OK to post!

Anusha Nagar, 10/16/2021, Homework 12

1. (i) $x \to x^3 - 6x^2 + 12x - 6$

$$x = x^3 - 6x^2 + 12x - 6$$

$$0 = x^3 - 6x^2 + 11x - 6$$

$x = 1 \Rightarrow$

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & 0 \end{array}$$

$$(x^2 - 5x + 6) \Rightarrow x = 2,\ x = 3$$

FP: $x = 1, 2, 3$

$$f'(x) = 3x^2 - 12x + 11$$

$f'(1) = 3 - 12 + 11 = 2 > 1 \Rightarrow$ unstable

$f'(2) = 12 - 24 + 11 \quad |-1| = 1 \Rightarrow$ stable

$f'(3) = 27 - 36 + 11 = 2 > 1 \Rightarrow$ unstable

$$\boxed{\begin{array}{l} \text{Fixed points: } x = 1, 2, 3 \\ \text{Stable fixed point: } x = 2 \end{array}}$$

(ii) $x \to x^4 - \frac{13x^2}{36} + x + \frac{1}{36}$

$$0 = x^4 - \frac{13}{36}x^2 + \frac{1}{36}$$

From calculator: $x = \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}$

$$f'(x) = 4x^3 - \frac{26}{36}x$$

$f'(\frac{1}{2}) = \frac{5}{36} < 1 \Rightarrow$ stable

$f'(-\frac{1}{2}) = \left|-\frac{5}{36}\right| < 1 \Rightarrow$ stable

$f'(\frac{1}{3}) = \frac{5}{54} < 1 \Rightarrow$ stable

$f'(-\frac{1}{3}) = \left|-\frac{5}{54}\right| < 1 \Rightarrow$ stable

$$\boxed{\begin{array}{l} \text{FP: } x = \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3} \\ \text{All FP are stable FP} \end{array}}$$

(2) $F(x) = F(x_0) + F'(x_0)(x - x_0)$

$F(x,y) = F(x_0,y_0) + F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)$

(i) $F(x,y) = \sqrt{x+4y}$ @ $1, 2$ $\Rightarrow (x_0,y_0) = (1,2)$ $\Rightarrow$ find $f(0.95, 1.02)$

$F(x,y) = 3 + \frac{1}{2} \cdot (\sqrt{x+4y})^{-1}(x-1) + 2 \cdot (\sqrt{x+4y})^{-1}(y-2)$

$$\boxed{F(x,y) = 3 + \frac{1}{6}(x-1) + \frac{2}{3}(y-2)}$$

@ $(0.95, 1.02) \Rightarrow 3 + \frac{1}{6}(0.05) + \frac{2}{3}(-0.98)$

$= 2.355$ $\Rightarrow$ approximate

Exact: 2.2428 $\Rightarrow$ pretty close

(ii) $F(x,y,z) = x^3 y^4 z^5$ @ $(1,1,1)$

$F(x,y,z) \approx F(x_0,y_0,z_0) + F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0)$

$= 1 + 3x_0^2(x-x_0) + 4y_0^3(y-y_0) + 5z_0^4(z-z_0)$

$F(x,y,z) = 1 + 3(x-1) + 4(y-1) + 5(z-1)$

@ $(1.01, 1.02, 0.99)$

$= 1 + 3(0.01) + 4(0.02) + 5(-0.01)$

$= 1.06$

Exact $= 1$

(iii) $F(x_1,x_2,x_3,x_4) = \sqrt{x_1+x_2+x_3+x_4}$ @ $(1,1,1,1)$

$F(\quad " \quad) = 2 + \frac{1}{2\sqrt{x_{10}+x_{20}+x_{30}+x_{40}}}(x_1-1) + \frac{1}{2\sqrt{x_{10}+x_{20}+x_{30}+x_{40}}}(x_2-1) + \frac{1}{2\sqrt{x_{10}+x_{20}+x_{30}+x_{40}}}(x_3-1) + \frac{1}{2\sqrt{x_{10}+x_{20}+x_{30}+x_{40}}}(x_4-1)$

$F(x_1,x_2,x_3,x_4) = 2 + \frac{1}{4}(x_1-1) + \frac{1}{4}(x_2-1) + \frac{1}{4}(x_3-1) + \frac{1}{4}(x_4-1)$

@ $(1.01, 1.01, 0.99, 0.99)$

$= 2 + \frac{1}{4}(0.01) + \frac{1}{4}(0.01) + \frac{1}{4}(-0.01) + \frac{1}{4}(-0.01)$

$= 2$ $\Rightarrow$ exactly equal!

Exact $= 2$

(3) $(x,y) \to \left(\underbrace{\frac{x}{y+1}}_{f(x,y)}, \underbrace{\frac{y}{x+1}}_{g(x,y)}\right)$ @ $(1,1)$

Jacobian Matrix: $\begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix}$

$f(x,y) = \frac{x}{y+1} \Rightarrow f_x = \frac{1}{y+1}$

$f_y = -\frac{x}{(y+1)^2}$

$g(x,y) = \frac{y}{x+1} \Rightarrow g_x = -\frac{y}{(x+1)^2}$

$g_y = \frac{1}{x+1}$

$\boxed{\text{Jacobian: } \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}}$

(4) $(x,y,z) \to (\underbrace{x+y+z}_{f}, \underbrace{x^2+y^2+z^2}_{g}, \underbrace{x^3+y^3+z^3}_{h})$

$f_x = f_y = f_z = 1$

$g_x = 2x,\ g_y = 2y,\ g_z = 2z$

$h_x = 3x^2,\ h_y = 3y^2,\ h_z = 3z^2$

$\boxed{\text{Jacobian: } \begin{bmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{bmatrix}}$

(5) In finding SFP, we take the derivatives of all of our functions & see if their absolute value is less than 1. When we have multivariable functions, we must have n functions for n variables. We present this in the Jacobian matrix. When our eigenvalues are less than 1, we are essentially performing the same analysis as before & seeing if the derivatives are below 1, as $A\vec{v} = \lambda\vec{v}$