Solutions to the Attendance Quiz for July 16, 2012 [Combinatorics Special Lecture]

General Comments Only three students got both questions completely, I am really impressed! Quite a few were on the right track, but I realized that I didn't describe all the steps of the Joyal mappings carefully. Hopefully the detailed solutions below would help clarify them.

One student asked me for a reference. The origial proof is in Joyal's paper (in French) in the journal *Advances in Mathematics* v.**42** (1981), pp 1-82, available via IRIS to Rutgers students and faculty.

1. Apply the Joyal bijection that inputs sequences in $\{1,\dots,n\}^n$ and outputs a **doubly rooted** labeled tree on $\{1,\dots,n\}$. To the following sequence (with $n=9$)

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Indicate clearly ROOT A and ROOT B. Draw the output, but also write it as a set of edges.

Sol. to 1: We first write the sequence in two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 1 & 6 & 1 & 1 & 8 & 2 & 7 \end{pmatrix}$$

Next we draw the graph of the function, i.e. a graph whose directed edges are

$$1 \to 4 \quad 2 \to 7 \quad 3 \to 1 \quad 4 \to 6 \quad 5 \to 1 \quad 6 \to 1 \quad 7 \to 8 \quad 8 \to 2 \quad 9 \to 7 \quad .$$

We see that we have two cycles:

$$1 \to 4 \to 6 \to 1 \qquad 2 \to 7 \to 8 \to 2$$

whose participants are $\{1,2,4,6,7,8\}$. This induces a permutation in two-line notation (remember that we have to **order** the top row:

$$\begin{pmatrix} 1 & 2 & 4 & 6 & 7 & 8 \\ 4 & 7 & 6 & 1 & 8 & 2 \end{pmatrix}$$

Going back to **one-line notation** we look at the bottom line, and see that the **skeleton** is:

$$4 \to 7 \to 6 \to 1 \to 8 \to 2 \quad .$$

This immediately gives that $A = 4$ and $B = 2$. Finally we reinstate the trees (possibly trivial) "hanging out" from our picture, the only non-trivial ones being

$$\{3 \to 1 \quad , \quad 5 \to 1\} \quad \textit{and} \quad \{9 \to 7\} \quad .$$

So the set of edges of labeled tree is

$$\{47, 67, 16, 18, 28, 13, 15, 79\}$$

Ans.: Set of Edges of the tree: (in lexicographic order) $\{13, 15, 16, 18, 28, 47, 67, 79\}$

Root A: 4 **Root B**: 2

2. Apply the Reverse Joyal bijection that inputs a **doubly rooted** labeled tree on $\{1, \ldots, n\}$ and outputs a sequence in $\{1, \ldots, n\}^n$, with $n = 9$.

Set of Edges of the tree: $\{13, 18, 19, 25, 34, 35, 36, 57\}$ Root $A = 1$, Root $B = 2$.

Sol. of 2: We first write down the **skeleton**

$$1 \rightarrow 3 \rightarrow 5 \rightarrow 2$$

and note the "hanging-trees" around its members:

Around 1: $\{9 \rightarrow 1, 8 \rightarrow 1\}$

Around 3: $\{6 \rightarrow 3, 4 \rightarrow 3\}$

Around 5: $\{7 \rightarrow 5\}$

Around 2: trivial tree only consisting of the vertex 2.

We now look at the skeleton 1352 and rewrite it in **two-line notation** where the top line is the **sorted** version of it

$$\begin{pmatrix} 1 & 2 & 3 & 5 \\ 1 & 3 & 5 & 2 \end{pmatrix}$$

We now draw the graph of this partial function with edges consisting of the cycles $1 \rightarrow 1$ and $2 \rightarrow 3 \rightarrow 5 \rightarrow 2$. Finally we reinstate the above trees at the right places, yielding the function from $\{1, \ldots, 9\}$ to $\{1, \ldots, 9\}$

$$f(1) = 1, f(2) = 3, f(3) = 5, f(4) = 3, f(5) = 2, f(6) = 3, f(7) = 5, f(8) = 1, f(9) = 1$$

i.e., in two-line notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 3 & 5 & 3 & 2 & 3 & 5 & 1 & 1 \end{pmatrix}$$

Finally, since the top line is $1 \ldots 9$ it can be dropped and we get the sequence

Ans.: 135323511 .