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> # Max Mekhanikov - HW 11 - Okay to post
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> # Question 1
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SFPe := proc(f, x) local f1, L, i :
f1 := diff(f, x) :
L := [solve(f = x, x)] :
[seq([L[i], normal(subs(x = L[i], f1))], i = 1 ..nops(L))] :

 end:
```

```
> SFPe(k*x*(1-x), x)
```

$$\left[[0, k], \left[\frac{k-1}{k}, -k+2\right]\right] \tag{1}$$

```
> fprime := diff(k·x·(1 − x), x)
```

$$fprime := k\,(1-x) - k\,x \tag{2}$$

```
> solve(fprime = (k − 1)/k, x)
```

$$\frac{k^2-k+1}{2\,k^2} \tag{3}$$

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> # Stable fixed point
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> # Question 2
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> Orb := proc(f, x, x0, K1, K2) local x1, i, L :
x1 := x0 :
for i from 1 to K1 do
 x1 := subs(x = x1, f) :
     #we don't record the first values of K1, since we are interested in the long-time behavior of the
     orbit
od:

L := [x1] :

for i from K1 to K2 do
 x1 := subs(x = x1, f) :   #we compute the next member of the orbit
 L := [op(L), x1] : #we append it to the list
od:

 L : #that's the output

end:
```

```
> Orb(3.1*x*(1-x), x, 0.5, 100, 110)
```

[0.5580141245, 0.7645665203, 0.5580141245, 0.7645665203, 0.5580141245, 0.7645665203, 0.5580141245, 0.7645665203, 0.5580141245, 0.7645665203, 0.5580141245, 0.7645665203] **(4)**

> $Orb(3.2*x*(1-x), x, 0.5, 100, 110)$

[0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906, 0.5130445091, 0.7994554906] **(5)**

> $Orb(3.3*x*(1-x), x, 0.5, 100, 110)$

[0.4794270198, 0.8236032832, 0.4794270198, 0.8236032832, 0.4794270198, 0.8236032832, 0.4794270198, 0.8236032832, 0.4794270198, 0.8236032832, 0.4794270198, 0.8236032832] **(6)**

> $Orb(3.4*x*(1-x), x, 0.5, 100, 110)$

[0.4519632747, 0.8421544082, 0.4519632271, 0.8421543926, 0.4519632634, 0.8421544045, 0.4519632359, 0.8421543956, 0.4519632566, 0.8421544024, 0.4519632407, 0.8421543973] **(7)**

> $Orb(3.5*x*(1-x), x, 0.5, 100, 110)$

[0.5008842111, 0.8749972637, 0.3828196827, 0.8269407062, 0.5008842111, 0.8749972637, 0.3828196827, 0.8269407062, 0.5008842111, 0.8749972637, 0.3828196827, 0.8269407062] **(8)**

> $Orb(3.51*x*(1-x), x, 0.5, 100, 110)$

[0.5067130559, 0.8773418215, 0.3777221554, 0.8250189317, 0.5067130559, 0.8773418215, 0.3777221554, 0.8250189317, 0.5067130559, 0.8773418215, 0.3777221554, 0.8250189317] **(9)**

> $Orb(3.52*x*(1-x), x, 0.5, 100, 110)$

[0.5120763623, 0.8794866485, 0.3730843903, 0.8233013466, 0.5120763623, 0.8794866485, 0.3730843903, 0.8233013466, 0.5120763623, 0.8794866485, 0.3730843903, 0.8233013466] **(10)**

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> # Based on these results, the second bifurcation point occurs at roughly k = 3.51 as shown above

  # Question 3
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> # c = lambda / alpha
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$SFPe\left(c \cdot x^{-b} \cdot x, x\right)$

$$\left[[0, 0], \left[e^{\frac{\ln(c)}{b}}, -c \left(e^{\frac{\ln(c)}{b}} \right)^{-b} b + c \left(e^{\frac{\ln(c)}{b}} \right)^{-b} \right] \right] \quad \textbf{(11)}$$

> $fprime1 := diff\left(c \cdot x^{-b} \cdot x, x\right)$

$$fprime1 := -c\, x^{-b}\, b + c\, x^{-b} \quad \textbf{(12)}$$

> $solve\left(fprime1 = e^{\frac{\ln(c)}{b}}, x\right)$

$$e^{-\frac{\ln\left(-\frac{1}{c\,(-1+b)}\right) b + \ln(c)}{b^2}} \quad \textbf{(13)}$$

> $solve\left(fprime1 = -c \left(e^{\frac{\ln(c)}{b}} \right)^{-b} b + c \left(e^{\frac{\ln(c)}{b}} \right)^{-b}, x\right)$

$$e^{\frac{\ln(c)}{b}} \quad \textbf{(14)}$$

> # -1 < $\mathrm{e}^{-\frac{\ln\left(-\frac{1}{c(-1+b)}\right) b+\ln(c)}{b^2}}$ < 1

-1 <$\mathrm{e}^{\frac{\ln(c)}{b}}$ < 1

 # Stability depends on -1<1-b<1, or 0<b<2

 # Equation 2

> $SFPe\left(x\cdot\exp\left(r\left(1-\frac{x}{k}\right)\right), x\right)$

$$[[0, \mathrm{e}^{r(1)}], [-RootOf(r(\_Z))\,k+k, \mathrm{e}^{r(RootOf(r(\_Z)))}\,(\mathrm{D}(r)(RootOf(r(\_Z)))\,RootOf(r(\_Z)) - \mathrm{D}(r)(RootOf(r(\_Z)))+1)]] \quad (15)$$

> $fprime2 := 1-r$

$$fprime2 := 1-r \quad (16)$$

> **# Stability is obtained when |1-r|<1 or 0<r<2**

>
 # Equation 3

> $SFPe\left(l\cdot x\cdot(1+a\cdot x)^{-b}, x\right)$

$$\left[[0, l], \left[\frac{\mathrm{e}^{\frac{\ln(l)}{b}}-1}{a}, -\frac{l\left(\mathrm{e}^{\frac{\ln(l)}{b}}\right)^{-b}\left(b\,\mathrm{e}^{\frac{\ln(l)}{b}}-\mathrm{e}^{\frac{\ln(l)}{b}}-b\right)}{\mathrm{e}^{\frac{\ln(l)}{b}}}\right]\right] \quad (17)$$

> $fprime3 := diff\left(l\cdot x\cdot(1+a\cdot x)^{-b}, x\right)$

$$fprime3 := l\,(x\,a+1)^{-b} - \frac{l\,x\,(x\,a+1)^{-b}\,b\,a}{x\,a+1} \quad (18)$$

> $solve\left(fprime3 = \frac{\mathrm{e}^{\frac{\ln(l)}{b}}-1}{a}, x\right)$

$$\frac{\mathrm{e}^{RootOf\left(\_Z\,b+\ln\left(-\frac{\mathrm{e}^{\frac{\ln(l)}{b}}-1}{l\,a\,(b\,\mathrm{e}^{\_Z}-b-\mathrm{e}^{\_Z})}\right)+\_Z\right)}-1}{a} \quad (19)$$

> $solve\left(fprime3 = -\frac{l\left(\mathrm{e}^{\frac{\ln(l)}{b}}\right)^{-b}\left(b\,\mathrm{e}^{\frac{\ln(l)}{b}}-\mathrm{e}^{\frac{\ln(l)}{b}}-b\right)}{\mathrm{e}^{\frac{\ln(l)}{b}}}, x\right)$

$$\frac{\mathrm{e}^{RootOf\left(\_Z\,b^2+b\ln\left(\frac{\left(l^{\frac{1}{b}}\right)^{-b}\left(b\,l^{\frac{1}{b}}-l^{\frac{1}{b}}-b\right)}{b\,\mathrm{e}^{\_Z}-b-\mathrm{e}^{\_Z}}\right)+\_Z\,b-\ln(l)\right)}-1}{a} \quad (20)$$

> # -1 < $\frac{e^{RootOf\left(\_Z b + \ln\left(-\frac{e^{\frac{\ln(l)}{b}} - 1}{l\, a\, (b\, e^{\_Z} - b - e^{\_Z})}\right) + \_Z\right)} - 1}{a}$ < 1

> # -1 < $\frac{e^{RootOf\left(\_Z b^2 + b \ln\left(\frac{\left(l^{\frac{1}{b}}\right)^{-b}\left(b\, l^{\frac{1}{b}} - l^{\frac{1}{b}} - b\right)}{b\, e^{\_Z} - b - e^{\_Z}}\right) + \_Z b - \ln(l)\right)} - 1}{a}$ < 1

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Question 4

```
> #Orbk(k,z,f,INI,K1,K2): Given a positive integer k, a letter (symbol), z, an expression f of z[1], ...,
     z[k] (representing a multi-variable function of the variables z[1],...,z[k]

     #a vector INI representing the initial values [x[1],..., x[k]], and (in applications) positive
     integres K1 and K2, outputs the

     #values of the sequence starting at n=K1 and ending at n=K2. of the sequence satisfying the
     difference equation
  ##x[n]=f(x[n-1],x[n-2],..., x[n-k+1]):

     #This is a generalization to  higher-order difference equation of procedure Orb(f,x,x0,K1,K2).
     For example
  #Orbk(1,z,5/2*z[1]*(1-z[1]),[0.5],1000,1010);  should be the same as
  #Orb(5/2*z[1]*(1-z[1]),z[1],[0,5],1000,1010);
   #Try:

   #Orbk(2,z,(5/4)*z[1]-(3/8)*z[2],[1,2],1000,1010);


  Orbk := proc(k, z, f, INI, K1, K2) local L, i, newguy :
  L := INI : #We start out with the list of initial values

  if not (type(k, integer) and type(z, symbol) and type(INI, list) and nops(INI) = k
     and type(K1, integer) and type(K2, integer) and K1 > 0 and K2 > K1) then
     #checking that the input is OK
   print(`bad input`) :
   RETURN(FAIL) :
  fi:

  while nops(L) < K2 do
   newguy := subs({seq(z[i] = L[-i], i = 1 ..k)}, f) :
     #Using what we know about the value yesterday, the day before yesterday, ... up to k days
     before yesterday we find the value of the sequence today
   L := [op(L), newguy] : #we append the new value to the running list of values of our sequence
  od:
```

$[op(K1..K2, L)]:$

end:

> $Orbk\left(2, z, \dfrac{z[1]+1*z[2]}{1\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] \quad (21)$$

> $Orbk\left(2, z, \dfrac{z[1]+1*z[2]}{2\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665, 0.6666666665] **(22)**

> $Orbk\left(2, z, \dfrac{z[1]+1*z[2]}{3\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[0.5000000002, 0.4999999998, 0.5000000002, 0.4999999998, 0.5000000002, 0.4999999998, 0.5000000002, 0.4999999998, 0.5000000002, 0.4999999998, 0.5000000002] **(23)**

> $Orbk\left(2, z, \dfrac{z[1]+1*z[2]}{4\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[0.4000000004, 0.3999999996, 0.4000000004, 0.3999999996, 0.4000000004, 0.3999999996, 0.4000000004, 0.3999999996, 0.4000000004, 0.3999999996, 0.4000000004] **(24)**

> $Orbk\left(2, z, \dfrac{z[1]+2*z[2]}{1\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000, 1.500000000] **(25)**

> $Orbk\left(2, z, \dfrac{z[1]+2*z[2]}{2\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000, 1.000000000] **(26)**

> $Orbk\left(2, z, \dfrac{z[1]+2*z[2]}{3\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[0.7500000002, 0.7499999994, 0.7500000002, 0.7499999997, 0.7500000002, 0.7499999994, 0.7500000002, 0.7499999997, 0.7500000002, 0.7499999994, 0.7500000002] **(27)**

> $Orbk\left(2, z, \dfrac{z[1]+2*z[2]}{4\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[0.6000000033, 0.5999999967, 0.6000000033, 0.5999999967, 0.6000000033, 0.5999999967, 0.6000000033, 0.5999999967, 0.6000000033, 0.5999999967, 0.6000000033] **(28)**

> $Orbk\left(2, z, \dfrac{z[1]+3*z[2]}{1\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000, 2.000000000] **(29)**

> $Orbk\left(2, z, \dfrac{z[1]+3*z[2]}{2\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[1.333333335, 1.333333332, 1.333333335, 1.333333332, 1.333333335, 1.333333332, **(30)** 1.333333335, 1.333333332, 1.333333335, 1.333333332, 1.333333335]

> $Orbk\left(2, z, \dfrac{z[1]+3*z[2]}{3\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[1.056238496, 0.9467817065, 1.056180901, 0.9468332585, 1.056123481, 0.9468846597, 1.056066236, 0.9469359103, 1.056009163, 0.9469870121, 1.055952263] **(31)**

> $Orbk\left(2, z, \dfrac{z[1]+3*z[2]}{4\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[1.577350271, 0.4226497305, 1.577350271, 0.4226497306, 1.577350271, 0.4226497305, 1.577350271, 0.4226497306, 1.577350271, 0.4226497305, 1.577350271] **(32)**

> $Orbk\left(2, z, \dfrac{z[1]+4*z[2]}{1\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[2.500000002, 2.499999998, 2.500000002, 2.499999998, 2.500000002, 2.499999998, 2.500000002, 2.499999998, 2.500000002, 2.499999998, 2.500000002] **(33)**

> $Orbk\left(2, z, \dfrac{z[1]+4*z[2]}{2\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[1.666666671, 1.666666662, 1.666666671, 1.666666662, 1.666666671, 1.666666662, 1.666666671, 1.666666662, 1.666666671, 1.666666662, 1.666666671] **(34)**

> $Orbk\left(2, z, \dfrac{z[1]+4*z[2]}{3\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[2.366025406, 0.6339745957, 2.366025405, 0.6339745958, 2.366025406, 0.6339745957, 2.366025405, 0.6339745958, 2.366025406, 0.6339745957, 2.366025405] **(35)**

> $Orbk\left(2, z, \dfrac{z[1]+4*z[2]}{4\cdot z[1]+z[2]}, [1.1, 5.3], 1000, 1010\right)$

[2.618033990, 0.3819660111, 2.618033989, 0.3819660110, 2.618033990, 0.3819660111, 2.618033989, 0.3819660110, 2.618033990, 0.3819660111, 2.618033989] **(36)**

>

$$X_n = \frac{X_{n-1} + a X_{n-2}}{b X_{n-1} + X_{n-2}}$$

$$X_n = X_{n-1} = X_{n-2} = Z$$

$$\frac{X_{n-1} + a X_{n-2}}{b X_{n-1} + X_{n-2}} = Z$$

$$\downarrow \; X_{n-1} = X_{n-2}$$

$$\frac{X_{n-1} + a X_{n-1}}{b X_{n-1} + X_{n-1}} = Z$$

$$\frac{(a+1)(X_{n-1})}{(b+1)(X_{n-1})} = Z$$

$$\frac{a+1}{b+1} = Z$$