Homework 16

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#1a $x(n) = x(n-1)\left(\frac{5}{3} - x(n-2)\right) \rightarrow$ $x(n) = x(n-1)(5/3 - y(n-1))$
$y(n) = x(n-1)$

$x = f(x,y) = x\left(\frac{5}{3} - y\right)$
$y = g(x,y) = x$ $y = x$ $x = 0$ and $y = 0$ or $y = 2/3$ $x = 2/3$

$J_a = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} - y & -x \\ 1 & 0 \end{bmatrix}$ FP's $= (0,0)$ and $(2/3, 2/3)$

$J_{a(0,0)} = \begin{bmatrix} 5/3 & 0 \\ 1 & 0 \end{bmatrix} \rightarrow \lambda_1 = 0$ & $\lambda_2 = 5/3$ since $|\lambda_2| > 1$, $(0,0)$ is not a stable fixed point

$J_{a\left(\frac{2}{3},\frac{2}{3}\right)}$ $\begin{bmatrix} 1 & -2/3 \\ 1 & 0 \end{bmatrix} \rightarrow (-\lambda)\cdot(1-\lambda) + 2/3$ $\lambda^2 - \lambda + 2/3$

$\lambda = \frac{1 \pm \sqrt{1 - 8/3}}{2} \rightarrow \frac{1}{2} \pm \frac{\sqrt{5/3}}{2} i$

$abs(\lambda) = \left(\frac{1}{4} + \frac{5/3}{4}\right)^{1/2} \rightarrow \left(\frac{5}{12} + \frac{3}{12}\right)^{1/2} \leq \left(\frac{8}{12}\right)^{1/2} < 1$ Stable!

$(2/3, 2/3)$ is the only stable fixed point of the given 2-D linear difference equation.

Note: When going from a matrix to a set of lambdas, I set the Matrix $- I_n \cdot \lambda = 0$ and solved for the determinant of the leftside.

1b $x(n) = x(n-1)(2 - x(n-2))$

$\downarrow$ $x(n) = x(n-1)(2 - y(n-1))$ $\qquad J = \begin{bmatrix} 2-y & -x \\ 1 & 0 \end{bmatrix}$

$y(n) = x(n-1)$

$f(x,y) = 2x - xy \quad g(x,y) = y \wedge f(x,y) = x \rightarrow \begin{matrix} x = 0 \\ y = 0 \end{matrix} \text{ or } \begin{matrix} y = 1 \\ x = 1 \end{matrix} \quad FP = (0,0) \wedge (1,1)$

$g(x,y) = x$

$J_{(0,0)} = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \quad \lambda_1 = 0 \ \& \ \lambda_2 = 2$ since $|\lambda_2| > 1$, $(0,0)$ is not a stable FP

$J_{(1,1)} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \rightarrow (-\lambda)(1-\lambda) + 1 \rightarrow \lambda^2 - \lambda + 1$

$\lambda = \frac{1 \pm \sqrt{1-4}}{2} \rightarrow \frac{1 \pm \sqrt{3}i}{2} \rightarrow \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$abs(\lambda) = \left(\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right)^{1/2} \rightarrow \left(\frac{1}{4} + \frac{3}{4}\right)^{1/2} = 1 \not< 1$

Since $|\lambda|$ is not less than 1, $(1,1)$ is also not stable.

Therefore, this difference equation has no stable fixed points

2. $x(n) = x(n-1)(a - x(n-2))$

$\downarrow$

$x(n) = x(n-1)(a - y(n-1))$ $\quad$ $f(x,y) = xa - yx$

$y(n) = x(n-1)$ $\quad$ $g(x,y) = x$

FP's $\rightarrow$ $f(x,y) = xa - yx = x \rightarrow x=0$ or $y = a-1$

$g(x,y) = x = y \rightarrow x = y$

$\downarrow$

$(0,0)$ & $(a-1, a-1)$ are Fixed points

Jacobian $= \begin{bmatrix} a-y & -x \\ 1 & 0 \end{bmatrix}$

At

$(0,0)$ $J = \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix} \rightarrow (-\lambda)(a-\lambda) = 0$ $\quad$ $\lambda_1 = a$ & $\lambda_2 = 0$

$(0,0)$ will be stable whenever $0 \le a < 1$, since we assume a to be positive and $|\lambda_1| < 1$ whenever a is less than 1.

At

$(a-1, a-1)$ $\quad$ $J = \begin{bmatrix} 1 & 1-a \\ 1 & 0 \end{bmatrix} \rightarrow (-\lambda)(1-\lambda) - (1-a) \rightarrow \lambda^2 - \lambda + (a-1)$

$$\lambda = \frac{1 \pm \sqrt{1-4a+4}}{2} = \frac{1}{2} \pm \frac{\sqrt{5-4a}}{2}$$

if $a > 1.25$, the eigenvalue is imaginary. Otherwise, it is real.

When $a = 1.25$

$$\lambda = \frac{1}{2} \pm \frac{\sqrt{0}}{2}, = \frac{1}{2} \text{ multiplicity } 2.$$

Since $|\lambda| < 1$, it seems that for $a = 1.25$, $(a-1, a-1)$ is stable.

When $a > 1.25$, λ is imaginary so we use the $abs(\lambda)$

$$abs(\lambda) = \left(\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{5-4a}}{2}\right)^2\right)^{1/2} \rightarrow \left(\frac{1}{4} + \frac{5-4a}{4}\right)^{1/2} \rightarrow \left(\frac{6-4a}{4}\right)^{1/2}$$

Since $a > 1.25$, I tried $a = 2$

or $a=2$ $\quad \lambda = \left(\frac{1}{4} + \frac{3}{4}\right) = 1 \quad$ $a=2$ is unstable since $\lambda = 1$

or $a=1.99$ $\quad \lambda = \left(\frac{1}{4} + \frac{2.97}{4}\right) = \frac{3.97}{4} < 1 \quad$ $a < 2$ is stable

So for all $1.25 < a < 2$, $(a-1, a-1)$ is stable. and $a = 1.25$ is stable

For $\quad a < 1.25$ eigenvalues are real

Eigenvalues: $\lambda_1 = \frac{1}{2} + \frac{\sqrt{5-4a}}{2} \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{5-4a}}{2}$

$$\left|\frac{1}{2} + \frac{\sqrt{5-4a}}{2}\right| < 1 \longrightarrow \left(\frac{1}{2} + \frac{\sqrt{5-4a}}{2} > -1\right) \wedge \left(\frac{1}{2} + \frac{\sqrt{5-4a}}{2} < 1\right)$$

$$a \le 5/4 \qquad 1 < a \le 5/4$$

Stable for $a \le 5/4$ and $a > 1$

$$\left|\frac{1}{2} - \frac{\sqrt{5-4a}}{2}\right| > -1 \longrightarrow \left(-1 < \frac{1}{2} - \frac{\sqrt{5-4a}}{2}\right) \wedge \left(1 > \frac{1}{2} - \frac{\sqrt{5-4a}}{2}\right)$$

$$-1 < a \le \frac{5}{4} \qquad a \le 5/4$$

Stable for: $-1 < a \le 5/4$

Combining the two, $1 < a \le 1.25$

Combining all 3, $(a-1, a-1)$ is stable for $1 < a < 2$

Question 3 optional

$x(n) = x(n-1)(a - x(n-2))(b - x(n-3))$

$x(n) = x(n-1)(a - y(n-1))(b - z(n-1))$ $\quad f(x,y,z) = x(a-y)(b-z)$

$y(n) = x(n-1)$ $\quad g(x,y,z) = x$

$z(n) = y(n-1)$ $\quad h(x,y,z) = y$

$x = x(a-y)(b-z)$ $\quad$ Let $y=z$

$x=y$, $y=z$ $> x=y=z$ $\quad 1 = (a-\cancel{y})(b-z) \Rightarrow 1 = ab - (b+a)z + z^2$

$x=y=z = \dfrac{b+a \pm \sqrt{a^2-2ab+b^2+4}}{2}$

FP's $\rightarrow (0,0,0)$ and $\left(\dfrac{b+a+\sqrt{a^2-2ab+b^2+4}}{2} = x,y,z\right)$ and $\left(\dfrac{b+a-\sqrt{a^2-2ab+b^2+4}}{2} = x,y,z\right)$

Jacobian $= \begin{bmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{bmatrix} = \begin{bmatrix} ab-az-by+yz & xz-bx & xy-ax \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Jacobian at $(0,0,0)$ $J = \begin{bmatrix} ab & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\det[J - I_3\lambda] = (-\lambda)\cdot(-\lambda)\cdot(ab-\lambda)$ $\quad ab\lambda^2 - \lambda^3 = 0$ $\quad \lambda_{1,2} = 0$ $\lambda_3 = ab$

At point $(0,0,0)$, whenever $\left(|ab| < 1\right)$ or $\left(a < \frac{1}{b} \text{ and } b < \frac{1}{a}\right)$

the fixed point is <u>stable</u>

Question 4: Find all equilibrium points of the 1st order ODEs
Then check the stability of each point
Confirm via maple plotting

a $x'(t) = x(t)(3-x(t))(5-x(t))$

$F(x) = x(3-x)(5-x) = 15x - 8x^2 + x^3$

i Equilibrium points are $x=0 \wedge x=3 \wedge x=5$

$F'(x) = 15 - 16x + 3x^2$ Stability is when $F'(x) < 0$

$F'(0) = 15$, Not stable
$F'(3) = -6$ Stable!
$F'(5) = 10$ Not Stable

ii $x=3$ is the only stable equilibrium point

b $x'(t) = x(t)^2(3-x(t))(5-x(t))(7-x(t))$

$F(x) = x^2(3-x)(5-x)(7-x) = -x^5 + 15x^4 - 71x^3 + 105x^2$

i Equilibrium Points are $x=0$, $x=3$, $x=5$, and $x=7$

$F'(x) = -5x^4 + 60x^3 - 213x^2 + 210x$

$F'(0) = 0$, not stable
$F'(3) = -72$, Stable!
$F'(5) = 100$, not stable
$F'(7) = -392$, stable!