OK to post

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1. (i)

$$x = x^3 - 6x^2 + 12x - 6$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

(by guessing roots)

$$= (x-1)(x-2)(x-3)$$

so $\boxed{1, 2, \text{and } 3 \text{ are fixed points.}}$

$$f'(x) = 3x^2 - 12x + 12$$

$|f'(1)| = 3$, so 1 is not stable.

$|f'(2)| = 12 - 24 + 12 = 0 < 1$, so 2 is stable.

$|f'(3)| = 27 - 36 + 12 = 3$ so 3 is not stable.

(ii) $x \to x^4 - \frac{13x^2}{36} + x + \frac{1}{36}$

$$\cancel{x} = x^4 - \frac{13x^2}{36} + \cancel{x} + 1/36$$

$$\Rightarrow \quad 0 = 36x^4 - 13x^2 + 1$$

$$\Rightarrow \quad 0 = 36x^4 - 9x^2 - 4x^2 + 1$$

$$\Rightarrow \quad 0 = 9x^2(4x^2-1) - 1(4x^2-1)$$

$$\Rightarrow \quad 0 = (9x^2-1)(4x^2-1)$$

$\Rightarrow$ $\boxed{x = \pm 1/3, \pm 1/2}$ are fixed pts.

$$f'(x) = 4x^3 - \frac{26}{36}x + 1$$

$|f'(1/3)| = \frac{49}{54} < 1$, so $1/3$ is stable.

$|f'(-1/3)| = \frac{59}{54} > 1$, so $-1/3$ is not stable.

$|f'(1/2)| = \frac{41}{36} > 1$, so $1/2$ is not stable.

$|f'(-1/2)| = \frac{31}{36} < 1$, so $-1/2$ is stable.

2)

(i) $f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0)$

$+ f_y(x_0,y_0)(y-y_0)$

$$f_x = 1/2\,(x+4y)^{-1/2} = \frac{1}{2\sqrt{x+4y}}$$

$$f_y = 1/2\,(x+4y)^{-1/2} \cdot 4 = \frac{2}{\sqrt{x+4y}}$$

$$f(x_0, y_0) = \sqrt{1+8} = 3$$

$$f_x(x_0, y_0) = \frac{1}{2\sqrt{9}} = 1/6$$

$$f_y(x_0, y_0) = \frac{2}{\sqrt{9}} = 2/3$$

$$g(x,y) = 3 + 1/6(x-1) + 2/3(y-2)$$

$$f(0.95, 1.02) = \sqrt{0.95 + 4 \cdot 1.02} = 2.428$$

$$g(0.95, 1.02) = 3 + 1/6(0.95-1) + 2/3(1.02-2)$$

$$= 2.338$$

(ii) $f(x,y,z) = x^3 y^4 z^5$ @ $(1,1,1)$

$$f_x = 3x^2 y^4 z^5$$

$f_y = 4x^3y^3z^5$

$f_z = 5x^3y^4z^4$

$f(1,1,1) = 1$

$f_x(1,1,1) = 3$

$f_y(1,1,1) = 4$

$f_z(1,1,1) = 5$

$$\boxed{g(x,y,z) = 1 + 3(x-1) + 4(x-1) + 5(x-1).}$$

$f(1.01, 1.02, 0.99) = (1.01)^3(1.02)^4(0.99)^5$

$= 1.061$

$g(1.01, 1.02, 0.99) = 1 + 3(0.01) + 4(0.02) + 5(-0.01)$

$= 1.06$

(iii) $f(x_1, x_2, x_3, x_4) = \sqrt{x_1 + x_2 + x_3 + x_4}$

$@ (1,1,1,1)$

$f_{x_1} = f_{x_2} = f_{x_3} = f_{x_4}$

$= \frac{1}{2}(x_1 + x_2 + x_3 + x_4)^{-1/2} = \frac{1}{2\sqrt{x_1+x_2+x_3+x_4}}$

$f(1,1,1,1) = 2$

$f_{x_1}(1,1,1,1) = 1/4$

$$\boxed{g(x_1, x_2, x_3, x_4) = 2 + 1/4(x_1 - 1) + 1/4(x_2 - 1) + 1/4(x_3 - 1) + 1/4(x_4 - 1)}$$

$$F(1.01, 1.01, 0.99, 0.99) = \sqrt{4} = 2$$

$$g(1.01, 1.01, 0.99, 0.99) = 2 + \frac{0.1 + 0.1 - 0.1 - 0.1}{4}$$

$$= 2$$

3) $$(x, y) \to \left(\frac{x}{y+1}, \frac{y}{x+1} \right)$$

$$J = \begin{pmatrix} f_{1x} & f_{1y} \\ f_{2x} & f_{2y} \end{pmatrix} = \begin{pmatrix} \frac{1}{y+1} & \frac{-x}{(y+1)^2} \\ \frac{-y}{(x+1)^2} & \frac{1}{x+1} \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} \frac{1}{2} & \frac{-1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}}$$

4) $(x,y,z) = (x+y+z,\ x^2+y^2+z^2,\ x^3+y^3+z^3)$

$$J = \begin{pmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}}$$

5. We can linearize f and g by writing

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$g(x,y) = g(x_0,y_0) + g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0).$$

Then, setting $f(x_0,y_0) = x_0$ and $g(x_0,y_0) = y_0$, we get:

$$f(x,y) - x_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$g(x,y) - y_0 = g_x(x_0,y_0)(x-x_0) + g_y(x_0,y_0)(y-y_0).$$

We can rewrite this as:

$$\begin{bmatrix} f(x,y) - x_0 \\ g(x,y) - y_0 \end{bmatrix} = \begin{bmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{bmatrix} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}.$$

Going through this transformation n times gives:

$$\begin{bmatrix} f(x_n, y_n) - x_0 \\ g(x_n, y_n) - y_0 \end{bmatrix} = \begin{bmatrix} f_x(x, y_0) & f_y(x_0, y_0) \\ g_x(x_0, y_0) & g_y(x_0, y_0) \end{bmatrix}^n \begin{bmatrix} x_n - x_0 \\ y_n - y_0 \end{bmatrix},$$

We can diagonalize the Jacobian J as $Q \lambda Q^{-1}$ where λ is the diagonal matrix consisting of eigenvalues of J. Then,

$$\begin{bmatrix} f(x_n, y_n) - x_0 \\ g(x_n, y_n) - y_0 \end{bmatrix} = (Q \lambda Q^{-1})^n \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}.$$

Then, $(Q\lambda Q^{-1})^n = Q\lambda Q^{-1} Q \lambda Q^{-1} Q \lambda Q^{-1} Q \lambda \ldots$

$$= Q \lambda^n Q^{-1}.$$

Thus,

$$\begin{pmatrix} f(x_n, y_n) - x_0 \\ g(x_n, y_n) - y_0 \end{pmatrix} = Q \Lambda^n Q^{-1} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$$

and $f(x_n, y_n) - x_0 < x - x_0$

and $g(x_n, y_n) - y_0 < y - y_0$

only if each eigenvalue gets smaller as $n \to \infty$, i.e. they are less than 1.