Classification of singular radial solutions to the σ_k Yamabe equation on annular domains

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Abstract

The study of the k-th elementary symmetric function of the Weyl-Schouten curvature tensor of a Riemannian metric, the so called σ_k curvature, has produced many fruitful results in conformal geometry in recent years. In these studies in conformal geometry, the deforming conformal factor is considered to be a solution of a fully nonlinear elliptic PDE. Important advances have been made in recent years in the understanding of the analytic behavior of solutions of the PDE. However, the singular behavior of these solutions, which is important in describing many important questions in conformal geometry, is little understood. This note classifies all possible radial solutions, in particular, the *singular* solutions of the σ_k Yamabe equation, which describes conformal metrics whose σ_k curvature equals a constant. Although the analysis involved is of elementary nature, these results

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should provide useful guidance in studying the behavior of singular solutions in the general situation.

Keywords: σ_k *curvature, Schouten curvature, singular radial solution, conformal metric, generalized Yamabe equation.*

1 Description of the results

This note is concerned with radial solutions to the equation

$$\sigma_k(A_g) = \text{constant}, \tag{1}$$

on domains of the form $\{ x \in \mathbb{R}^n : r_1 < |x| < r_2 \}$, where A_g is the Weyl-Schouten tensor of the conformal metric $g = v^{-2}(|x|)|dx|^2$:

$$A_g = \frac{1}{n-2}\{Ric - \frac{R}{2(n-1)} g\},$$

$\sigma_k(A_g)$ denotes the k-th elementary symmetric function of the eigenvalues of A_g with respect to g, and $0 \leq r_1 < r_2 \leq \infty$. In this note we classify all possible radial solutions, in particular, the *singular* solutions. The main motivation for our study is to use these results as guidance in studying the behavior of singular solutions in the general situation.

There are at least two kinds of geometric considerations that lead to the study of singular solutions of equations of the above type. The first kind is related to the characterization of the size of the limit set of the image domain in $\mathbb{S}^n$ of the developing map of a locally conformally flat n-manifold. More specifically, one is led to find necessary/sufficient conditions on a domain $\Omega \subset \mathbb{S}^n$ so that it admits a conformal metric $g = v^{-2}(x)|dx|^2$ which is complete, and with its Weyl-Schouten tensor A_g in the $\Gamma_k^{\pm}$ class, *i.e.*, the eigenvalues, $\lambda_1 \leq \cdots \leq \lambda_n$, of A_g at each $x \in \Omega$ satisfy $\sigma_j(\lambda_1, \cdots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$, in the case of Γ_k^+; and $(-1)^j \sigma_j(\lambda_1, \cdots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$, in the case of Γ_k^-. For $k \geq 2$, it is natural to restrict to metrics whose Weyl-Schouten tensor is in the $\Gamma_k^{\pm}$ class, because, for a metric in such a class, (1) becomes a *fully nonlinear* PDE in v that is *elliptic* . In the case of $k = 1$, $\sigma_1(A_g)$ is simply a constant multiple of the scalar curvature of g; so A_g in the $\Gamma_k^{\pm}$ class is a generalization of the notion that the scalar curvature R_g of g having a fixed $\pm$ sign. For the positive scalar curvature case, in [38], Schoen-Yau proved that if a complete metric $g = v^{-2}(x)|dx|^2$ exists on a domain $\Omega \subset \mathbb{S}^n$ with $\sigma_1(A_g)$ having a positive lower bound, then the Hausdorff dimension of $\partial\Omega$ has to be $\leq \frac{n-2}{2}$. Later Mazzeo and Pacard [31] proved that if $\Omega \subset \mathbb{S}^n$ is a domain such that $\mathbb{S}^n \setminus \Omega$ consists a finite number of smooth submanifolds of dimension $1 \leq k \leq \frac{n-2}{2}$, then one can find a complete metric $g = v^{-2}(x)|dx|^2$ on Ω with its scalar curvature identical to $+1$. For the negative scalar curvature case, the work of Loewner-Nirenberg [30], Aviles [1], and Veron [39] implies that if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric with negative constant scalar curvature, then the Hausdorff dimension of $\partial\Omega > \frac{n-2}{2}$. Loewner-Nirenberg [30] also proved that if $\Omega \subset \mathbb{S}^n$ is a domain with smooth boundary $\partial\Omega$ of dimension $> \frac{n-2}{2}$, then there exists a complete metric $g = v^{-2}(x)|dx|^2$ on Ω with $\sigma_1(A_g) = -1$. This result was later generalized by D. Finn [11] to the case of $\partial\Omega$ consisting of smooth submanifolds of dimension $> \frac{n-2}{2}$ and with boundary. So the consideration of singular solutions of equations of type (1) can be considered as a natural generalization of these known results. In fact, in [8], Chang, Hang, and Yang proved that if $\Omega \subset \mathbb{S}^n$ $(n \geq 5)$ admits a complete, conformal metric g with

$$\sigma_1(A_g) \geq c_1 > 0, \quad \sigma_2(A_g) \geq 0, \quad \text{and}$$

$$|R_g| + |\nabla_g R|_g \leq c_0, \tag{2}$$

then $\dim(\mathbb{S}^n \setminus \Omega) < \frac{n-4}{2}$. This has been generalized by M. Gonzalez to the case of $2 < k < n/2$: if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric g with

$$\sigma_1(A_g) \geq c_1 > 0, \quad \sigma_2(A_g), \ \cdots \ \sigma_k(A_g) \geq 0, \quad \text{and}$$

(2), then $\dim(\mathbb{S}^n \setminus \Omega) < \frac{n-2k}{2}$. See also the work of Guan, Lin and Wang [15].

From a more broad perspective, one main reason for the attention to the σ_k-curvature $(k > 1)$ in conformal geometry is that these curvatures place a much stronger control on the curvature tensor. In dimension 4, the σ_2 curvature comes into play in the Chern-Gauss-Bonnet formula. Chang, Gursky and Yang [4] observed that if $\sigma_1(A_g), \sigma_2(A_g) > 0$ at a point on a 4-dimensional manifold, then the Ricci tensor of g is positive definite at that point. This algebraic relation has been generalized to higher dimensions by Guan, Viaclovsky, and Wang [16]. The first important application of the σ_k-curvature to conformal geometry is the main theorem in [4], where the authors proved that if (i) $\int_{M^4} \sigma_2(A_g) d\ vol$, which is conformally invariant on M^4, is positive; and (ii) the Yamabe class of (M^4, g) is positive, then there is a conformal metric $\tilde{g} = e^{2w} g$ on M^4 such that $A_{\tilde{g}} \in \Gamma_2^+$.

The second kind of consideration for studying singular solutions of (1) is due to a basic phenomenon of solutions of the more general equation, $\sigma_k(A_g) = f(x, v)$: for a solution v with its $A_g \in \Gamma_k^+$, there is higher derivative estimates for the solution v in terms of the C^0 estimates of v [40] [17]; while for a solution v with its $A_g \in \Gamma_k^-$, there is a lack of second derivative estimates for v, even when the C^0 norm of v is under control. The singular radial solutions exhibit this behavior explicitly.

Let us formulate our tasks more explicitly: the problem of the classification of radial solutions of (1) consists of determining (a) the maximal domain of definition of each solution (a finite ball, a finite punctured ball, an annulus, $\mathbb{R}^n \setminus \{0\}$, or the entire $\mathbb{R}^n$?); (b) the limiting behavior of each solution upon approaching the limits of its domain of definition and the geometric meanings of such behavior (completeness vs incompleteness, etc); and (c) whether the solution is the the Γ^+ or Γ^- class.

First, let us work out (1) more explicitly in the case of radial solutions. Let

$$g = v^{-2}(|x|)|dx|^2, \quad \text{and} \quad |x| = r.$$

Then

$$\begin{aligned} A_{ij} &= \tfrac{v_{ij}}{v} - \tfrac{|\nabla v|^2}{2v^2}\delta_{ij} \\ &= \lambda \delta_{ij} + \mu \tfrac{x_i x_j}{|x|^2}, \end{aligned}$$

with $\lambda = \frac{v_r}{rv}(1 - \frac{r v_r}{2v})$ and $\mu = \frac{v_{rr}}{v} - \frac{v_r}{rv}$. The eigenvalues of A with respect to $|dx|^2$ are λ with multiplicity $(n-1)$, and $\lambda + \mu$ with multiplicity 1. The formula for $\sigma_k(A_g)$ can be found easily by the binomial expansion of $(x-\lambda)^{n-1}(x-\lambda-\mu)$:

$$\sigma_k(A_g) = c_{n,k} v^{2k} \lambda^{k-1}(n\lambda + k\mu), \tag{3}$$

where $c_{n,k} = \frac{(n-1)!}{k!(n-k)!}$.

First, the following observation follows directly from (3):

Remark 1. *If $k > 1$ and $v(r) > 0$ is a C^2 function on $r_1 < r < r_2$ such that $\sigma_k(A_{g_v})$ has a fixed sign for $r_1 < r < r_2$, then $v(r)$ is strictly monotone on $r_1 < r < r_2$. More specifically $\lambda = \frac{v_r}{rv}(1 - \frac{r v_r}{2v})$ never changes sign on $r_1 < r < r_2$.*

Next we introduce new variables, $t = \ln r$, $x = r\omega$ with $\omega \in \mathbb{S}^{n-1}$, and $\xi(t)$ such that

$$g = v^{-2}(r)|dx|^2 = e^{-2\xi}(dt^2 + d\omega^2),$$

then

$$\xi + t = \ln v, \qquad v_r = e^{\xi}(\xi_t + 1) = (\xi_t + 1)v/r,$$

and

$$v_{rr} = e^{\xi - t}[\xi_{tt} + \xi_t(\xi_t + 1)] = [\xi_{tt} + \xi_t(\xi_t + 1)]ve^{-2t}.$$

Thus

$$\lambda = \frac{1 - \xi_t^2}{2e^{2t}}, \qquad \text{and} \qquad \mu = e^{-2t}(\xi_{tt} + \xi_t^2 - 1).$$

(3) then becomes

$$\sigma_k(A_g) = c_{n,k}e^{2k(\xi+t)}\frac{(1-\xi_t^2)^{k-1}}{2^{k-1}e^{2(k-1)t}}\left[n\frac{1-\xi_t^2}{2e^{2t}}+k\frac{\xi_{tt}+\xi_t^2-1}{e^{2t}}\right]$$

$$= c'_{n,k}(1-\xi_t^2)^{k-1}\left[\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)\right]e^{2k\xi}, \qquad (4)$$

where $c'_{n,k} = nc_{n,k}2^{1-k} = 2^{1-k}\binom{n}{k}$.

The next proposition summarizes some further elementary properties of solutions of (4).

Proposition 1. *Consider any C^2 solution of* (4) *on an interval (t_-, t_+) with $k > 1$.*

1. *If σ_k has a fixed sign on the interval (t_-, t_+), then either $1-\xi_t^2 > 0$ or $1-\xi_t^2 < 0$ on (t_-, t_+).*
2. *If $\sigma_k > 0$ and $1-\xi_t^2 > 0$ on (t_-, t_+), then $v^{-2}(|x|)|dx|^2$ automatically stays in the Γ_k^+ class,* i.e., *it also satisfies $\sigma_l > 0$ for any $1 \le l < k$.*
3. *If $\sigma_k > 0$ and $1-\xi_t^2 < 0$ on (t_-, t_+), and k is even, then $v^{-2}(|x|)|dx|^2$ automatically stays in the Γ_k^- class,* i.e., *it also satisfies $(-1)^l\sigma_l > 0$ for any $1 \le l < k$.*
4. *If $\sigma_k < 0$ and $1-\xi_t^2 < 0$ on (t_-, t_+), and k is odd, then $v^{-2}(|x|)|dx|^2$ automatically stays in the Γ_k^- class.*
5. *if $k > 1$, $0 < r_* < \infty$ is a limit point of the domain of definition of a solution v of* (3) *with $\sigma_k(A_g) =$ constant, and v stays bounded away from 0 and ∞ upon approaching r_*, then v approaches a positive finite limit, and its first derivative has either $v_r \to 0$, or $rv_r/v \to 2$, but its second derivative v_{rr} blows up.*

Equation (4), for $\sigma_k \equiv$ constant, has a first integral, *i.e.*, a conserved quantity which reduces (4) to a first order ODE. After we completed our work, we realized that in the case of $2k \neq n$, this fact was already pointed out by Viaclovsky in [40] based on his variational characterization of the solutions. In fact, the first integral for (4), when σ_k is a constant, can be obtained in all cases in a straightforward manner: simply multiplying both sides of (4) by $2ne^{-n\xi}\xi_t$, one has, assuming σ_k is normalized to be $2^{-k}\binom{n}{k}$,

$$[e^{(2k-n)\xi}(1-\xi_t^2)^k - e^{-n\xi}]_t \equiv 0.$$

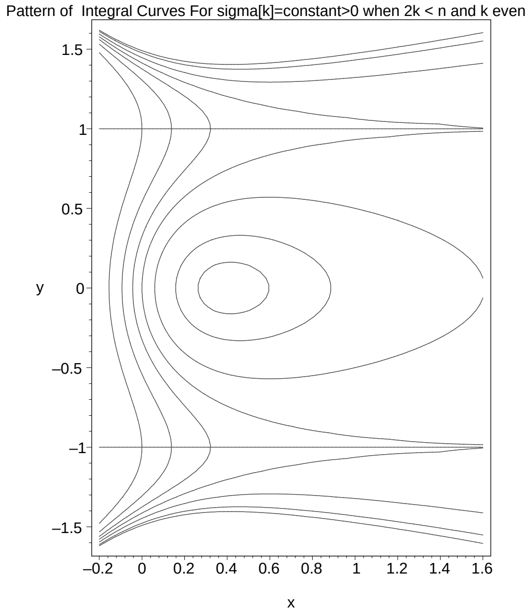
So $e^{(2k-n)\xi}(1-\xi_t^2)^k - e^{-n\xi}$ is a constant along any solution. The remaining analysis of the behavior of the radial solutions is elementary: one uses the first integral to investigate the maximum domain of definition of each solution and its asymptotic behavior upon approaching the end points of its domain; one could also use the phase plane portrait of (4) in the ξ-ξ_t plane as a guidance for the global behavior of solutions of (4). The phase plane portrait of (4), when σ_k is a constant, depends on the relation between $2k$ and n, as well as on whether k is odd or even.

To help understand the different situations that can possibly occur, the patterns of the phase portraits of (4), when σ_k is a *positive* constant, are displayed on the next page. Note that the x and y axes in the displayed phase portraits stand for the ξ and ξ_t axes, respectively. Despite the elementary nature of the analysis involved, the catalog of different behaviors of the radial solutions is of potential reference value, so we provide a full catalog of the behaviors of the radial solutions in the following three theorems. For any C^2 solution of (4), let $-\infty \le t_- < t < t_+ \le \infty$ be its maximum domain of definition, and correspondingly $r_- = e^{t_-}$, and $r_+ = e^{t_+}$. Note that (4), when σ_k is a constant, is invariant under the reflection $t \mapsto -t$, which, in terms of $|x|$, corresponds to symmetry under inversion. The results in the following theorems are stated subject to this inversion.

Theorem 1. *The behavior of the radial solutions of* (4)*, when* $k > 1$ *and* σ_k *is a* positive *constant, normalized to be* $2^{-k}\binom{n}{k}$*, are cataloged as follows. Recall that, along any solution, either* $1-\xi_t^2 < 0$*, or* $1-\xi_t^2 > 0$*; and* $e^{(2k-n)\xi}(1-\xi_t^2)^k - e^{-n\xi}$ *is a constant. Denote this constant by* h*.*

Case I. $1-\xi_t^2 > 0$*. Recall that all such solutions are in the* Γ_k^+ *class. These solutions fall into one of the following three categories.*

1. *If* $h = 0$*, then the domain of definition of* $v(|x|)$ *is the entire* $\mathbb{R}^n$*, and* $v(|x|)^{-2} = \left(\frac{2\rho}{|x|^2+\rho^2}\right)^2$ *for some positive parameter* ρ*. So these solutions give rise to the round spherical metric on* $\mathbb{R}^n \cup \{\infty\} = \mathbb{S}^n$*.*
2. *If* $h < 0$*, then the domain of definition of* $v(|x|)$ *is given by* $0 < r_- < |x| < r_+ < \infty$*.* v *and* v_r *stay bounded on* $[r_-, r_+]$*, in fact,* $v_r \to 0$ *as* $r \to r_-$ *and* $rv_r/v \to 2$ *as* $r \to r_+$*, but* v_{rr} *blows up at both ends of the interval* $[r_-, r_+]$*.*
3. *If* $h > 0$*, then the behavior of* v *is classified according to the relation between* $2k$ *and* n*:*
   (a) *If* $2k < n$*, then* h *has the further restriction* $h \le h^* = \frac{2k}{n-2k}\left(\frac{n-2k}{n}\right)^{\frac{n}{2k}}$ *and the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < \infty$*. In fact,* $\xi(t)$ *is a periodic function of* t*, giving rise to a metric* $g = \frac{e^{-2\xi(\ln|x|)}}{|x|^2}|dx|^2$ *on*


Pattern of Integral Curves For sigma[k]=constant>0 when 2k < n and k even
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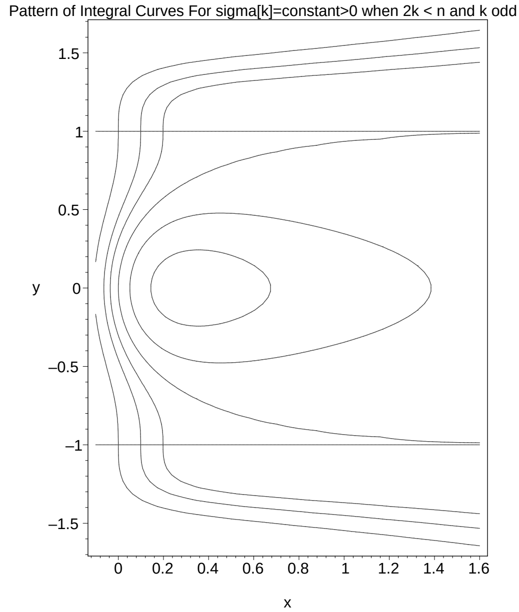
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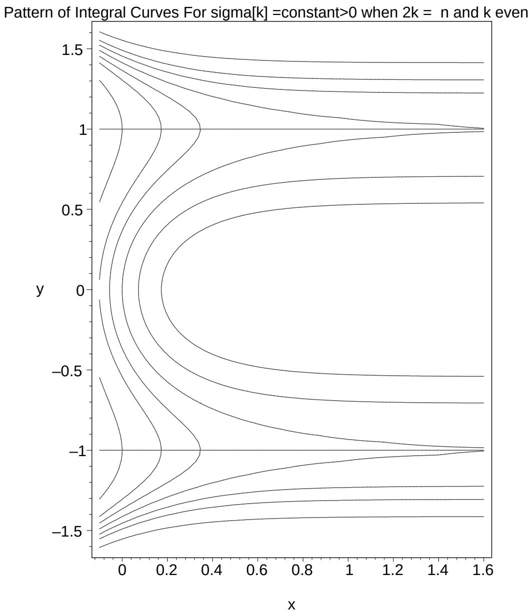
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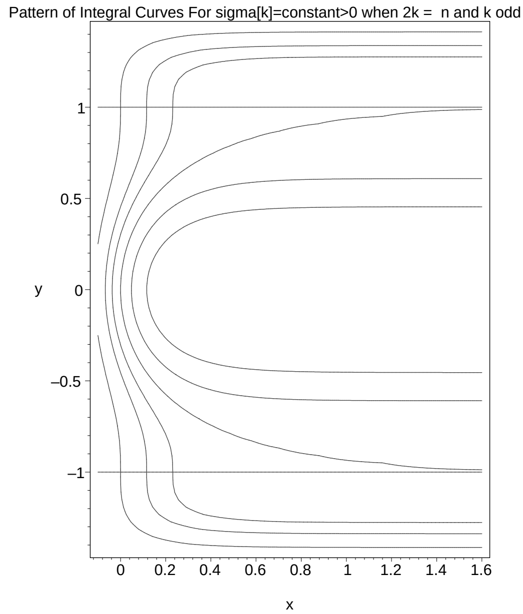
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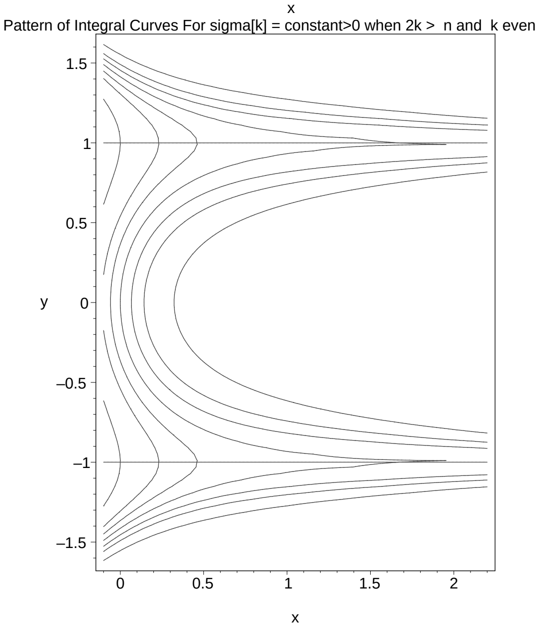
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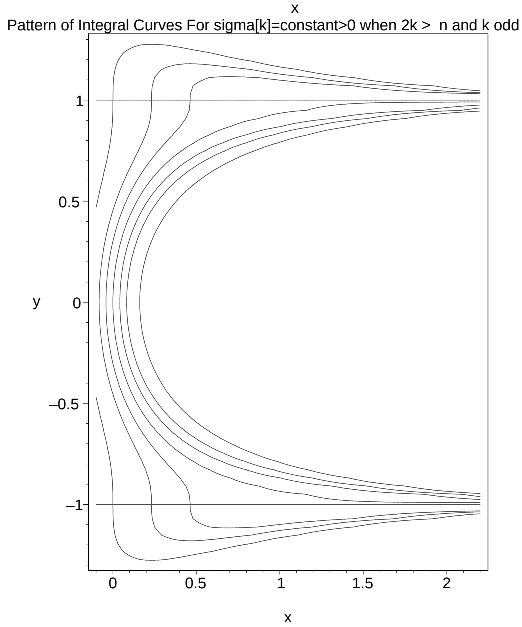

Pattern of Integral Curves For sigma[k]=constant>0 when 2k > n and k odd
y
x

$\mathbb{R}^n \setminus \{0\}$ *which is complete. Note that the case* $h = h^*$ *gives rise to the cylindrical metric* $\frac{|dx|^2}{|x|^2}$ *on* $\mathbb{R}^n \setminus \{0\}$.

(b) *If* $2k = n$, *then* h *satisfies the further restriction* $h < 1$ *and the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < \infty$. *As* $|x| \to 0$, $v^{-2}(|x|)|dx|^2$ *has the asymptotic* $g \sim |x|^{-2(1-\sqrt{1-\sqrt[k]{h}})}|dx|^2$, *and as* $|x| \to \infty$, $v^{-2}(|x|)|dx|^2$ *has the asymptotic* $g \sim |x|^{-2(1+\sqrt{1-\sqrt[k]{h}})}|dx|^2$. *Thus* g *gives rise to a metric on* $\mathbb{R}^n \setminus \{0\}$ *singular at* 0 *and at* ∞ *which behaves like the cone metric, is incomplete with finite volume.*

(c) *If* $2k > n$, *then the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < \infty$. $v^{-2}(|x|)$ *has an asymptotic expansion of the form*

$$v^{-2}(|x|) = \rho^{-2}\{1 - \sqrt[k]{h}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\}$$

as $|x| \to 0$, *where* $\rho > 0$ *is a positive parameter, thus* $v(|x|)$ *has a positive, finite limit, but* $v_{rr}(|x|)$ *blows up at* $|x| \to 0$. *The behavior of* v *as* $|x| \to \infty$ *can be described similarly. Putting together, we conclude that* $v^{-2}|dx|^2$ *extends to a* $C^{2-\frac{n}{k}}$ *metric on* $\mathbb{S}^n$.

Case II. $1-\xi_t^2 < 0$ *and* k *even. Recall that all such solutions are in the* Γ_k^- *class. Subject to an inversion, these solutions are defined for* $0 \le r_- < (\text{ or } \le)\,|x| < r_+ < \infty$, *and has the asymptotic expansion* $v^{-2} \sim (r_+ - r)^{-2}$ *as* $|x| \to r_+$. *Their behavior as* $|x| \to r_-$ *falls into one of the following three categories.*

1. *If* $h = 0$, *then the domain of definition of* $v(|x|)$ *is* $|x| < r_+ < \infty$. *These solutions define the hyperbolic metric defined on* $|x| < r_+$.
2. *If* $h < 0$, *then the domain of definition of* $v(|x|)$ *is* $0 < r_- < |x| < r_+ < \infty$. *As* $|x| \to r_-$, $v(|x|)$ *has a positive, finite limit,* $v_r(|x|) \to 0$, *but* $v_{rr}(|x|)$ *blows up. These solutions define metrics on* $r_- < |x| < r_+$, *which is complete near* $|x| = r_+$, *and has its second derivative blowing up as* $|x| \to r_-$.
3. *If* $h > 0$, *then the behavior of* v *as* $|x| \to r_-$ *is classified according to the relation between* $2k$ *and* n:

   (a) *If* $2k < n$, *then the domain of definition of* $v(|x|)$ *is* $0 < r_- < |x| < r_+ < \infty$. *The corresponding metric has the following degeneracy at* r_-: $g \sim (r - r_-)^{\frac{4k}{n-2k}}|dx|^2$ *as* $|x| \to r_-$, *and is complete as* $|x| \to r_+$.

(b) *If* $2k = n$, *then the domain of definition of* $v(|x|)$ *is* $0 < |x| < r_+ < \infty$.

The metric has the conical degeneracy $g \sim |x|^{2(\sqrt{1+\sqrt[k]{h}}-1)}|dx|^2$ *as* $|x| \to 0$, *and is complete as* $|x| \to r_+$.

(c) *If* $2k > n$, *then the domain of definition of* $v(|x|)$ *is* $0 < |x| < r_+$. $v^{-2}(|x|)$ *has an asymptotic expansion of the form*

$$v^{-2}(|x|) = \rho^{-2}\{1 + \sqrt[k]{h}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\}$$

as $|x| \to 0$, *where* $\rho > 0$ *is a positive parameter, thus, as* $|x| \to 0$, $v(|x|)$ *has a positive, finite limit, but* $v_{rr}(|x|)$ *blows up.* $v^{-2}(|x|)|dx|^2$ *is complete as* $|x| \to r_+$.

Case III. $1 - \xi_t^2 < 0$ *and* k *odd. In this case* $h < 0$. *Subject to an inversion, these solutions are defined for* $0 \le r_- <$ (*or* $\le$) $|x| < r_+ < \infty$, *and as* $r \to r_+$, v *and* v_r *stay bounded, but* v_{rr} *blows up. The behavior of* v *as* $|x| \to r_-$ *is classified according to the relation between* $2k$ *and* n:

1. *If* $2k < n$, *then the domain of definition of* $v(r)$ *is* $0 < r_- < r < r_+ < \infty$. *As* $r \to r_-$, $v^{-2}|dx|^2$ *has the degeneracy:* $g \sim (r - r_-)^{\frac{4k}{n-2k}}dx|^2$.
2. *If* $2k = n$, *then the domain of definition of* $v(|x|)$ *is* $0 < |x| < r_+ < \infty$. *As* $|x| \to 0$, $v^{-2}|dx|^2$ *has the conical degeneracy* $g \sim |x|^{2(\sqrt{1+\sqrt[k]{|h|}}-1)}|dx|^2$.
3. *If* $2k > n$, *then the domain of definition of* $v(|x|)$ *is* $0 < |x| < r_+ < \infty$. *As* $|x| \to 0$, v^{-2} *has the asymptotic expansion of the form*

$$v^{-2}(|x|) = \rho^{-2}\{1 + \sqrt[k]{|h|}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\}$$

as $|x| \to 0$, *where* $\rho > 0$ *is a positive parameter, thus* $v(|x|)$ *stays bounded, but* $v_{rr}(|x|)$ *blows up both as* $|x| \to 0$ *and as* $|x| \to r_+$.

Remark 2. *At an end point* $0 < r_* < \infty$ *of the domain of definition of* $v(r)$ *which corresponds to* v *having a positive, finite limit, the proof for Theorem 1 will give the rate of blowing up of* v_{rr} *as proportional to* $|r - r_*|^{-1+\frac{1}{k}}$.

The behavior of the radial solutions of (4), when $k > 1$ and σ_k is a *negative* constant, normalized to be $-2^{-k}\binom{n}{k}$, is cataloged in the following theorem, and the phase plane pattern in the ξ-ξ_t plane is displayed on the next page.

Theorem 2. *Recall that, along any solution of (4), when $k > 1$ and σ_k is a* negative *constant, normalized to be $-2^{-k}\binom{n}{k}$, we have that either $1 - \xi_t^2 < 0$, or $1 - \xi_t^2 > 0$; and*

$e^{(2k-n)\xi}(1 - \xi_t^2)^k + e^{-n\xi}$ is a constant. Denote this constant by h.

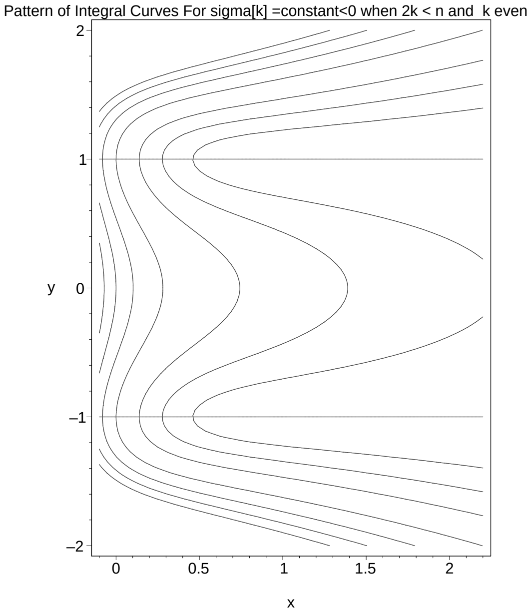
Case I. $1 - \xi_t^2 > 0$. *All such solutions have $h > 0$ and fall into one of the following three categories.*

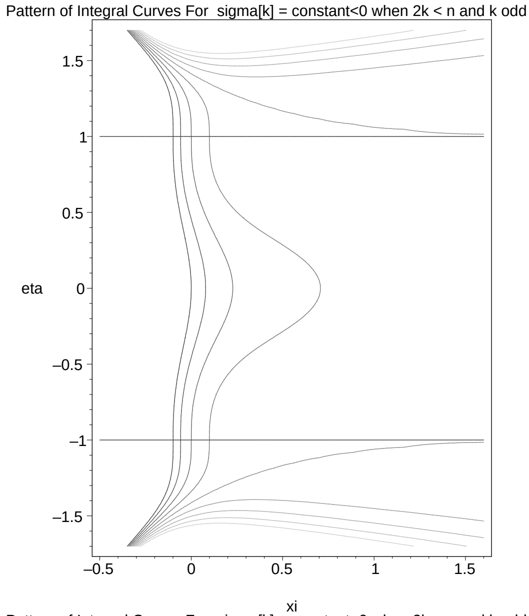
1. *If $2k < n$, then the domain of definition of $v(|x|)$ is given by $0 < r_- < |x| < r_+ < \infty$. The second derivatives of these solutions blow up at both ends of the interval $[r_-, r_+]$.*
2. *If $2k = n$, then v or its inversion falls into one of the following three subcases:*
   (a) *If $0 < h < 1$, then the domain of definition of $v(|x|)$ is given by $0 < |x| < r_+$. As $|x| \to r_+$, $v(|x|)$ has a positive, finite limit, $v_r(|x|) \to 0$, but $v_{rr}(|x|)$ blows up; as $|x| \to 0$, $v^{-2}(|x|)$ has the asymptotic $v^{-2}(|x|) \sim$*

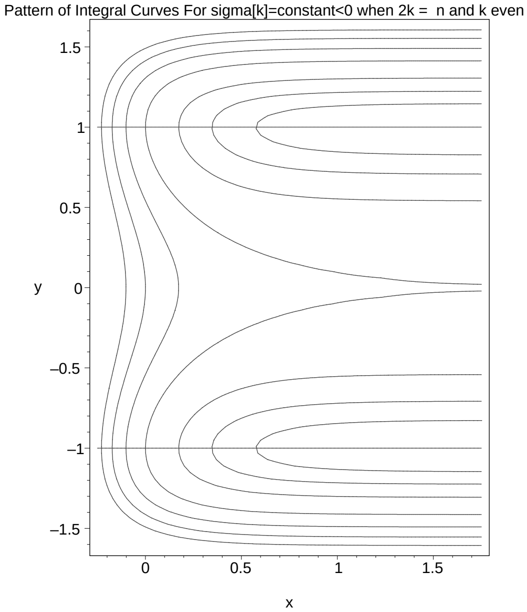
   $|x|^{-2(1-\sqrt{1-\sqrt[k]{h}})}$. So the metric $v^{-2}|dx|^2$ behaves like an incomplete, conic metric near 0.
   (b) *If $h = 1$, then the domain of definition of $v(|x|)$ is given by $0 < |x| < r_+$. As $|x| \to r_+$, $v(|x|)$ has a positive, finite limit, $v_r(|x|) \to 0$, but $v_{rr}(|x|)$ blows up; as $|x| \to 0$, $v^{-2}(|x|)$ has the asymptotic $v^{-2}(|x|) \sim$*

   $|x|^{-2}(\ln \frac{1}{|x|})^{-\frac{2}{k}}$. So the metric $v^{-2}|dx|^2$ can be thought of as a complete metric near 0.
   (c) *If $h > 1$, then the domain of definition of $v(|x|)$ is given by $0 < r_- < |x| < r_+ < \infty$. The second derivatives of these solutions blow up at both ends of the interval $[r_-, r_+]$.*
3. *If $2k > n$, then v or its inversion falls into one of the following six sub-*

   cases. Let $h^* = \frac{2k}{2k-n}\left(\frac{2k-n}{n}\right)^{\frac{n}{2k}}$. Note that $M(h) = \max_{\xi\in\mathbb{R}}\{e^{(n-2k)\xi}h - e^{-2k\xi}\}$ is strictly monotone in h, and h^* is the unique positive number such that $M(h^*) = 1$.
   (a) *If $h < h^*$, then the domain of definition of $v(|x|)$ is given by $0 < |x| < r_+$. As $|x| \to r_+$, $v(|x|)$ has a positive, finite limit, $v_r(|x|) \to 0$, but $v_{rr}(|x|)$*


Pattern of Integral Curves For sigma[k] =constant<0 when 2k < n and k even
y
x


Pattern of Integral Curves For sigma[k] = constant<0 when 2k < n and k odd
eta
xi


Pattern of Integral Curves For sigma[k]=constant<0 when 2k = n and k even
y
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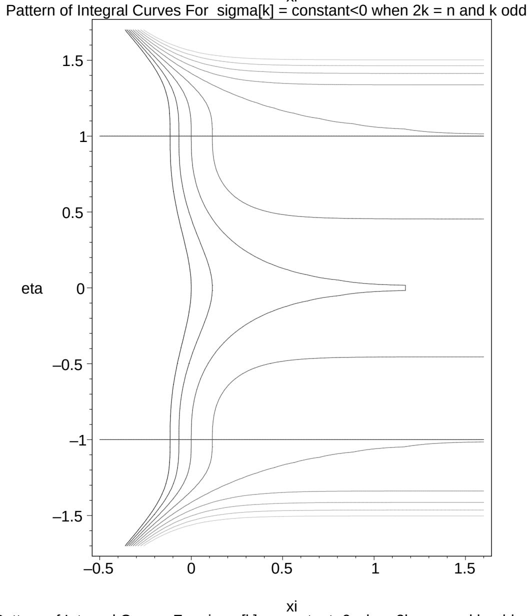
Pattern of Integral Curves For sigma[k] = constant<0 when 2k = n and k odd
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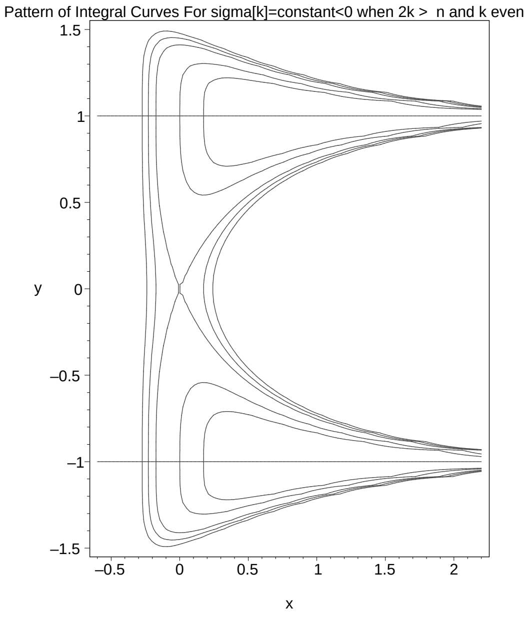
Pattern of Integral Curves For sigma[k]=constant<0 when 2k > n and k even
y
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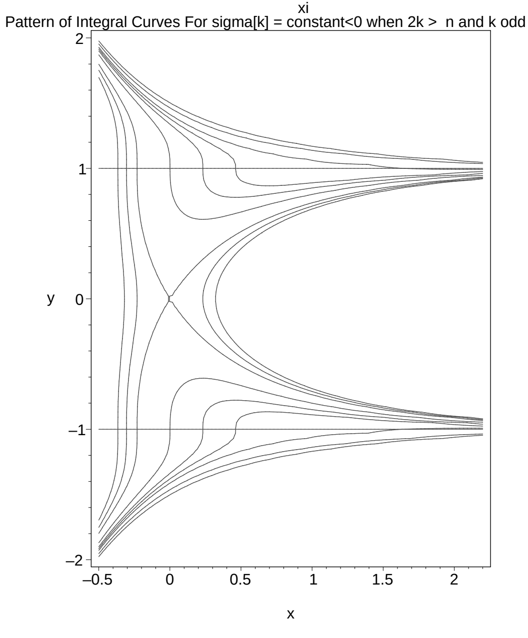
Pattern of Integral Curves For sigma[k] = constant<0 when 2k > n and k odd
y
x

blows up; as $|x| \to 0$, $v^{-2}(|x|)$ *has the asymptotic*

$$v^{-2}(|x|) = \rho^{-2}\{1 - \sqrt[k]{h}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\},$$

where $\rho > 0$ *is a positive parameter. This case corresponds to the integral curves in the phase portrait that intercept and are also asymptotic to the lines* $\xi_t = \pm 1$.

(b) *If* $h = h^*$ *and* $\xi_{tt} < 0$, *then the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < r_+$. *As* $|x| \to r_+$, $v(|x|)$ *has a positive, finite limit,* $v_r(|x|) \to 0$, *but* $v_{rr}(|x|)$ *blows up; as* $|x| \to 0$, $v^{-2}(|x|)|dx|^2$ *has the cylindrical metric* $|x|^{-2}|dx|^2$ *as the asymptotic. This case corresponds to the integral curves in the phase portrait that are asymptotic to the unique equilibrium point* $\xi = \xi^* = \frac{1}{n}\ln[\frac{2k}{(2k-n)h}]$, $\xi_t = 0$ *and intercept the lines* $\xi_t = \pm 1$.

(c) *If* $h = h^*$ *and* $\xi_{tt} > 0$, *then the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < \infty$. *As* $|x| \to \infty$, $v^{-2}(|x|)|dx|^2$ *has the cylindrical metric* $|x|^{-2}|dx|^2$ *as the asymptotic. As* $|x| \to 0$, $v^{-2}(|x|)$ *has the asymptotic*

$$v^{-2}(|x|) = \rho^{-2}\{1 - \sqrt[k]{h}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\},$$

where $\rho > 0$ *is a positive parameter. This case corresponds to the integral curves in the phase portrait that are asymptotic to both the unique equilibrium point* $\xi = \xi^* = \frac{1}{n}\ln[\frac{2k}{(2k-n)h}]$, $\xi_t = 0$ *and the lines* $\xi_t = \pm 1$.

(d) *If* $h = h^*$ *and* $\xi_{tt} \equiv 0$, *then* $\xi \equiv \xi^*$ *and the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < \infty$. *The metric* $v^{-2}(|x|)|dx|^2$ *is the cylindrical metric* $|x|^{-2}|dx|^2$.

(e) *If* $h > h^*$ *and* $\xi_{tt} < 0$, *then the domain of definition of* $v(|x|)$ *is given by* $0 < r_- < |x| < r_+ < \infty$. *The second derivatives of these solutions blow up at both ends of the interval* $[r_-, r_+]$. *This case corresponds to the integral curves in the phase portrait that intercepts the lines* $\xi_t = \pm 1$ *at both ends.*

(f) *If* $h > h^*$ *but* $\xi_{tt} > 0$, *then the domain of definition of* $v(|x|)$ *is given by* $0 < |x| < \infty$, *and as* $|x| \to 0$, $v^{-2}(|x|)$ *has the same asymptotic as in case (a) above; as* $|x| \to \infty$, $v^{-2}(|x|)$ *has a similar asymptotic. One* *may think of* $v^{-2}(|x|)|dx|^2$ *as extending to a* $C^{2-\frac{n}{k}}$ *metric on* $\mathbb{S}^n$. *This case corresponds to the integral curves in the phase portrait that are asymptotic to the lines* $\xi_t = \pm 1$ *at both ends.*

Case II. *k odd and $1-\xi_t^2 < 0$. Recall that all such solutions are in the Γ_k^- class. Subject to an inversion, these solutions are defined for $0 \le r_- < (\text{ or } \le)\, |x| < r_+ < \infty$, and has the asymptotic expansion $v^{-2} \sim (r_+ - r)^{-2}$ as $|x| \to r_+$. Their behavior as $|x| \to r_-$ falls into one of the following three categories.*

1. *If $h = 0$, then the domain of definition of $v(|x|)$ is $|x| < r_+ < \infty$. These solutions define the hyperbolic metrics defined on $|x| < r_+$.*
2. *If $h > 0$, then the domain of definition of $v(|x|)$ is $0 < r_- < |x| < r_+ < \infty$. As $|x| \to r_-$, $v(|x|)$ has a positive limit, $v_r(|x|) \to 0$, but $v_{rr}(|x|)$ blows up.*
3. *If $h < 0$, then the behavior of v as $|x| \to r_-$ is classified according to the relation between $2k$ and n:*
   (a) *If $2k < n$, then the domain of definition of $v(|x|)$ is $0 < r_- < |x| < r_+ < \infty$. The metric $v^{-2}|dx|^2$ has the degeneracy at r_-: $g \sim (r - r_-)^{\frac{4k}{n-2k}}|dx|^2$ as $|x| \to r_-$, and is complete as $|x| \to r_+$.*
   (b) *If $2k = n$, then the domain of definition of $v(|x|)$ is $0 < |x| < r_+ < \infty$. The metric $v^{-2}|dx|^2$ has the conical degeneracy $g \sim |x|^{2(\sqrt{1+\sqrt[k]{|h|}}-1)}|dx|^2$ as $|x| \to 0$, and is complete as $|x| \to r_+$.*
   (c) *If $2k > n$, then the domain of definition of $v(|x|)$ is $0 < |x| < r_+ < \infty$. As $|x| \to 0$, v^{-2} has the asymptotic expansion of the form*

$$v^{-2}(|x|) = \rho^{-2}\{1 + \sqrt[k]{|h|}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\}$$

as $|x| \to 0$, where $\rho > 0$ is a positive parameter.

Case III. *k even and $1 - \xi_t^2 < 0$. In this case $h > 0$. All such solutions, or their inversion, are defined for $0 \le r_- < (\text{or} \le)\, |x| < r_+ < \infty$, and as $r \to r_+$, v and v_r stay bounded, but v_{rr} blows up. The behavior of v as $|x| \to r_-$ is classified according to the relation between $2k$ and n:*

1. *If $2k < n$, then the domain of definition of $v(r)$ is $0 < r_- < r < r_+ < \infty$. As $r \to r_-$, $v^{-2}|dx|^2$ has the degeneracy: $g \sim (r - r_-)^{\frac{4k}{n-2k}}|dx|^2$.*

2. *If $2k = n$, then the domain of definition of $v(|x|)$ is $0 < |x| < r_+ < \infty$. As $|x| \to 0$, $v^{-2}|dx|^2$ has the conical degeneracy $g \sim |x|^{2(\sqrt{1+\sqrt[k]{h}}-1)}|dx|^2$.*
3. *If $2k > n$, then the domain of definition of $v(|x|)$ is $0 < |x| < r_+ < \infty$. As $|x| \to 0$, v^{-2} has the asymptotic expansion of the form*

$$v^{-2}(|x|) = \rho^{-2}\{1 + \sqrt[k]{h}\frac{k}{2k-n}\left(\frac{|x|}{\rho}\right)^{2-\frac{n}{k}} + \cdots\}$$

as $|x| \to 0$, where $\rho > 0$ is a positive parameter, thus $v(|x|)$ stays bounded, but $v_{rr}(|x|)$ blows up both as $|x| \to 0$ and as $|x| \to r_+$.

The classification of solutions of (4) when $\sigma_k(A_g) \equiv 0$ is easily done by integrating out the equation.

Theorem 3. *Any radial solution of (4), when $\sigma_k(A_g) \equiv 0$, is one of the following three forms.*

1. $\xi \equiv \pm 1$;
2. $\xi(t) = (1 - \frac{n}{2k})^{-1} \ln \left|\sinh\left[(1-\frac{n}{2k})(t-t_0)\right]\right| + c$ *for some constants* t_0, c;
3. $\xi(t) = (1 - \frac{n}{2k})^{-1} \ln \cosh\left[(1-\frac{n}{2k})(t-t_0)\right] + c$ *for some constants* t_0, c.

2 Indication of Proofs.

Here is an elementary proof of Proposition 1.

Proof. First, the conclusion in part 1 follows readily from (4). To prove part 2, notice that $\sigma_k > 0$ and $1 - \xi_t^2 > 0$ imply that $\frac{k}{n}\xi_{tt} + (\frac{1}{2} - \frac{k}{n})(1 - \xi_t^2) > 0$. Then it follows, for

$1 \le l < k$, that

$$\begin{aligned}
\sigma_l(A_g) &= c'_{n,l}(1-\xi_t^2)^{l-1}\left[\frac{l}{n}\xi_{tt}+(\frac{1}{2}-\frac{l}{n})(1-\xi_t^2)\right]e^{2l\xi} \\
&= \frac{l}{k}c'_{n,l}(1-\xi_t^2)^{l-1}\left[\frac{k}{n}\xi_{tt}+(\frac{k}{2l}-\frac{k}{n})(1-\xi_t^2)\right]e^{2l\xi} \\
&= \frac{l}{k}c'_{n,l}(1-\xi_t^2)^{l-1}\left[\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)+(\frac{k}{2l}-\frac{1}{2})(1-\xi_t^2)\right]e^{2l\xi} \\
&> 0.
\end{aligned}$$

Similarly, in the situation for part 3, $\sigma_k > 0$, $1-\xi_t^2 < 0$, and k is even, then $\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2) < 0$, and

$$\begin{aligned}
(-1)^l\sigma_l(A_g) &= (-1)^l\frac{l}{k}c'_{n,l}(1-\xi_t^2)^{l-1}\left[\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)+(\frac{k}{2l}-\frac{1}{2})(1-\xi_t^2)\right]e^{2l\xi} \\
&> 0.
\end{aligned}$$

Finally, in the situation for part 4, $\sigma_k < 0$, $1-\xi_t^2 < 0$, and k is odd, then $\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2) < 0$, and

$$\begin{aligned}
(-1)^l\sigma_l(A_g) &= (-1)^l\frac{l}{k}c'_{n,l}(1-\xi_t^2)^{l-1}\left[\frac{k}{n}\xi_{tt}+(\frac{1}{2}-\frac{k}{n})(1-\xi_t^2)+(\frac{k}{2l}-\frac{1}{2})(1-\xi_t^2)\right]e^{2l\xi} \\
&> 0.
\end{aligned}$$

For part 5, recall that the equation in this case has a first integral: when $\sigma_k(A_g)$ is normalized to be $\pm 2^{-k}\binom{n}{k}$, in terms of $\xi = \ln(v/r)$ we have that $e^{(2k-n)\xi}(1-\xi_t^2)^k \pm e^{-n\xi}$ is a constant along any solution. The conclusion of part 5 follows then easily, noting the relation $1-\xi_t^2 = (2-rv_r/v)rv_r/v$. ☐

For the proof to Theorems 1 and 2, we will offer enough details for the case $\sigma_k = 2^{-k}\binom{n}{k}$, but leave out the details for the case of $\sigma_k = -2^{-k}\binom{n}{k}$.

Proof of Theorem 1. Recall that, in this case, for each solution $\xi(t)$, there is a constant h such that

$$(1-\xi_t^2)^k = e^{-2k\xi} + he^{(n-2k)\xi}, \tag{5}$$

along the solution. Set $D(\xi) = e^{-2k\xi} + he^{(n-2k)\xi}$. Recall also that either $1 - \xi_t^2 > 0$ or $1 - \xi_t^2 < 0$ along the solution.

Case I. $1 - \xi_t^2 > 0$. With the phase plane portrait as a guidance, it is routine to see that:

1. If $h = 0$, then $1 - \xi_t^2 = e^{-2\xi}$, which can be integrated out to produce $\xi = \ln\cosh(t - c)$. In terms of v, we see that $v^{-2} = (\frac{2e^c}{|x|^2+e^{2c}})^2$. So $v^{-2}|dx|^2$ gives rise to the round spherical metric on $\mathbb{S}^n$ in this case.

2. If $h < 0$, then $\xi^- \leq \xi < \xi^+$, where $\xi^\pm$ are (unique) finite numbers determined by $D(\xi^-) = 1$ and $D(\xi^+) = 0$. As $\xi \to \xi^+$, $\xi_t^2 \to 1$, which implies that t tends to finite limits, thus the maximum domain of definition of v is $r_- < |x| < r_+$ for some $0 < r_- < r_+ < \infty$, and because $\xi_t^2 \to 1$ and $\xi \to \xi^+$ as $|x| \to r_\pm$, v and v_r stay bounded, but v_{rr} blows up as $|x| \to r_\pm$. Also as $r \to r_+$, $\xi_t \to 1$, thus $rv_r/v \to 2$; while as $r \to t_-$, $\xi_t \to -1$, thus $v_r \to 0$.

3. If $h > 0$, then the analysis depends on the relation between $2k$ and n:

   (a) If $2k < n$, then $\xi^- \leq \xi \leq \xi^+$, where $\xi^\pm$ are the two roots of $D(\xi) = 1$. This case requires that $h \leq h^*_{n,k}$, where $h^*_{n,k}$ is the unique positive number such that $\min_{\mathbb{R}} D(\xi) = 1$. In this case, the phase portrait shows that the solution curve $(\xi(t), \xi_t(t))$ is a periodic orbit bounded between ξ^- and ξ^+, thus is defined for all t. So the metric $g = \frac{e^{-2\xi(\ln|x|)}}{|x|^2}|dx|^2$ is a complete metric on $\mathbb{R}^n \setminus \{0\}$.

   (b) If $2k = n$, then $\xi^- \leq \xi$, where $D(\xi^-) = 1$, and as $\xi \to \infty$, $D(\xi) \to h$, so $0 < h < 1$ and $\xi_t^2 \to 1 - \sqrt[k]{h}$ as $\xi \to \infty$. Using $D(\xi) = h + e^{-2k\xi}$ and the relation

$$dt = \pm \frac{d\xi}{\sqrt{1 - \sqrt[k]{D(\xi)}}}, \tag{6}$$

   which follows from (5), we conclude that

   (i) $\xi(t)$ is defined for $-\infty < t < \infty$, and

   (ii) $t \pm \frac{\xi}{\sqrt{1-\sqrt[k]{h}}}$ has a finite limit as $\xi \to \infty$.

   Therefore, near $|x| \sim 0$, we have

$$g \sim |x|^{-2(1-\sqrt{1-\sqrt[k]{h}})}|dx|^2,$$

and near $|x| \sim \infty$, we have

$$g \sim |x|^{-2(1+\sqrt{1-\sqrt[k]{h}})}|dx|^2.$$

These are incomplete, finite volume metrics on $\mathbb{R}^n \setminus \{0\}$, corresponding to conical metrics on $\mathbb{S}^n \setminus \{0, \infty\}$.

(c) If $2k > n$, then $\xi^- \leq \xi$, where $D(\xi^-) = 1$, and as $\xi \to \infty$, $D(\xi) \to 0$

and $\xi_t^2 \to 1$. Using (6) and the asymptotic expansion $\sqrt[k]{D} \sim \sqrt[k]{h}e^{\frac{n-2k}{k}\xi}$, we conclude that

(i) $\xi(t)$ is defined for $-\infty < t < \infty$, and

(ii) $\xi \pm t = c + \frac{\sqrt[k]{h}}{2}\frac{k}{2k-n}e^{-\frac{2k-n}{k}\xi} + h.o.t.$ as $\xi \to \infty$,

for some constant c, from which we conclude that as $|x| \to 0$, we take the + sign in (ii) and

$$v^{-2} = e^{-2c}\{1 - \sqrt[k]{h}\frac{k}{2k-n}e^{-\frac{2k-n}{k}c}|x|^{\frac{2k-n}{k}} + h.o.t.\}$$

as $|x| \to 0$. The analysis near x at ∞ can be carried out in a similar way. Thus we conclude that $v^{-2}|dx|^2$ extends to a $C^{2-\frac{n}{k}}$ metric on $\mathbb{S}^n$.

The analysis for the case $1 - \xi_t^2 < 0$ depends on whether k is odd or even.

Case II. $1 - \xi_t^2 < 0$ and k even.

1. If $h = 0$, then $\xi_t^2 = 1 + e^{-2\xi}$. From this, we obtain

$$dt = \pm\frac{d\xi}{\sqrt{1+e^{-2\xi}}}.$$

It is easy to conclude from here that $v^{-2}|dx|^2$ provides the hyperbolic metric on either $\{|x| < r\}$ or $\{|x| > r\}$.

2. If $h < 0$, then $\xi < \xi^+$, where ξ^+ is the unique root of $D(\xi) = 0$. As $\xi \to \xi^+$, $\xi_t^2 \to 1$ and t has a finite limit. In terms of v and $|x|$, this produces a boundary

point where v_{rr} blows up. As $\xi \to -\infty$, $D(\xi) \to \infty$. In fact $\sqrt[k]{D(\xi)} \sim e^{-2\xi}$. Using this and

$$dt = \pm \frac{d\xi}{\sqrt{1 + \sqrt[k]{D(\xi)}}}, \tag{7}$$

we conclude that t also has a finite limit. Taking the $-$ sign, for instance, and letting t_+ denote this (upper) limit of t, we obtain

$$t_+ - t = e^{\xi} + h.o.t.$$

as $\xi \to -\infty$, so that

$$g \sim (t_+ - t)^{-2}|dx|^2 = (\ln \frac{r_+}{|x|})^{-2}|dx|^2$$

as $|x| \to r_+$. In conclusion, we obtain a conformal metric $v^{-2}|dx|^2$ on $r_- < |x| < r_+$ which has second derivative blow up near r_- and is complete near r_+.

3. If $h > 0$, the phase portrait indicates that the range for ξ is the entire real line. As $\xi \to -\infty$, $D(\xi) \to \infty$. In fact $\sqrt[k]{D(\xi)} \sim e^{-2\xi}$. Using this and (7), we conclude that t has a finite limit as $\xi \to -\infty$, and again denoting the corresponding (upper) limit of $|x|$ as r_+, we have

$$g \sim (t_+ - t)^{-2}|dx|^2 = (\ln \frac{r_+}{|x|})^{-2}|dx|^2,$$

as $|x| \to r_+$. so g is complete as $|x| \to r_+$. The analysis as $\xi \to \infty$ depends on the relation between $2k$ and n.

   (a) If $2k < n$, then $\sqrt[k]{D(\xi)} \sim \sqrt[k]{h} e^{\frac{n-2k}{k}\xi}$ as $\xi \to \infty$. Using this and (7), we conclude that t has a finite limit as $\xi \to \infty$. Taking the $-$ sign case, for instance, and denoting this limit as t_-, we have

$$t - t_- \sim e^{-\frac{n-2k}{2k}\xi}$$

   as $\xi \to \infty$. So

$$g \sim (t - t_-)^{\frac{4k}{n-2k}}|dx|^2 = (\ln \frac{|x|}{r_-})^{\frac{4k}{n-2k}}|dx|^2,$$

and the metric $v^{-2}|dx|^2$ is defined on $r_- < |x| < r_+$, and as $|x| \to r_-$, it has the above degeneracy; as $|x| \to r_+$, it is complete.

(b) If $2k = n$, then $\sqrt[k]{D(\xi)} \sim \sqrt[k]{h}$ as $\xi \to \infty$. Using this and (7), we conclude that $t \to \pm\infty$ as $\xi \to \infty$. In fact, $t \sim \pm\frac{\xi}{\sqrt{1+\sqrt[k]{h}}}$. Taking the $-$ sign, for instance, we obtain

$$g \sim |x|^{2(\sqrt{1+\sqrt[k]{h}}-1)}|dx|^2.$$

So the metric $v^{-2}|dx|^2$ is defined on $0 < |x| < r_+$, and as $|x| \to 0$, it has the above degeneracy; as $|x| \to r_+$, it is complete.

(c) If $2k > n$, then $\sqrt[k]{D(\xi)} \to 0$ as $\xi \to \infty$. In fact, $\sqrt[k]{D(\xi)} \sim \sqrt[k]{h}e^{\frac{n-2k}{k}\xi}$. Using this and (7), we conclude that $t \to \pm\infty$ as $\xi \to \infty$. In fact, for some constant c,

$$\xi \pm t = c - \frac{\sqrt[k]{h}}{2}\frac{k}{2k-n}e^{-\frac{2k-n}{k}\xi} + h.o.t.$$

as $\xi \to \infty$, from which we conclude that

$$v^{-2} = e^{-2c}\{1 + \sqrt[k]{h}\frac{k}{2k-n}e^{-\frac{2k-n}{k}c}|x|^{\frac{2k-n}{k}} + h.o.t.\}$$

as $|x| \to 0$. Thus the metric $v^{-2}|dx|^2$ defined on $0 < |x| < r_+$ is complete as $|x| \to r_+$, and extends to be a $C^{2-\frac{n}{k}}$ metric on $|x| < r_+$.

Case III. $1 - \xi_t^2 < 0$ and k odd. In this case $h < 0$ and all such solutions have a finite limit point where $\xi_t = \pm 1$, which corresponds to a limit point $0 < r_* < \infty$ where $v(|x|)$ has a positive finite limit, but v_{rr} blows up. By inversion, we may take $r_+ = r_*$. The behavior of $v(|x|)$ as $|x| \to r_-$ depends on the relation between $2k$ and n.

1. If $2k < n$, then $\sqrt[k]{D(\xi)} \sim \sqrt[k]{h}e^{\frac{n-2k}{k}\xi}$ as $\xi \to \infty$. Using this and (7), we conclude that t has a finite limit as $\xi \to \infty$. Taking the $-$ sign case, for instance, and denoting this limit as t_-, we have

$$t - t_- \sim e^{-\frac{n-2k}{2k}\xi}$$

as $\xi \to \infty$, so

$$g \sim (t - t_-)^{\frac{4k}{n-2k}} |dx|^2 = (\ln \frac{|x|}{r_-})^{\frac{4k}{n-2k}} |dx|^2,$$

and the metric $v^{-2}|dx|^2$ is defined on $r_- < |x| < r_+$, and as $|x| \to r_-$, it has the above degeneracy.

2. If $2k = n$, then $\sqrt[k]{D(\xi)} \sim \sqrt[k]{h}$ as $\xi \to \infty$. Using this and (7), we conclude that $t \to \pm\infty$ as $\xi \to \infty$. In fact, $t \sim \pm \frac{\xi}{\sqrt{1+\sqrt[k]{|h|}}}$. Taking the $-$ sign, for instance, we obtain

$$g \sim |x|^{2(\sqrt{1+\sqrt[k]{|h|}}-1)} |dx|^2,$$

so the metric $v^{-2}|dx|^2$ is defined on $0 < |x| < r_+$, and as $|x| \to 0$, it has the above degeneracy.

3. If $2k > n$, then $\sqrt[k]{D(\xi)} \sim \sqrt[k]{h} e^{\frac{n-2k}{k}\xi}$ as $\xi \to \infty$. Using this and (7), we conclude that $t \to \pm\infty$ as $\xi \to \infty$. In fact, for some constant c,

$$\xi \pm t = c - \frac{\sqrt[k]{|h|}}{2} \frac{k}{2k-n} e^{-\frac{2k-n}{k}\xi} + h.o.t.$$

as $\xi \to \infty$, from which we conclude that

$$v^{-2} = e^{-2c}\{1 + \sqrt[k]{|h|} \frac{k}{2k-n} e^{-\frac{2k-n}{k}c} |x|^{\frac{2k-n}{k}} + h.o.t.\}$$

as $|x| \to 0$. Thus the metric $v^{-2}|dx|^2$ defined on $0 < |x| < r_+$ and extends to be a $C^{2-\frac{n}{k}}$ metric on $|x| < r_+$.

□

Proof of Theorem 3. When $\sigma_k(A_g) \equiv 0$, any radial solution of (4) satisfies either $\xi_t^2 = 1$ or $\xi_{tt} = (1 - \frac{n}{2k})(1 - \xi_t^2)$. Either case can be integrated out easily. For instance, in the second case, set $\eta = \xi_t$, then $\eta_t = (1 - \frac{n}{2k})(1 - \eta^2)$. It follows that either $\eta \equiv \pm 1$ or

$$\eta = \frac{ae^{(2-\frac{n}{k})t} + 1}{ae^{(2-\frac{n}{k})t} - 1}$$

for some (non-zero) constant a. Integrating one more time concludes Theorem 3. $\square$

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