> *#OK to post*
> *#Julian Herman, November 1st, 2021, Assignment 16*

> **read** *`/Users/julianherman/Documents/Rutgers/Fall 2021/Dynamical Models In Biology/HW/M15.txt`*

>

> *#1)a)*

> $Help11(\ )$

SFPe(f,x), Orbk(k,z,f,INI,K1,K2) **(1)**

> $Orbk\left(2, z, z[1]\cdot\left(\frac{5}{3} - z[2]\right), [0.5, 0.5], 1000, 1020\right)$

[0.6666666666, 0.6666666666] **(2)**

> $Orbk\left(2, z, z[1]\cdot\left(\frac{5}{3} - z[2]\right), [0.7, 0.8], 1000, 1020\right)$

[0.6666666668, 0.6666666668] **(3)**

> *#$\frac{2}{3}$ is a STABLE equilibrium point*
> $Orbk\left(2, z, z[1]\cdot\left(\frac{5}{3} - z[2]\right), [0, 0], 1000, 1020\right)$

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] **(4)**

> $Orbk\left(2, z, z[1]\cdot\left(\frac{5}{3} - z[2]\right), [0.1, 0.1], 1000, 1020\right)$

[0.6666666667, 0.6666666667] **(5)**

> *#0 is an UNSTABLE equilibrium point*

> *#b)*

> $Orbk(2, z, z[1]\cdot(2 - z[2]), [0, 0], 1000, 1020)$

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] **(6)**

> $Orbk(2, z, z[1]\cdot(2 - z[2]), [0.1, 0.1], 1000, 1020)$

[1.041271875, 1.060759433, 1.016979902, 0.9551887798, 0.9389697679, 0.9810461487, 1.040919623, 1.060649058, 1.017247698, 0.9555525834, 0.9390715010, 0.9808108036, 1.040570134, 1.060537838, 1.017511676, 0.9559137190, 0.9391740677, 0.9805787596, 1.040223377, 1.060425805, 1.017771898] **(7)**

> *#0 is unstable*

> $Orbk(2, z, z[1]\cdot(2 - z[2]), [1, 1], 1000, 1020)$

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \quad (8)$$

> $Orbk(2, z, z[1]\cdot(2 - z[2]), [1.0000001, .99999], 1000, 1020)$

[0.9999998999, 1.000010000, 1.000010100, 1.000000100, 0.9999900000, 0.9999899000, 0.9999998999, 1.000010000, 1.000010100, 1.000000100, 0.9999900000, 0.9999899000, 0.9999998999, 1.000010000, 1.000010100, 1.000000100, 0.9999900000, 0.9999899000, 0.9999998999, 1.000010000, 1.000010100] **(9)**

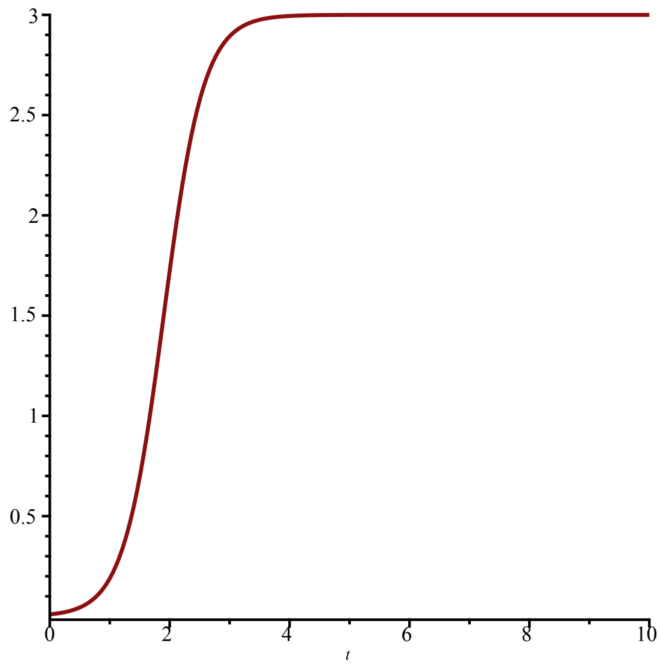
> $convert(\%, set)$

{0.9999899000, 0.9999900000, 0.9999998999, 1.000000100, 1.000010000, 1.000010100} **(10)**

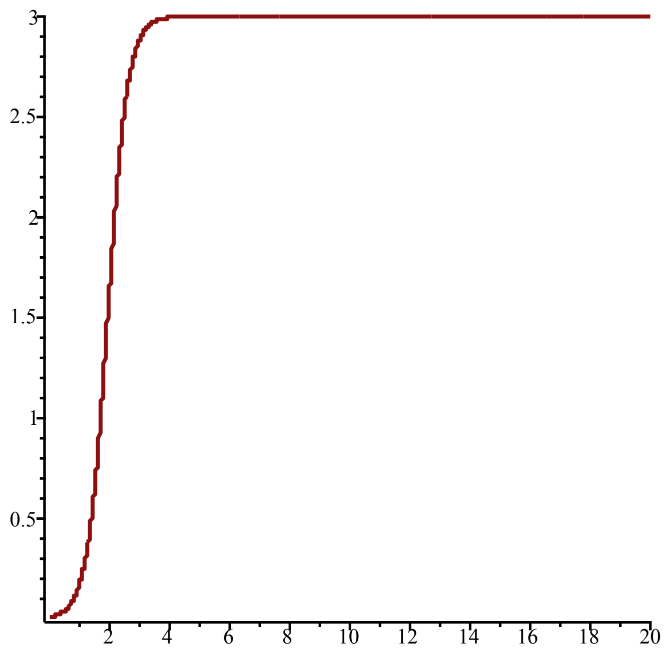
> *#Oscillates, 1 is unstable as well as it is an edge case for the condition: the |eigenvalues| are =1*

> *#4) example)*

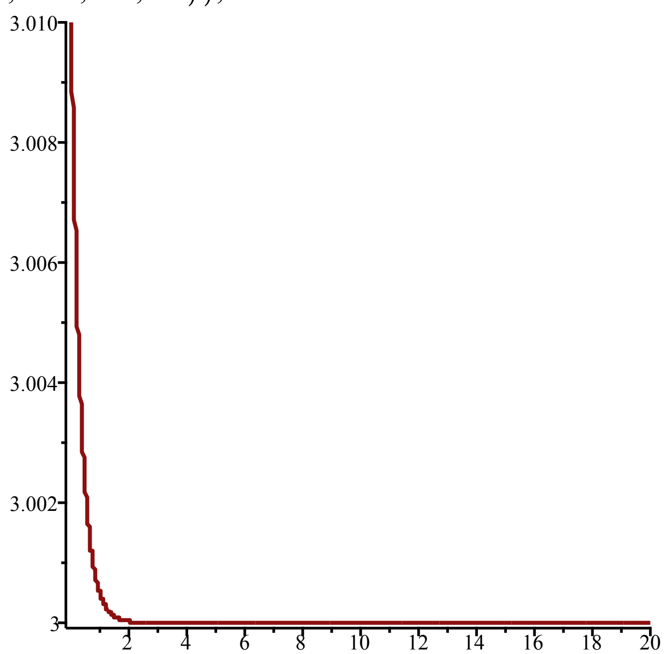
> $plot(op(2, dsolve(\{diff(x(t), t) = x(t) * (3-x(t)), x(0) = 0.01\}, x(t))), t = 0..10)$



> $plot(Dis1(y * (3-y), y, 0.01, 0.01, 20));$



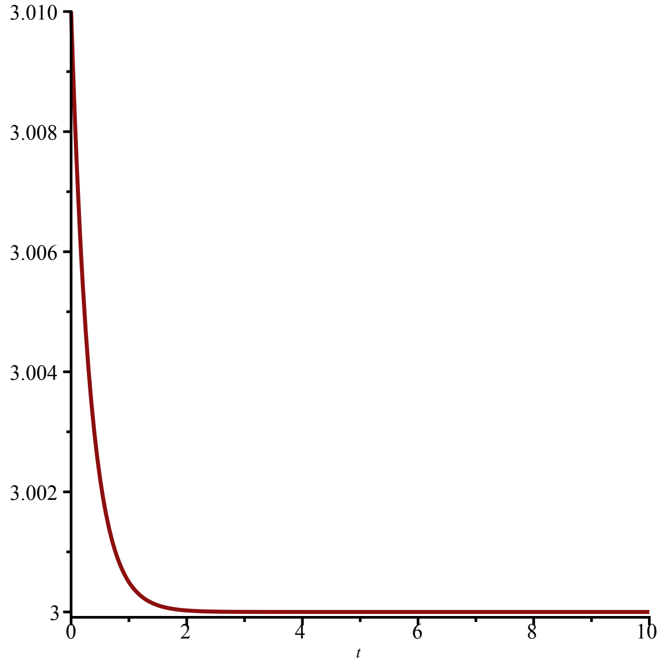
> $plot(Dis1(y*(3-y), y, 3.01, .01, 20));$



> $Help15(\,)$

HW3(u,v,w), HW2(u,v) , Dis1(F,y,y0,h,A), ToSys(k,z,f,INI) **(11)**

> $plot(op(2, dsolve(\{diff(x(t), t) = x(t)*(3-x(t)), x(0) = 3.01\}, x(t))), t = 0..10)$



> #a)ii)

> $F := diff(x \cdot (3 - x) \cdot (5 - x), x)$:

> $subs(x = 0, F)$

$$15 \qquad (12)$$

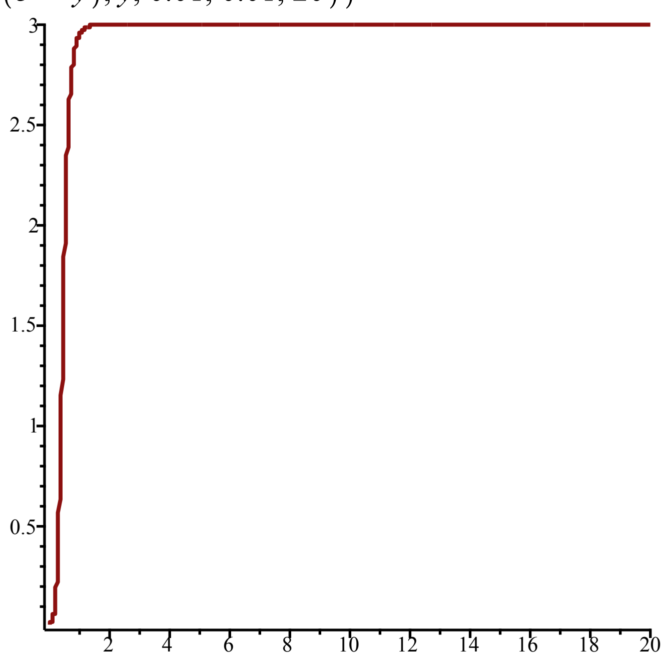
> $subs(x = 3, F)$

$$-6 \qquad (13)$$

> $subs(x = 5, F)$

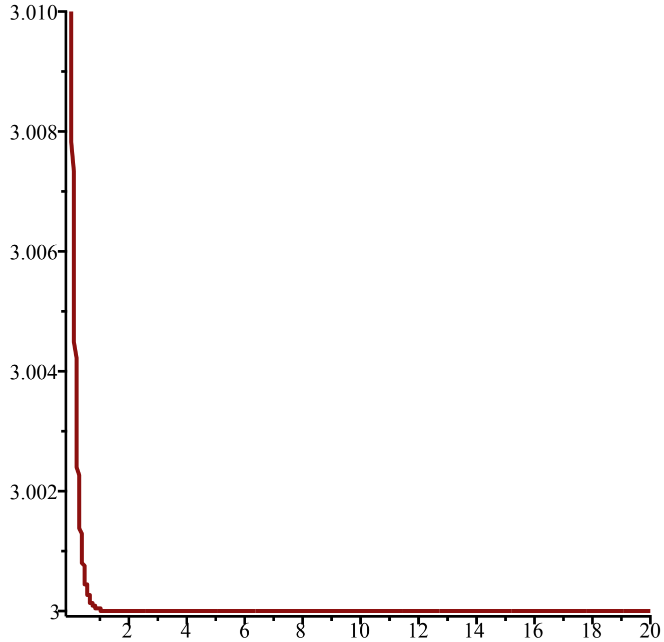
$$10 \qquad (14)$$

> $plot(Dis1(y \cdot (3 - y) \cdot (5 - y), y, 0.01, 0.01, 20))$



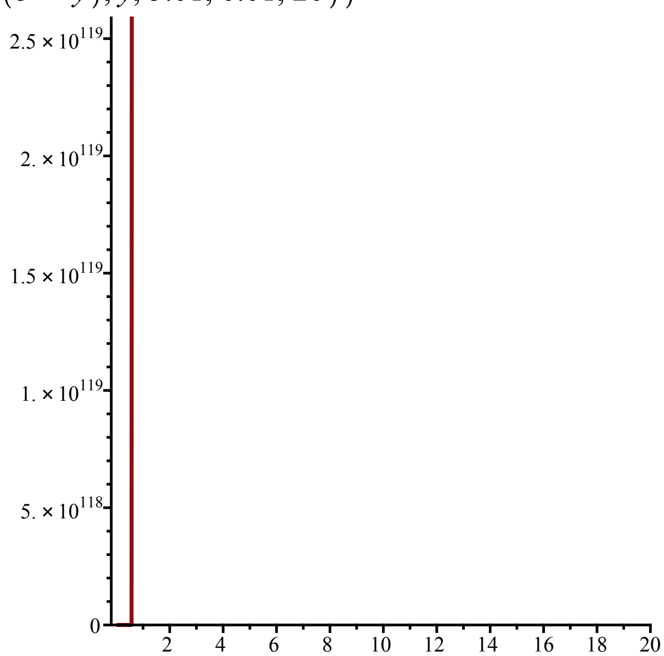
> #x=0 is UNSTABLE because starting at 0.01, it goes to x=3

> $plot(Dis1(y \cdot (3 - y) \cdot (5 - y), y, 3.01, 0.01, 20))$



> #x=3 is STABLE because starting at 3.01, it goes back to x=3

> $plot(Dis1(y \cdot (3 - y) \cdot (5 - y), y, 5.01, 0.01, 20))$



> #x=5 is UNSTABLE because starting at 5.01, it goes to x=infinity

> #b)ii)

> $F := diff(x^2 \cdot (3 - x) \cdot (5 - x) \cdot (7 - x), x) :$

> $subs(x = 0, F)$

$$0 \quad \textbf{(15)}$$

> $subs(x = 3, F)$

$$-72 \quad \textbf{(16)}$$

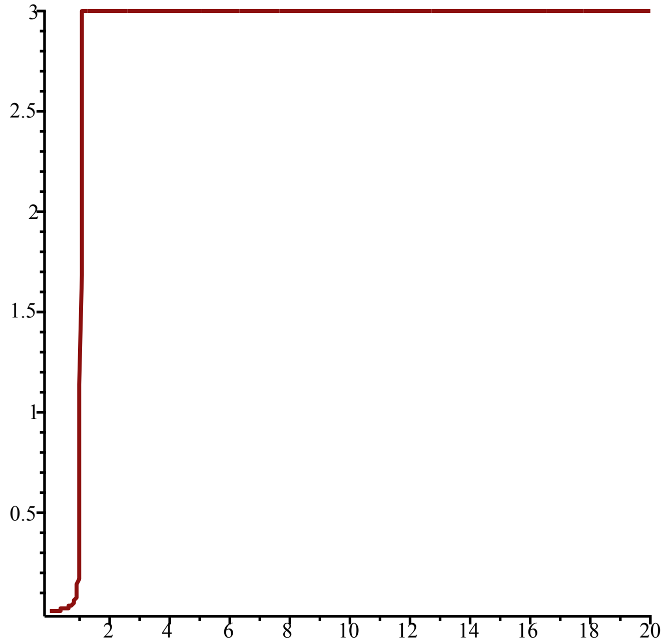
> $subs(x = 5, F)$

$$100 \quad \textbf{(17)}$$

> $subs(x = 7, F)$

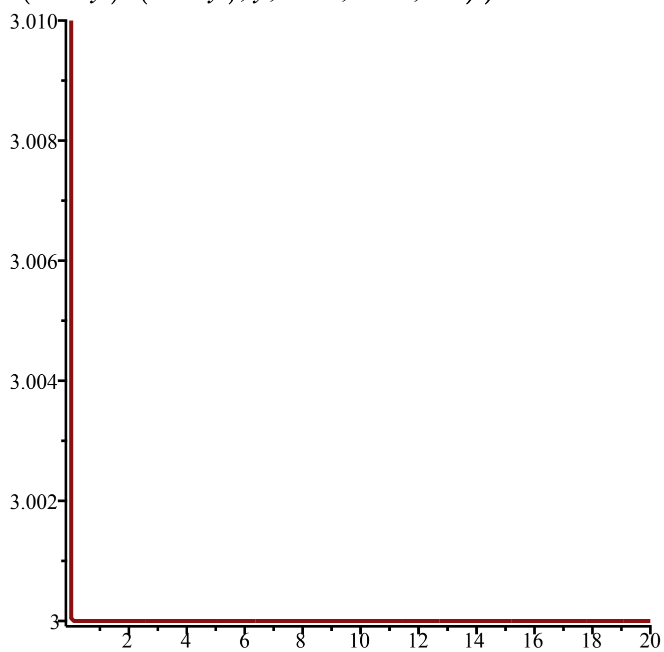
$$-392 \tag{18}$$

> $plot\left(Dis1\left(y^2 \cdot (3-y) \cdot (5-y) \cdot (7-y), y, 0.01, 0.01, 20\right)\right)$



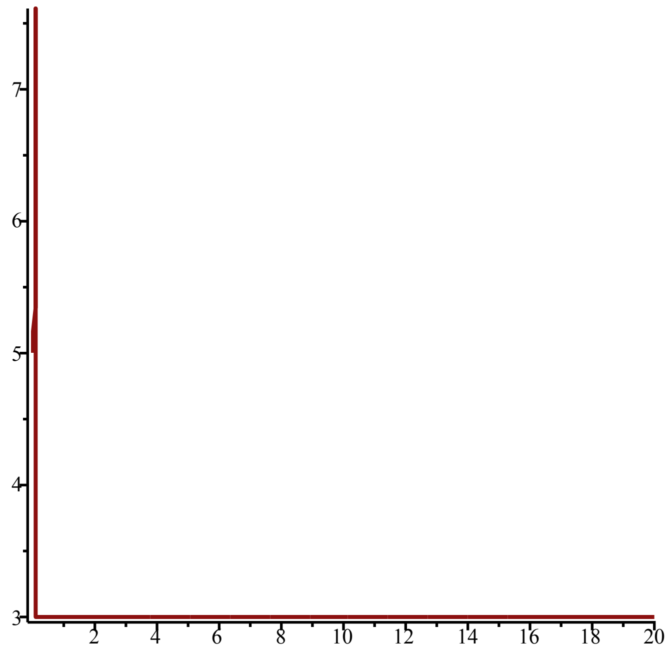
> #x=0 is UNSTABLE because starting at 0.01, it goes to x=3

> $plot\left(Dis1\left(y^2 \cdot (3-y) \cdot (5-y) \cdot (7-y), y, 3.01, 0.01, 20\right)\right)$



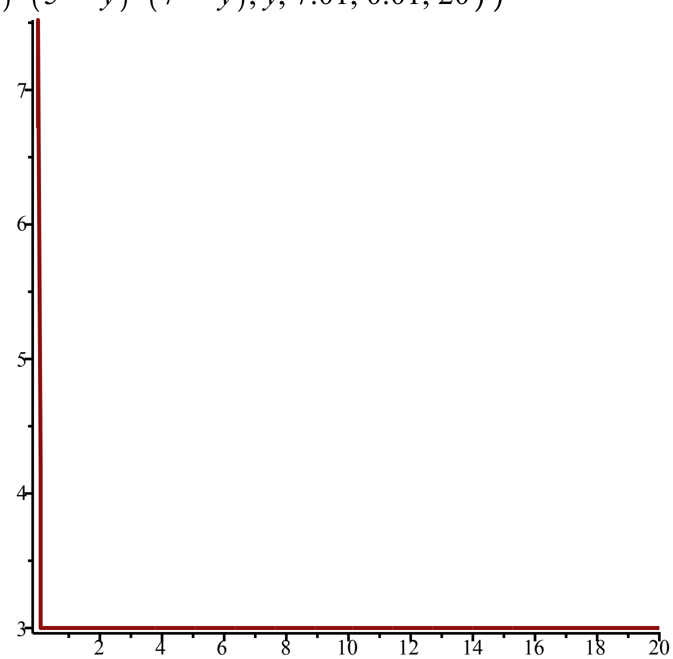
> #x=3 is STABLE because starting at 3.01, it goes back to x=3

> $plot\left(Dis1\left(y^2 \cdot (3-y) \cdot (5-y) \cdot (7-y), y, 5.01, 0.01, 20\right)\right)$

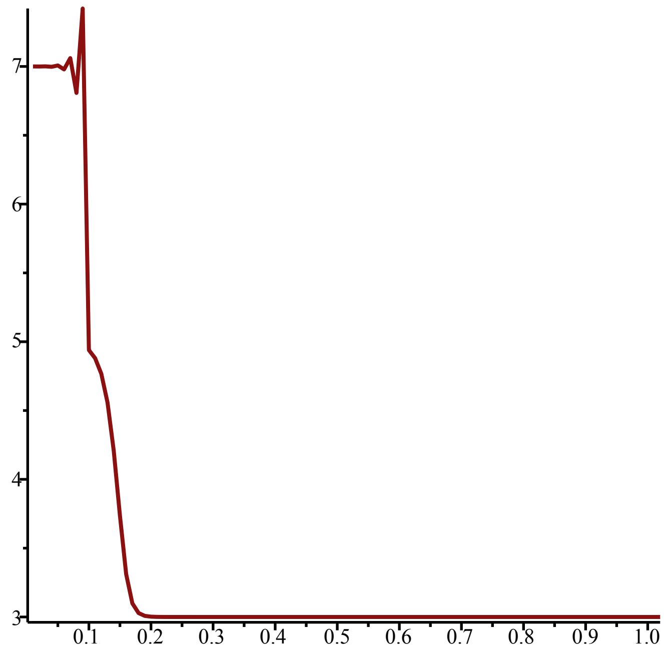


> #x=5 is UNSTABLE because starting at 5.01, it spikes up > 7 and ultimately goes to x=3

> $plot\left(Dis1\left(y^2 \cdot (3-y) \cdot (5-y) \cdot (7-y), y, 7.01, 0.01, 20\right)\right)$



> $plot\left(Dis1\left(y^2 \cdot (3-y) \cdot (5-y) \cdot (7-y), y, 7.0001, 0.01, 1\right)\right)$



> *#x=7 is STABLE (locally)*

> *#x=3 appears to be a global equilibrium point*

>

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Julian Herman, 11/1/21, Assignment 16

1.) a) $x(n) = x(n-1)\left(\frac{5}{3} - x(n-2)\right)$

Let: $x_1(n) = x(n)$

$x_2(n) = x_1(n-1)$

Then: $\begin{cases} x_1(n) = x_1(n-1)\left(\frac{5}{3} - x_2(n-1)\right) \\ x_2(n) = x_1(n-1) \end{cases}$

Solving for eq. points $(x_1(n) = x_2(n) = x)$:

$$x = x\left(\frac{5}{3} - x\right)$$

$$x^2 - \frac{2}{3}x = 0$$

$$x\left(x - \frac{2}{3}\right) = 0$$

$$\boxed{\text{EQ. pts: } x = 0, \quad x = \frac{2}{3}}$$

CHECK STABILITY:

$$[x_1(n), x_2(n)] \to \left[x_1(n-1)\left(\frac{5}{3} - x_2(n-1)\right), x_1(n-1)\right]$$

REPLACE $x_1(n) \to z_1$, $x_2(n) \to z_2$

$$[z_1, z_2] \to \left[z_1\left(\frac{5}{3} - z_2\right), z_1\right]$$

Let $f(z_1, z_2) = z_1\left(\frac{5}{3} - z_2\right)$, $g(z_1, z_2) = z_1$

$$J = \begin{bmatrix} \frac{\partial f}{\partial z_1} , & \frac{\partial f}{\partial z_2} \\ \frac{\partial g}{\partial z_1} , & \frac{\partial g}{\partial z_2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} - z_2 , & -z_1 \\ 1 , & 0 \end{bmatrix}$$

Point $(z_1, z_2) = (0, 0)$:

$$J = \begin{bmatrix} 5/3 , & 0 \\ 1 , & 0 \end{bmatrix} \qquad \det\left(\begin{bmatrix} 5/3 - \lambda , & 0 \\ 1 , & -\lambda \end{bmatrix}\right) = 0$$

$$-\frac{5}{3}\lambda + \lambda^2 = 0 \qquad |\lambda_1| < 0 \ \checkmark$$

$$\lambda\left(\lambda - \frac{5}{3}\right) = 0 \qquad |\lambda_2| \not< 0 \ \times$$

$$\lambda_1 = 0, \quad \lambda_2 = 5/3$$

$$\boxed{\Rightarrow (0,0) \text{ is an UNSTABLE EQ. PT.}}$$

Point $\left(\frac{2}{3}, \frac{2}{3}\right)$:

$$J = \begin{bmatrix} 1 , & -\frac{2}{3} \\ 1 , & 0 \end{bmatrix} \qquad \det\left(\begin{bmatrix} 1 - \lambda , & -\frac{2}{3} \\ 1 , & -\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 - \lambda + \frac{2}{3} = 0$$

$$\lambda = \frac{1 \pm \sqrt{1 - 8/3}}{2} = \frac{1}{2} \pm \frac{i\sqrt{5/3}}{2}$$

$$|\lambda_1| = |\lambda_2| = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{5/3}}{2}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{5}{12}} = \sqrt{\tfrac{2}{3}}$$

$$\Rightarrow \boxed{\sqrt{\tfrac{2}{3}} = 0.8164 < 1 \Rightarrow \left(\tfrac{2}{3}, \tfrac{2}{3}\right) \text{ is STABLE}}$$

$\Rightarrow$ $x = 0$ is UNSTABLE, $x = \frac{2}{3}$ is STABLE

b) $x(n) = x(n-1)\left(2 - x(n-2)\right)$

Let: $x_1(n) = x(n)$
$x_2(n) = x_1(n-1)$

Then:
$$\begin{cases} x_1(n) = x_1(n-1)\left(2 - x_2(n-1)\right) \\ x_2(n) = x_1(n-1) \end{cases}$$

Eq. pt's: $x = x(2-x)$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$\boxed{x = 0, \quad x = 1}$$

Let $x_1 = z_1$, $x_2 = z_2$:

$$(z_1, z_2) \to \left(z_1(2 - z_2), z_1\right)$$

$$J = \begin{bmatrix} 2-z_2, & -z_1 \\ 1, & 0 \end{bmatrix}$$

at pt: $(z_1, z_2) = (0,0)$ $\quad \det\left(\begin{bmatrix} 2-\lambda, & 0 \\ 1, & -\lambda \end{bmatrix}\right) = 0$

$\lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda_1 = 0, \quad \lambda_2 = 2$

$|\lambda_1| = 0 < 1$ ✓

$|\lambda_2| = 2 \not< 1$ X

(0,0) is UNSTABLE

at pt: $(1,1)$ $\quad \det\left(\begin{bmatrix} 1-\lambda, & -1 \\ 1, & -\lambda \end{bmatrix}\right) = 0$

$\lambda^2 - \lambda + 1 = 0$

$\lambda = \dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$

$|\lambda_1| = |\lambda_2| = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$ edge case

(1,1) is UNSTABLE, $|1| = 1 \not< 1$ X

$\Rightarrow$ $\boxed{0, 1 \text{ are both UNSTABLE EQ. PT'S}}$

2) $x(n) = x(n-1)\,(a - x(n-2))$

EQ PT's: $x = x\,(a-x)$

$$x^2 + (1-a)\,x = 0$$

$$x\,(x + (1-a)) = 0$$

$$x = 0, \quad x = -(1-a) = (a-1)$$

$$\boxed{\text{EQ. PT's: } x = 0, \quad x = a-1}$$

For $x=0$ to be stable:

$$J = \begin{bmatrix} a - z_2, & -z_1 \\ 1, & 0 \end{bmatrix} \qquad \det\left(\begin{bmatrix} a-\lambda & 0 \\ 1, & -\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 - a\lambda = 0$$

$$\lambda(\lambda - a) = 0$$

$$\lambda_1 = 0, \quad \lambda_2 = a$$

$$\boxed{\begin{array}{l}\text{For } x=0 \text{ to be stable:} \\ |a| < 1\end{array}}$$

For $x = (a-1)$ to be stable:

$$J = \begin{bmatrix} a-(a-1), & -(a-1) \\ 1, & 0 \end{bmatrix} \qquad \det\left(\begin{bmatrix} 1-\lambda, & (1-a) \\ 1, & -\lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 - \lambda - (1-a) = 0$$

$$\lambda^2 - \lambda + (a-1) = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4(a-1)}}{2} = \frac{1}{2} \pm \frac{\sqrt{5-4a}}{2}$$

1^{st} CASE: $(5-4a) < 0 \rightarrow 5 < 4a \rightarrow a > \frac{5}{4}$:

$$|\lambda_1| = |\lambda_2| = \sqrt{\frac{1}{4} + \frac{|5-4a|}{4}} < 1$$

$$\frac{1}{4} + \frac{|5-4a|}{4} < 1$$

$$|5-4a| < 3$$

$5-4a < 3$ $\qquad$ $5-4a > -3$

$-4a < -2$ $\qquad$ $-4a > -8$

$a > \frac{1}{2}$ $\qquad$ $a < 2$

$\Rightarrow$ IF $a > \frac{5}{4}$, it is stable when: $\frac{1}{2} < a < 2$, but "a" must be greater than $\frac{5}{4}$ so: $\boxed{\frac{5}{4} < a < 2}$

2^{nd} CASE: $(5-4a) > 0 \rightarrow 5 > 4a \rightarrow a < \frac{5}{4}$:

$$\left|\frac{1}{2} + \frac{\sqrt{5-4a}}{2}\right| < 1 \quad \text{AND} \quad \left|\frac{1}{2} - \frac{\sqrt{5-4a}}{2}\right| < 1$$

$\frac{1}{2} + \frac{\sqrt{5-4a}}{2} < 1$ $\qquad$ $\frac{1}{2} - \frac{\sqrt{5-4a}}{2} < 1$

$\sqrt{5-4a} < 1$ $\qquad$ $-\sqrt{5-4a} < 1$

$5-4a < 1$ $\qquad$ SAME $\qquad$ $5-4a < 1$

$-4a < -4$

$a > 1$ $\rightarrow$ $x = (a-1)$ is stable for $1 < a < \frac{5}{4}$

$\boxed{1 < a < \frac{5}{4}}$ $\qquad$ $\frac{5}{4} < a < 2$

$= 1 < a < 2$

For $x = 0$ to be stable: $|a| < 1$

$x = (a-1)$ to be stable: $1 < a < 2$

4.) a.) i) $F(x) = x(3-x)(5-x)$

$x(3-x)(5-x) = 0$

EQ. Pt's: $x = 0, \quad x = 3, \quad x = 5$

ii) $F'(0) = 15 \Rightarrow x = 0$ UNSTABLE, $15 \nless 0$

$F'(3) = -6 \Rightarrow x = 3$ is STABLE, $-6 < 0$

$F'(5) = 10 \Rightarrow x = 5$ is UNSTABLE, $10 \nless 0$

b) i) $F(x) = x^2(3-x)(5-x)(7-x)$

$$x^2(3-x)(5-x)(7-x) = 0$$

EQ Pts: $x=0$, $x=3$, $x=5$, $x=7$

ii) $F'(0) = 0 \Rightarrow x=0$ is UNSTABLE

$F'(3) = -72 \Rightarrow x=3$ is STABLE

$F'(5) = 100 \Rightarrow x=5$ is UNSTABLE

$F'(7) = -392 \Rightarrow x=7$ is STABLE