OK to post

Anne Somalwar, 10/10/2021, hw11

#4) (ii) $z = \frac{z + az}{bz + z}$

$$\boxed{z = \frac{1+a}{b+1}}$$

#5) I followed the procedure given in Keshet 2.7 for systems of nonlinear difference equations.

$\left(x_n = \frac{x_{n-1} + a x_{n-2}}{b x_{n-1} + x_{n-2}} \right)$ can be rewritten as the system:

$$\Rightarrow \begin{cases} x_n = \dfrac{x_{n-1} + a y_{n-1}}{b x_{n-1} + y_{n-1}} \\ y_n = x_{n-1} \end{cases}$$

$$f(x,y) = \frac{x+ay}{bx+y}$$

$$g(x,y) = x$$

$$\partial f / \partial x = \frac{bx+y - (x+ay)(b)}{(bx+y)^2}$$

$$= \frac{y-aby}{(bx+y)^2}$$

$$\partial f / \partial y = \frac{(a)(bx+y) - (x+ay)}{(bx+y)^2}$$

$$= \frac{abx - x}{(bx+y)^2} = \frac{abx-x}{b^2x^2+2bxy+y^2}$$

$$\bar{x} = \bar{y} = \frac{a+1}{b+1}$$

$$a_{11} = \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}} = \frac{\bar{y} - ab\bar{y}}{b^2\bar{x}^2 + 2b\bar{x}\bar{y} + \bar{y}^2}$$

$$= \frac{1-ab}{b^2\bar{x} + 2b\bar{x} + \bar{x}} = \frac{1-ab}{(\bar{x})(b+1)^2}$$

$$= \frac{1-ab}{\left(\frac{a+1}{b+1}\right)(b+1)^2} = \frac{1-ab}{(a+1)(b+1)}$$

$$a_{12} = \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}} = \frac{ab\bar{x} - \bar{x}}{b^2\bar{x}^2 + 2b\bar{x}\bar{y} + \bar{y}^2}$$

$$= \frac{ab-1}{b^2\bar{x} + 2b\bar{x} + \bar{x}} = \frac{ab-1}{(a+1)(b+1)}$$

$$a_{21} = \left.\frac{\partial g}{\partial x}\right|_{\bar{x}\bar{y}} = 1$$

$$a_{22} = \left.\frac{\partial g}{\partial y}\right|_{\bar{x}\bar{y}} = 0$$

$$A = \begin{pmatrix} \frac{1-ab}{(a+1)(b+1)} & \frac{ab-1}{(a+1)(b+1)} \\ 1 & 0 \end{pmatrix}$$

$$\det(A - \lambda I)$$

$$= \det \begin{pmatrix} \frac{1-ab}{(a+1)(b+1)} - \lambda & \frac{ab-1}{(a+1)(b+1)} \\ 1 & -\lambda \end{pmatrix}$$

$$= \left(\frac{1-ab}{(a+1)(b+1)} - \lambda\right)(-\lambda) - \left(\frac{ab-1}{(a+1)(b+1)}\right)$$

$$= \frac{ab-1}{(a+1)(b+1)} \cdot \lambda + \lambda^2 - \frac{ab-1}{(a+1)(b+1)}$$

$$= \lambda^2 + \frac{ab-1}{(a+1)(b+1)} \lambda - \frac{ab-1}{(a+1)(b+1)} = 0$$

According to the text,

The fixed point is stable when:

$$\boxed{2 > 1 - \frac{ab-1}{(a+1)(b+1)} > \left|\frac{ab-1}{(a+1)(b+1)}\right|}$$