Charles Griebell
Final

```
> read `C:/Users/cgrie/Dynam Models Bio/Final Exam/DMB.txt`
```

First Written: Nov. 2021

This is DMB.txt, A Maple package to explore Dynamical models in Biology (both discrete and continuous)
accompanying the class Dynamical Models in Biology, Rutgers University. Taught by Dr. Z. (Doron Zeilbeger)

The most current version is available on WWW at:
http://sites.math.rutgers.edu/~zeilberg/tokhniot/DMB.txt .
Please report all bugs to: DoronZeil at gmail dot com .

For general help, and a list of the MAIN functions,
type "Help();". For specific help type "Help(procedure_name);"

For a list of the supporting functions type: Help1();
For help with any of them type: Help(ProcedureName);

For a list of the functions that give examples of Discrete-time dynamical systems (some famous), type: HelpDDM();
For help with any of them type: Help(ProcedureName);

For a list of the functions continuous-time dynamical systems (some famous) type: HelpCDM();
For help with any of them type: Help(ProcedureName);

------------------------------ **(1)**

1. Day 0 there is 1 rabbit
   Day 1 there is 1 rabbit
   Day 2 there is 2 rabbit

If the number of rabbits on day n , $r(n)$, is expressed as
$r(n) = 2\,r(n-1) - r(n-3)$

where $r(n-1)$ represents the number of rabbits yesterday
$r(n-3)$ represents the number of rabbits three days ago

What is the decimal value of $\frac{r(1000)}{r(999)}$?

Answer to 1: We will use rsolve

```
> rabbit := rsolve({r(n)= 2*r(n-1)-r(n-3),r(0)=1,r(1)=1,r(2)=2},{r}
);
```

$$rabbit := \left\{ r(n) = \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n + \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n \right\} \quad \textbf{(2)}$$

and then

```
> rabbit_1000 := subs(n=1000,rabbit[1]);
  rabbit_999  := subs(n=999,rabbit[1]);
```

$$rabbit\_1000 := r(1000) = \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{1000} + \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{1000}$$

$$rabbit\_999 := r(999) = \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^{999} + \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right)\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^{999} \quad \textbf{(3)}$$

And our answer for Problem 1 is:

```
> evalf(rabbit_1000/rabbit_999);
```

$$\frac{r(1000)}{r(999)} = 1.618033988 \quad \textbf{(4)}$$

2. A certain species with carrying capacity 1 has a rate of change that is equal to

$$\frac{dQ}{dt} = \frac{5}{2}\cdot(Q)\cdot(1-Q)\cdot\left(1-\frac{1}{2}\cdot Q\right)$$

(a) Find all of the equilibrium solutions

Ans to (a): By inspection, Q = 0 and Q = 1 are the only equilibrium solutions of the species. We **disregard** Q=2 as an equilibrium solution because the carrying capacity is 1.

(b) Find all of the stable equilibrium solutions

Ans to (b) First, the underlying transformation of the differential equation is:

```
> UT := f(Q)=(5/2)*Q*(1-Q)*(1-(1/2)*Q);
```

$$UT := f(Q) = \frac{5\,Q\,(1-Q)\left(1-\frac{Q}{2}\right)}{2} \quad \textbf{(5)}$$

Then, differentiate the transformation to get the

```
> UT_prime:= diff(UT,Q);
```

$$UT\_prime := \frac{d}{dQ} f(Q) = \frac{5\,(1-Q)\left(1-\frac{Q}{2}\right)}{2} - \frac{5\,Q\left(1-\frac{Q}{2}\right)}{2} - \frac{5\,Q\,(1-Q)}{4} \quad \textbf{(6)}$$

For the continuous case, whenever $f\,''\cdot(Q) < 0$ we have a stable equilibrium. Thus for values $Q=0$ and $Q=1$ we get:

```
> print(`for Equilibrium Q = 0`);
  subs(Q=0,UT_prime);
  print(`which is greater than 0, which is unstable`);
  print(`and for equilibrium Q=1`);
  subs(Q=1,UT_prime);
  print(`which is less than 0, which is stable`);
  print(`THEREFORE Q=1 is only stable equilibrium`);
```

for Equilibrium Q = 0

UT_prime

which is greater than 0, which is unstable

and for equilibrium Q=1

UT_prime

which is less than 0, which is stable

THEREFORE Q=1 is only stable equilibrium **(7)**

(c) If its value at t=0 is 0.1 whats its value at t=1000.

```
> ode:= diff(Q(t),t)=(5/2)*Q(t)*(1-Q(t))*(1-(1/2)*Q(t));
  particular := dsolve({ode,Q(0)=0.1});
  answer := evalf(subs(t=1000,particular));
```

$$ode := \frac{d}{dt} Q(t) = \frac{5\,Q(t)\,(1-Q(t))\left(1-\frac{Q(t)}{2}\right)}{2}$$

$$particular := Q(t) = \frac{19\,e^{\frac{5t}{2}}}{81\left(-\frac{1}{\sqrt{1+\frac{19\,e^{\frac{5t}{2}}}{81}}}-1\right)\left(-\frac{19\,e^{\frac{5t}{2}}}{81}-1\right)}$$

$$answer := Q(1000) = 0.9999999997 \quad \textbf{(8)}$$

Problem 3. A certain species with carrying capacity 1 such that its quantity today is

$$x(n) = \frac{5}{2}x(n-1)\cdot(1-x(n-1))\cdot\left(1-\frac{1}{2}x(n-1)\right)$$

(a) Find all the equilibrium solutions

First, the underlying transformation is:

$$f(x) = \frac{5}{2}x(1-x)\left(1-\frac{1}{2}x\right)$$

For the discrete case, an equilibrium point occurs at $x = c$ whenever $f(x) = x$.

Therefore, we use FP

#FIXED POINTS

```
> fixed:=FP([(5/2)*x*(1-x)*(1-(1/2)*x)],[x]);
  print(`but obviously`);
  fixed[3];
  print(`is not a solution because it exceeds the carrying
  capacity`);
```

$$fixed := \left\{[0], \left[\frac{3}{2} - \frac{\sqrt{105}}{10}\right], \left[\frac{3}{2} + \frac{\sqrt{105}}{10}\right]\right\}$$

but obviously

$$\left[\frac{3}{2} + \frac{\sqrt{105}}{10}\right]$$

is not a solution because it exceeds the carrying capacity **(9)**

(b) Find the stable equilibrium solutions (DISCRETE)

```
> SFP([(5/2)*x*(1-x)*(1-(1/2)*x)],[x]);
  print(`which is the floating point version of`);
  fixed[2];
```

$$\{[0.475304923]\}$$

which is the floating point version of

$$\left[\frac{3}{2} - \frac{\sqrt{105}}{10}\right]$$ **(10)**

(c) if at day zero its value is 0.1 what is its value at day 1000 with 10 decimal accuricy

```
> print(`answer to c`);
  Orb([(5/2)*x*(1-x)*(1-(1/2)*x)],[x],[0.1],1000,1000)[1];
```

answer to c

$[0.4753049232]$ **(11)**

Problem 4. Ath the first generations there are equal proportions of people with genotypes *AA*, *Aa*, and *aa*
Under the Hardy-Weinberg Hypothesis (Law)

First, Create the hardy-Weinberg Matrix
#USE HW3 instead of HW

```
> equalFreq:= HW3(1/3,1/3,1/3);
  #I see so HW outputs a 2x2 matrix
  Orb(HW3(u,v,w),[u,v,w],evalf([1/3,1/3,1/3]),2,2);
```

$$equalFreq := \left[\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right]$$

$$[[0.2500000000, 0.5000000000, 0.2500000000]] \quad \textbf{(12)}$$

(a) What Proportion of the Second Generation would have genotype *Aa*?
ans: Assuming tot is the same as survival

Ans: 50% (from above)

(b) Ans: 50% (code below)

```
> Orb(HW3(u,v,w),[u,v,w],evalf([1/3,1/3,1/3]),1000,1000);
```

$$[[0.2500000000, 0.5000000000, 0.2500000000]] \quad \textbf{(13)}$$

HWg(u,v,M): The Generalized Hardy-Weinberg unerlying transformation with (u,v), M is the survival matrix. Based on Ann Somalwar's HW3g(u,v,w) (only retain the first two components and replace w by 1-u-v)

Try:

HWg(u,v,[[1,2,1],[2,3,4],[1,3,2]]); **(14)**

Problem 5:
Ath the first generations there are equal proportions of people with genotypes *AA*, *Aa*, and *aa* **However** *AA* females are twice as likely to mate with *Aa* males than all the other 8 mating combinations

We will use HWg

```
> hw_p5 := HW3g(u,v,w,[[1,2,1],[1,1,1],[1,1,1]]);
```

$$hw\_p5 := \left[\frac{u^2 + \frac{3}{2} v u + \frac{1}{4} v^2}{u^2 + 3 v u + 2 u w + v^2 + 2 v w + w^2}, \frac{\frac{3}{2} v u + 2 u w + \frac{1}{2} v^2 + v w}{u^2 + 3 v u + 2 u w + v^2 + 2 v w + w^2}, \frac{\frac{1}{4} v^2 + v w + w^2}{u^2 + 3 v u + 2 u w + v^2 + 2 v w + w^2} \right] \quad \textbf{(15)}$$

(a) For generation 2:

```
> g2 := Orb(hw_p5,[u,v,w],evalf([1/3,1/3,1/3]),2,2);
  print(`ans:`);
  g2[1][2];
```

$$g2 := [[0.3027472527, 0.4989010989, 0.1983516484]]$$

ans:

$$0.4989010989 \quad \textbf{(16)}$$

(b) for generation 1000

```
> g1000:= Orb(hw_p5,[u,v,w],evalf([1/3,1/3,1/3]),1000,1000);
  print(`ans`);
  g1000[1][2];
```

$$g1000 := [[0.5512669093, 0.3974661800, 0.05126690975]]$$

ans

$$0.3974661800 \quad \textbf{(17)}$$

Problem 6: Suppose that

$$x(n) = \frac{1 + x(n-1) + y(n-1)}{2 + x(n-1) + 3\,y(n-1)}$$
$$y(n) = \frac{1 + x(n-1) + 3\,y(n-1)}{3 + x(n-1) + 2\,y(n-1)}$$

If $x(0) = 100$ and $y(0) = 1000$ what do you think that $y(100000000000000000000000000000000000000000000000000000)$ would be equal to? (rounded to 10 decimal places)

Ans: The smartest approach is to first find the equilibrium solutions of the system - in case there
This is because maple might not know how to do huge calculations like this. Maybe it can without the numbers getting super corrupted

```
> T := [(1+x+y)/(2+x+3*y),(1+x+3*y)/(3+x+2*y)];
  FP(T,[x,y]);
  print(`fixed points`);
  evalf(FP(T,[x,y]));
  print(`stable fixed point`);
  SFP(T,[x,y]);
```

$$T := \left[\frac{1+x+y}{2+x+3\,y}, \frac{1+x+3\,y}{3+x+2\,y}\right]$$

$$\left\{\left[RootOf\left(\_Z^4+7\,\_Z^3+\_Z^2-1\right), \frac{3\,RootOf\left(\_Z^4+7\,\_Z^3+\_Z^2-1\right)^3}{10} + \frac{11\,RootOf\left(\_Z^4+7\,\_Z^3+\_Z^2-1\right)^2}{5} + \frac{7\,RootOf\left(\_Z^4+7\,\_Z^3+\_Z^2-1\right)}{10} - \frac{1}{10}\right]\right\}$$

fixed points

$$\{[0.4705902280, 0.7478789082]\}$$

stable fixed points

$$\{[0.4705902280, 0.7478789082]\} \quad \textbf{(18)}$$

We see that eventually, many solutions will eventually reach the stable orbit shown above because having only 1 fixed point also being stable. To verify, use Orb to 1000th term

```
> Orb(T,[x,y],[100.,1000.],1000,1010);
```

[[0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080], [0.4705902280, 0.7478789080]] **(19)**

ANSWER to 6: 0.7478789082 Pretty close to what Orb says, but Orb isists 080 are last three digits instead of 082 is the fixed point every time

Problem 7: In the SIRS model

Let population $N = 1000$
Let paramater $\gamma = 0.5$
Let paramater $\nu = 100$

Suppose the start has 300 infected 300 Susceptible and hence 400 removed

(a) if β=0.05 in the long run, how many removed individuals will there be?

NOTE: In the SIRS model, removed individuals can return to the susceptible class. Therefore, if we have a fixed $N = 1000$
Then removed will be $R_{longTerm} = N - \left(S_{longTerm} + I_{longTerm}\right)$

Ans to (a): Long term, there should be 0 removed indivuals

```
> #The evidence below shows humanity lives, and the population the
  disease decays
  print(`from information below removed is 0`);
  lr:=SEquP(SIRS(s,i,0.05,0.5,100,1000),[s,i]);
  removed:=1000-(lr[1][1]+lr[1][2]);
```

from information below removed is 0

$$lr := \{[1000., 0.]\}$$

$$removed := 0. \quad \textbf{(20)}$$

(b) If β=1.4 in the long run, how many removed individuals would there be

```
> lr:= SEquP(SIRS(s,i,1.4,0.5,100,1000),[s,i]);
  removed := 1000-(lr[1][1]+lr[1][2]);
  print(`which rounds down to 923`)
```

$$lr := \{[71.42857143, 4.619758351]\}$$

$$removed := 923.9516702 \quad \textbf{(21)}$$

(c) What value of β is the cut off of when there would start to be a perpetual amount of infected people?

Ans: the cutoff occurs when $\beta > \frac{\nu}{N}$ which in this case is $\beta > \frac{100}{1000} = 0.1$

Question 8 GeneNet

(a) find to an accuracy of 10 decimals the exact height of the horiontal asymptote (aka the stable equilibrium is the same value for all 6 proteins) I can use any initial conditions (presumably nonnegative)!

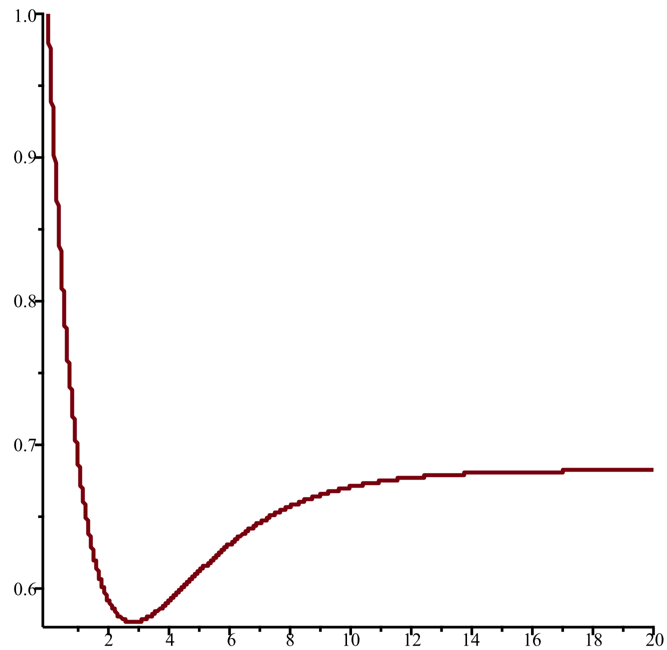
ans to (a): 0.6823278038

```
> print(GeneNet);
  TimeSeries(GeneNet(0,1,0.2,2,a,b,c,d,e,f),[a,b,c,d,e,f],[1.,1.,1.
  ,1.,1.,1.],0.01,20,1);
  print(`answer to (a) below using SEquP`);
  SEquP(GeneNet(0,1,0.2,2,a,b,c,d,e,f),[a,b,c,d,e,f]);
```

proc$(a0, a, b, n, m1, m2, m3, p1, p2, p3)$

$$[-m1 + a/(1 + p3\^{}n) + a0, -m2 + a/(1 + p1\^{}n) + a0, -m3 + a/(1 + p2\^{}n) + a0, -b*(p1 - m1), -b*(p2 - m2), -b*(p3 - m3)]$$

end proc



answer to (a) below using SEquP

{[0.6823278038, 0.6823278038, 0.6823278038, 0.6823278038, 0.6823278038, 0.6823278038]} **(22)**

(b) When changing α from 1 to 3, do I still have a horizontal asymptote? yes its ans to (b) 1.213411663

```
> SEquP(GeneNet(0,3,0.2,2,a,b,c,d,e,f),[a,b,c,d,e,f]);
```

{[1.213411663, 1.213411663, 1.213411663, 1.213411663, 1.213411663, 1.213411663]} **(23)**

(c) find a value of α when there is still a stable equilibrium

```
> SEquP(GeneNet(0,7.39,0.2,2,a,b,c,d,e,f),[a,b,c,d,e,f]);
```

{[1.777450148, 1.777450148, 1.777450148, 1.777450148, 1.777450148, 1.777450148]} **(24)**

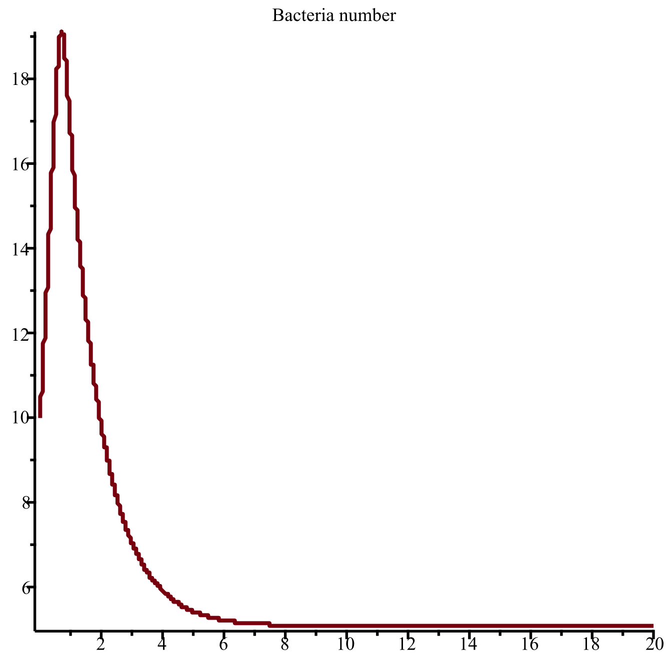
Ans to (c): alpha = 7.39 is the largest 2-decimal value that still leads to a horizontal asymptote

Question 9: In the chemostat model, Look at equations 19a and 19b with $\alpha_1 = 2.5$ and $\alpha_2 = 2.7$
*NOTE: Nutrient Concentration is denoted by "C" and bacteria concentration is denoted by "N"

(a) What would be the value of the Bacteria population after a very long time?

ANS: 5.083333333

```
> #ANS: we see the eventual bacteria
  TimeSeries(ChemoStat(N,C,2.5,2.7),[N,C],[10.,10.],0.01,20,1);
  Help(SEquP);
  #SEquP is the best way to go
  SEquP(ChemoStat(N,C,2.5,2.7),[N,C]);
```



SEquP(F,x): Given a transformation F in the list of variables finds all the Stable Equilibrium points of the continuous-time dynamical system x'(t)=F(x(t))

*SEquP([5/2*x*(1-x)],[x]);*

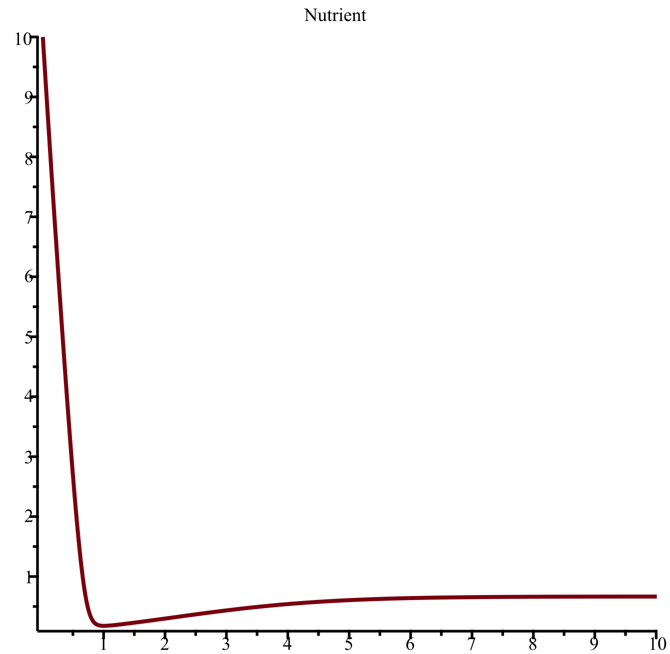
SEquP([y(1-x-y),x*(3-2*x-y)],[x,y]);*

$$\{[5.083333333, 0.6666666667]\} \quad \textbf{(25)}$$

(b) What would be the value of the Nutrient concentration after a very longtime?

ANS: 0.6666666667

```
> TimeSeries(ChemoStat(N,C,2.5,2.7),[N,C],[10.,10.],0.01,10,2);
```



Question 10: Consider a mini-internet with websites 1,2,... 9
A random surfer Who is currently on either site 1,2, or 3 will stay at their website with a probability of 0.2. The chance leaving the sight is equally distributed (in this case, 0.1 for each webpage because 8 left
A random surfer Who is currently on either site 4,5, or 6 will stay at their website with a probability of 0.4. The chance leaving the sight is equally distributed (in this case, 0.6/8 for each webpage because 8 left
A random surfer Who is currently on either site 7,8, or 9 will stay at their website with a probability of 0.6. The chance leaving the sight is equally distributed (in this case, 0.05 for each webpage because 8 left

In the long run (a)What is the probability that a surfer will be at page 1?
(b)What is the probability that a surfer will be at page 9?

Solution: construct a system of linear equations

```
> f1 := s[1]=0.2*s[1]+0.1*(s[2]+s[3])+(0.6/8)*(s[4]+s[5]+s[6])
  +0.05*(s[7]+s[8]+s[9]);
  f2 := s[2]=0.2*s[2]+0.1*(s[1]+s[3])+(0.6/8)*(s[4]+s[5]+s[6])
  +0.05*(s[7]+s[8]+s[9]);
  f3 := s[3]=0.2*s[3]+0.1*(s[1]+s[2])+(0.6/8)*(s[4]+s[5]+s[6])
  +0.05*(s[7]+s[8]+s[9]);
  #
  f4 := s[4]=0.1*(s[1]+s[2]+s[3])+0.4*s[4]+(0.6/8)*(s[5]+s[6])
  +0.05*(s[7]+s[8]+s[9]);
  f5 := s[5]=0.1*(s[1]+s[2]+s[3])+0.4*s[5]+(0.6/8)*(s[4]+s[6])
  +0.05*(s[7]+s[8]+s[9]);
  f6 := s[6]=0.1*(s[1]+s[2]+s[3])+0.4*s[6]+(0.6/8)*(s[4]+s[5])
  +0.05*(s[7]+s[8]+s[9]);
```

```
#
f7 := s[7]=0.1*(s[1]+s[2]+s[3])+(0.6/8)*(s[4]+s[5]+s[6])+0.05*(s
[9]+s[8])+0.6*s[7];
f8 := s[8]=0.1*(s[1]+s[2]+s[3])+(0.6/8)*(s[4]+s[5]+s[6])+0.05*(s
[7]+s[9])+0.6*s[8];
f9 := s[9]=0.1*(s[1]+s[2]+s[3])+(0.6/8)*(s[4]+s[5]+s[6])+0.05*(s
[7]+s[8])+0.6*s[9];
total := s[1]+s[2]+s[3]+s[4]+s[5]+s[6]+s[7]+s[8]+s[9] = 1;
```

$f1 := s_1 = 0.2\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_8 + 0.05\, s_9$

$f2 := s_2 = 0.2\, s_2 + 0.1\, s_1 + 0.1\, s_3 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_8 + 0.05\, s_9$

$f3 := s_3 = 0.2\, s_3 + 0.1\, s_1 + 0.1\, s_2 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_8 + 0.05\, s_9$

$f4 := s_4 = 0.1\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.4\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_8 + 0.05\, s_9$

$f5 := s_5 = 0.1\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.4\, s_5 + 0.07500000000\, s_4 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_8 + 0.05\, s_9$

$f6 := s_6 = 0.1\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.4\, s_6 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.05\, s_7 + 0.05\, s_8 + 0.05\, s_9$

$f7 := s_7 = 0.1\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_9 + 0.05\, s_8 + 0.6\, s_7$

$f8 := s_8 = 0.1\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_9 + 0.6\, s_8$

$f9 := s_9 = 0.1\, s_1 + 0.1\, s_2 + 0.1\, s_3 + 0.07500000000\, s_4 + 0.07500000000\, s_5 + 0.07500000000\, s_6 + 0.05\, s_7 + 0.05\, s_8 + 0.6\, s_9$

$$total := s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8 + s_9 = 1 \quad \textbf{(26)}$$

```
> solve({f1,f2,f3,f4,f5,f6,f7,f8,f9,total},[s[1],s[2],s[3],s[4],s
[5],s[6],s[7],s[8],s[9]]);
```

$$[[s_1 = 0.07692307692, s_2 = 0.07692307692, s_3 = 0.07692307692, s_4 = 0.1025641026, s_5 = 0.1025641026, s_6 = 0.1025641026, s_7 = 0.1538461538, s_8 = 0.1538461538, s_9 = 0.1538461538]] \quad \textbf{(27)}$$

Ans to (a) 0.07692307692 is the steady state of s1
Ans to (b) 0.1538461538 is the steady state of s9