OK to post

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1.) i) $x = x^3 - 6x^2 + 12x - 6$

$$x^3 - 6x^2 + 11x - 6 = 0$$

$$x^3 - x^2 - 5x^2 + 5x + 6x - 6 = 0$$

$$x^2(x-1) - 5x(x-1) + 6(x-1) = 0$$

$$(x-1)(x^2 - 5x + 6) = 0$$

$$(x-1)(x-3)(x-2) = 0$$

$$\boxed{\text{Fixed points: } x = 1,\ x = 2,\ x = 3}$$

$$f(z) = z^3 - 6z^2 + 12z - 6$$

$$f'(z) = 3z^2 - 12z + 12$$

$f'(1) = 3 - 12 \cancel{+12} = |3| \nless 1$: $\boxed{x = 1 \text{ is NOT stable}}$

$f'(2) = 12 - 24 + 12 = |0| < 1$ ✓ : $\boxed{x = 2 \text{ is STABLE}}$

$f'(3) = 27 - 36 + 12 = |3| \nless 1$: $\boxed{x = 3 \text{ is NOT stable}}$

ii) $x = x^4 - \frac{13x^2}{36} + x + \frac{1}{36}$

$$x^4 - \frac{13}{36}x^2 + \frac{1}{36} = 0$$

$$\left(x^2 - \frac{4}{36}\right)\left(x^2 - \frac{9}{36}\right) = 0$$

$$x^2 = \frac{4}{36} \qquad x^2 = \frac{9}{36}$$

$$x = \pm\frac{2}{6} = \pm\frac{1}{3} \qquad x = \pm\frac{3}{6} = \pm\frac{1}{2}$$

$$\boxed{\text{Fixed points: } x = \tfrac{1}{3},\ x = -\tfrac{1}{3},\ x = \tfrac{1}{2},\ x = -\tfrac{1}{2}}$$

$$f(z) = z^4 - \frac{13}{36}z^2 + z + \frac{1}{36}$$

$$f'(z) = 4z^3 - \frac{13}{18}z + 1$$

$f'(\frac{1}{3}) = \left|\frac{49}{54}\right| < 1$ ✓ $\therefore$ $\boxed{x = \frac{1}{3} \text{ is STABLE}}$

$f'(-\frac{1}{3}) = \left|\frac{59}{54}\right| \not< 1$ $\therefore$ $\boxed{x = -\frac{1}{3} \text{ is NOT stable}}$

$f'(\frac{1}{2}) = \left|\frac{41}{36}\right| \not< 1$ $\therefore$ $\boxed{x = \frac{1}{2} \text{ is NOT stable}}$

$f'(-\frac{1}{2}) = \left|\frac{31}{36}\right| < 1$ ✓ $\therefore$ $\boxed{x = -\frac{1}{2} \text{ is STABLE}}$

2) i) $L = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$f(1,2) = \sqrt{1+8} = \sqrt{9} = 3$

$f_x(1,2) = \frac{1}{2}(x+4y)^{-1/2} = \frac{1}{2}(1+8)^{-1/2} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

$f_y(1,2) = \frac{1}{2}(x+4y)^{-1/2}\cdot 4 = 2\cdot(1+8)^{-1/2} = \frac{2}{3}$

$\boxed{L = 3 + \frac{1}{6}\cdot(x-1) + \frac{2}{3}(y-2)}$

$f(.95, 1.02) = \sqrt{.95 + 4(1.02)} = 2.242766$

$L(.95, 1.02) = 3 + \frac{1}{6}(.95-1) + \frac{2}{3}(1.02-2) = 2.338333$

ii) $L = f(1,1,1) + f_x(1,1,1)(x-1) + f_y(1,1,1)(y-1) + f_z(1,1,1)(z-1)$

$f(1,1,1) = 1$

$f_x(1,1,1) = 3y^4z^5x^2 = 3(1)^4(1)^5(1)^2 = 3$

$f_y(1,1,1) = 4x^3z^5y^3 = 4(1)^3(1)^5(1)^3 = 4$

$f_z(1,1,1) = 5x^3y^4z^4 = 5(1)^3(1)^4(1)^4 = 5$

$\boxed{L = 1 + 3(x-1) + 4(y-1) + 5(z-1)}$

$f(1.01, 1.02, .99) = 1.06057$

$L(1.01, 1.02, .99) = 1 + 3(1.01-1) + 4(1.02-1) + 5(.99-1)$

$= 1.06$

iii) $L = f(1,1,1,1) + f_{x_1}(1,1,1,1)(x_1-1) + f_{x_2}(1,1,1,1)(x_2-1)$
$+ f_{x_3}(1,1,1,1)(x_3-1) + f_{x_4}(1,1,1,1)(x_4-1)$

$f(1,1,1,1) = \sqrt{4} = 2$

$f_{x_1}(1,1,1,1) = f_{x_2}(\ldots) = f_{x_3}(\ldots) = f_{x_4}(\ldots)$ by symmetry

$= \frac{1}{2}(x_1+x_2+x_3+x_4)^{-1/2} = \frac{1}{2}(4)^{-1/2} = \frac{1}{4}$

$$\boxed{L = 2 + \frac{1}{4}(x_1-1) + \frac{1}{4}(x_2-1) + \frac{1}{4}(x_3-1) + \frac{1}{4}(x_4-1)}$$

$f(1.01, 1.01, 0.99, 0.99) = 2$

$L(1.01, 1.01, 0.99, 0.99) = 2 + \frac{1}{4}(.01) + \frac{1}{4}(.01) + \frac{1}{4}(-.01) + \frac{1}{4}(-.01)$

$= 2$

3.) $(x,y) \to \left(\frac{x}{y+1}, \frac{y}{x+1}\right)$ Let: $f(x,y) = \frac{x}{y+1}$, $g(x,y) = \frac{y}{x+1}$

$= x(y+1)^{-1}$ $\quad$ $= y(x+1)^{-1}$

$$J = \begin{bmatrix} \frac{\partial f}{\partial x}(1,1), & \frac{\partial f}{\partial y}(1,1) \\ \frac{\partial g}{\partial x}(1,1), & \frac{\partial g}{\partial y}(1,1) \end{bmatrix}$$

$\frac{\partial f}{\partial x}(1,1) = \frac{1}{y+1} = \frac{1}{2}$, $\quad \frac{\partial f}{\partial y}(1,1) = -x(y+1)^{-2} = -1(2)^{-2}$

$= -\frac{1}{2^2} = -\frac{1}{4}$

$$\frac{\partial g}{\partial x}(1,1) = -y(x+1)^{-2} = -1(2)^{-2} = \frac{-1}{4}$$

$$\frac{\partial g}{\partial y}(1,1) = \frac{1}{x+1} = \frac{1}{2}$$

$$\boxed{\text{Jacobian} = \begin{bmatrix} 1/2 , & -1/4 \\ -1/4 , & \frac{1}{2} \end{bmatrix}}$$

4) $(x, y, z) \to (\underbrace{x+y+z}_{f}, \underbrace{x^2+y^2+z^2}_{g}, \underbrace{x^3+y^3+z^3}_{q})$

$$J = \left.\begin{bmatrix} \frac{\partial f}{\partial x}, & \frac{\partial f}{\partial y}, & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x}, & \frac{\partial g}{\partial y}, & \frac{\partial g}{\partial z} \\ \frac{\partial q}{\partial x}, & \frac{\partial q}{\partial y}, & \frac{\partial q}{\partial z} \end{bmatrix}\right\} \text{each partial being evaluated at } (1,1,1)$$

$$\frac{\partial f}{\partial x}(1,1,1) = \frac{\partial f}{\partial y}(..) = \frac{\partial f}{\partial z}(..) = 1 \text{ by symmetry}$$

$$\frac{\partial g}{\partial x}(1,1,1) = \frac{\partial g}{\partial y}(..) = \frac{\partial g}{\partial z}(..) = 2(1) = 2 \text{ by symmetry}$$

$\frac{\partial q}{\partial x}(1,1,1) = \frac{\partial q}{\partial y}(\ldots) = \frac{\partial q}{\partial z}(\ldots) = 3(1)^2 = 3$ by symmetry

$$\boxed{J = \begin{bmatrix} 1, & 1, & 1 \\ 2, & 2, & 2 \\ 3, & 3, & 3 \end{bmatrix}}$$

5) A fixed point (x_0, y_0) of some transformation (x,y) → (f(x,y),g(x,y)) is a STABLE fixed point if all of the eigenvalues of the Jacobian matrix of the transformation have absolute value less than 1. This makes sense because a fixed point is defined to be stable IF when you plug points (or vectors) in the same local neighborhood as the fixed point (just slightly deviating) the transformation eventually returns back to the fixed point (or vector). One can approximate the transformation functions, f(x,y) and g(x,y), through linearization and use these linearizations to determine the criteria for a fixed point to be stable. This criteria is determined by plugging in some (x,y) in the local neighborhood of the fixed point (x_0, y_0) and observing what must happen for the linearization to eventually return the fixed point (after some iterations). These specific linearizations are defined by the value of the function evaluated at the fixed point plus the product of the partial derivatives of each variable (evaluated at the fixed point) with the corresponding difference between some input (x,y) and the fixed point (for each variable). In order for the transformation to point back to the fixed point after some amount of iterations, it is apparent that the product of the partial derivatives with their corresponding changes in x,y,.. must shrink upon each iteration, that way the linearization evaluated at the point (x, y) near (x_0, y_0) is going to eventually equal (x_0, y_0). In order for these values to shrink upon each iteration, all of the eigenvalues of the matrix of the partial derivatives (Jacobian) must have absolute value less than 1, meaning that the transformation this matrix applies to the vector (x,y) shrinks in its magnitude each iteration.