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Open Problems and Conjectures

Edited by Gerry Ladas

In this section we present some open problems and conjectures about some interesting types of difference equations. Please submit your problems and conjectures with all relevant information to G. Ladas.

Convergence to Periodic Solutions

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It was shown in [5] that every solution of the rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + Bx_n}, \quad n = 0, 1, \ldots \tag{1}$$

with nonnegative initial conditions and

$$B \in (0, \infty) \quad \text{and} \quad \alpha, \beta, \gamma, A \in [0, \infty) \tag{2}$$

converges to a period two solution if and only if

$$\gamma = \beta + A. \tag{3}$$

The character of solutions of Eq. (1) when either $\gamma > \beta + A$ or $\gamma < \beta + A$ was also investigated in [5]. See also [4].

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Our aim here is to pose several open problems and conjectures about rational difference equations with the property that all their solutions converge to periodic solutions with the same period.

Throughout this paper, when we say that "every solution of a certain difference equation converges to a periodic solution with period p" we mean that every solution converges to a, not necessarily prime, solution with period p and furthermore the set of solutions of the equation with prime period p is nonempty.

First we pose some conjectures about the following third order rational difference equations with nonnegative initial conditions x_{-2}, x_{-1}, and x_0:

$$x_{n+1} = \frac{x_{n-1}}{x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots \tag{4}$$

$$x_{n+1} = \frac{x_n + x_{n-2}}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{5}$$

$$x_{n+1} = \frac{1 + x_{n-2}}{x_n}, \quad n = 0, 1, \ldots \tag{6}$$

$$x_{n+1} = \frac{1 + x_n}{x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots . \tag{7}$$

CONJECTURE 1 *Every positive solution of Eq.* (4) *converges to a period two solution of Eq.* (4) *of the form*:

$$\ldots, \phi, 1 - \phi, \ldots$$

with $0 \leq \phi \leq 1$.

CONJECTURE 2 *Every positive solution of Eq.* (5) *converges to a period four solution of Eq.* (5) *of the form*:

$$\ldots, \phi, \psi, \frac{\phi + \psi^2}{\phi\psi - 1}, \frac{\phi^2 + \psi}{\phi\psi - 1}, \ldots$$

with $\phi, \psi \in (0, \infty)$ *and* $\phi\psi > 1$.

CONJECTURE 3 (= **$20**[1]) *Every positive solution of Eq.* (6) *converges to a period five solution of Eq.* (6) *of the form*:

$$\ldots, \phi, \psi, \frac{1+\phi}{\phi\psi - 1}, \phi\psi - 1, \frac{1+\psi}{\phi\psi - 1}, \ldots$$

with $\phi, \psi \in (0, \infty)$ *and* $\phi\psi > 1$.

CONJECTURE 4 (= **$20**[1]) *Every positive solution of Eq.* (7) *converges to a period six solution of Eq.* (7) *of the form*:

$$\ldots, \phi, \psi, \frac{\psi}{\phi}, \frac{1}{\phi}, \frac{1}{\psi}, \frac{\phi}{\psi}, \ldots$$

with $\phi, \psi \in (0, \infty)$.

Note that our conjecture implies that every positive solution of Eq. (7) converges to a positive solution of the difference equation

$$y_{n+1} = \frac{y_n}{y_{n-1}}, \quad n = 0, 1, \ldots .$$

More generally we conjecture that for every $p \in (0, \infty)$, every positive solution of the difference equation

$$x_{n+1} = \frac{(1/p^2) + x_n}{p x_{n-1} + x_{n-2}}, \quad n = 0, 1, \ldots \tag{8}$$

converges to a period six solution of Eq. (8) of the form:

$$\ldots, \phi, \psi, \frac{\psi}{p\phi}, \frac{1}{p^2\phi}, \frac{1}{p^2\psi}, \frac{\phi}{p\psi}, \ldots$$

with $\phi, \psi \in (0, \infty)$, that is, to a positive solution of the difference equation

$$y_{n+1} = \frac{y_n}{p y_{n-1}}, \quad n = 0, 1, \ldots .$$

- What is it that makes every solution of a difference equation converge to a solution of another difference equation?

[1]To the first person who submits a correct solution to G. Ladas by 1/1/2002.

This is a question of paramount importance for the development of the basic theory of difference equations. Some related questions concerning periodicity are the following.

- What is it that makes every solution of a difference equation periodic with the same period?
- What is it that makes every solution of a difference equation converge to a period k solution, for some fixed positive integer k?

In this direction we pose some open problems for the general third order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + Bx_n + Cx_{n-1} + Dx_{n-2}}, \quad n = 0, 1, \dots \tag{9}$$

with nonnegative parameters and nonnegative initial conditions x_{-2}, x_{-1}, x_0 such that

$$A + Bx_n + Cx_{n-1} + Dx_{n-2} > 0 \quad \text{for all } n \geq 0.$$

For some known results on some special cases of Eq. (9) dealing with second order rational difference equations, see [1], [3–9] and [11]. See also [2, 3] and [10] for difference equations of the form

$$x_{n+1} = \sum_{i=0}^{k} \frac{A_i}{x_{n-i}}, \quad n = 0, 1, \dots$$

which by the change of variables $x_n = (1/y_n)$ reduce to rational difference equations of the form

$$y_{n+1} = \frac{1}{\sum_{i=0}^{k} A_i y_{n-i}}, \quad n = 0, 1, \dots .$$

Open Problem 1 (= **\$30**[1]) Find all special cases of nonlinear difference equations of the form of Eq. (9) with the property that every solution of each of these equations converges to a periodic solution with period 2.

Computer observations indicate that, in addition to Eq. (4), the following equations offer a partial answer to the above open problem.

Here we only present examples for which an analytic proof is neither known nor obvious.

(i)

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_{n-2}}, \quad n = 0, 1, \dots . \tag{10}$$

Every solution of Eq. (10) converges to a period two solution of the form

$$\dots, \phi, \frac{1}{\phi}, \dots$$

with $\phi > 0$.

(ii)

$$x_{n+1} = \frac{x_n + x_{n-1}}{x_n + x_{n-2}}, \quad n = 0, 1, \dots . \tag{11}$$

Every solution of Eq. (11) converges to a period two solution of the form

$$\dots, \phi, \frac{\phi}{2\phi - 1}, \dots$$

with $\phi > (1/2)$.

(iii)

$$x_{n+1} = \frac{x_{n-1} + x_{n-2}}{x_n + x_{n-2}}, \quad n = 0, 1, \dots . \tag{12}$$

Every solution of Eq. (12) converges to a period two solution of the form

$$\dots, \phi, \frac{\phi}{2\phi - 1}, \dots$$

with $\phi > (1/2)$.

(iv)

$$x_{n+1} = \frac{x_{n-1} + x_{n-2}}{x_{n-2}}, \quad n = 0, 1, \dots . \tag{13}$$

Every solution of Eq. (13) converges to a period two solution of the form

$$\ldots, \phi, \frac{\phi}{\phi - 1}, \ldots$$

with $\phi > 1$.

(v)

$$x_{n+1} = \frac{1 + x_{n-1}}{1 + x_n + x_{n-2}}, \quad n = 0, 1, \ldots . \tag{14}$$

Every solution of Eq. (14) converges to a period two solution of the form

$$\ldots, \phi, \frac{1}{2\phi}, \ldots$$

with $\phi > 0$.

(vi)

$$x_{n+1} = \frac{1 + x_{n-1} + x_{n-2}}{x_{n-2}}, \quad n = 0, 1, \ldots . \tag{15}$$

Every solution of Eq. (15) converges to a period two solution of the form

$$\ldots, \phi, \frac{1 + \phi}{\phi - 1}, \ldots$$

with $\phi > 1$.

One can see that the character of prime period two solutions that Eq. (9) possesses depends on whether or not $C = 0$. For Eq. (9) to have a unique period two solution

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{16}$$

it is necessary (but not sufficient) that $C > 0$. On the other hand when $C = 0$ and $B + D > 0$, Eq. (9) possesses prime period two solution if and only if

$$\gamma = \beta + A + \delta. \tag{17}$$

Furthermore in this case, the values ϕ and ψ of all prime period two solutions (16) of Eq. (9) are given by

$$(B+D)\phi\psi = \alpha + (\beta+\delta)(\phi+\psi)$$

with

$$\phi, \psi \in [0, \infty) \quad \text{and} \quad \phi \neq \psi.$$

The character of solutions of Eq. (9) when either

$$\gamma < \beta + A + \delta \tag{18}$$

or

$$\gamma > \beta + A + \delta \tag{19}$$

is also very important to investigate.

Open Problem 2 (= **\$30**[1]) Assume that

$$C = 0 \quad \text{and} \quad B + D > 0.$$

(a) In addition to (17), what other conditions are needed so that every positive solution of Eq. (9) converges to a period two solution?
(b) In addition to (18), what other conditions are needed so that every positive solution of Eq. (9) converges to the equilibrium of the equation?
(c) In addition to (19), what other conditions are needed so that Eq. (9) possesses unbounded solutions?

Open Problem 3 Find all special cases of Eq. (9) with the property that every solution of the equation converges to a periodic solution with period $k \in \{3, 4, 5, 6\}$.

Open Problem 4 Assume $k \geq 7$. Find all special cases of Eq. (9) with the property that every solution of the equation converges to a periodic solution with period k.

Open Problem 5 Assume that for some choice of the parameters and for some $k \geq 2$, Eq. (9) has a unique locally asymptotically stable prime period k solution and a unique positive equilibrium. Is, in this case, the equilibrium always a saddle point?

The following nonlinear difference equations are special cases of Eq. (9) with the property that every positive solution of each of these equations is periodic with the same period.

$$\text{Period } 2: \quad x_{n+1} = \frac{1}{x_n}, \quad n = 0, 1, \ldots \tag{20}$$

$$\text{Period } 4: \quad x_{n+1} = \frac{1}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{21}$$

$$\text{Period } 5: \quad x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{22}$$

$$\text{Period } 6: \quad x_{n+1} = \frac{1}{x_{n-2}}, \quad n = 0, 1, \ldots \tag{23}$$

$$\text{Period } 6: \quad x_{n+1} = \frac{x_n}{x_{n-1}}, \quad n = 0, 1, \ldots \tag{24}$$

$$\text{Period } 8: \quad x_{n+1} = \frac{1 + x_n + x_{n-1}}{x_{n-2}}, \quad n = 0, 1, \ldots. \tag{25}$$

Open Problem 6 In addition to Eqs. (20)–(25), are there other, essentially different, special cases of Eq. (9) with the property that every solution is periodic with the same period?

An open problem of paramount importance for Eq. (1) is the following.

Open Problem 7 (= **\$20**[1]) Assume that (2) and (3) hold.

(a) Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of Eq. (1). Determine, in terms of the initial conditions x_{-1} and x_0, the period two solution of Eq. (1)

$$\ldots, \phi, \psi, \phi, \psi, \ldots \tag{26}$$

to which $\{x_n\}_{n=-1}^{\infty}$ converges.

(b) Let (26) be a period two solution of Eq. (1). Determine the set of initial conditions x_{-1} and x_0 for which the corresponding solution $\{x_n\}_{n=-1}^{\infty}$ of Eq. (1) converges to (26).

The change of variables $x_n = (1/y_n)$ reduces the difference equation

$$x_{n+1} = \frac{x_{n-1}}{x_n + x_{n-1}}, \quad n = 0, 1, \ldots \tag{27}$$

to

$$y_{n+1} = 1 + \frac{y_{n-1}}{y_n}, \quad n = 0, 1, \ldots$$

and so it follows from [1] that every positive solution of Eq. (27) converges to a period two solution of Eq. (27).

It is interesting that this result generalizes to the following.

THEOREM 1 *Let k be a positive integer. Then every positive solution of the difference equation*

$$x_{n+1} = \frac{x_{n-k}}{x_n + x_{n-1} + \cdots + x_{n-k}}, \quad n = 0, 1, \ldots \tag{28}$$

converges to a period $(k+1)$ *solution of Eq.* (28).

Proof Observe that

$$x_{n+1} - x_{n-k} = \frac{x_{n-k}}{x_n + \cdots + x_{n-k}}(1 - x_n - \cdots - x_{n-k}) \quad \text{for } n \geq 0$$

and that the quantity

$$J_n = 1 - x_n - \cdots - x_{n-k}$$

has a constant sign because

$$J_{n+1} = \frac{x_n + \cdots + x_{n-k+1}}{x_n + \cdots + x_{n-k}} J_n \quad \text{for } n \geq 0.$$

The proof is now a consequence of the above observations and the fact that every positive solution of Eq. (28) is bounded. ■

The following open problem is asking for an extension of Theorem 1.

Open Problem 8 Assume that k is a positive integer and that

$$A, B_0, B_1, B_2, \ldots, B_k \in [0, \infty) \quad \text{with } \sum_{i=0}^{k} B_i > 0.$$

Obtain necessary and sufficient conditions on the parameters A and $B_0, \ldots, B_k$ so that every positive solution of the difference equation

$$x_{n+1} = \frac{x_{n-k}}{A + B_0 x_n + \cdots + B_k x_{n-k}}, \quad n = 0, 1, \ldots \tag{29}$$

converges to a period $(k+1)$ solution of Eq. (29).

Note that the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \quad n = 0, 1, \ldots \tag{30}$$

is a special case of Eq. (1) with $\alpha = \beta = 0$, $\gamma = 1$, and $A = B = 1$, for which condition (3) is satisfied. Therefore every nonnegative solution of Eq. (30) converges to a (not necessarily prime) period two solution of Eq. (30), namely,

$$\ldots, 0, \phi, 0, \phi, \ldots$$

with $\phi \geq 0$.

For the more general equation

$$x_{n+1} = \frac{x_{n-k}}{1 + x_n + \cdots + x_{n-k+1}}, \quad n = 0, 1, \ldots \tag{31}$$

where k is a positive integer one can easily show that every nonnegative solution of Eq. (31) converges to a period $(k+1)$ solution of the form

$$\ldots, \underbrace{0, 0, \ldots, 0}_{k-\text{terms}}, \phi, \ldots \tag{32}$$

with $\phi \geq 0$.

Open Problem 9 (**=\$20**[1])

(a) Does Eq. (31) possess a positive solution which converges to zero?
(b) Can we determine, in terms of the nonnegative initial conditions $x_{-k}, \ldots, x_0$, the value of ϕ in (32) which corresponds to the nonnegative solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (31)?

(c) Can we determine the set of all nonnegative initial conditions $x_{-k}, \ldots, x_0$ for which the corresponding solution $\{x_n\}_{n=-k}^{\infty}$ of Eq. (31) converges to a period $(k+1)$ solution

$$\ldots, \underbrace{0, 0, \ldots, 0}_{k-\text{terms}}, \phi, \ldots$$

with a given $\phi \geq 0$?

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