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#Julian Herman, 10/11/21, Assignment 10

1.) $x(n) = \dfrac{x(n-1)}{x(n-1)+c}$

To be a stable fixed point, the absolute value of the derivative evaluated at the fixed point must be < 1.

$f(x) = \dfrac{x}{x+c}$

$f'(x) = \dfrac{x+c-x}{(x+c)^2} = \dfrac{c}{(x+c)^2}$

$|f'(x=0)| < 1$

$\left|\dfrac{c}{(0+c)^2}\right| < 1$

$\left|\dfrac{c}{c^2}\right| < 1$

$\left|\dfrac{1}{c}\right| < 1$

$\dfrac{1}{|c|} < 1$

$1 < |c|$

same as

$|c| > 1$

is true for

$(-\infty, -1), (1, \infty)$

For $c < -1$ or $c > 1$, the point $x = 0$ is a stable fixed point.

$(-\infty, -1), (1, \infty)$

2) i.) $(x, y) \to \left(-\frac{16}{3}x + 5y,\ -7x + \frac{13}{2}y\right)$

* For some mapping, such as the above, a fixed point is stable iff the absolute value of each and every eigenvalue of the jacobian matrix evaluated at the fixed point is less than 1 . . . This comes from the fact that the Linearization (Taylor expansion) of the equations representing the original mapping evaluated at some point near fixed point (x,y) are equal to the function evaluated at the fixed point plus the respective partial derivatives. If the |eigenvalues| of the jacobian evaluated at the fixed point < 1 then the partial derivative terms disappear (eventually $= 0$) and the mapping points to itself... see below:

Let $f(x,y) = -\frac{16}{3}x + 5y$

Let $(\bar{x}, \bar{y})$ be fixed points

Linearization at some point (x', y') near $(\bar{x}, \bar{y})$:

$$f(\bar{x}+x', \bar{y}+y') = f(\bar{x},\bar{y}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x},\bar{y}} x' + \left.\frac{\partial f}{\partial y}\right|_{\bar{x},\bar{y}} y' + \ldots$$

This small deviation (x', y') from $(\bar{x}, \bar{y})$ must still result in $(\bar{x}, \bar{y})$ in order for $(\bar{x}, \bar{y})$ to be <u>stable</u>. So when the $|\text{eignvalues of jacobian}| < 1$, these partial derivative terms, after multiple iterations, have a smaller and smaller effect, eventually going to 0 resulting in the function <u>still</u> mapping to itself, hence, the fixed point $(\bar{x}, \bar{y})$ is stable:

$$f(\bar{x}+x', \bar{y}+y') = f(\bar{x}, \bar{y}) + (\approx 0) = f(\bar{x}, \bar{y})$$

i.) $$\det \begin{bmatrix} -\frac{16}{3} - \lambda, & 5 \\ -7, & \frac{13}{2} - \lambda \end{bmatrix} = \left(-\frac{16}{3} - \lambda\right)\left(\frac{13}{2} - \lambda\right) + 35 = 0$$

$$= -\frac{104}{3} + \frac{16}{3}\lambda - \frac{13}{2}\lambda + \lambda^2 + 35 = 0$$

$$\lambda^2 - \frac{7}{6}\lambda + \frac{1}{3} = 0$$

$$6\lambda^2 - 7\lambda + 2 = 0$$

$$6\lambda^2 - 3\lambda - 4\lambda + 2$$

$$3\lambda(2\lambda - 1) - 2(2\lambda - 1)$$

$$(3\lambda - 2)(2\lambda - 1) = 0$$

$$\boxed{\lambda_1 = \frac{2}{3} \quad \lambda_2 = \frac{1}{2}}$$

$$\left(\left|\frac{2}{3}\right| \cap \left|\frac{1}{2}\right|\right) < 1 \quad \checkmark \qquad \underline{(0,0) \text{ is STABLE}}$$

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ii) $$\det\begin{bmatrix} \frac{92}{3}-\lambda, & -25 \\ 35, & -\frac{57}{2}-\lambda \end{bmatrix} = \left(\frac{92}{3}-\lambda\right)\left(-\frac{57}{2}-\lambda\right) + 875 = 0$$

$$= -874 - \frac{13}{6}\lambda + \lambda^2 + 875 = \lambda^2 - \frac{13}{6}\lambda + 1 = 0$$

$$= 6\lambda^2 - 13\lambda + 6 = 0$$

$$(2\lambda-3)(3\lambda-2) = 0$$

$$\boxed{\lambda_1 = \frac{3}{2} \quad \lambda_2 = \frac{2}{3}}$$

$$\left(\left|\frac{3}{2}\right| \cap \left|\frac{2}{3}\right|\right) \overset{?}{<} 1$$

$$\downarrow$$

$$1.5 \not< 1, \quad \underline{(0,0) \text{ is NOT STABLE!}}$$

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iii) $$\det\begin{bmatrix} -\frac{177}{4}-\lambda, & \frac{75}{2} \\ -\frac{105}{2}, & \frac{89}{2}-\lambda \end{bmatrix} = \frac{-15753}{8} - \frac{1}{4}\lambda + \lambda^2 + \frac{7875}{4} = 0$$

$$= 8\lambda^2 - 2\lambda - 3 = 0$$

$$8\lambda^2 + 4\lambda - 6\lambda - 3$$

$$4\lambda(2\lambda+1) - 3(2\lambda+1)$$

$$(4\lambda-3)(2\lambda+1) = 0$$

$$\boxed{\lambda_1 = \frac{3}{4}, \ \lambda_2 = \frac{-1}{2}}$$

$$\left(\left|\frac{3}{4}\right| \cap \left|\frac{-1}{2}\right|\right) < 1 \quad \checkmark$$

$$\underline{(0,0) \text{ is STABLE!}}$$