

$$x(0) = \begin{bmatrix} \left(\frac{a}{-\frac{1}{2} + \frac{\sqrt{109}}{2}}\right)A + \left(\frac{a}{-\frac{1}{2} - \frac{\sqrt{109}}{2}}\right)B \\ A + B \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Using maple $\Rightarrow$ $A = \frac{12\sqrt{109}}{109}$, $B = -\frac{12\sqrt{109}}{109}$

$$x(t) = \left(\frac{12\sqrt{109}}{109}\right)\left(\frac{a}{-\frac{1}{2} + \frac{\sqrt{109}}{2}}\right) e^{\left(\frac{1}{2} + \frac{\sqrt{109}}{2}\right)t} + \left(\frac{12\sqrt{109}}{109}\right)\left(\frac{a}{-\frac{1}{2} - \frac{\sqrt{109}}{2}}\right) e^{\left(\frac{1}{2} - \frac{\sqrt{109}}{2}\right)t}$$

$$y(t) = \frac{12\sqrt{109}}{109} e^{(1/2 + \frac{\sqrt{109}}{2})t} + \frac{-12\sqrt{109}}{109} e^{(1/2 - \frac{\sqrt{109}}{2})t}$$

(I checked w maple that this is equivalent to the result in (i))

(iii) see maple code

3) $$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

see maple code.