LECTURE NOTES OF AN INTRODUCTORY GRADUATE PDE COURSE

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Preface

These notes grew out of teaching graduate level introductory PDE courses regularly over the last 20 years. There are several good graduate level PDE texts on the market; my reasons for producing these notes include at least the following: (i) many students in my courses have not had an undergraduate PDE course—while it is possible to teach an introductory graduate level PDE course to such students using some existing texts, it is important to provide some discussion of elementary methods as covered in a typical undergraduate level PDE course dealing with prototypical PDE examples, and *relate* these methods to the more abstract and systematic methods typically presented in a graduate course; (ii) many existing texts seem to aim to train specialists in PDE at the PhD level, and tend to choose the material and presentation for this audience, often leaving out discussions on background and motivation, and not necessarily providing enough discussions relating the more abstract and advanced methods with the more elementary ones; (iii) while many students in my courses have an interest in applying the PDE tools in their research, but not all are necessarily interested in becoming a specialist in PDE per se, and it's more important for them to understand the heuristic ideas behind the (often abstract) methods than to focus on the formal (and unmotivated to an uninitiated) arguments. I have also found, based on my experience sitting on numerous oral qualifying exams, that many students have had too little experience working hands-on with prototypical PDE examples, as a result many students can memorize proofs of general theorems, but seem to have difficulty with recognizing the scope and limitations of the general theorems they have learned, and with making necessary modifications for the particular problems that they may face. These notes have been prepared with addressing these issues in mind.

It is important to emphasize to students in a PDE course that they shouldn't expect that a few general theorems would suffice to "solve" the PDEs that they may encounter — no such theorems exist; and this is a feature of the vast and important subject of PDEs. Many modern mathematics texts are influenced by the Bourbaki style, emphasizing a formal logical structure, making its coverage as general as possible, and presenting only polished proofs. Such a style is certainly more efficient and adequate for a reader with sufficient background and experience, but seems difficult for a reader with little experience in this vast field: it does not give much clue on how the methods and theories of PDEs developed from tackling some instructive examples. It is more beneficial for students to learn by working with prototypical examples, doing *hands-on computations*, and *experiencing first hand* how a solution method may succeed in a certain context but may encounter difficulties in other contexts, and through this learning process gradually develop the experience and intuition to deal with more general cases and abstract theories. Students who understand thoroughly how methods, techniques and theories work or fail to work in well chosen prototypical examples will be rewarded immensely in tackling more general cases and abstract theories.

Here are several features that I emphasize in these notes:

- Instead of following typical mathematical presentation from the general theory to the concrete cases (or only general theory), I always use examples to illustrate how an idea is (often naturally) born, and then look for features in the examples that can be extended to more general contexts. Fourier series and Fourier transforms arise in such a fashion in applying the elementary method of separation of variables. Our employment of the elementary solution methods is not for teaching rote procedures, but is aimed at helping students build intuition on what free parameters may enter in the elementary construction of solutions to different kinds of equations, and how they may affect the behavior of solutions and in satisfying initial and/or boundary conditions; and we emphasize on looking for **a priori** estimates of solutions, instead of only representation formulae, and emphasize on using the estimates, their suggested approximation procedure, and the constructed explicit solutions (in prototypical examples) to produce more general solutions.

- I don't aim to present the most polished or concise treatment; rather, I often treat the same problem multiple times, using different approaches, perhaps not giving a thorough treatment in the first round, but adding additional ideas in later rounds.

- I make an effort to minimize the technical prerequisites expected of the students. The development in the notes does not rely in any essential way on the technical aspects of Lebesgue's integration theory, other than teaching the students how to work with L^p norms (mostly L^2 norms) and to accept that the space of L^p

integrable functions is complete. The notes motivate the need for Fourier series and Fourier transforms early on and begin to make limited use of some of their properties before a summary of their main properties is provided at appropriate places.

- I try to make connections with methods used to solve ODEs, pointing out relations as well as differences. The elementary construction of solutions—most notably separation of variables—often reduces the problem to solving some sets of ODEs; more importantly, understanding the role played by the free parameters in the construction of ODE solutions makes it easier for students to understand the issue of well-posedness, the role of initial or/and boundary conditions in a well-posed PDE problem. Many basic concepts, such as Duhamel principle, eigenfunction expansion, and the spectrum of a differential operator, are easiest to understand in the context of ODE problems.

- I emphasize the need to make any reasonable limit of classical solutions as a generalized solution and often motivate notions of generalized/weak solutions through such a limiting process and show the need, fruitfulness, and some flexibility for such notions.

- I include a good number of exercises and problems for students to practice their trade—*this is an essential aspect of learning.* Just as a student of painting can't become a painter by just learning the theory of perspectives but not spending thousands of hours practicing painting, a student of PDE can't learn the skills of the trade by only reading the general proofs and theories. In particular, I have included a number of exercises involving solutions to some classical differential equations: Bessel equations, Legendre equations, and their generalizations; they arise in constructing eigenfunctions for the Laplace operator in the flat Euclidean space, in the round sphere, as well as in the hyperbolic space.

- I gradually introduce more advanced tools and approaches. Chapters 1 through 3 are mostly on elementary solution methods, with an emphasis on constructing solutions converging in appropriate norms depending on the contexts. Chapter 4 introduces the maximum principle, energy method, and variational method in the simplest context. Chapter 5 covers the basics of the Laplace and Poisson Equations. Chapters 6 through 9 introduce some more general tools and approaches, and provide an introduction to several topics that one may encounter in studying initial or boundary value problems. Some of the topics (e.g. those in Chapters 7 and 9) may not get covered in a one semester course, and this would not cause difficulty for a student to directly to the material after Chapter 9, after having studied the first five chapters and relevant sections in Chapter 6. The material of Chapters 10 and 11 differs not too much from that of the existing textbooks, with the exception of perhaps some part of sections 2 and 3 of Chapter 11.

Since these notes are produced to meet the needs of the typical students in my graduate courses, they are more limited in their scope and depth of coverage in their current form compared with many of the existing textbooks. I hope that students who have used these notes will find it easier to study their interested topics more systematically and in more depth from some of the existing textbooks.

I have only included in the bibliography published textbooks and surveys that I have consulted over the years. I also benefited from the lectures and course notes on PDEs by my teachers Louis Nirenberg and Sergiu Klainerman when I was a graduate student at the Courant Institute.

In my fall 2017 graduate PDE course at Rutgers, David Herrera and Parker Hund provided detailed lists of corrections and suggestions, and Parker Hund continued to do additional proofreading after the course. This has been very valuable in cutting down the number of typos and clarifying the writing of many topics. I would like to thank them and other students in my courses for their helpful input.

In addition to using earlier versions or portions of these notes in graduate PDE courses at Rutgers, I also used a portion of an earlier draft of these notes in giving select PDE lectures to graduate students at the North University of China in summer 2015 and spring 2018 when I was a visiting professor there; I also used my time there to expand, revise, and edit these notes. I warmly acknowledge the support and friendly atmosphere provided by colleagues and students of the North University of China.

These notes are my attempts to help students learn this fascinating but somewhat difficult subject. They began as supplementary course notes, and have not been constructed as a formal text, although they contain more than enough material for a one year introductory course. Since I have made an effort to explain the ideas behind various solution methods, I have not avoided repeating certain explanations when I see a need; as a result these notes have not been as concise as I had initially hoped for. At this point they may serve as a supplement, not as a replacement, for mature existing texts. I do have additional notes that I intend to include after some revision. A lot of feedback and input from students on the selection of material, level, arrangement, and style of presentation will be needed to improve these notes and make them helpful to students. Please feel free to send me your comments, suggestions or criticism, and I would appreciate them greatly.

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Part I

Introduction and Elementary Solution Methods

Chapter 1

Formulation of Some Prototype PDE Problems and Initial Comparisons with ODE Problems

The goals of the first Chapter are:

- to discuss the setting up process of some prototype PDEs, in particular, to make students be aware of the connections and differences between the physical and geometric assumptions that have been made in the setting up process on the one hand, and the technical mathematical assumptions that have been made on the other hand;
- to discuss the need for notions of solutions with varying smoothness assumptions;
- to discuss typical boundary/initial conditions;
- to make an initial discussion on the well-posedness issue in general terms;
- to make some general comparisons on approaches and results with those from the theory of ODEs; and
- to introduce the methods of separation of variables, which lead to Fourier's series and integrals.

1.1 Formulation of Some Prototype PDE Problems

In this section we will use various examples to illustrate

- how to set up a PDE in a physical or geometrical context by translating the relevant physical or geometrical principle into an equation or equations;
- how the equations initially set up (often in integral form) are reduced to differential equations under appropriate regularity assumptions on the solutions;
- the distinction between physical/geometrical assumptions and mathematical assumptions (mostly on the smoothness of the solutions) in setting up the PDEs;
- the method of calculus of variations in formulating PDE problems;
- the needs for initial/boundary conditions to determine a solution; and
- the needs for varying degrees of smoothness of solutions depending on the context and property of the PDEs.

Many PDEs that we encounter are set up by translating the relevant physical or geometrical principle into equations. However, most physical and geometrical principles are initially formulated in terms of integral equations, not as differential equations directly; we often need to make some smoothness assumptions on the relevant physical or geometrical quantities to reduce the integral equation(s) to PDE(s).

Other times direct, faithful translation of the physical principle may produce a PDE which is too difficult to manage, so we may need to make some simplifying physical or mathematical assumptions to obtain a simpler PDE as a first approximation to the physical process. Of course, whether the simplified PDE gives a good description to the physical process is subject to verification with data. But that is not our main task here; our main task is to learn how to analyze a PDE. As we will learn, however, having some physical and geometrical intuition will be of great help in the analysis of PDEs.

The set up of a PDE to model a physical or geometrical problem needs to be on as firm a ground as possible; but the standards of physical justifications are often different from those of mathematical justifications: we need to be mindful of the assumptions that are used to set up the problem, but the modeling part *may not call for mathematical justifications at each step.* So you will see varying degree of mathematical rigor in the setting up process.

1.1.1 Formulation of Some Prototype PDEs From Their Integral Forms

We will first illustrate the set up process using an example in one spatial dimension, then discuss examples in more than one spatial dimensions, where the concept of flux and the divergence theorem will play a prominent role. Whenever a problem in multi-dimension is difficult to comprehend, always return to the one-spatial-dimension case and try to understand it first.

Example 1.1 (Equation for the motion of a fluid along a one-dimensional tube)**.** Here, the relevant physical quantities are: the (linear) density $\rho(x,t)$ of the fluid at location x and time t defined as mass per unit length, where x is the coordinate along the tube (as a one-dimensional object), and the velocity $v(x,t)$ (in the direction of x-axis here) of the fluid at location x and time t. The first relevant physical principle is the **law of conservation of mass**, which can be expressed as

the rate of change of mass in any section of the tube
= the rate of fluid flowing in across the ends of the tube
− the rate of fluid flowing out across the ends of the tube.

More quantitatively, for any $x_1 < x_2$, and any moment t,

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t). \tag{1.1}$$

Here, the right hand side terms come from the mass of the section to the left of x_1 with length $v(x_1,t)$ and density $\rho(x_1,t)$ entering the section between x_1 and x_2 per unit time, and the mass of the section to the left of x_2 with length $v(x_2,t)$ and density $\rho(x_2,t)$ leaving the section between x_1 and x_2 per unit time; and we have already made some *tentative mathematical assumptions* (integrability and differentiability) on the quantities $\rho(x,t)$ and $v(x,t)$ to make sense of the relation above.

(1.1) is not a PDE yet, and is not easy to work with. We next make the *more definite mathematical assumption* that ρ and v are C^1 functions of x and t everywhere, so that

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho(x,t)\,dx = \int_{x_1}^{x_2} \frac{\partial\,\rho(x,t)}{\partial t}\,dx,$$

$$\rho(x_1,t)v(x_1,t) - \rho(x_2,t)v(x_2,t) = -\int_{x_1}^{x_2} \frac{\partial\,[\rho(x,t)v(x,t)]}{\partial x}\,dx.$$

Then (1.1) becomes

$$\int_{x_1}^{x_2} \frac{\partial\,\rho(x,t)}{\partial t}\,dx = -\int_{x_1}^{x_2} \frac{\partial\,[\rho(x,t)v(x,t)]}{\partial x}\,dx. \tag{1.2}$$

Since, at any moment t, (1.2) holds for *any* $x_1 < x_2$, and we have assumed the integrands to be continuous, we arrive at

$$\frac{\partial\,\rho(x,t)}{\partial t} = -\frac{\partial\,[\rho(x,t)v(x,t)]}{\partial x} \quad \text{at any } (x,t),$$

which is often referred to as the equation of continuity, and is written as

$$\frac{\partial\,\rho(x,t)}{\partial t} + \frac{\partial\,[\rho(x,t)v(x,t)]}{\partial x} = 0. \tag{1.3}$$

By a similar process using the **law of conservation of momentum**, we arrive at

$$\frac{\partial[\rho(x,t)v(x,t)]}{\partial t} + \frac{\partial\,[\rho(x,t)v^2(x,t)]}{\partial x} = -\frac{\partial p(x,t)}{\partial x}, \tag{1.4}$$

here, $p(x,t)$ is the pressure at (x,t) (pressure normally is defined as force per unit area, but in this set up of a thin tube $p(x,t)$ is taken as the force of one section of the fluid exerted on the other section at location x and time t, so has the unit of force), and we have made the *physical assumption* that there is no external force acting on the fluid, and the *mathematical assumption* that the quantities involved have enough differentiability to allow differentiation under the integral sign to carry through.

(1.3) and (1.4) are two equations involving three unknowns: $\rho(x,t), v(x,t)$, and $p(x,t)$. For an ideal isentropic fluid, $p(x,t)$ is determined through the density $\rho(x,t)$: $p(x,t) = p(\rho(x,t))$. Then (1.3) and (1.4) form a system of two equations in ρ and v. For many gases, $p = A\rho^\gamma$ for some $A > 0$ and $\gamma \geq 1$. As we will see later, an important assumption based on physical principle is that $p'(\rho) > 0$ for $\rho > 0$.

In some physical situation, a solution may exhibit a sharp change of value across an interface. Mathematically such solutions are modeled by piecewise continuous solutions with a **jump discontinuity**. The following remark discusses how to modify the above derivation to take into account of a jump discontinuity in a solution, and to derive a corresponding condition along the interface of discontinuity.

Remark 1.1. * Suppose that $\rho(x,t)$ and $v(x,t)$ have a jump discontinuity along the curve $\{(x,t) : x = \xi(t)\}$, namely, $\rho(x,t)$ and $v(x,t)$ are C^1 in $\{(x,t) : x \leq \xi(t)\}$ and $\{(x,t) : x \geq \xi(t)\}$, respectively, with limiting values of $\rho^-(\xi(t),t)$, $v^-(\xi(t),t)$, and $\rho^+(\xi(t),t)$, $v^+(\xi(t),t)$, along the left and right sides of the curve $x = \xi(t)$, respectively. Then in reducing (1.1) to a PDE, nothing needs to be changed if $x_1 < x_2 < \xi(t)$, or $\xi(t) < x_1 < x_2$, so (1.2) holds as long as $x \neq \xi(t)$. When $x_1 < \xi(t) < x_2$, in carrying

*May be skipped on a first reading.

Figure 1.1: Include a figure, depicting the line of discontinuity and the left and right limiting values of the functions on either side

through differentiation through the integral on the left side of (1.1), we have to be mindful of the discontinuity of $\rho(x,t)$ at $x = \xi(t)$, and modify the procedure as

$$\frac{d}{dt}\left\{\int_{x_1}^{\xi(t)} + \int_{\xi(t)}^{x_2}\right\}\rho(x,t)\,dx = \int_{x_1}^{x_2}\frac{\partial\rho(x,t)}{\partial t}\,dx + \left[\rho^-(\xi(t),t) - \rho^+(\xi(t),t)\right]\xi'(t),$$

assuming $\xi(t)$ to be differentiable in t. This follows from a variant of the following lemma.

Lemma 1.1. *Suppose that $\rho(x,t)$ is a C^1 function defined on $[x_1,x_2]\times[t_1,t_2]$, $\xi(t)\in C^1[t_1,t_2]$ such that $x_1 < \xi(t) < x_2$ for all $t\in[t_1,t_2]$. Then $G(t) := \int_{x_1}^{\xi(t)}\rho(x,t)\,dx$ is C^1 in $[t_1,t_2]$, and*

$$\frac{d}{dt}\left(\int_{x_1}^{\xi(t)}\rho(x,t)\,dx\right) = \int_{x_1}^{\xi(t)}\rho_t(x,t)\,dx + \rho(\xi(t),t)\xi'(t). \tag{1.5}$$

This is a simple consequence of the chain rule after recognizing that $G(t) = F(\xi(t),t)$, where $F(y,t) := \int_{x_1}^{y}\rho(x,t)\,dx$ is C^1 in $[x_1,x_2]\times[t_1,t_2]$.

But some preliminary set up is needed to apply this Lemma, as, in our context, the integrand $\rho(x,t)$ is expected to have a discontinuity at $x = \xi(t)$ for $t_1 \le t \le t_2$, but otherwise is a C^1 function defined on $\{(x,t) : t_1 \le t \le t_2, x_1 \le x \le \xi(t)\}$; this is not quite the set up to apply the above Lemma directly. This will be left as an exercise, with some guidance provided.

The right hand side of (1.1) is also expected to experience a discontinuity at $x = \xi(t)$, and is treated by applying the Fundamental Theorem of Calculus to $\rho(x,t)v(x,t)$ in the x variable between $[x_1,\xi(t)]$, and $[\xi(t),x_2]$, separately:

$$\begin{aligned}
&\left[\rho(x_1,t)v(x_1,t) - \rho^-(\xi(t),t)v^-(\xi(t),t)\right] + \left[\rho^+(\xi(t),t)v^+(\xi(t),t) - \rho(x_2,t)v(x_2,t)\right]\\
&\quad + \rho^-(\xi(t),t)v^-(\xi(t),t) - \rho^+(\xi(t),t)v^+(\xi(t),t)\\
&= -\int_{x_1}^{\xi(t)}\frac{\partial\left[\rho(x,t)v(x,t)\right]}{\partial x}\,dx - \int_{\xi(t)}^{x_2}\frac{\partial\left[\rho(x,t)v(x,t)\right]}{\partial x}\,dx\\
&\quad + \rho^-(\xi(t),t)v^-(\xi(t),t) - \rho^+(\xi(t),t)v^+(\xi(t),t)\\
&= -\int_{x_1}^{x_2}\frac{\partial\left[\rho(x,t)v(x,t)\right]}{\partial x}\,dx + \rho^-(\xi(t),t)v^-(\xi(t),t) - \rho^+(\xi(t),t)v^+(\xi(t),t).
\end{aligned}$$

Since $\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial[\rho(x,t)v(x,t))]}{\partial x}$ is already established at $x \neq \xi(t)$, so the integrals on the two sides cancel, and we can conclude that

$$[\rho^-(\xi(t),t) - \rho^+(\xi(t),t)]\,\xi'(t) = \rho^-(\xi(t),t)v^-(\xi(t),t) - \rho^+(\xi(t),t)v^+(\xi(t),t). \tag{1.6}$$

Note that $[\rho^-(\xi(t),t) - \rho^+(\xi(t),t)]$ is the jump of $\rho(x,t)$ across $x = \xi(t)$, while $\rho^-(\xi(t),t)v^-(\xi(t),t) - \rho^+(\xi(t),t)v^+(\xi(t),t)$ is the jump of $\rho(x,t)v(x,t)$ across $x = \xi(t)$. (1.6) relates the speed of the point of discontinuity, $\xi'(t)$, with the jumps of ρ and ρv across the discontinuity; and is a case of the so called **Rankine-Hugoniot** conditions. This leads to a solution $(\rho(x,t), v(x,t))$, which satisfies (1.3) except along the interface $x = \xi(t)$, with a jump discontinuity along $x = \xi(t)$, and the jumps of ρ and ρv satisfy (1.6) along $x = \xi(t)$.

Solutions with such kind of jump discontinuity are called **shock wave solutions**. Note that $\xi(t)$, $\rho^\pm(\xi(t),t)$, and $v^\pm(\xi(t),t)$ may be part of the unknowns here.

A physicist may derive (1.6) as follows. We account for how much mass has crossed $x = \xi(t)$ per unit time: the mass of the section to the left of $\xi(t)$ with length $v^-(\xi(t),t) - \xi'(t)$ and density $\rho^-(\xi(t),t)$ has crossed $\xi(t)$, so the mass crossed is $\rho^-(\xi(t),t)[v^-(\xi(t),t) - \xi'(t)]$; and once crossing $\xi(t)$, the mass moves at the speed of $v^+(\xi(t),t)$, occupying the section to the right of $\xi(t)$ with length $v^+(\xi(t),t) - \xi'(t)$ and with density $\rho^+(\xi(t),t)$. Thus we must have

$$\rho^-(\xi(t),t)[v^-(\xi(t),t) - \xi'(t)] = \rho^+(\xi(t),t)[v^+(\xi(t),t) - \xi'(t)],$$

which is equivalent to (1.6).

A situation as described here may arise when a piston is pushed with speed of v^- along a uniform tube filled with still air. Here, we may assume that the air right in front of the piston moves with (constant) speed v^-, and take $v^+ = 0$, then (1.6) would take the form $[\rho^- - \rho^+]\,\xi'(t) = \rho^- v^-$. This equation alone can't determine the shock wave speed $\xi'(t)$, as ρ^- is to be determined; one also needs to establish a Rankine-Hugoniot type condition based on the momentum equation (1.4) to determine $\xi'(t)$.

Example 1.2 (Equation for heat conduction)**.** Here, the relevant physical quantities are the temperature $u(x,t)$ at location x and time t, the specific heat c of the medium, i.e., the amount of heat needed for each unit increase of temperature per unit mass, and the density $\rho(x,t)$ of the medium. Unless we deal with an inhomogeneous medium, we assume c and ρ are constants across the medium and are independent of t. There is no macroscopic movement of material in this set up; the changes in $u(x,t)$ can be attributed to the transfer of energy due to microscopic movement of molecules. The basic law describing the conduction of heat is expressed as

$$\begin{aligned} &\text{the rate of change of heat in a region} \\ =\ &\text{heat flux across the boundary of the region} \\ &+ \text{ the rate of heat production within the region.} \end{aligned}$$

If we let $\vec{q}(x,t)$ denote the heat flux vector, i.e., $\vec{q}(x,t)\cdot\vec{n}$ gives the heat transferred per unit time across a unit area with unit normal vector $\vec{n}$, and let $f(x,t)$ denote the heat produced per unit time per unit mass within the region, then, for any region Ω with piecewise C^1 boundary* (so that the divergence theorem can be applied to $\overline{\Omega}$; we can take $\overline{\Omega}$ to be balls or boxes whose closure are contained in our domain of consideration), the above law translates into

$$\int_\Omega c\rho \frac{\partial u}{\partial t}\,dx = -\int_{\partial\Omega} \vec{q}(x,t)\cdot\vec{n}(x)\ d\sigma + \int_\Omega \rho f(x,t)\,dx, \tag{1.7}$$

where $\vec{n}(x)$ denotes the exterior unit normal to $\partial\Omega$ at x, and $d\sigma$ denotes the area element of $\partial\Omega$.

One often used postulate on the heat flux vector is **Fourier's law**: $\vec{q}(x,t) = -k\nabla u(x,t)$, where $k > 0$ is the thermal conductivity of the medium, assumed to

*A domain Ω with C^1 boundary is an open set of $\mathbb{R}^n$ such that any point $P \in \partial\Omega$ has a neighborhood V in $\mathbb{R}^n$ with the property that $\partial\Omega \cap V$ is a C^1 hypersurface; and that $\Omega \cap V$ stays on one side of this hypersurface. A domain (also called a region) with piecewise C^1 boundary is modeled on polyhedrons such as a cube or a prism. Here is an analytical definition. A domain Ω has piecewise C^1 boundary if any point $P \in \partial\Omega$ has a neighborhood V in $\mathbb{R}^n$ and C^1 functions $F_i(\mathbf{x})$ defined in V for $1 \le i \le k$ and some $k \ge 1$ such that (i). $\Omega \cap V = \{\mathbf{x} \in V : F_i(\mathbf{x}) > 0, i = 1,\cdots,k\}$; (ii). $F_i(P) = 0$ for $i = 1,\cdots,k$ and $\partial\Omega \cap V = \cup_I \{\mathbf{x} \in V : F_i(\mathbf{x}) = 0 \text{ for } i \in I \text{ and } F_j(\mathbf{x}) > 0 \text{ for } j \notin I\}$, where I runs over the set of non-empty subsets of $\{1,\cdots,k\}$; and (iii) The Jacobian matrix $[DF_1(P),\cdots,DF_k(P)]$ has full rank. Conditions (ii) and (iii) allow a stratification of $\partial\Omega \cap V$ as union of $\Sigma_m = \cup_{|I|=m}\{\mathbf{x} \in V : F_i(\mathbf{x}) = 0 \text{ for } i \in I \text{ and } F_j(\mathbf{x}) > 0 \text{ for } j \notin I\}$, $m = 1,\cdots,\min\{k,n\}$, where Σ_m is a codimension m manifold in $\mathbb{R}^n$ as a consequence of the implicit function theorem, which allows us to treat $u_i = F_i(\mathbf{x}), i = 1,\cdots,k$ (taking $k \le n$ for simplicity) as part of the coordinates to reparametrize V and regard $\Omega \cap V$ as the image of a neighborhood of $\mathbf{u} = 0$ in $\{\mathbf{u} : u_i > 0, i = 1,\cdots,k\}$. More concretely, any point Q on Σ_m has a neighborhood U in $\mathbb{R}^n$ such that $\Sigma_m \cap U$ is represented as the graph of a $\mathbb{R}^m$-valued C^1 function defined in a neighborhood of a point in $\mathbb{R}^{n-m}$. Condition (i) gives the notion of Ω staying on one side of Σ_1 and a continuous choice of unit exterior/interior normal vector on Σ_1. These conditions also imply that, if $S_1 = \{\mathbf{x} \in U : F_1(\mathbf{x}) = 0, F_i > 0 \text{ for } i \ne 1\}$ and $S_2 = \{\mathbf{x} \in U : F_2(\mathbf{x}) = 0, F_i > 0 \text{ for } i \ne 2\}$ are two components in Σ_1 such that $\overline{S_1} \cap \overline{S_2}$ contains a non-empty codimension 2 piece, namely, $\{\mathbf{x} \in U : F_1(\mathbf{x}) = F_2(\mathbf{x}) = 0, F_i(\mathbf{x}) > 0 \text{ for other } i's\}$ is not empty, then this codimension 2 piece is in the shared boundary of *only* S_1 and S_2; in fact, any codimension 2 piece of $\partial\Omega$ must be contained in the shared boundary of two codimension 1 pieces. This rules out the possibility of $\partial\Omega$ having a cross or triple-junction piece if $\Omega \subset \mathbb{R}^2$, as well as Ω equals the three dimensional open ball removing a line segment or a closed loop, or isolated points.

Figure 1.2: Include a figure, depicting a region with piecewise C^1 boundary surface and the heat flux vector field

be a constant for a uniform medium. To reduce (1.7) into a PDE, we make the mathematical assumption that k is uniform across the medium, and $u(x,t)$ is twice continuously differentiable in $x \in \bar{\Omega}$, so that by the Divergence Theorem

$$\begin{aligned} & -\int_{\partial\Omega} \vec{q}(x,t) \cdot \vec{n(x)}\ d\sigma \\ =& k\int_{\partial\Omega} \nabla u(x,t) \cdot \vec{n(x)}\ d\sigma \\ =& k\int_{\Omega} \operatorname{div}(\nabla u(x,t))\ dx \\ =& k\int_{\Omega} \Delta u(x,t)\, dx, \end{aligned}$$

where $\Delta u(x,t) = \operatorname{div}(\nabla u(x,t)) = \sum_{i=1}^{3} \frac{\partial^2 u(x,t)}{\partial x_i^2}$ is called the **Laplacian** of u. Thus (1.7) reduces to

$$\int_{\Omega} c\rho \frac{\partial u}{\partial t}\, dx = \int_{\Omega} [k\Delta u + \rho f(x,t)]\ dx.$$

Since this relation holds on any (reasonably regular) region Ω* and we have assumed the integrands to be continuous, we arrive at

$$\frac{\partial u}{\partial t} = \gamma \Delta u + \frac{1}{c} f(x,t). \tag{1.8}$$

with $\gamma = \frac{k}{c\rho} > 0$ denoting the thermal diffusivity of the medium.

Remark 1.2. If Fourier's law needs to be modified in a certain situation, such as when k depends on (x,t) or $u(x,t)$, or both: $k = k(u,x,t)$, or $\vec{q}(x,t) = -\mathbf{A}(x,t)\nabla u(x,t)$, where $\mathbf{A}(x,t)$ is an 3×3 matrix with certain properties, for example, positive definite, then (1.8) is modified in the first case as

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho} \nabla \left(k(u,x,t)\nabla u(x,t)\right) + \frac{1}{c} f(x,t),$$

and in the second case as

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho} \nabla \left(\mathbf{A}(x,t)\nabla u(x,t)\right) + \frac{1}{c} f(x,t).$$

*For our purposes it suffices to restrict Ω to balls or rectangular boxes.

An example of the first case is when the medium is a mix of two or more material with different specific heat, or density, or thermal conductivity. A rod made up of two kinds of material welded at $x = 0$ is a specific example. Due to potential discontinuities of some of the terms, we have to re-examine the derivation process. In this 1-D case, the heat flux term across $[x_1, x_2]$ is $-q(x_2, t) + q(x_1, t)$. If $q(x, t)$ had a discontinuity in x, then we would need to make modifications when expressing this difference in terms of an integral of its x-derivative. However, physical consideration makes it reasonable to assume that $u(x, t)$, the temperature, as well as $q(x, t)$ should not experience a discontinuity. Thus at the welding point $x = 0$, the continuity of $q(x, t)$ leads to $k_- u_x(0-, t) = k_+ u_x(0+, t)$, where $k_\mp$ are the thermal conductivity coefficient of the medium to the left and right of $x = 0$, respectively, and $u_x(0\mp, t)$ are the left and right x-derivative of $u(x, t)$ at $(0, t)$, respectively. When $k_- \neq k_+$, this leads to solutions with a discontinuity in $u_x(x, t)$ at $(0, t)$. Away from $x = 0$, appropriate forms of the heat equation hold.

The same equation arises in many other processes of diffusion, i.e. the process of equalization of the concentration in a medium with an initially non-homogeneous distribution of some substance (such as the dilution of a dye in a medium), where $\vec{q}(x, t)$ would have a natural interpretation as a flux, and the law corresponding to Fourier's law is often referred to as Fick's Law. The equation also arises in describing Brownian motion, which underlies many diffusion processes. For this reason, (1.8) is also often called a diffusion equation.

Boundary Conditions (BC); Initial Conditions (IC); Initial-Boundary Value Problems (IBVP); Initial Value Problems (IVP); and Boundary Value Problems (BVP)

To determine $u(x, t)$ in a fixed region $x \in D$ and $t > 0$, in addition to (1.8), we also need to know the initial data $u(x, 0)$ for $x \in D$, and information on the boundary data $u(x, t)$ for $x \in \partial D$ and $t > 0$. The simplest type of boundary condition is the homogeneous **Dirichlet** boundary condition $u(x, t) = 0$ for $(x, t) \in \partial D \times \mathbb{R}^+$. In such a case we look for a solution $u(x, t)$ to

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma \Delta u + \dfrac{1}{c} f(x, t) \quad \text{for } (x, t) \in D \times \mathbb{R}^+, \\ u(x, t) = 0 \quad \text{for } (x, t) \in \partial D \times \mathbb{R}^+, \\ u(x, 0) = g(x) \quad \text{for } x \in D, \text{ for a given function } g. \end{cases} \tag{1.9}$$

This is an example of an initial-boundary value problems (IBVP).

One may also encounter a **Neumann** type boundary condition, prescribing $\frac{\partial u}{\partial \vec{n}}(x,t)$ along $(x,t) \in \partial D \times \mathbb{R}^+$. Since $\vec{q}(x,t) \cdot \vec{n}(x,t) = -k\frac{\partial u}{\partial \vec{n}}(x,t)$ under the assumption of Fourier's law, prescribing $\frac{\partial u}{\partial \vec{n}}(x,t)$ along $(x,t) \in \partial D \times \mathbb{R}^+$ has the interpretation of prescribing the heat flux across the boundary of D. The corresponding IBVP would be

$$\begin{cases} \dfrac{\partial u}{\partial t} = \gamma \Delta u + \dfrac{1}{c} f(x,t) \quad \text{for } (x,t) \in D \times \mathbb{R}^+, \\ u(x,0) = g(x) \quad \text{for } x \in D, \text{ for a given function } g, \\ \dfrac{\partial u}{\partial \vec{n}}(x,t) = h(x,t) \quad \text{for } (x,t) \in \partial D \times \mathbb{R}^+, \text{ and a given function } h \text{ on } \partial D \times \mathbb{R}^+. \end{cases}$$

$h \equiv 0$ would be appropriate when ∂D is totally insulated. When the heat flux across the boundary of D depends on $u(x,t)$ there in a linear fashion, we encounter a **Robin** type boundary condition: $\frac{\partial u}{\partial \vec{n}}(x,t) + au(x,t) = h(x,t)$ for $(x,t) \in \partial D \times \mathbb{R}^+$ for some a.

In some situations, we take D to be the entire $\mathbb{R}^n$, so there is only the initial condition but no explicit boundary condition (but for the heat equation, there will be implicit condition that the solution not grow too fast as $x \to \infty$). Such an initial value problem (IVP) is called a **Cauchy** problem (compare with the Cauchy problem in ODEs).

A natural formal definition for a solution of an IBVP such as (1.9) would require a function $u(x,t)$, such that $u(x,t)$, together with $u_t(x,t)$, $\partial^2_{x_i x_j} u(x,t)$ be continuous on the closure of the domain $D \times \mathbb{R}^+$, and the PDE itself holds on this closure as well. But such requirements can be too restrictive. For instance, if we demand that the PDE in (1.9) to hold on the closure of the domain $D \times \mathbb{R}^+$, this would require to evaluate $\Delta u(x,0)$ by $\Delta g(x)$, so g would have to be C^2 in $\overline{D}$—we will learn that we can construct a solution of (1.9) even for an initial data g which is only continuous. Furthermore, the continuity of u, u_t, and Δu along $\partial D \times \{0\}$, together with the initial and boundary conditions in (1.9) would require $g(x) = 0$, $u_t(x,0) = 0$, and $0 = \gamma \Delta g(x) + \frac{1}{c} f(x,0)$ for $x \in \partial D$. These are called **boundary compatibility** conditions when we consider solutions to the above IBVP such that u, u_t, and Δu are continuous on the closure of the domain $D \times \mathbb{R}^+$. We may demand these conditions in certain approaches, but may not demand them in others—this often happens when we find it advantageous to require the boundary or initial conditions be taken on in an integral (i.e. average) sense, in stead of in a point wise sense.

When D is a two or higher dimensional region, in particular when it does not have a sufficiently smooth boundary, such as when it is a domain with corners or edges, it's not always straightforward to define the meaning of functions such as $\partial^2_{x_i x_j} u(x,t)$ on the boundary, and require them to be continuous on the closure of the domain, and it's possible that for certain such domains, the size of $\partial^2_{x_i x_j} u(x,t)$ of a solution of (1.9) may grow unbounded as x approaches a corner or edge point of the boundary, so it may not make sense to demand that the solution and all the terms in the PDE be continuous on the closure of the boundary, and the PDE be valid on the closure of the boundary in such cases.

Based such consideration, we often require the PDE in an IBVP, such as in (1.9), to hold only in the interior of the (space or space-time) region, not necessarily on its boundary; but would require $u(x,t)$ itself (and $u_x(x,t)$, if need be, as in the case of the Neumann or Robin type boundary conditions) to be continuous on $\overline{D} \times \mathbb{R}^+$*, or on $D \times [0,\infty)$, to make sense of the boundary and initial conditions there.

For a similar reason we could require the initial condition $u(x,0) = g(x)$ to hold on the closure $\overline{D}$ of D, but may not insist it—in other words, we may ignore the boundary compatibility conditions in some approaches, although we will make a study of this issue in some cases later on. As we will see, we often expect a solution u to (1.9) to be continuous on $\overline{D \times \mathbb{R}^+}$ in some sense (often in an integral sense), but not necessarily in a point-wise sense; so we may allow g to be not continuous.

If we consider evolution equations such as (1.9) on a finite time interval $0 < t < T$, we often include $D \times \{T\}$ to be part of the domain on which the PDE holds; in other words, the PDE is supposed to hold on $D \times (0,T]$.

(1.8) is an example of a **parabolic** PDE. For an IBVP for a parabolic PDE such as (1.9) on a finite time interval $(0,T]$, the initial and boundary conditions are imposed on only a portion of the boundary of $D \times (0,T]$: $\partial D \times [0,T] \cup \overline{D} \times \{0\}$. This portion is called the **parabolic boundary** of $D \times (0,T]$, and is often denoted either as $\partial'(D \times (0,T])$ or as $\partial_p(D \times (0,T])$.

Since (1.9) involves different orders of differentiation of the unknown u in x and t, it's natural to look for a solution which reflects different requirements on the order of differentiation of the unknown u in x and t. We often use $C^{2,1}_{x,t}$ to denote the class of functions which are twice continuously differentiable in x and once continuously differentiable in t in appropriate domains.

*The continuity of $u_x(x,t)$ on $\overline{D} \times \mathbb{R}^+$ is in the sense that $u_x(x,t)$ has a continuous extension to $\overline{D} \times \mathbb{R}^+$. In general $C^k(\overline{\Omega})$ denotes the space of functions in $C^k(\Omega)$ such that for any $l \le k$, $\partial^l_x u$ has a continuous extension to $\overline{\Omega}$. A piecewise C^k function in Ω refers to a function u such that Ω can be partitioned into the non-overlapping union of a finite number of domains Ω_i with piecewise C^k boundary: $\Omega = \cup_i \overline{\Omega_i}$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \ne j$, and the restriction of u to each Ω_i is C^k and has an extension as a function of $C^k(\overline{\Omega_i})$.

Example 1.3 (Equation of one dimensional waves)**.** The equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.10}$$

describes the small transverse vibration of a perfectly elastic string, where $u(x,t)$ denotes the transverse displacement from x at time t. c^2 is determined by the property of the medium, and is equal to $\frac{T}{\rho}$, with T being the (lateral) tension of the string and ρ being the density of the string. As we will learn soon that c represents the speed of propagation of wave.

When a perfectly elastic string is stretched it has a tension force tangential to the string. Denote the magnitude of the tension at $(x, u(x,t))$ as $T(x,t)$. Then the vertical and lateral components have magnitude $T(x,t)\sin\theta(x,t)$ and $T(x,t)\cos\theta(x,t)$ respectively, where $\theta(x,t)$ is the angle of inclination of the string at $(x, u(x,t))$, so $\tan\theta(x,t) = \partial_x u(x,t)$.

For small transverse vibration of a perfectly elastic string, we consider any lateral movement to be negligible, so the lateral component $T(x,t)\cos\theta(x,t) = T$ is a constant. Thus the vertical component has magnitude $T\partial_x u(x,t)$.

If we neglect gravity, then for any $x_1 < x_2$, we have

$$\frac{d}{dt}\left(\int_{x_1}^{x_2} \rho \partial_t u(x,t)\, dx\right) = T\partial_x u(x_2,t) - T\partial_x u(x_1,t) = T\int_{x_1}^{x_2} \partial^2_{xx} u(x,t)\, dx,$$

if we assume that $\partial_x u(x,t)$ has continuous derivative in x. The wave equation (1.10) then follows from this.

(1.10) also represents longitudinal waves and comes from its integral form, based on Newton's second law of motion

$$\frac{d}{dt}\left(\int_{x_1}^{x_2} \rho u_t\, dx\right) = T\left[u_x(x_2,t) - u_x(x_1,t)\right], \quad \text{for any } x_1 < x_2, \tag{1.11}$$

where $u_x(x_2,t)$ and $u_x(x_1,t)$ account for local deformation of the elastic medium at x_2 and x_1, respectively, and $Tu_x(x_2,t)$ and $Tu_x(x_1,t)$ account for the elastic forces exerted to the ends of the section between x_1 and x_2 due to tension according to Hooke's law, with $T > 0$ proportional to the Young's modulus of the medium. Here, we assume the medium has a constant $T > 0$; when the Young's modulus of the medium depends on the position coordinate x (for instance, when the medium consists of two materials with uniform but different Young's modulus welded at a contact point, T would be a piecewise constant function of x), the right hand side of (1.11) should be replaced by $T(x_2)u_x(x_2,t) - T(x_1)u_x(x_1,t)$.

When T has a jump discontinuity at a finite number of points, then a solution's derivatives may experience **jump discontinuities** at these points and points along the **characteristic lines**—to be introduced later—issued from these points, and we expect (1.10) to hold away from these points, and add the supplementary condition that $u(x,t)$ be continuous and $u_x(x,t)$ be piecewise continuous such that $T(x)u_x(x,t)$ becomes continuous.

The higher dimensional version of (1.10) is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0, \tag{1.12}$$

where $\Delta = \sum_{i=1}^{n} \partial_{x_i}^2$ is the **Laplace operator** in dimension n. (1.12) in the case of $n = 3$ describes the propagation of sound waves and electro-magnetic waves in space.

A natural IBVP for the wave equation (1.10) is

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & \text{for } (x,t) \in (0,l) \times \mathbb{R}^+, \\ u(x,0) = g(x) & \text{for } x \in (0,l), \text{ for a given function } g, \\ u_t(x,0) = h(x) & \text{for } x \in (0,l), \text{ for a given function } h, \\ u(0,t) = k_1(t),\ u(l,t) = k_2(t) & \text{for } t \in \mathbb{R}^+, \text{ and given functions } k_1 \text{ and } k_2 \text{ on } \mathbb{R}^+. \end{cases}$$

One or both of the boundary conditions above could also be of Neumann type.

A typical IVP problem for (1.12) is

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n, \text{ for a given function } g, \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^n, \text{ for a given function } h. \end{cases}$$

Example 1.4 (**Laplace** equation)**.** All three examples above are **evolution** equations, as the unknowns depend on the time variable. In such settings solutions that are not dependent on the time variable are called **equilibrium (stationary)** states (or solutions)—such solutions may exist only if no term in the equation depends on the time variable explicitly; for example, one can't talk about an equilibrium solution of (1.8) unless the f term has no explicit dependence on t.

In (1.8) when $f \equiv 0$ and in (1.12), the equilibrium (stationary) states satisfy $\Delta u = 0$. This equation is called the Laplace equation. A solution satisfying this equation is called a **harmonic function**. This equation also arises from a great number of other situations. For instance, the real and imaginary parts of a complex analytic function satisfy the Laplace equation.

Although the most commonly encountered boundary values problems for equations like the Laplace equation involve the Dirichlet, or Neumann, or Robin boundary conditions, there may be other types of boundary conditions. For instance, if $u(x)$ represents the electrostatic potential in vacuum outside a finite number of conductors C_k, $1 \le k \le K$, where the total charge Q_k on each conductor C_k is prescribed, then the physical problem of determination of $u(x)$ is the following boundary value problem for u:

$$\begin{cases} \Delta u(x) = 0 & \text{for } x \in \Omega := \mathbb{R}^3 \setminus \cup_{k=1}^{K} C_k, \\ \displaystyle\int_{\partial C_k} \frac{\partial u(x)}{\partial \vec{n}(x)} d\sigma(x) = Q_k & \\ u(x) = u_k & \text{on } \partial C_k,\ 1 \le k \le K, \text{ for constants } u_k \text{ to be determined,} \\ u(x) \to 0 & x \to \infty. \end{cases}$$

Exercises

Exercise 1.1.1. Let $u(x)$ be a continuous function on $[0,1]$.

(a). If $\int_0^1 u(x)v(x)\,dx = 0$ for all continuous functions v over $[0,1]$ with $v(0) = v(1) = 0$, prove that $u \equiv 0$ over $[0,1]$.

(b). If $\int_0^1 u(x)v(x)\,dx = 0$ for all continuous functions v over $[0,1]$ with $\int_0^1 v(x)\,dx = 0$, prove that u is a constant over $[0,1]$.

(c). If $\int_0^1 u(x)v'(x)\,dx = 0$ for all C^1 functions v over $[0,1]$ with $v(0) = v(1) = 0$, prove that u is a constant over $[0,1]$.

Exercise 1.1.2. Model on the derivation of (1.3) in **Example 1.1** to derive (1.4). Also derive that if $\rho(x,t), v(x,t)$, and $p(x,t)$ are piecewise C^1 in both $\{(x,t) : x \le \xi(t)\}$ and $\{(x,t) : x \ge \xi(t)\}$, with $\rho^-(\xi(t),t), v^-(\xi(t),t), p^-(\xi(t),t)$, and $\rho^+(\xi(t),t), v^+(\xi(t),t), p^+(\xi(t),t)$ denoting the limiting values of these variables to the left and right of $x = \xi(t)$, respectively, then we also have

$$\begin{aligned} &\left(\rho^-(\xi(t),t)v^-(\xi(t),t) - \rho^+(\xi(t),t)v^+(\xi(t),t)\right)\xi'(t) \\ =&p^-(\xi(t),t) + \rho^-(\xi(t),t)\left(v^-(\xi(t),t)\right)^2 - p^+(\xi(t),t) - \rho^+(\xi(t),t)\left(v^+(\xi(t),t)\right)^2. \end{aligned}$$

Exercise 1.1.3. In the derivation for (1.8), if a portion Γ of the boundary of the domain is completely insulated, derive that the boundary condition on Γ should be $\frac{\partial u(x,t)}{\partial n(x)} = 0$, where $n(x)$ denotes the exterior unit normal to Γ at x. If, on the other hand, Γ is immersed in a (large amount of) medium which maintains a temperature of u_0, then the boundary condition on Γ should be $k\frac{\partial u(x,t)}{\partial n(x)}+h(u(x,t)-u_0)=0$, where $h>0$ is some constant.

Exercise 1.1.4. Provide a proof for Lemma 1.1. Then provide the necessary adaptation to apply it to derive (1.5). Note that a natural approach would be to examine the limit of

$$h^{-1}\left(\int_{x_1}^{\xi(t+h)}\rho(x,t+h)\,dx-\int_{x_1}^{\xi(t)}\rho(x,t)\,dx\right)$$

as $h\to 0$ via

$$\begin{aligned}&h^{-1}\left(\int_{x_1}^{\xi(t+h)}\rho(x,t+h)\,dx-\int_{x_1}^{\xi(t)}\rho(x,t)\,dx\right)\\=&h^{-1}\left(\int_{x_1}^{\xi(t)}\left[\rho(x,t+h)-\rho(x,t)\right]\,dx+\int_{\xi(t)}^{\xi(t+h)}\rho(x,t+h)\,dx\right).\end{aligned}$$

But since $\rho(x,t+h)$ is defined only for $x_1\le x\le\xi(t+h)$, and the order between $\xi(t+h)$ and $\xi(t)$ may vary with h — $\rho(x,t+h)$ may be undefined if $\xi(t+h)<x<\xi(t)$. Below is a suggested work around to avoid this issue.

Since differentiation is an infinitesimal problem, it suffices to modify t_1 and t_2, if necessary, and find some $x_2>x_1$ such that $x_2>\xi(t)$ for all $t\in[t_1,t_2]$, and a C^1 extension $\widehat{\rho}(x,t)$ of $\rho(x,t)$ onto $[x_1,x_2]\times[t_1,t_2]$, so that we can apply Lemma 1.1 to $\widehat{\rho}(x,t)$ to obtain the desired result.

Another possible approach is to make a proof for $\int_{x_1}^{\xi(t)-\epsilon}\rho(x,t)dx$, for each $\epsilon>0$, and then consider the limit as $\epsilon\searrow 0$.

Exercise 1.1.5. Here we provide some guidance on constructing a C^1 extension $\widehat{\rho}(x,t)$ to $[x_1,x_2]\times[t_1,t_2]$ of $\rho(x,t)$ for some appropriate choice of t_1, t_2, and x_2, where $\rho(x,t)$ is the function referred to in **Remark 1.1**, and is assumed to be C^1 in the region $\{(x,t):t_1\le t\le t_2, x_1\le x\le\xi(t)\}$, and $\xi(t)\in C^1[t_1,t_2]$ with $\xi(t)>x_1$ for $t\in[t_1,t_2]$. Note that $\delta:=\min\{\xi(t)-x_1:t\in[t_1,t_2]\}>0$. For any $t^*\in(t_1,t_2)$, we can find $t_1\le t_1^*<t^*<t_2^*\le t_2$ such that $|\xi(t)-\xi(t^*)|<\delta/4$ for all $t\in[t_1^*,t_2^*]$. Define $h(y,t)=\rho(y+\xi(t),t)$. Then $\rho(x,t)=h(x-\xi(t),t)$, and it suffices to find a C^1 extension of $h(y,t)$ to $[-\delta,\delta]\times[t_1^*,t_2^*]$. $h(y,t)$ is defined for $x_1-\xi(t)\le y\le 0$ and is C^1 there. For $0<y<\delta$, we define $h(y,t)=ah(-y,t)+bh(-y/2,t)$ for some constants a and b. Prove that one can choose a and b such that the extended function is C^1 in $-\delta<y<\delta$, $t_1^*<t<t_2^*$. Then prove that $\rho(x,t)$ has a C^1 extension to $[x_1,x_2]\times[t_1^*,t_2^*]$ of $\rho(x,t)$ for some appropriate choice of x_2.

1.1.2 Formulation of Some Prototype PDEs Using Calculus of Variations

Next we illustrate the method of the **calculus of variations** through the derivation for the equation of **Minimal Surfaces**.

Example 1.5 (Equation of minimal surfaces)**.** Fix a region Ω in $\mathbb{R}^n$ and fix a boundary value function $\phi(x)$ defined for $x \in \partial\Omega$. Among all graphs over Ω with the boundary value $\phi(x)$, we look for one with minimal area. More specifically, let

$$M_\phi = \left\{ u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = \phi \right\}.$$

we want to find

$$\min_{u \in M_\phi} \iint_\Omega \sqrt{1 + |\nabla u|^2}\, dx,$$

and find a $u \in M_\phi$ that attains the above minimum value.

The existence of a graph attaining the minimum creates some issues, as we are looking for an element in an infinite dimensional space M_ϕ which attains the minimum of a functional. Suppose for now that such a graph u exists. Then for any $v \in M_0$, and any $\epsilon \in \mathbb{R}$ near 0, $u + \epsilon v \in M_\phi$, and

$$A(\epsilon) := \iint_\Omega \sqrt{1 + |\nabla u + \epsilon \nabla v|^2}\, dx$$

has a minimum at $\epsilon = 0$ and is differentiable in ϵ, so

$$A'(0) = \iint_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}}\, dx = 0 \quad \text{for any } v \in M_0. \tag{1.13}$$

This is an integral version of the minimal surface equation. If we further assume that this u is $C^2(\overline{\Omega})$, then we can use the divergence theorem to integrate by parts in the above to arrive at

$$\begin{aligned} 0 &= \iint_\Omega \left\{ \operatorname{div}\left(\frac{v(x)\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v(x) \right\} dx \\ &= -\iint_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v(x)\, dx + \int_{\partial\Omega} \frac{\nabla u(x) \cdot \vec{n}(x)}{\sqrt{1 + |\nabla u(x)|^2}} v(x) d\sigma(x) \\ &= -\iint_\Omega \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) v(x)\, dx. \end{aligned} \tag{1.14}$$

In the above we applied the divergence theorem $\int_{\partial\Omega} \vec{P}(x)\cdot\vec{n}(x)\,d\sigma(x) = \int_{\Omega} \mathrm{div}\vec{P}(x)\,dx$ to the $C^1(\bar{\Omega})$ vector field $\vec{P}(x) = \frac{v(x)\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}$, as

$$\mathrm{div}\left(\frac{v(x)\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\right) = \frac{\nabla u\cdot\nabla v}{\sqrt{1+|\nabla u|^2}} + \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)v(x).$$

Since (1.14) holds for arbitrary $v \in M_0$, we conclude that

$$\begin{cases} \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

In fact, if we assume only $u \in C^2(\Omega)$ instead of $u \in C^2(\overline{\Omega})$, we can still carry out this integration by parts properly by limiting the choice of v to those with compact support* in Ω, namely, $v \in C^1_c(\Omega)$, as then all the integrands become 0 near the boundary and the integration by parts are justified; but this limitation on v is harmless for our argument to (1.15), as $C^1_c(\Omega)$ is dense in M_0 in the $C^0(\overline{\Omega})$ norm as well as in $L^p(\Omega)$ norm for any $1 \le p \le \infty$.

It is not easy to find directly a $C^2(\Omega)\cap C^1(\overline{\Omega})$ function achieving the minimum area, but we will discuss later that that it is relatively easy to find a less regular function satisfying (1.13), and that there is a theory which proves that any appropriately defined solution of (1.13) is automatically $C^2(\Omega)$ and satisfies (1.15).

Remark 1.3. In contrast to the equations in **Examples 1.2, 1.3, 1.4**, (1.15) is **nonlinear** in the unknown u (as is (1.3)): if we move the terms involving the unknown(s) in (1.8) and (1.12) to the left hand side, and regard the left hand side as an operator L acting on the unknown(s) $u(x,t)$, $L[u]$, then L satisfies

$$L[a_1u_1 + a_2u_2] = a_1L[u_1] + a_2L[u_2]$$

for any coefficients a_1 and a_2, and any functions u_1 and u_2; while the operators in (1.15) and (1.3) do not satisfy this property.

A **linear** PDE satisfies the **superposition principle**: if u_1 and u_2 both satisfy $L[u] = 0$, then so does $a_1u_1 + a_2u_2$ for any coefficients a_1 and a_2; and if $L[u_1] = f_1$ and $L[u_2] = f_2$, then $L[a_1u_1 + a_2u_2] = a_1f_1 + a_2f_2$. This superposition principle simplifies the construction of general solutions to linear PDEs enormously.

*The support of a function v is the closure of the set $\{x : v(x) \neq 0\}$, and is sometimes denoted as *supp*(v). We say v has compact support in Ω, if *supp*(v) is a closed and bounded subset of Ω. This implies that there is a neighborhood U of $\partial\Omega$ in which $v(x) = 0$, and as a consequence, the derivatives of v also vanish in this neighborhood.

Example 1.6 (Variational characterization of harmonic functions)**.** Harmonic functions have a variational characterization similar to that for the minimal surface equation. Using the same notation as in the previous example, if $u \in M_\phi$ attains

$$\min_{u \in M_\phi} \iint_\Omega |\nabla u|^2 \, dx,$$

then for any $v \in M_0$, the function of t,

$$E[u + tv] \overset{\text{def}}{=} \iint_\Omega |\nabla(u + tv)|^2 \, dx$$

would attain its minimum at $t = 0$, therefore $\frac{d}{dt}\Big|_{t=0} E[u + tv] = 0$. But

$$\frac{d}{dt}\Big|_{t=0} E[u + tv] = 2 \iint_\Omega \nabla u(x) \cdot \nabla v(x) \, dx,$$

so we arrive at

$$\iint_\Omega \nabla u(x) \cdot \nabla v(x) \, dx = 0 \quad \text{for any } v \in M_0. \tag{1.16}$$

(1.16) is called the **weak (integral) form** for harmonic functions, as it is formulated using only the first derivatives of u. If we further assume that this u is $C^2(\Omega) \cap C^1(\overline{\Omega})$, then we can use the divergence theorem to integrate by parts in the above

$$\begin{aligned}
&\iint_\Omega \left[\Delta u(x) v(x) + \nabla u(x) \cdot \nabla v(x)\right] \, dx \\
=&\iint_\Omega \operatorname{div}\left(v(x) \nabla u(x)\right) \, dx \\
=&\int_{\partial\Omega} v(x) \, \vec{n}(x) \cdot \nabla u(x) \, d\sigma(x) \\
=&0, \qquad \text{using } v(x) = 0 \text{ for } x \in \partial\Omega,
\end{aligned}$$

and arrive at

$$-\iint_\Omega \Delta u(x) v(x) \, dx = \iint_\Omega \nabla u(x) \cdot \nabla v(x) \, dx = 0,$$

for all $v \in M_0$ (in fact, we require v to have compact support in Ω), therefore $\Delta u(x) = 0$ in Ω and $u(x)$ is harmonic in the classical sense. The above computation clearly also shows that a harmonic function in the classical sense also satisfies (1.16), the weak form.

Later we will see that it is relatively easy to find a function u that attains the minimum of $E[u]$ and therefore satisfies (1.16), *as long as one enlarges the class M_ϕ appropriately.* Again, what remains is to prove that a function satisfying (1.16) is also a harmonic function in the classical sense.

Remark 1.4. A solution $u(x,t)$ to the heat equation (1.8) is also related to the variational characterization in the sense that, if $u(x,t)$ solves $u_t(x,t) = \Delta_x u(x,t)$ in $\Omega \times (0,T]$ and $u(x,t) = 0$ for all $(x,t) \in \partial\Omega \times (0,T]$, and $u(x,t)$ is sufficiently smooth in $\Omega \times [0,T]$ to justify the differentiation of the integral and integration by parts below, then, with $\tilde{E}[u(\cdot,t)] \overset{\text{def}}{=} \iint_\Omega |u(x,t)|^2\,dx$, and $E[u(\cdot,t)] = \iint_\Omega |\nabla u(x,t)|^2\,dx$, we have, for $t \in (0,T]$,

$$\begin{aligned}
&\frac{d}{dt}\tilde{E}[u(\cdot,t)] = 2\iint_\Omega u(x,t)u_t(x,t)\,dx \\
=&2\iint_\Omega u(x,t)\Delta u(x,t)\,dx \\
=&2\iint_\Omega \left[\operatorname{div}(u(x)\nabla u(x)) - |\nabla u(x,t)|^2\right]\,dx \\
=&-2\iint_\Omega |\nabla u(x,t)|^2\,dx \le 0 \quad \text{using } u(x,t) = 0 \text{ when } x \in \partial\Omega,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{dt}E[u(\cdot,t)] = 2\iint_\Omega \nabla u(x,t)\cdot\nabla u_t(x,t)dt \\
=&-2\iint_\Omega \Delta u(x,t)u_t(x,t)\,dx + 2\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial n(x)}u_t(x,t)\,d\sigma(x) = -2\int_\Omega |\Delta u(x,t)|^2\,dx \le 0,
\end{aligned}$$

where in the last step, we have used $u_t(x,t) = 0$ for $(x,t) \in \partial\Omega \times (0,T]$ due to the boundary condition $u(x,t) = 0$ there. So $E[u(\cdot,t)]$ and $\tilde{E}[u(\cdot,t)]$ both decrease along a solution of the homogeneous heat equation. This can be used to prove the uniqueness of solution of (1.9).

Corollary 1.2. *For a bounded domain D with piecewise C^1 boundary,* (1.9) *can have at most one solution in the class $C(\overline{D} \times [0,\infty)) \cap C^{2,1}_{x,t}(\overline{D} \times (0,\infty))$.*

Proof. Suppose that v and w are two solutions to (1.9) satisfying the regularity assumptions here. Then $u = v - w$ is a solution of the homogeneous version of (1.9), namely, with $f \equiv 0$ and $g \equiv 0$. Under our regularity assumptions, we can justify the computations above to conclude that $\frac{d}{dt}\tilde{E}[u(\cdot,t)] \le 0$ at all $t > 0$. So $\tilde{E}[u(\cdot,t)]$ is a non-increasing function of $t \in (0,\infty)$. But $\tilde{E}[u(\cdot,t)]$ is continuous for $t \in [0,\infty)$ under our assumption that $u \in C(\overline{D} \times [0,\infty))$, and $\tilde{E}[u(\cdot,0)] = 0$. Thus we conclude that $\tilde{E}[u(\cdot,t)] \equiv 0$ for all $t \in (0,\infty)$. This implies that $u(x,t) = 0$ for all $(x,t) \in D \times (0,\infty)$. This proves that $v \equiv w$ for all $(x,t) \in D \times (0,\infty)$. In more advanced theory, one can prove, when $\partial D \in C^2$, that a solution u to the homogeneous version of (1.9) in the class $C(\overline{D} \times [0,\infty)) \cap C^{2,1}_{x,t}(D \times (0,\infty))$ is automatically in $C(\overline{D}\times[0,\infty))\cap C^{2,1}_{x,t}(\overline{D}\times(0,\infty))$, as is the case here, therefore, uniqueness holds in the class $C(\overline{D}\times[0,\infty))\cap C^{2,1}_{x,t}(D\times(0,\infty))$. Later on we will also give a proof of uniqueness in this class using the maximum principle.

□

Remark 1.5. Note that $E[v(\cdot,t)]$ may not be continuous at $t=0$ under only the assumption that $v\in C(\overline{D}\times[0,\infty))\cap C^{2,1}_{x,t}(\overline{D}\times(0,\infty))$; in fact, it may even approach ∞ as $t\searrow 0$, if the boundary value of $v(x,0)$ does not have the right regularity; part (iii) of Exercise 2.3.3 will exhibit an example with this behavior. If one would like to use the monotone decreasing property of $E[u(\cdot,t)]$ to prove the uniqueness of (1.9), one has to make additional assumptions or to prove that $E[u(\cdot,t)]$ has the needed continuity—care has to be taken in using formal computations.

Remark 1.6. Many students have found it a challenge to work with curvilinear coordinates; variational formulation often provides a natural and simplified way to derive formulae in curvilinear coordinates. The formula for the Laplace operator in polar coordinates in two dimensions $\Delta u=\frac{1}{r}(ru_r)_r+\frac{1}{r^2}u_{\theta\theta}$ can be derived by the change of variables formula; yet it is easier to derive it based on the calculations in the variational approach as follows—the method can be easily adapted to other situations. Based on computations that lead to (1.16) and subsequent computation, for any u is $C^2(\Omega)\cap C^1(\overline{\Omega})$—not just for harmonic functions, we have

$$\left.\frac{d}{dt}\right|_{t=0}E[u+tv]=2\iint_\Omega\nabla u(x)\cdot\nabla v(x)\,dx=-2\iint_\Omega\Delta u(x)v(x)\,dx \tag{1.17}$$

for $v\in M_0$. Using $|\nabla u|^2=u_r^2+\frac{1}{r^2}u_\theta^2$ and $dx=r\,drd\theta$, it is straightforward to prove that, for any $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$ and $v\in C^2_c(\Omega)\subset M_0$

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0}E[u+tv]=&\left.\frac{d}{dt}\right|_{t=0}\iint\left\{(u_r+tv_r)^2+\frac{1}{r^2}(u_\theta+tv_\theta)^2\right\}r\,drd\theta\\ =&2\iint\left\{u_rv_r+\frac{1}{r^2}u_\theta v_\theta\right\}r\,drd\theta\\ =&-2\iint v\left\{\frac{1}{r}(ru_r)_r+\frac{1}{r^2}u_{\theta\theta}\right\}r\,drd\theta\\ =&-2\iint_\Omega v\left\{\frac{1}{r}(ru_r)_r+\frac{1}{r^2}u_{\theta\theta}\right\}dx.\end{aligned}$$

In the third line above we have evaluated the double integral by iterated integrals and applied integration by parts in one variable calculus, see comments below for why we don't need to be too concerned about the geometry of the domain in carrying out such iterated integrals. Comparing with (1.17), we conclude that

$$\iint_\Omega \Delta u(x)v(x)\,dx = \iint_\Omega v\left\{\frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}\right\}dx \quad \text{for any } v \in C_c^2(\Omega),$$

which then leads to $\Delta u = \frac{1}{r}(ru_r)_r + \frac{1}{r^2}u_{\theta\theta}$—this is a point-wise relation, and we only need to carry out the above computation in appropriate subdomains surrounding the point of concern which would justify the above computations in a routine way, or take only v supported near the point of concern; the same computation also shows that the center for the polar coordinates can be chosen arbitrarily.

Exercises

Exercise 1.1.6. Let $f \in C[0,l]$. Prove that $u \in C^2[0,l]$ is a solution of

$$\begin{cases} u''(x) + f(x) = 0 \quad \text{for } x \in (0,l), \\ u'(0) = a,\ u'(l) = b \end{cases}$$

iff $\frac{d}{dt}\big|_{t=0} E(u+tv) = 0$ for all $v \in C^1[0,l]$, where $E(u) = \int_0^l \left(\frac{1}{2}|u'(x)|^2 - f(x)u(x)\right)dx + au(0) - bu(l)$. In addition, prove that a necessary condition for this problem to have a solution is that $\int_0^l f(x)\,dx = a - b$.

Exercise 1.1.7. Prove that for any $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$, there holds

$$\iint_\Omega [\Delta u(x)v(x) + \nabla u(x)\cdot\nabla v(x)]\,dx = \int_{\partial\Omega} v(x)\,\frac{\partial u(x)}{\partial \vec{n}(x)}\,d\sigma(x).$$

Exercise 1.1.8. Let P be an $n \times n$ orthogonal matrix, $u \in C^2(\mathbf{R}^n)$. Prove that

(a). $\nabla[u(Px)] = \nabla u(Px)P$, if both $\nabla[u(Px)]$ and $\nabla u(Px)$ are treated as row vectors;

(b). $\|\nabla[u(Px)]\| = \|\nabla u(Px)\|$;

(c). $\Delta_x[u(Px)] = \Delta_x u(Px)$.

This shows that if one makes a rotation of axes, $\Delta_x u(x)$ is unaffected.

Exercise 1.1.9. This exercise illustrates how to use variational characterization to derive the Laplace operator in 3-dimensional spherical coordinates.

(a). Prove that $\sum_{i=1}^{n} |\nabla_{\mathbf{e}_i} u(x)|^2$ is independent of the choice of an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^{n}$ at x.

(b). Using the relation between rectangular and spherical polar coordinates

$$\begin{cases} x = & r\sin\theta\cos\phi \\ y = & r\sin\theta\sin\phi \\ z = & r\cos\theta \end{cases}$$

to verify that

$$\begin{cases} \mathbf{e}_r = & (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \\ \mathbf{e}_\theta = & (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta), \\ \mathbf{e}_\phi = & (-\sin\phi, \cos\phi, 0) \end{cases}$$

forms an orthonormal basis at x, and $\nabla_{\mathbf{e}_r} u = \frac{\partial u}{\partial r}$, $\nabla_{\mathbf{e}_\theta} u = r^{-1}\frac{\partial u}{\partial \theta}$, and $\nabla_{\mathbf{e}_\phi} u = (r\sin\theta)^{-1}\frac{\partial u}{\partial \phi}$.

(c). Based on (a) and (b),

$$|\nabla u(x)|^2 = |\frac{\partial u}{\partial r}|^2 + r^{-2}|\frac{\partial u}{\partial \theta}|^2 + (r\sin\theta)^{-2}|\frac{\partial u}{\partial \phi}|^2.$$

Use this, the change of variables formula for triple integral

$$dxdydz = r^2\sin\theta dr d\theta d\phi,$$

and (1.17) to prove that

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right].$$

$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$ is called the **spherical Laplace operator**, a natural generalization of the Laplace operator on functions defined on the unit round sphere. This is based on the following computation and in analogy to (1.17)

$$\begin{aligned} &\iint_{\mathbb{S}^2}\left\{\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right]v(\theta,\phi)\right\}\sin\theta d\theta d\phi \\ =& -\iint_{\mathbb{S}^2}\left(\frac{\partial u}{\partial\theta}\frac{\partial v}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial u}{\partial\phi}\frac{\partial v}{\partial\phi}\right)\sin\theta d\theta d\phi \\ =& -\iint_{\mathbb{S}^2}\nabla_{\mathbb{S}^2}u\cdot\nabla_{\mathbb{S}^2}v\; d\sigma, \end{aligned}$$

with $\nabla_{\mathbb{S}^2}u = (\nabla_{\mathbf{e}_\theta}u)\,\mathbf{e}_\theta + \left(\nabla_{\mathbf{e}_\phi}u\right)\mathbf{e}_\phi$ denoting the spherical gradient of u as a function of θ and ϕ on the unit sphere $\mathbb{S}^2$, and $d\sigma = \sin\theta\; d\theta d\phi$ denoting the area element on the unit sphere.

1.2 Some Initial Comparisons and Relations with ODEs

Here we make some initial, general comments on the comparison in the construction and behavior of solutions between of ODEs and PDEs.

1.2.1 Superposition Principle and Separation of Variables

The "general" solution of an ODE (or a system of ODEs) depends on a finite number of parameters and forms a "finite dimensional manifold", where the parameters can often be adjusted to satisfy some additional side conditions such as initial or boundary conditions (one way to obtain the general solution is to produce the solution in terms of the initial values). In the case of a system of **linear homogeneous** ODEs, this is particularly simple: the solution space is linearly spanned by **a finite number of basis solutions**. In fact, a basic common property of **linear homogeneous** ODEs or PDEs is the **linear superposition principle**: if $u_1, \cdots, u_N$ are solutions to the (same) linear homogeneous ODE or PDE, then so is $a_1u_1 + a_2u_2 + \cdots + a_Nu_N$ for any coefficients $a_1, a_1, \cdots, a_N$.

One difference, however, is that even for simple linear PDEs such as the ones we derived above, the solution space (without imposing boundary or initial conditions) is often **infinite dimensional**. One of our initial focuses will be to learn how to construct an infinite dimensional "basis" of solutions for many linear PDEs (often with constant coefficients) and use them to construct general solutions. One main task will be to make the right sense of the convergence of "linear superposition" of infinitely many basis solutions.

Remark 1.7. Our approach will often be to start by understanding a concept or method in the simplest setting, then examine to what extent the simplest examples represent prototypical situations, and try to extend our method to be applicable in a more general setting. So we will not be afraid of looking for special solutions in special situations first to gain intuition.

One of the simplest and most useful methods for constructing particular solutions of (mostly) linear PDEs with constant coefficients is **separation of variables**: one looks for solutions of the form $u(x,t) = T(t)X(x)$ (sometimes in the form of $T(x) + X(x)$) to reduce the construction of such solutions of a PDE to some set of ODEs.

Example 1.7. Let's look for some sample solutions of (1.8) when x is one dimensional and takes values in the entire $\mathbb{R}$ (so that we don't have to deal with boundary value for now). We will first look for solutions of the form $u(x,t) = T(t)X(x)$. The direct application of separation of variables works when $f = 0$. For simplicity of computation, we also take $\gamma = 1$. Then we need to find $X(x)$ and $T(t)$ such that

$$T'(t)X(x) = T(t)X''(x) \quad \text{for all } (x,t).$$

We deduce that whenever $X(x) \neq 0$ and $T(t) \neq 0$, we must have

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}.$$

From this we conclude that $X''(x)/X(x)$ and $T'(t)/T(t)$ must be a constant, independent of (x,t), for it does not allow $\frac{X''(x_1)}{X(x_1)} \neq \frac{X''(x_2)}{X(x_2)}$ for some $x_1 \neq x_2$ with $X(x_1) \neq 0, X(x_2) \neq 0$. Call this constant $-\lambda$. Then we have a set of ODEs

$$X''(x) = -\lambda X(x), \quad x \in \mathbb{R}; \tag{1.18}$$

$$T'(t) = -\lambda T(t), \quad t \in \mathbb{R}. \tag{1.19}$$

In most situations we expect $X(x)$ and $T(t)$ to be continuous, so (1.18) and (1.19) should continue to hold even when $X(x) = 0$ or $T(t) = 0$. Our remaining task is to find λ for which the above two equations have non-trivial solutions ($X \equiv 0$ and $T \equiv 0$ are always solutions, called trivial solutions).

Note that for each value of the parameter ξ, $X(x) = e^{ix\xi}$ is a solution of (1.18) with $\lambda = \xi^2$, and $T(t) = e^{-\xi^2 t}$ solves (1.19) for the same λ. So $u_\xi = e^{ix\xi - \xi^2 t}$ is a solution of (1.18) and (1.19) with $\lambda = \xi^2$. By the superposition principle, any finite linear combination of such solutions

$$\sum_{\xi \in \text{a finite set}} c(\xi) e^{ix\xi - \xi^2 t}$$

is a solution of (1.8).

Question: Does this construction generate all solutions to the Cauchy problem for the homogeneous version of the heat equation (1.8)?

Note that at this point, ξ can be any scalar in $\mathbb{C}$, so we have a continuum of parameters to be used to construct a solution of $u_t - u_{xx} = 0$. If we take ξ to be a real parameter, then the solutions obtained above all correspond to $\lambda = \xi^2 \geq 0$. For other choice of the parameter value λ there are also solutions. E.g. for non-real complex valued ξ, $e^{ix\xi - \xi^2 t}$ is a solution of the homogeneous heat equation, but such solutions grow exponentially in x as x goes to one end of infinity. Since we construct solutions defined for x on the entire $\mathbb{R}$, physical consideration often rules out solutions that grow exponentially in a space variable, thus restricts $\xi \in \mathbb{R}$ (see an exercise of this section, however, for an application using a model defined on the half real line where ξ is allowed to take on non-real values).

If we now try to construct a solution of the IVP

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}^+, \\ u(x,0) = g(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.20}$$

to satisfy a fairly arbitrary initial data, say, for any g bounded and continuous on $\mathbb{R}$ (i.e., in $C_b(\mathbb{R})$), or continuous on $\mathbb{R}$, with the additional property that $g(x) \to 0$ as $x \to \pm\infty$ (i.e., in $C_0(\mathbb{R})$), or in $L^p(\mathbb{R})$, the space of functions whose pth power is integrable on $\mathbb{R}$, then it is not enough to take only finite linear combinations of solutions of the form $e^{ix\xi - \xi^2 t}$. * It turns out that an "infinite linear combination" in the form of an integral

$$\int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi, \tag{1.21}$$

where $c(\xi)$'s are the coefficients of the "infinite linear combination", provides such a general solution. The main issue is the convergence of such integrals for $(x,t) \in \mathbb{R}^1 \times \mathbb{R}^+$; another issue is how to choose $c(\xi)$ to satisfy a given initial data $u(x,0)$.

Note that we now restrict the domain of t to $\mathbb{R}^+$, as for $t < 0$, the exponentially fast growth of $e^{-\xi^2 t}$ in ξ would put severe decay requirement on $c(\xi)$ in the construction of a solution of the form $\int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi$, which would put severe restriction on the kind of initial data g for which this construction would produce a solution of (1.20).

At least formally, for real ξ, the factor $e^{-\xi^2 t}$ helps with the convergence of the integral for $t > 0$, but not for $t < 0$; also we expect

$$u(x,0) = \int_{\xi \in \mathbb{R}} c(\xi) e^{ix\xi} d\xi. \tag{1.22}$$

We recognize that the relation between $u(x,0)$ and $c(\xi)$ is that between a function and its Fourier transform—we do not need any systematic theory of Fourier transforms at

*This is based on properties of Fourier transforms. For any finite measure μ on $\mathbb{R}$, define its Fourier transform $\hat{\mu}$ by $\hat{\mu}(\xi) = \int_{\mathbb{R}} e^{-ix\xi}\, d\mu(x)$; in particular, for any $g \in L^1(\mathbb{R})$, denote $\hat{g}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} g(x) dx$. Then $\hat{g}(\xi) \in C_0(\mathbb{R})$ for every $g \in L^1(\mathbb{R})$. Most relevant properties here are (a) if $\hat{\mu}(\xi) = \hat{\nu}(\xi)$ for all $\xi \in \mathbb{R}$, then $\mu = \nu$; (b) if $g(x) = \int_{\mathbb{R}} e^{ix\xi}\, d\mu(\xi)$ for some finite measure μ on $\mathbb{R}$, and $g \in L^1(\mathbb{R})$, then $\mu = \hat{g}(\xi) d\xi$ for some $\hat{g} \in C_0(\mathbb{R})$. A function in $C^\infty(\mathbb{R})$ decaying faster than any power of $|x|$, together with any of its finite order derivatives, as $|x| \to \infty$, can not be represented in the form of a finite, or infinite but discrete, sum of the form $\sum_j c_j e^{ix\xi_j}$.

this point; we will provide a formula relating $c(\xi)$ to $u(x,0)$ when needed and prove it rigorously later on. The purpose here is to illustrate that the homogeneous heat equation has an infinite parameter family of solutions, and how this family may be used to construct a general solution. More systematic study of Fourier transforms, including their mapping properties between various function spaces, will be needed for more in-depth study of the theory of PDEs.

If one is interested in solving the homogeneous heat equation on a finite interval $[0,l]$ with the simplest boundary conditions $u(0,t) = u(l,t) = 0$ for all $t > 0$, for instance, then in our separable solutions, we want $X(0) = X(l) = 0$. So the choice of $X(x)$ is limited to satisfy a boundary value problem of ODEs:

$$\begin{cases} X''(x) = -\lambda X(x), \\ X(0) = X(l) = 0. \end{cases} \tag{1.23}$$

$X(x) \equiv 0$ is always a solution; we are interested in non-trivial solutions, namely, solutions that are not identically 0. For certain values of λ such as $\lambda = \left(\frac{n\pi}{l}\right)^2$, for $n = 1, 2, \cdots$, (1.23) has non-trivial solutions: all solutions are scaler multiples of $X_n(x) = \sin\left(\frac{n\pi x}{l}\right)$. Furthermore, these λ values are the only ones for which (1.23) has non-trivial solutions. This is worked out by applying the boundary conditions in (1.23) to the general solutions to the first equation of (1.23)—one may either write $\lambda = \xi^2$ for some $\xi \in \mathbb{C}$, and write the general solution in the form of $c_1 e^{i\xi x} + c_2 e^{-i\xi x}$ in the case $\xi \neq 0$, or work out the general solution depending on whether λ is 0, a positive real, a negative real, or is in the remaining cases.

Note that a boundary value problem of ODEs such as (1.23) exhibits different behavior from an IVP of ODEs: while an IVP of (well behaved) ODEs always has a unique solution for any given initial data; a BVP of ODEs may fail the uniqueness, and in fact, may fail to have a solution for certain boundary data. For example, the BVP $\{X''(x) + X(x) = 0, X(0) = 0, X(\pi) = 1\}$ has no solution at all.

Back to our problem. At this point

$$\sum_{n \in \text{a finite set}} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$$

is a genuine solution of the homogeneous heat equation with the boundary condition $u(0,t) = u(l,t) = 0$ for all $t > 0$, with initial data $u(x,0) = \sum_{n \in \text{a finite set}} c_n \sin\left(\frac{n\pi x}{l}\right)$. To satisfy an arbitrarily given initial data $u(x,0)$, we again need to make an infinite sum. Formally, we need to choose c_n such that

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = u(x,0) \quad \text{on } (0,l) \text{ in an appropriate sense.} \tag{1.24}$$

We need to figure out how to determine c_n's in terms of the given $u(x,0)$, and in what sense the constructed series converges at $(x,0)$ to $u(x,0)$ (in fact we need to show that the constructed series solution $u(y,t)$ converges to $u(x,0)$ as $(y,t) \to (x,0)$), and converges at (x,t), $t>0$, to a solution of the heat equation (1.20). The study of such problems was initiated by J. Fourier in the early 1800's.

Again, the factor $e^{-\left(\frac{n\pi}{l}\right)^2 t}$ will help with the convergence of $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$ for $t>0$, but not for $t<0$.

This brings up the issue of **well-posedness**.

1.2.2 Well-posedness

For the IVP of an ODE system of the form $\frac{du}{dt} = f(u(t),t), u(t_0) = u_0$, the **Cauchy-Picard** theorem gives us the existence of a unique solution on a (maybe short) time interval $[t_0-\delta, t_0+\delta]$, for a right hand side f that is Lipschitz* in u; and the solution depends continuously on the initial data. For a boundary/initial value problem for a PDE, we also need to address the same issues. We say a boundary/initial value problem for a PDE is well-posed if

- there exists a solution for all data in a reasonable (often closed) set of a function space;
- the solution is unique (or depends at most on a finite number of free parameters);
- the solution depends on the data in a continuous way.

In the above the existence of a solution for data in a closed set means that if for a sequence of data g_j, there exists a solution u_j, and $g_j \to g_\infty$ in appropriate sense, then we expect u_j to converge in appropriate sense to a solution with g_∞ as data.

The meaning of "continuous way" has some flexibility in the sense that we may need to measure the variation of solution in norms that are appropriate for the problem, not necessarily the familiar norms that are used to measure uniform convergence.

Our calculations for the sample solutions of the heat equation above suggest that the Cauchy problem (or the boundary/initial value problem) is probably well-posed in the above sense for forward time $t>0$, but not for backward time $t<0$. The latter can be seen easily, if we use uniform convergence to measure variation of solutions:

*$f(u,t)$ is said to be Lipschitz in u for $(u,t) \in U$, if there exists some $L>0$ such that $|f(u,t) - f(v,t)| \le L|u-v|$ whenever $(u,t),(v,t) \in U$. A basic property of such a Lipschitz function is that $f(u,t)$ is differentiable in u except on a negligible set, called a set of measure 0.

one way to formulate continuous dependence of solution for $t \in [T_1, T_2]$ in this setting of *linear equation* is that there exists a constant $C > 0$ depending on the equation, and possibly on T_1, T_2, such that for all solutions $u(x,t)$ to the homogeneous heat equation which are continuous and bounded in $\mathbb{R} \times [T_1, T_2]$, namely, in the space $C_b(\mathbb{R} \times [T_1, T_2])$,

$$||u(\cdot, t)||_{C_b(\mathbb{R})} \leq C||u(\cdot, 0)||_{C_b(\mathbb{R})} \quad \text{for all } t \in [T_1, T_2]$$

where $||u(\cdot,t)||_{C_b(\mathbb{R})}$ is defined by $\sup\{|u(x,t)| : x \in \mathbb{R}\}$, and $C_b(\mathbb{R})$ (respectively $C_b(\mathbb{R} \times [T_1, T_2])$) denotes the space of bounded continuous functions on $\mathbb{R}$ (respectively $\mathbb{R} \times [T_1, T_2]$). This critirion would then imply that, for any two such solutions u and v,

$$||u(\cdot, t) - v(\cdot, t)||_{C_b(\mathbb{R})} \leq C||u(\cdot, 0) - v(\cdot, 0)||_{C_b(\mathbb{R})} \quad \text{for all } t \in [T_1, T_2].$$

This is the sense in which we say that the solution $u(\cdot, t)$ varies continuously in $C_b(\mathbb{R})$ with $u(\cdot, 0)$ in the same norm. If we are interested in learning how other norms of the solutions depend on appropriate norms of initial data, we can formulate a corresponding estimate (the norms of the solutions can be different from those of the data).

If the heat equation were well-posed in this formulation on $[-\delta, 0]$ for some $\delta > 0$, we would expect the above estimate hold for $[T_1, T_2] = [-\delta, 0]$. But all the solutions $e^{ix\xi - \xi^2 t}$ for real ξ have norm in $C_b(\mathbb{R})$ equal to 1 at time $t = 0$, yet at any negative time $t = -\delta$, the $C_b(\mathbb{R})$ norm of this solution is equal to $e^{\xi^2\delta}$, which can be made as large as one wants by taking large real ξ, thus there is no C for which the above estimate can hold. This phenomenon of well-posedness possibly only in one direction is very different from the behavior of ODEs.

Remark 1.8. We remarked earlier that physical consideration typically rules out a solution that would grow too fast in a space variable, but a solution that grows in the time variable is not to be discarded without further consideration. The formulation of continuous dependence in the context of an evolution equation is done for any given **finite** time interval, as illustrated above: the constant C above may depend on T_1 and T_2, and may grow, if T_1 or T_2 grows. This is true even in the ODE setting: the IVP $\{u'(t) = u(t), u(0) = u_0\}$, is well-posed, even though its solution $u(t) = u_0 e^t$ grows exponentially in t for $t > 0$; in the mean time, for any given $T > 0$, $\max\{|u(t) - v(t)| : T_1 \leq t \leq T\} \leq e^T |u(0) - v(0)|$ for any two solutions to this IVP.

In some context there is a notion of **large time stability** of certain (special) solutions. Some students may confuse the notion of continuous dependence of solution on data (over a finite time interval) with the notion of large time stability.

We will learn in this course that the IVP and IBVP for the wave equation (1.12) discussed in the last section are well-posed; but a similarly formulated IVP or IBVP for the Laplace equation is not well-posed. An example of a well-posed problem for the Laplace equation is the boundary value problem (BVP)

$$\begin{cases} \Delta u(x) = 0 & \text{for } x \in D, \\ u(x) = \phi(x) & \text{for } x \in \partial D, \text{ and a given function } \phi(x) \text{ on } \partial D. \end{cases}$$

Here the continuous dependence of u on ϕ would mean that the size of u in D, measured in an appropriate norm, depends on the size of ϕ on ∂D in a continuous way. In fact, we will see that $||u||_{C(\overline{D})} \leq ||\phi||_{C(\partial D)}$.

There is no direct generalization of the Cauchy-Picard local existence theorem for a general Cauchy problem for a PDE. For the Cauchy problem for PDEs which are **analytic** in the unknowns (namely, all the terms have convergent power series expansions in their arguments) and have analytic initial data on an analytic initial surface which is **non-characteristic** (to be defined later), there is a partial generalization of the Cauchy-Picard theorem, called **Cauchy-Kowalevskaya** theorem. We will discuss this theorem and the relevant notions of characteristics of PDEs later on.

Remark 1.9. Our examples earlier may suggest that PDEs (at least linear ones) may have too big a solution space that we have to impose appropriate boundary/initial conditions to have a well-posed problem. So it was a surprise when, in 1957, H. Lewy constructed a first order linear PDE that has no solution *anywhere*.

1.2.3 Additional Comments on Separation of Variables

Separation of variables of constant coefficient second order PDEs often lead to an ODE similar to (1.18), whose solutions are generated by solutions of the form $e^{i\xi x}$, with ξ being an appropriate parameter, often called Fourier frequency or Fourier mode. One often directly looks for a solution of a constant coefficient PDE in the form of $e^{i\xi x}T(t)$, where $T(t)$ is to be solved in terms of ξ and t; conditions on ξ, if any, would be dictated by the boundary conditions. There are also times where one looks for solutions for the form $u(x,t) = e^{i\nu t}X(x)$, a solution with time frequency ν (such a solution is often called a time-harmonic solution, or a standing wave solution); one can find an example in the next section when discussing solutions of a Schrödinger equation. The same remark also applies to higher order PDEs with constant coefficients.

In the majority of situations, the PDEs we encounter are formulated in terms of real valued scalars or vectors and we are interested in solutions as real valued scalars or vectors. But our construction of particular solutions using separation of variables, such as for (1.8), often uses complex valued solutions such as $e^{i\xi x}$, as they are often easier to work with and give a cleaner representation. For example, one could also formulate a version of (1.22) in terms of the real and imaginary parts of $e^{i\xi x}$, but the resulting relation would not be as clean.

A solution of the form $u(x,t) = X(x)T(t)$, when $X(x), T(t)$ are real valued, often can be interpreted as representing a standing wave, as its profile as a function of x is determined by $X(x)$, with $T(t)$ only playing the role of modulating the range of the $u(x,t)$; the set of points where $X(x) = 0$ (often called the nodal set) does not change with t. When $X(x)$ and $T(t)$ are complex valued, the above interpretation is not directly applicable; for example, a solution of the form $e^{i\xi x}e^{ic|\xi|t}$ for real valued c, ξ represents a **traveling wave** at a speed of c, and the same interpretation applies to its real and imaginary parts, $\cos(\xi x + c|\xi|t), \sin(\xi x + c|\xi|t)$.

Example 1.8. If one applies separation of variables to construct solutions to the one dimensional case of (1.10): $u_{tt} - c^2 u_{xx} = 0$ in the form of $u(x,t) = X(x)T(t)$, one would need to find $X(x), T(t)$ and constants λ such that

$$X''(x) = -\lambda X(x), \quad T''(t) = c^2 \lambda T(t).$$

It turns out that only when λ is a non-negative real number, will the non-zero solutions $X(x)$ not grow too fast as $x \to \infty$ and $-\infty$. For such λ's one can work out the solutions in either real or complex forms and then apply the superposition principle to construct additional solutions. In this way one can find both standing wave and traveling wave solutions. The details will be assigned as an exercise.

Example 1.9. If we apply separation of variables to construct solutions to $u_t - u_{xxxx} = 0$, it would lead to the set of ODEs

$$\begin{aligned} X''''(x) + \lambda X(x) = 0 \quad &\text{for } x \in \mathbb{R}, \\ T'(t) = -\lambda T(t) \quad &\text{for } t \in \mathbb{R}, \end{aligned}$$

for some constant λ. We could work with solutions to $X''''(x) + \lambda X(x) = 0$, which form a 4-dimensional vector space for each λ, and involve the fourth order roots of λ; but the remark above meant that we could directly look for a solution of the PDE of the form $u(x,t) = e^{i\xi x}T(t)$. If we plug $u(x,t) = e^{i\xi x}T(t)$ directly into $u_t - u_{xxxx} = 0$, we would get

$$T'(t)e^{i\xi x} - \xi^4 T(t) e^{i\xi x} = 0,$$

from which we conclude that $T'(t) = \xi^4 T(t)$, so $T(t) = T(0)e^{\xi^4 t}$. Here we directly used that $X(x) = e^{i\xi x}$ satisfies $X''''(x) = -\lambda X(x)$ with $-\lambda = \xi^4$, but did not write out the equation $X''''(x) + \lambda X(x) = 0$ explicitly. To conclude, $e^{i\xi x + \xi^4 t}$ is a solution of $u_t - u_{xxxx} = 0$ for any parameter ξ. For any $t > 0$, this solution grows exponentially in ξ for real valued ξ, and, so we anticipate difficulty for convergence in forming solutions of the form $\int_{\mathbb{R}} c(\xi)e^{i\xi x+\xi^4 t}d\xi$ for $t > 0$—the issue here is the exponential growth of $e^{i\xi x+\xi^4 t}$ in ξ, its growth in t is not relevant for this issue.

The computations here suggest that the IVP for $u_t - u_{xxxx} = 0$ is not well-posed for $t > 0$, but seems well-posed for $t < 0$. A similar computation suggests that the IVP for $u_t + u_{xxxx} = 0$ is well-posed for $t > 0$.

If we consider solutions to $u_t + u_{xxxx} = 0$ for x on a finite interval $(0, l)$, then a well-posed problem would need to impose some boundary conditions at $x = 0$ and $x = l$. Examples of such boundary conditions include $\{u(0,t) = 0, u(l,t) = 0, u_x(0,t) = 0, u_x(l,t) = 0 \text{ for all } t > 0\}$ or $\{u(0,t) = 0, u(l,t) = 0, u_{xx}(0,t) = 0, u_{xx}(l,t) = 0 \text{ for all } t > 0\}$.

If we use the latter boundary conditions, then separation of variables would lead to the boundary value problem $\{X''''(x) = -\lambda X(x), 0 < x < l; X(0) = X''(0) = 0, X(l) = X''(l) = 0\}$, and the questions we need to address include *(a). For which* λ *does this problem possess non-trivial solutions? and (b). Whether we can use (infinite) linear combinations of such solutions to represent an arbitrary function on* $(0, l)$ *corresponding to the initial data?*

Here, if $\lambda = -\xi^4$ for some ξ, then the general solution of $X''''(x) = -\lambda X(x)$ is given by

$$X(x) = c_1 e^{i\xi x} + c_2 e^{-i\xi x} + c_3 e^{\xi x} + c_4 e^{-\xi x},$$

assuming $\lambda \neq 0$, and the four boundary conditions would produce a homogeneous system of four linear equations in c_1, c_2, c_3, c_4. Unless the determinant of that linear system is equal to 0, the only solution is $c_1 = c_2 = c_3 = c_4 = 0$.

So only those ξ's which make the determinant of that linear system equal to 0 can lead to non-trivial solutions $X(x)$. For each such ξ, we would need to determine the dimension of solutions to the linear homogeneous system in c_1, c_2, c_3, c_4—in the second order case $X''(x) + \lambda X(x) = 0$, the dimension turns out to be always equal to 1. The determinant of that linear system is a holomorphic function of ξ, so it can have at most countable number of solutions and they have no finite accumulation point.

Suppose that for a countable number of ξ_k, the dimension of solutions is also 1, say, c_2, c_3, c_4 are solved in terms of c_1: $c_2 = b_2(\xi_k)c_1$, $c_3 = b_3(\xi_k)c_1$, $c_4 = b_4(\xi_k)c_1$, then, setting $c_1 = a_k$, and $X_k = e^{i\xi_k x} + b_2(\xi_k)e^{-i\xi_k x} + b_3(\xi_k)e^{\xi_k x} + b_4(\xi_k)e^{-\xi_k x}$, we would try to use $\sum_k a_k X_k(x)e^{-\xi_k^4 t}$ as a general solution—for this to work in $t > 0$ it requires that $\Re(\xi_k^4) \to -\infty$ does not happen along a subsequence. The analysis here is more complicated than that for the IVP or for a second order PDE. An answer to (b) would often lead to the study of the spectrum of a Sturm-Liouville boundary value problem. We will study some Sturm-Liouville boundary value problem later on.

Remark 1.10. The solutions to many problems arising from the separation of variables often carry one or more arbitrary constants, due to the equations being homogeneous. However, there are situations where the constants are subject to some constraints, such as the case in constructing solutions to the Maxwell's equations, which are the equations governing electric and magnetic fields. In the vacuum, it takes on the following form

$$\begin{cases} \text{div } \mathbf{E} = 0 \\ \text{div } \mathbf{B} = 0 \\ \dfrac{\partial \mathbf{E}}{\partial t} = c^2 \text{curl}\, \mathbf{B} \\ \dfrac{\partial \mathbf{B}}{\partial t} = -\text{curl}\, \mathbf{E} \end{cases}$$

where $\text{curl}\, \mathbf{B}$ and $\text{curl}\, \mathbf{E}$ are the curl of $\mathbf{B}$ and $\mathbf{E}$ respectively. In rectangular coordinates, $\mathbf{B}$ is given by

$$\text{curl}\, \mathbf{B} = (\frac{\partial B_3}{\partial x_2} - \frac{\partial B_2}{\partial x_3}, \frac{\partial B_1}{\partial x_3} - \frac{\partial B_3}{\partial x_1}, \frac{\partial B_1}{\partial x_2} - \frac{\partial B_2}{\partial x_1}).$$

For any vector parameter $\boldsymbol{\xi}$, we look for a solution of the form

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i\boldsymbol{\xi}\cdot\mathbf{x}} T(t), \quad \mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 e^{i\boldsymbol{\xi}\cdot\mathbf{x}} T(t)$$

for some constant vectors $\mathbf{E}_0, \mathbf{B}_0$. The first two equations give rise to

$$\mathbf{E}_0 \cdot \boldsymbol{\xi} = 0, \quad \mathbf{B}_0 \cdot \boldsymbol{\xi} = 0.$$

Thus both $\mathbf{E}_0$ and $\mathbf{B}_0$ need to be orthogonal to $\boldsymbol{\xi}$. We compute by definition to find curl $\left(\mathbf{E}_0 e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\right) = i\mathbf{E}_0 \times \boldsymbol{\xi}$, and curl $\left(\mathbf{B}_0 e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\right) = i\mathbf{B}_0 \times \boldsymbol{\xi}$, so the last two equations lead to

$$\begin{cases} \mathbf{E}_0 T'(t) = ic^2 T(t)\mathbf{B}_0 \times \boldsymbol{\xi} \\ \mathbf{B}_0 T'(t) = -iT(t)\mathbf{E}_0 \times \boldsymbol{\xi} \end{cases}$$

Thus $T'(t) = i\lambda T(t)$ holds for all t and some constant real parameter λ, and $\mathbf{E}_0 \perp \mathbf{B}_0$. Now using $\|\mathbf{B}_0 \times \boldsymbol{\xi}\| = \|\mathbf{B}_0\|\|\boldsymbol{\xi}\|$ and $\|\mathbf{E}_0 \times \boldsymbol{\xi}\| = \|\mathbf{E}_0\|\|\boldsymbol{\xi}\|$, we find that

$$\lambda = \pm c\|\boldsymbol{\xi}\|, \quad \|\mathbf{E}_0\| = c\|\mathbf{B}_0\|.$$

The above give the constraints on $\mathbf{E}_0, \mathbf{B}_0$, and the solution will then take the form

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i(\boldsymbol{\xi}\cdot\mathbf{x} \pm c\|\boldsymbol{\xi}\|t)},$$
$$\mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 e^{i(\boldsymbol{\xi}\cdot\mathbf{x} \pm c\|\boldsymbol{\xi}\|t)}.$$

These represent **plane waves**, as for any real parameter d, $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ each remains a constant vector along the family of solutions to $\boldsymbol{\xi} \cdot \mathbf{x} \pm c\|\boldsymbol{\xi}\|t = d$, which represents planes with $\boldsymbol{\xi}$ as a normal vector, progressing in the direction of $\boldsymbol{\xi}$ at a speed of c when $\lambda = -c\|\boldsymbol{\xi}\|$, and progressing in the opposite direction of $\boldsymbol{\xi}$ at a speed of c when $\lambda = c\|\boldsymbol{\xi}\|$. If we will take $\lambda = -c\|\boldsymbol{\xi}\|$, then

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{E}_0 e^{i(\boldsymbol{\xi}\cdot\mathbf{x} - c\|\boldsymbol{\xi}\|t)}, \mathbf{B}(\mathbf{x}, t) = \mathbf{B}_0 e^{i(\boldsymbol{\xi}\cdot\mathbf{x} - c\|\boldsymbol{\xi}\|t)}, \quad \mathbf{E}(\mathbf{x}, t) = -c\mathbf{B}(\mathbf{x}, t) \times \left(\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|}\right).$$

Note that the solutions here $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ exhibit the same phase in the sense that, in real form, $\mathbf{E}_0 \cos(\boldsymbol{\xi} \cdot \mathbf{x} - c\|\boldsymbol{\xi}\|t)$ and $\mathbf{B}_0 \cos(\boldsymbol{\xi} \cdot \mathbf{x} - c\|\boldsymbol{\xi}\|t)$ oscillate in the same phase.
Question: What kind of solutions does one get if one tries to construct a solution of the form $\mathbf{E}(\mathbf{x}, t) = \mathbf{E}(\mathbf{x})T(t)$ and $\mathbf{B}(x, t) = \mathbf{B}(\mathbf{x})T(t)$, either restricting to real-valued solutions or allowing complex-valued solutions?

Below is an example where the separation of variable appears in the form of a solution only depending on one variable; there is an exercise in which one looks for a solution as the sum of a function of one variable and another function of a different variable.

Example 1.10. The Navier-Stokes equations model the motion of fluids. When the fluid is incompressible, namely, when the volume of the fluid does not experience change as it flows, the equations take the form of

$$\begin{cases} \operatorname{div} \mathbf{u}(\mathbf{x}, t) = 0, \\ \dfrac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} + \mathbf{u}(\mathbf{x}, t) \cdot \nabla \mathbf{u}(\mathbf{x}, t) = -\rho^{-1} \nabla p(\mathbf{x}, t) + \nu \Delta \mathbf{u}(\mathbf{x}, t), \end{cases} \tag{1.25}$$

where $\mathbf{u}(\mathbf{x}, t)$ represents the velocity of the fluid at location $\mathbf{x}$ and time t in a rectangular coordinate system, $p(\mathbf{x}, t)$ represents the pressure there, ρ represents the density of the fluid, which is taken as a positive constant here, and ν represents the kinetic viscosity of the fluid.

These are among the most challenging PDEs to study. We will look for particular steady state solutions modeling a fluid flow in an infinite channel, namely, the domain is $\{(x, y, z) : -a \le y \le a, x, z \in \mathbb{R}\}$, and we assume that $\mathbf{u}(\mathbf{x}, t)$ takes the form of $(U(y), 0, 0)$ for some $U(y)$ to be determined and $p(\mathbf{x}, t)$ will also be independent of t.

Then the equation $\operatorname{div} \mathbf{u}(\mathbf{x}, t)$ is satisfied automatically, and the second set of equations takes the form of

$$\begin{cases} 0 = -\rho^{-1}\dfrac{\partial p}{\partial x} + \nu \dfrac{\partial^2 U(y)}{\partial y^2} \\ 0 = -\rho^{-1}\dfrac{\partial p}{\partial y} \\ 0 = -\rho^{-1}\dfrac{\partial p}{\partial z} \end{cases}$$

Thus p is a function of x alone, and we get

$$0 = -\rho^{-1} p'(x) + \nu U''(y) \text{ for all } x \in \mathbb{R}, -a < y < a.$$

This forces $-p'(x) = G$ for some constant G, and $\nu U''(y) = -\rho^{-1} G$. Thus we find $p(x) = p_0 - Gx$ and $U(y) = -\frac{Gy^2}{2\rho\nu} + by + c$ for some constants b, c.

A typical boundary condition is the so called no-slip condition, namely, the fluid particles on the boundary stick to the boundary: $\mathbf{u}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{x} = (x, y, z)$ on the boundary. This implies that $U(\pm a) = 0$, which then implies that

$$U(y) = \frac{G(a^2 - y^2)}{2\rho\nu}.$$

Exercises

Exercise 1.2.1. Prove that $\{(\frac{n\pi}{l})^2 : n \in \mathbb{N}\}$ provides the set of all eigenvalues to the BVP (1.23), and that for each $\lambda = (\frac{n\pi}{l})^2$, the space of eigenfunctions associated with λ is 1-dimensional, and is spanned by $\{\sin(\frac{n\pi x}{l})\}$.

Exercise 1.2.2. Prove that $\{(\frac{n\pi}{l})^2 : n = 0, 1, \cdots\}$ provides the set of all eigenvalues to the BVP

$$\begin{cases} X''(x) = -\lambda X(x), \\ X'(0) = X'(l) = 0. \end{cases} \tag{1.26}$$

and that for each $\lambda = (\frac{n\pi}{l})^2$, the space of eigenfunctions associated with λ is 1-dimensional, and is spanned by $\{\cos(\frac{n\pi x}{l})\}$.

Exercise 1.2.3. Construct separable solutions to the homogeneous heat equation $u_t(x, t) - u_{xx}(x, t) = 0$ over $(0, l) \times (0, \infty)$ with the Neumann boundary condition $u_x(0, t) = u_x(l, t) = 0$ for all $t > 0$. Then use superposition principle to construct a formal solution to the above problem with the initial data $u(x, 0) = g(x)$ for $0 < x < l$.

Exercise 1.2.4. When ξ is not real, separable solutions $u(x,t) = e^{i\xi x - \xi^2 t}$ to the homogeneous heat equation $u_t(x,t) - u_{xx}(x,t) = 0$ can be used to model the temperature variation underground the surface of the earth, as discussed in 7.1 of Partial Differential Equations by Fritz John. More specifically, the solution when $\xi = -a + ai$ for $a > 0$ has its real part as $u(x,t) = \cos(2a^2 t - ax)e^{-ax}$. If we model the shallow layer of the earth as a portion of the flat space corresponding to $x \geq 0$, and use $-u(x,t)$ to model the temperature variation underground the surface of the earth due to its seasonal variation at the $x = 0$ level: $-u(x,0) = -\cos(2\pi t)$ (here we choose $a = \sqrt{\pi}$ and use year as unit of t to model the annual seasonal variation), work out the range of temperature variations at depth $x > 0$ and time-delay of its highest and lowest temperature in relation to those at $x = 0$.

Exercise 1.2.5. Construct separable solutions to the Laplace equation $\Delta u(x,y) = 0$ over $\mathbb{R}^2$ of the form $e^{ix\xi}Y(y)$, where ξ is a real parameter. Does this method give a family of harmonic functions that are bounded over the entire $\mathbb{R}^2$? Does it give a family of harmonic functions that are bounded over the upper half plane $\mathbb{R}^2_+$ or the lower half plane $\mathbb{R}^2_-$?

Exercise 1.2.6. Prove that the BVP

$$\begin{cases} u''(x) + u(x) = 0 & \text{for } x \in (0,\pi), \\ u(0) = a, \; u(\pi) = b \end{cases}$$

has a solution iff $a + b = 0$; and that the BVP

$$\begin{cases} u''(x) + u(x) = f(x) & \text{for } x \in (0,\pi), \\ u(0) = 0, \; u(\pi) = 0 \end{cases}$$

has a solution iff $\int_0^\pi f(x) \sin x \, dx = 0$.

Exercise 1.2.7. Construct separable solutions to the homogeneous wave equation $u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0$ over $\mathbb{R}^2$ of the form $X(x)T(t)$. Work out solutions in both real or complex forms. Identify the solution as a standing wave or traveling wave solution, then apply the superposition principle to construct standing wave solutions if the initial construction only gives directly traveling wave solution, or vice versa.

Exercise 1.2.8. Apply separation of variables/Fourier series to construct solutions to the initial/boundary value problem to the one dimensional wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{on } (x,t) \in [0,l] \times \mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l]. \end{cases}$$

Exercise 1.2.9. Use superposition to construct some standing wave solutions to the Maxwell's equations. Are they in the same phase?

Exercise 1.2.10. Apply separation of variables/Fourier series to investigate the well-posedness of the initial/boundary value problem

$$\begin{cases} u_{tt} + u_{xx} = 0, & \text{on } (x,t) \in (0,\pi) \times \mathbb{R}^+, \\ u(0,t) = u(\pi,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in (0,\pi), \\ u_t(x,0) = h(x), & \text{for } x \in (0,\pi). \end{cases}$$

Exercise 1.2.11. Apply separation of variables/Fourier series to investigate the well-posedness of the boundary value problem

$$\begin{cases} u_{tt} + u_{xx} = 0, & \text{on } (x,t) \in (0,\pi) \times (0,H), \\ u(0,t) = u(\pi,t) = 0, & \text{for } H > t > 0, \\ u(x,0) = g(x), & \text{for } x \in (0,\pi), \\ u(x,H) = h(x), & \text{for } x \in (0,\pi). \end{cases}$$

Exercise 1.2.12. Apply separation of variables/Fourier series to investigate the well-posedness of the boundary value problem

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{on } (x,t) \in (0,\pi) \times (0,2\pi), \\ u(0,t) = u(\pi,t) = 0, & \text{for } t \in (0,\pi), \\ u(x,0) = g(x), & \text{for } x \in (0,\pi), \\ u(x,2\pi) = h(x), & \text{for } x \in (0,\pi). \end{cases}$$

In particular, investigate the set of solutions to the problem for $g \equiv h \equiv 0$, and conditions on g and h needed even to find formal solutions.

Exercise 1.2.13. Apply separation of variables, or its variant as described in Remark 1.7, to construct solutions to $u_t - u_{xxx} = 0$ on $(x,t) \in \mathbb{R} \times [0,\infty)$. Discuss the role of solutions to $X'''(x) + \lambda X(x) = 0$ in the construction of solutions to the PDE here.

Exercise 1.2.14. Construct steady state solutions of the incompressible Navier-Stokes equations in an infinite circular pipe of radius R, where one assumes that the velocity field takes the form of $\mathbf{u}(\mathbf{x},t) = (0,0,U(r))$ for some function $U(r)$ to be determined, and $r = \sqrt{x^2+y^2}$, and that the no-slip boundary condition holds.

Exercise 1.2.15. Construct solutions of minimal surface equation (1.15) of the form $u(x,y) = f(x) + g(y)$ when $\Omega = \mathbb{R}^2$ (there is no need to consider the boundary condition then). The resulting solutions give rise to Scherk's surfaces, and in fact become infinite along a discrete set of lines.

1.3 Additional Prototype PDE Problems, Linearization, and Dimensional Analysis

The previous sections provide a few examples of prototype IVPs, or BVPs, or IBVPs, which occur often in applications. Here we mention a few other kinds of PDE problems that also arise often from physical or geometrical applications.

1.3.1 Eigenvalue and Eigenfunction Problems

These problems already arise in applying the method of separation of variables. For example, we were led to study (1.23) when attempting to construct solutions to on a finite interval for the homogeneous case of (1.9). The key question we need answer for is whether the eigenvalue problem has sufficiently many linearly independent solutions such that any (reasonable) arbitrary function can be "spanned" in terms of these eigenfunctions.

If we make an analogy with linear algebra, thinking of $-\frac{d^2}{dx^2}$ as a linear operator acting on a class of functions with appropriate boundary conditions such as in (1.23), the question becomes whether such an operator can be completely diagonalized. Even in the context of linear algebra, not all linear operators defined in terms of matrix multiplication can be diagonalized. However, some important classes of linear operators, including those defined in terms of matrix multiplication by real symmetric matrices, Hermitian matrices, and unitary matrices, can be diagonalized; furthermore, one can diaganolize such operators via a set of eigenvectors which are orthonormal. It turns out that many eigenvalue problems arising from PDE contexts have structures similar to those for eigenvalue problems for the above mentioned linear operators in finite dimensions. The key is to find a mechanism to extend the arguments to the infinite dimensional settings. These are studied in the abstract under the umbrella of self-adjoint operators, the simplest subclass of which is the compact self-adjoint operators. These problems in the PDE contexts are called Sturm-Liouville problems, and were initially studied by more classical DE methods. We will study the Sturm-Liouville problems later on.

These eigenvalue problems also arise in the study of Schrödinger equations, which are used to describe a physical system in which quantum effects are significant. One simple form of the Schrödinger equation takes the following form

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2\mu}\Delta + V(x,t)\right]\Psi(x,t), \tag{1.27}$$

where $\hbar$ is the reduced Planck constant $h/(2\pi)$, μ the particle's reduced mass, and $V(x,t)$ the potential energy of the field in which the particle moves.

When $V(x,t) = V(x)$ is independent of t, one is often interested in standing wave solutions of the form $\Psi(x,t) = e^{-iEt/\hbar}W(x)$. Then $W(x)$ satisfies

$$\left[-\frac{\hbar^2}{2\mu}\Delta + V(x)\right] W(x) = EW(x). \tag{1.28}$$

Here one question of concern is the existence and distribution of E's for which (1.28) has solutions that are square integrable in x over the entire space and not identically 0. The difference between this eigenvalue boundary value problem and those of the previous section is that the boundary conditions here at spatial infinity are implicit: we require the eigenfunctions to be square integrable in x over the entire space, which says that, in some sense, the eigenfunctions tend to 0 as $x \to \infty$; but these two conditions are not equivalent.

If $V(x) \to \infty$ as $x \to \infty$, $V(x)$ is said to have a confinement effect, as it tends to "favor" solutions which $\to 0$ as $x \to \infty$—technically, one asks for a solution which is square integrable in x over the entire space. In such cases (1.28) typically has only discrete number of eigenvalues with square integrable eigenfunctions.

If $V(x) \to 0$ as $x \to \infty$, then the field is considered to have weak effect at spatial infinity, and it's possible to have solutions which do not decay fast enough to make them square integrable in x over the entire space. Such situations arise when describing **scattering** of particles.

In dimension 1, (1.28) becomes an ODE. If, further, $V(x) \equiv 0$, then the equation describes a free particle. The solutions to (1.28) in this case with $E > 0$ are

$$W(x) = c_1 e^{i\sqrt{2\mu E}x/\hbar} + c_2 e^{-i\sqrt{2\mu E}x/\hbar},$$

so

$$\Psi(x,t) = c_1 e^{i(\sqrt{2\mu E}x - Et)/\hbar} + c_2 e^{-i(\sqrt{2\mu E}x + Et)/\hbar},$$

represents the superposition of two free traveling waves. These $W(x)$ are not square integrable in x over the entire space. But there is still mathematical reason to define $E > 0$ to be in the (continuous) spectrum of the operator $-\frac{\hbar^2}{2\mu}\Delta$ on $L^2(\mathbb{R})$. We hope to say more about this topic later on.

One often used simple model of $V(x)$ in describing scattering is a function with compact support. Outside the support of V, the wave function $W(x)$ satisfies the same equation as the one for a free particle. One problem describing the scattering of a particle in such a situation needs to find a wave function $W(x)$ to (1.28) such that $W(x) = c_1' e^{i\sqrt{2\mu E}x/\hbar}$ for x large; one hopes to determine the solution elsewhere, especially to the left of the support of V. Such a wave function would describe a wave $e^{i(\sqrt{2\mu E}x - Et)/\hbar}$ to the right of the support of V, which is traveling to the right, and can be interpreted as representing the transmitted wave after the scattering by the potential V. Note that such a wave function has the form $c_1 e^{i\sqrt{2\mu E}x/\hbar} + c_2 e^{-i\sqrt{2\mu E}x/\hbar}$ to the left of the support of V. $e^{i\sqrt{2\mu E}x/\hbar}$ will represent an incidence wave $e^{i(\sqrt{2\mu E}x-Et)/\hbar}$, while $e^{-i\sqrt{2\mu E}x/\hbar}$ will represent the reflected wave $e^{-i(\sqrt{2\mu E}x+Et)/\hbar}$. Mathematically one needs to find the relation between c_1, c_2, and c_1'. The exercises below aim to build some hands-on experience in constructing eigenfunctions of these problems.

Exercises

Exercise 1.3.1. This exercise shows how an eigenvalue problem arises in constructing a solution of (1.9) on a two-dimensional (unit) disk D.

(i) Look for a separable solution of the first equation in (1.9) (assuming $f \equiv 0$) in the form of $u(x,y,t) = T(t)X(x,y)$, where $(x,y) \in D$. Deduce that $X(x,y)$ must satisfy

$$\begin{cases} \Delta X(x,y) = -\lambda X(x,y) \quad (x,y) \in D, \\ X(x,y) = 0 \quad (x,y) \in \partial D \end{cases}$$

for some constant λ.

(ii) Look for solutions to the eigenvalue problem above in the form of $X = R(r)\Theta(\theta)$. Deduce that $R(r)$ and $\Theta(\theta)$ must satisfy

$$\left[R''(r) + \frac{R'(r)}{r}\right]\Theta(\theta) + \frac{R(r)}{r^2}\Theta''(\theta) = -\lambda R(r)\Theta(\theta),$$

which can be rewritten as

$$r^2 R^{-1}(r)\left[R''(r) + \frac{R'(r)}{r}\right] + \lambda r^2 + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0,$$

whenever $R(r) \neq 0$ and $\Theta(\theta) \neq 0$. It now follows that $\Theta''(\theta) + \mu\Theta(\theta) = 0$ for some constant μ and all $\theta \in [0, 2\pi]$, and that

$$\begin{cases} R''(r) + \frac{R'(r)}{r} + \left[\lambda - \frac{\mu}{r^2}\right] R(r) = 0 \quad \text{for } 0 < r < 1, \\ R(r) \in C^1[0,1] \text{ with } R'(0) = 0 \text{ and } R(1) = 0 \quad \text{at } r = 1. \end{cases} \tag{REV}$$

Figure 1.3: Insert a figure illustrating a potential well and a possible eigenfunction

It's natural to impose the condition that $u(r, 0) = u(r, 2\pi)$ and $u_\theta(r, 0) = u_\theta(r, 2\pi)$ for all $0 < r < 1$, which leads to the condition that $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. Thus we must look for those constants μ for which the problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad \text{for all } \theta \in [0, 2\pi], \\ \Theta(0) = \Theta(2\pi) \text{ and } \Theta'(0) = \Theta'(2\pi) \end{cases} \tag{ΘEV}$$

has non-trivial solutions.

It remains to understand whether (ΘEV) has a set of eigenfunctions which can span the set of all continuous functions, and answer a similar problem on D incorporating the solutions to (REV).

Exercise 1.3.2. This problem considers the spectrum of the operator $L = -\frac{d^2}{dx^2} + V(x)$ on $L^2(\mathbb{R})$, where $V(x)$ is a piecewise constant function defined as

$$V(x) = \begin{cases} V_0 & \text{for } |x| < l, \\ 0 & \text{for } |x| \geq l. \end{cases}$$

A solution here is understood to be a function $\psi(x)$ absolutely continuous over $\mathbb{R}$ with absolutely continuous derivative $\psi'(x)$ over $\mathbb{R}$ such that $-\psi''(x) + V(x)\psi(x) = E\psi(x)$ on $\mathbb{R}$ except at $x = \pm l$. The problem is long, and may appear to be tedious. But it can be worked out using elementary ODE techniques, and the solutions can provide guidance for more complicated problems.

(a). Prove that for any $E > 0$ the problem has no non-trivial eigenfunctions in $L^2(\mathbb{R})$; however the problem has non-trivial bounded solutions on $\mathbb{R}$. Such $E > 0$ are said to be in the **continuous spectrum** of the operator (partly because the set of such E's forms a continuum, in contrast to situations such as (1.23), where the eigenvalues are isolated); $L - E$ does not have a bounded inverse on $L^2(\mathbb{R})$, namely, there exists no constant $C > 0$ such that for any $f \in L^2(\mathbb{R})$ one can find a (unique) solution u to $(L - E)u = f$ such that $u \in L^2(\mathbb{R})$, and $||u||_{L^2(\mathbb{R})} \leq C||f||_{L^2(\mathbb{R})}$. (In this context, for any $f \in L^2(\mathbb{R})$, one can construct a solution u to $(L - E)u = f$ by solving an IVP for the ODE, but u may not lie in $L^2(\mathbb{R})$.)

(b). The case $V_0 < 0$ is considered to be a potential well. This part will show that there are finite number of eigenvalues E in the range $V_0 < E < 0$. Set $\mu = \sqrt{E - V_0}$.

(i). Prove that if $E \leq V_0$, then the only bounded solutions on $\mathbb{R}$ are identically 0.

(ii). Prove that any $L^2(\mathbb{R})$ eigenfunction ψ with $V_0 < E < 0$ must have the form

$$\psi(x) = \begin{cases} Ae^{\lambda x} & \text{for } x \leq -l, \\ B\cos(\mu x) + C\sin(\mu x) & \text{for } |x| \leq l \\ De^{-\lambda x} & \text{for } x > l, \end{cases}$$

where $\lambda = \sqrt{|E|}$.

(iii). $\psi(x)$ must satisfy the conditions that $\psi(x)$ and $\psi'(x)$ be continuous at $x = -l$ and at $x = l$. Set $A = 1$. Verify that these conditions lead to

$$\begin{aligned} B &= e^{-\lambda l}\left[\frac{\lambda}{\mu}\sin(\mu l) + \cos(\mu l)\right], \\ C &= e^{-\lambda l}\left[\frac{\lambda}{\mu}\cos(\mu l) - \sin(\mu l)\right], \\ e^{-\lambda l}D &= B\cos(\mu l) + C\sin(\mu l), \\ e^{-\lambda l}D &= \frac{\mu}{\lambda}\left[B\sin(\mu l) - C\cos(\mu l)\right]. \end{aligned}$$

(iv). Prove that the above system has a solution iff

$$2\cot(2\mu l) = \frac{2E - V_0}{\mu\sqrt{|E|}}.$$

(v). Note that the right hand side above, considered as a function of μ (noting $|E| = |V_0| - \mu^2$), is defined for $0 < \mu < \sqrt{|V_0|}$, is monotone increasing in μ for $0 < \mu < \sqrt{|V_0|}$, and $\to -\infty$ as $\mu \to 0+$, $\to \infty$ as $\mu \to \sqrt{|V_0|}$. Using this to prove that the above equation has a finite number of solutions in $0 < \mu < \sqrt{|V_0|}$.

(vi). Setting $\frac{\mu}{\sqrt{|V_0|}} = \sin\theta$ for some $0 < \theta < \frac{\pi}{2}$, verify that the above equation is equivalent to $\cot(2\mu l) = -\cot(2\theta)$, from which it follows that $\mu = -\frac{\theta}{l} + \frac{k\pi}{2l}$ for some $k \in \mathbb{Z}$, which can be written as $\sqrt{|V_0|}\sin\theta = -\frac{\theta}{l} + \frac{k\pi}{2l}$. Prove that this equation has m solutions in the range $0 < \theta < \frac{\pi}{2}$, where $m = [\frac{2l\sqrt{|V_0|}}{\pi}] + 1$, with the first $\mu_1 < \min\{\frac{\pi}{2l}, \sqrt{|V_0|}\}$.

(vii). Verify that $\sqrt{|V_0|}\mu_1^{-1}e^{-l\sqrt{|V_0|-\mu_1^2}}\cos(\mu_1 x)$ provides an eigenfunction over $|x| < l$ for μ_1.

(c). The case $V_0 > 0$ is called a potential barrier. Part (a) already establishes that no $E > 0$ can have an $L^2(\mathbb{R})$ eigenfunction. The interest here is to understand the behavior of solutions that describe scattering by the potential, in particular, when $0 < E < V_0$. Namely, we examine solutions $\psi(x)$ such that $\psi(x) = c_1' e^{i\sqrt{E}x}$ for $x > l$. Note that such a solution has the form $c_1 e^{i\sqrt{E}x} + c_2 e^{-i\sqrt{E}x}$ for $x < -l$ for some constants c_1 and c_2. Our aim is to find the relation between c_1, c_2, and c_1'. Due to linearity, we may normalize $c_1' = 1$.

(i). Let $\psi(x)$ be any solution of $(L - E)\psi(x) = 0$ on $\mathbb{R}$. Prove that the Wronskian $W(\psi, \overline{\psi}) = \psi(x)\overline{\psi}'(x) - \overline{\psi}(x)\psi'(x)$ is a constant independent of $x \in \mathbb{R}$.

(ii). Let $\psi(x)$ be the solution of $(L - E)\psi(x) = 0$ on $\mathbb{R}$ such that $\psi(x) = e^{i\sqrt{E}x}$ for $x > l$. Verify that $W(\psi, \overline{\psi}) = -2\sqrt{E}i$.

(iii). In the setting of (ii), let $\psi(x) = c_1 e^{i\sqrt{E}x} + c_2 e^{-i\sqrt{E}x}$ for $x < -l$ for some constants c_1 and c_2. Prove that $W(\psi, \overline{\psi}) = -2\sqrt{E}i\,[|c_1|^2 - |c_2|^2]$, and conclude that $|c_1|^2 - |c_2|^2 = 1$. In the general case, we will have $|c_1|^2 - |c_2|^2 = |c_1'|^2$. It is customary to normalize $c_1 = 1$, then this relation turns into $1 = |c_2|^2 + |c_1'|^2$. c_2 is called the coefficient of reflection, and c_1' is called the coefficient of transmission.

(iv). The argument in (i)–(iii) works for any potential with compact support. Try to determine c_2 and c_1' for the specific potential barrier here.

1.3.2 Linearization

Nonlinear differential equations are in general much more difficult to study. One often used tool is to try to construct a solution near a known solution, through the use of Inverse or Implicit Function Theorem in appropriate function spaces. The first step is to *linearize* a nonlinear DE at a given function. We will see that the linearization leads to a linear PDE (often with variable coefficients which arise from the given function).

Here we introduce the concept of linearization through an example, and then provide guidance on working out the linearization of the equations of motion of ideal isentropic gas in one dimension.

We define the operator $\mathcal{M}(u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$, which arises in the PDE for minimal surface equations. Then the linearization of $\mathcal{M}$ at u is defined to be

$$\mathcal{M}'_u[v] := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \mathcal{M}(u + \epsilon v).$$

In our case, it turns out

$$\mathcal{M}'_u[v] = \sum_{i,j=1}^{n} \nabla_i \left[\frac{(1+|\nabla u|^2)\delta_{ij} - \nabla u_i \nabla u_j}{(1+|\nabla u|^2)^{3/2}} \nabla v_j\right],$$

and thus

$$\mathcal{M}'_0[v] = \Delta v,$$

and in general

$$\mathcal{M}'_u[v] = \sum_{i,j=1}^{n} \nabla_i \left[a_{ij}(x)\nabla v_j\right], \quad \text{with } a_{ij}(x) = \frac{(1+|\nabla u|^2)\delta_{ij} - \nabla u_i \nabla u_j}{(1+|\nabla u|^2)^{3/2}}.$$

Note that for any vector $\xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j = \frac{(1+|\nabla u|^2)|\xi|^2 - |\nabla u \cdot \xi|^2}{(1+|\nabla u|^2)^{3/2}},$$

so

$$\frac{|\xi|^2}{(1+|\nabla u|^2)^{3/2}} \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \frac{|\xi|^2}{(1+|\nabla u|^2)^{1/2}}.$$

Suppose that u is a solution of (1.15) for a certain g, and one is interested in whether (1.15) has a solution for some $g + \epsilon h$ near g, then, as a first approximation, one needs to construct a v such that $M'_u[v] = 0$ in Ω, and $v = h$ on $\partial\Omega$. The PDE $M'_u[v] = 0$ is a linear second order variable coefficient PDE, and due to the positive definiteness of $(a_{ij}(x))$, it is called an elliptic PDE.

For the equations of motion for ideal isentropic gas in one dimension, as described in **Example 1.1**, the left hand sides of the equations can be incorporated into the nonlinear operator

$$N(\rho, v) = \begin{pmatrix} \rho_t + (\rho v)_x \\ (\rho v)_t + (\rho v^2)_x + p'(\rho)\rho_x \end{pmatrix}.$$

Notice that for $\rho = \rho_0$, a constant, we have $N(\rho_0, 0) = \vec{0}$. The linearized operator of N at $(\rho_0, 0)$ is

$$N'_{(\rho_0,0)}\begin{pmatrix} \rho^* \\ v^* \end{pmatrix} = \begin{pmatrix} \rho^*_t + \rho_0 v^*_x \\ \rho_0 v^*_t + p'(\rho_0)\rho^*_x \end{pmatrix}.$$

So as a first approximation, if we want to solve $N'_{(\rho_0,0)}\begin{pmatrix}\rho^* \\ v^*\end{pmatrix} = \vec{0}$, we have

$$\begin{cases} \rho_t^* + \rho_0 v_x^* = 0, \\ \rho_0 v_t^* + p'(\rho_0)\rho_x^* = 0. \end{cases}$$

This is a system of linear equations for ρ^* and v^*, and one can easily eliminate one variable to arrive at

$$\rho_{tt}^* - c^2 \rho_{xx}^* = 0, \qquad \text{with } c^2 = p'(\rho_0).$$

This confirms that small disturbances in a one dimensional ideal isentropic gas obeys the wave equation in **Example 1.3**.

Exercises

Exercise 1.3.3. Verify that the linearization of the nonlinear operator

$$N(\rho, v) = \begin{pmatrix} \rho_t + (\rho v)_x \\ (\rho v)_t + (\rho v^2)_x + p'(\rho)\rho_x \end{pmatrix}$$

at $(\rho_0, 0)$, with $\rho_0 > 0$ being a constant, is

$$N'_{(\rho_0,0)}\begin{pmatrix}\rho^* \\ v^*\end{pmatrix} = \begin{pmatrix} \rho_t^* + \rho_0 v_x^* \\ \rho_0 v_t^* + p'(\rho_0)\rho_x^* \end{pmatrix}.$$

Exercise 1.3.4. Suppose that $\rho^*(x,t)$ and $v^*(x,t)$ are C^2 solutions to

$$\begin{cases} \rho_t^* + \rho_0 v_x^* = 0, \\ \rho_0 v_t^* + p'(\rho_0)\rho_x^* = 0. \end{cases}$$

Prove that

$$\rho_{tt}^* - c^2 \rho_{xx}^* = 0, \qquad \text{with } c^2 = p'(\rho_0).$$

Exercise 1.3.5. Verify that the linearization of the incompressible Navier-Stokes equations at a solution $(\mathbf{u}_0(\mathbf{x},t), p_0(\mathbf{x},t))$ is

$$\begin{cases} \operatorname{div} \mathbf{v}(\mathbf{x},t) = 0, \\ \dfrac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{u}_0(\mathbf{x},t)\cdot\nabla\mathbf{v}(\mathbf{x},t) + \mathbf{v}(\mathbf{x},t)\cdot\nabla\mathbf{u}_0(\mathbf{x},t) = -\rho\nabla p(\mathbf{x},t) + \nu\Delta\mathbf{v}(\mathbf{x},t). \end{cases}$$

In particular, when $\mathbf{u}_0(\mathbf{x},t) = (U(y),0,0)$ is the steady state solution in the infinite plate $-a \le y \le a$, the linearized system takes on the more specific form

$$\begin{cases} \operatorname{div} \mathbf{v}(\mathbf{x}, t) = 0, \\ \dfrac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} + U(y)\dfrac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial x} + v_2(\mathbf{x}, t)\begin{bmatrix} U'(y) \\ 0 \\ 0 \end{bmatrix} = -\rho \nabla p(\mathbf{x}, t) + \nu \Delta \mathbf{v}(\mathbf{x}, t). \end{cases}$$

1.3.3 Dimensional Analysis and Scaling*

Dimensional analysis and scaling are often used in dealing with the analysis of PDEs. Here are a few brief comments on these two topics.

Dimensional analysis refers to the requirement that all summands in a physical law must have the same dimensions, therefore its validity is unaffected by a change of unit in one or more of the physical quantities. Dimensional analysis is used extensively by physicists, chemists, and engineers in the analysis of their problems, and in checking the validity of the formula that they face.

An underlying principle is that every physical quantity is assigned a dimension; and once a unit for this dimension is chosen, the physical quantity can now be assigned a numerical value. For instance, the length of a one-dimensional object is assigned the dimension of *length*, the area of a two-dimensional region is assigned the dimension of $(length)^2$, and the dimension of speed is *length/time*. We have a sense for the length of the edge of a page (in relation to a familiar reference length), but it does not have an intrinsic numerical value; it is given a numerical value only after a unit for measuring the length has been chosen. Two different units for the same dimension are related by a scaler; the ratio of the numerical values of two quantities of the same dimension (with respect to a common unit) is invariant when the unit for the dimension is changed, and is an example of a *dimensionless* quantity.

The choice of a unit for physical quantities often depends on the context: there is often a *characteristic unit* for quantities of the same dimension in a given context (one often used guiding principle is that the numerical values measured in this characteristic unit would not be too huge or too tiny). For instance, in describing the ripple of water waves, *meter* seems a reasonable choice as characteristic unit for length, and *second* a reasonable choice as characteristic unit for time. If we choose to use *hour* as characteristic unit for time, then the numerical value of the wave speed is amplified by a factor of 3600, but the typical time related to the evaluation of the wave motion is measured in the thousandth of an hour; this need to deal with both

*Skipping dimensional analysis will not disrupt the reading of the rest of the material.

a huge numerical value of wave speed and a tiny numerical value of time is not as convenient as using meter and second in this context.

Scientists often carry out a *non-dimensionalization* of the equation to facilitate its analysis. One chooses a characteristic unit for each dimension, and expresses all involved quantities as numerical values measured against the chosen units, and works with the resulting reduced equation. For instance, in using the heat equation in a given setting, one first chooses a characteristic unit T for time variable t, and a characteristic unit L for length variable x. Note that ν, the thermal diffusivity, has the dimension of *(length)*^{2}*/time*. If one rescales the variables $x = \xi L$, $t = \tau T$, where ξ is the measurement of length against L, and τ is the measurement of time against T, and choose as a characteristic scale U for u, and set $u = \Upsilon U$. The substitution $u(\xi L, \tau T) = \Upsilon(\xi, \tau)U$ turns the equation $u_t - \nu u_{xx} = 0$ into $\Upsilon_\tau - \frac{\nu T}{L^2}\Upsilon_{\xi\xi} = 0$. This is the non-dimensionalized form of the equation, where ξ, τ, Υ are all dimensionless. The dimensionless quantity $\kappa = \frac{\nu T}{L^2}$ is now the only parameter describing the heat diffusion in this context.

One can look up that the thermal diffusivity of Silicone Dioxide is $0.83mm^2/s$. So if one looks at a problem with $T = 10s$, $L = 10mm$, it would lead to $\kappa = 0.083$. But if one is interested in studying heat diffusion in the setting of microchips, it would be reasonable to choose $T = 1s$, $L = 10^{-2}mm$, then $\kappa = 8,300$ in the dimensionless form.

In principle, the arguments for the transcendental functions are dimensionless, as transcendental functions are defined by power series, and it does not make sense to sum terms of different dimensions, if the argument carries a physical dimension. But we have encountered many solutions formulae involving transcendental functions with arguments which seem to carry dimension, or even different dimensions, for instance, in (1.21). This is due either to our having assigned a numerical value to a dimensional constant so its dimension is not explicitly visible, or to our working with non-dimensionalized form of the equation. In the case of (1.21), we have normalized the coefficient of thermal diffusivity γ to be 1, which carries the dimension of (length)^2/(time); it we put γ back and trace its role, we will find it appears in the form γt in place of t. If x takes the dimension of length, then ξ would need to take the dimension of (length)$^{-1}$, and $x\xi$ and $\xi^2\gamma t$ would both be dimensionless. The same comment applies to other representations of solutions to (1.8): if we need a solution formula for a general γ, we should use the *dimensionless* quantities, such as $(\frac{n\pi}{l})^2\gamma t$ or $|x|^2/(\gamma t)$, to replace $(\frac{n\pi}{l})^2 t$, or $|x|^2/t$ (in (2.18)), which are derived assuming $\gamma = 1$. To avoid having to keep track of the dimensions, the easiest approach is to work with non-dimensionalized form of the equation. We will see more examples later on where such dimensional analysis would help to make sure that the parameters are placed in the correct place in the solution representation.

Two different physically scaled problems may reduce to the same non-dimensionalized equation. Scientists have often used such relations to reduce the study of a physical problem that is large and expensive to do experiment with to a smaller scale problem that shares identical non-dimensionalized equation. For instance, suppose a different context has a different choice of a characteristic unit $\hat{L}$ for length, a characteristic unit $\hat{T}$ for time, and a different $\hat{\nu}$, but it turns out that $\kappa = \frac{\nu T}{L^2} = \frac{\hat{\nu}\hat{T}}{\hat{L}^2}$. Then both $\Upsilon_1(\xi,\tau) := u(\xi L, \tau T)/U$ and $\Upsilon_2(\xi,\tau) := \hat{u}(\xi\hat{L}, \tau\hat{T})/\hat{U}$ satisfy the same diffusion equation. If they also satisfy the same initial and boundary conditions in (ξ,τ) coordinates, then we expect $\Upsilon_1(\xi,\tau) = \Upsilon_2(\xi,\tau)$ and can use the relation

$$u(\xi L, \tau T)/U = \hat{u}(\xi\hat{L}, \tau\hat{T})/\hat{U} = \Upsilon_1(\xi,\tau) = \Upsilon_2(\xi,\tau)$$

to relate $u(x,t)$ to $\Upsilon_1(\xi,\tau)$, or to $\hat{u}(x,t)$. In such a situation, $\hat{u}(x,t) = u(\frac{Lx}{\hat{L}}, \frac{Tt}{\hat{T}})\hat{U}/U$ can be used to draw conclusions on $\hat{u}(x,t)$ through information on u.

One can use such analysis to get information on a solution of (1.20) when the initial data g is a *point source*, concentrated at 0, say. A point source function is best defined as a limit, in appropriate sense, of a sequence of non-negative functions, having unit total integral, and concentrating its mass (total integral) at 0 in the limit. More specifically, if $h(x) \geq 0$ on $\mathbb{R}$, with its support in $\{x : |x| \leq 1\}$, and $\int_{\mathbb{R}^n} h(x)dx = 1$, then, for $\lambda_j \to 0$, as $j \to \infty$, and $h_j(x) := \lambda_j^{-n} h(x/\lambda_j)$, we see that $\int_{\mathbb{R}^n} h_j(x)dx = 1$, and for any $\eta \in C(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h_j(x)\eta(x)dx = \eta(0), \quad \text{as } j \to \infty.$$

It is in this sense that $\{h_j(x)\}$ converges to a limit function $\delta(x)$, which is called the unit point source function at 0, or Dirac's function.

One can see that, for any $\lambda > 0$, $\{\lambda^{-n} h_j(x/\lambda)\}$ would have the same limit, so one expects $\lambda^{-n}\delta(x/\lambda) = \delta(x)$. Now, if $u(x,t)$ is a solution of (1.20) with $u(x,0) = \delta(x)$, then, using the same computation as in the dimension analysis $u_\lambda(x,t) := \lambda^{-n} u(x/\lambda, t/\lambda^2)$ satisfies the same diffusion equation, with the same initial data. It is reasonable to expect $u_\lambda(x,t) = u(x,t)$ for any (x,t) and $\lambda > 0$. The relation $\lambda^{-n} u(x/\lambda, t/\lambda^2) = u(x,t)$ for any (x,t) and $\lambda > 0$ would give us lead to find an explicit form for $u(x,t)$. We will do that in a later section.

In addition to the relatively routine applications mentioned above, dimensional analysis has been used by insightful scientists to derive interesting solution formulas in some situations, avoiding lengthy computations which a routine derivation would need. See some easily accessible notes on this by John Hunter at https://www.math.ucdavis.edu/~hunter/m280_09/applied_math.html.

In the analysis of PDEs, we often need to use certain inequalities that are applicable to all functions in a linear function space and involve integrable norms of the functions and their derivatives. Mathematicians often use dimensional analysis, or in a form called **scaling**, to find the correct dependence of the constants in the inequalities.

Here are two examples to illustrate how scaling is done. We are interested in the validity of the inequality

$$\exists C > 0 \text{ depending on } p, q \geq 1, \text{ and } l \text{ such that for all } u \in C^1[0,l] \text{ with } u(0)=0,$$
$$\left(\int_0^l |u(x)|^q dx\right)^{1/q} \leq C \left(\int_0^l |u'(x)|^p dx\right)^{1/p}. \tag{1.29}$$

In applications, one needs to know, as much as possible, how C depends on p, q, and l, especially when one or more of them varies.

The following elementary inequalities for $u \in C^1[0,l]$ with $u(0)=0$:

$$\max_{0\leq x\leq l} |u(x)|^2 \leq l \int_0^l |u'(x)|^2\,dx, \quad \int_0^l |u(x)|^2\,dx \leq l^2 \int_0^l |u'(x)|^2\,dx, \tag{1.30}$$

are special cases of (1.29), and can be proved easily by using $u(x) = \int_0^x u'(y)\,dy$; but the coefficients in (1.30) depend on l in a different scale and can be found by rescaling the spatial variable $0 < x < l$ to $x = lz$, so $0 < z < 1$, and consider $v(z) = u(lz)$. After we establish $\max_{0\leq z\leq 1}|v(z)| \leq \int_0^1 |v'(z)|^2\,dz$, and $\int_0^1 |v(z)|^2\,dz \leq \int_0^1 |v'(z)|^2\,dz$, a change of variables $x = lz$ would give back (1.30).

Dimensional analysis would predict the same scaling: if one treats u as dimensionless and x as having the dimension of length, then $|u'(x)|^2$ has the dimension of $(\text{length})^{-2}$, and $\int_0^l |u'(x)|^2\,dx$ has the dimension of $(\text{length})^{-1}$; so to balance with the dimension of the left hand side, one would need the factor l in front of $\int_0^l |u'(x)|^2\,dx$ for the first inequality, and the factor l^2 for the second inequality in (1.30). In particular, this analysis shows that it is impossible to get a constant $C > 0$ independent of l, such that $\int_0^l |u(x)|^2\,dx \leq C \int_0^l |u'(x)|^2\,dx$ holds for all $u \in C^1[0,l]$ with $u(0)=0$, and for *all* $l > 0$.

Similar analysis can be done for (1.29): If (1.29) has been established for $l = 1$, then for a general $l > 0$, and $u \in C^1[0,l]$ with $u(0) = 0$, then we have $v(z) = u(lz) \in C^1[0,1]$ with $v(0) = 0$, and expect (1.29) to hold for v with a constant

$C_1 > 0$ depending only on p and q. But $||v||_{L^q[0,1]} = l^{-1/q}||u||_{L^q[0,l]}$, and $||v'||_{L^p[0,1]} = l^{1-1/p}||u'||_{L^p[0,l]}$. Thus we have

$$||u||_{L^q[0,l]} \leq C_1 l^{1-1/p+1/q}||u'||_{L^p[0,l]},$$

conditioned on having established (1.29) for $l = 1$. Thus $C = C_1 l^{1-1/p+1/q}$ would be how the constant in (1.29) depends on l.

Here is a more complicated (interpolation) inequality which can be analyzed in a similar fashion.

For appropriate choices of $1 \leq p, q, r < \infty$, and $0 \leq \alpha, \beta \leq 1$,
$\exists C > 0$ depending on p, q, r, the dimension n and the radius R of the ball
$B_R(0)$ in $\mathbb{R}^n$, and α, β such that for all $u \in C^2(B_R(0))$ with $u(x) = 0$ on $\partial B_R(0)$,

$$\left(\int_{B_R(0)} |\nabla u(x)|^q dx\right)^{1/q} \leq C \left(\int_{B_R(0)} |u(x)|^p dx\right)^{\alpha/p} \left(\int_{B_R(0)} |\nabla^2 u(x)|^r dx\right)^{\beta/r} \tag{1.31}$$

We use this example to illustrate scaling on both the range and the domain. If (1.31) is to hold for all admissible u, then it should hold with $u(x)$ replaced by $\lambda u(x)$ for any λ. This would create a factor of $|\lambda|$ on the left hand side, and a factor of $|\lambda|^{\alpha+\beta}$ on the right hand side. In order for the resulting inequality to hold for all λ, it's necessary that $\alpha + \beta = 1$. This corresponds to scaling on the range.

Next, we consider scaling on the domain: if (1.31) is to hold for $R = 1$, we can scale the general $R > 0$ case to the $R = 1$ case by considering $v(z) = u(Rz)$. Noting

$$||\nabla v||_{L^q(B_1(0))} = R^{1-n/q}||\nabla u||_{L^q(B_R(0))},$$
$$||v||_{L^p(B_1(0))} = R^{-n/p}||u||_{L^p(B_R(0))},$$
$$||\nabla^2 v||_{L^r(B_1(0))} = R^{2-n/r}||\nabla^2 u||_{L^r(B_R(0))},$$

the inequality (1.31) for v on $B_1(0)$ gives

$$||\nabla u||_{L^q(B_R(0))} \leq C_1 R^{n/q-1+\alpha(-n/p)+\beta(2-n/r)}||u||^{\alpha}_{L^p(B_R(0))}||\nabla^2 u||^{\beta}_{L^r(B_R(0))},$$

implying that the dependence of the constant C on R in (1.31) is

$$C = C_1 R^{n/q-1+\alpha(-n/p)+\beta(2-n/r)}.$$

Another way to account for the scaling is to make each term scaling invariant, namely, $R^{1-n/q}||\nabla u||_{L^q(B_R(0))}$, $R^{-n/p}||u||_{L^p(B_R(0))}$, and $R^{2-n/r}||\nabla^2 u||_{L^r(B_R(0))}$ are each scaling invariant (with respect to scaling of the independent variable x): the change of variables $x = \lambda y$, $u(x) = u(\lambda y) \stackrel{\text{def}}{=} w(y)$, $\widehat{R} = \lambda^{-1}R$, would produce $R^{1-n/q}||\nabla_x u||_{L^q(B_R(0))} = \widehat{R}^{1-n/q}||\nabla_y w||_{L^q(B_{\widehat{R}}(0))}$, etc. We would then write (1.31) in the form of

$$R^{1-n/q}||\nabla u||_{L^q(B_R(0))} \leq C_1 \left\{R^{-n/p}||u||_{L^p(B_R(0))}\right\}^{\alpha} \left\{R^{2-n/r}||\nabla^2 u||_{L^r(B_R(0))}\right\}^{\beta}.$$

Note that

$$R^{1-n/q}||\nabla u||_{L^q(B_R(0))} = \left(R^{-n}\int_{B_R(0)} |R\nabla_x u(x)|^q dx\right)^{1/q},$$

$$R^{-n/p}||u||_{L^p(B_R(0))} = \left(R^{-n}\int_{B_R(0)} |u(x)|^p dx\right)^{1/p},$$

$$R^{2-n/r}||\nabla^2 u||_{L^r(B_R(0))} = \left(R^{-n}\int_{B_R(0)} |R^2\nabla_x^2 u(x)|^r dx\right)^{1/r},$$

so each is the scaled (averaged) integral norm of the corresponding scaled integrand.

Again, this discussion is conditioned on establishing (1.31) on $B_1(0)$. Scaling analysis also provides some condition on the relation between p, q, r, α, β. Let u have compact support in $B_1(0)$, then for any $r > 1$, $v(z) = u(rz)$ also has compact support in $B_1(0)$. Thus (1.31) would hold for v on $B_1(0)$. But the same scaling computation above shows that $||\nabla v||_{L^q(B_1(0))} = r^{1-n/q}||\nabla u||_{L^q(B_1(0))}$ and respective scaling laws for the other two norms. This leads to

$$r^{1-n/q}||\nabla u||_{L^q(B_1(0))} \leq C_1 r^{\alpha(-n/p)+\beta(2-n/r)}||u||^{\alpha}_{L^p(B_1(0))}||\nabla^2 u||^{\beta}_{L^r(B_1(0))}$$

for all $r > 1$. This implies that a necessary condition for (1.31) to hold on $B_1(0)$ is $1 - n/q \leq \alpha(-n/p) + \beta(2 - n/r)$.

For those encountering such kind of analysis for the first time, the case $\alpha = 0$ and $\beta = 1$ would present a relatively simple case of (1.31): a necessary condition is that $1 + n/q - n/r \geq 0$, which is often written in the form of $1/q \geq 1/r - 1/n$, or $q \leq \frac{rn}{n-r}$ when $1 \leq r < n$. Note that such scaling analysis only produces some necessary conditions, and additional analysis is needed to determine whether the inequality (or equality) actually holds. For example, $\alpha = 1$, $\beta = 0$, and $1 - n/q = -n/p$ satisfy the scaling analysis, but the corresponding (1.31) does not hold; in fact, (1.31) with the inequality reversed holds in this case.

Exercises

Exercise 1.3.6. Let $u(x)$ be a smooth function on $B_R(0)$. Define $u_R(z) := u(Rz)$ for $z \in B_1(0)$. Verify that

$$||u_R||_{L^p(B_1(0))} = R^{-\frac{n}{p}}||u||_{L^p(B_R(0))}, \quad ||\partial_z^k u_R||_{L^p(B_1(0))} = R^{k-\frac{n}{p}}||\partial_x^k u||_{L^p(B_R(0))}.$$

In particular, $||\Delta_z u_R||_{L^p(B_1(0))} = R^{2-\frac{n}{p}}||\Delta_x u||_{L^p(B_R(0))}$. Later we will prove that there exists $C = C(n) > 0$ such that if u is a harmonic function in $B_1(0)$, then $||\nabla u||_{C(\overline{B_{\frac{1}{2}}(0)})} \leq C||u||_{C(\overline{B_1(0)})}$. Use the scaling computations here to show that if $u(x)$ is a bounded harmonic function on $\mathbb{R}^n$, then it must be a constant on $\mathbb{R}^n$.

Exercise 1.3.7. Let $u(x,t)$ be a smooth function on $(x,t) \in B_R(0) \times [0, R^2]$, and let $f(x,t) := (\partial_t - \Delta_x)u(x,t)$. Define $u_R(\xi,\tau) := u(R\xi, R^2\tau)$ for $(\xi,\tau) \in B_1(0) \times [0,1]$. Verify that

$$\begin{aligned} ||u_R||_{L^p(B_1(0)\times[0,1])} &= R^{-\frac{(n+2)}{p}}||u||_{L^p(B_R(0)\times[0,R^2])}, \\ ||\partial_\xi^k \partial_\tau^l u_R||_{L^p(B_1(0)\times[0,1])} &= R^{k+2l-\frac{(n+2)}{p}}||\partial_x^k \partial_t^l u||_{L^p(B_R(0)\times[0,R^2])}. \end{aligned}$$

Also verify that $(\partial_\tau - \Delta_\xi)u_R(\xi,\tau) = R^2 f(R\xi, R^2\tau)$.

1.4 Supplement: Local Solvability of Certain Geometric PDEs*

While most PDE problems in applications involve initial or boundary conditions, or both, certain geometric problems can be formulated as a local solvability problem of certain associated geometric PDEs, with no initial or boundary conditions. We discuss such an example involving geometry in this section. The method of reduction to be discussed below may seem ad hoc, but serves as a good example to illustrate how an initial formulation, which does not seem to fit any standard PDE theory, can be reduced to a form where some of the standard theories become applicable.

Let $\vec{r} = \vec{r}(x,y)$ be a local parametrization of a piece of a surface M. Then the metric properties of M, namely, the lengths of tangents and angles between tangents on M are determined by the **first fundamental form**

$$\begin{aligned} ds^2 = |d\vec{r}|^2 &= d\vec{r}\cdot d\vec{r} \\ &= (\vec{r}_x\,dx + \vec{r}_y\,dy)\cdot(\vec{r}_x\,dx + \vec{r}_y\,dy) \\ &= \vec{r}_x\cdot\vec{r}_x\,dx^2 + 2\vec{r}_x\cdot\vec{r}_y\,dxdy + \vec{r}_y\cdot\vec{r}_y\,dy^2 \\ &= E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2, \end{aligned}$$

*Optional material.

where $E(x,y) = \vec{r}_x \cdot \vec{r}_x$, $F(x,y) = \vec{r}_x \cdot \vec{r}_y$, and $G(x,y) = \vec{r}_y \cdot \vec{r}_y$. For instance, the length of a piece of a curve $\vec{r} \circ C(s)$ on M given through the parametrization, $C(s) = (x(s), y(s)) \in \Omega, a \le s \le b$, where Ω is a domain in the parameter x-y plane, is computed as

$$\int_a^b \sqrt{E(x(s),y(s))\left(\frac{dx}{ds}\right)^2 + 2F(x(s),y(s))\frac{dx}{ds}\frac{dy}{ds} + G(x(s),y(s))\left(\frac{dy}{ds}\right)^2}\, ds,$$

and the area of the piece of surface on M corresponding to the domain Ω under the representation $\vec{r}$ is $\int_\Omega \sqrt{E(x,y)G(x,y) - F^2(x,y)}dxdy$.

Note that the matrix

$$\begin{bmatrix} E(x,y) & F(x,y) \\ F(x,y) & G(x,y) \end{bmatrix}$$

is positive definite. If we introduce a set of new parameters (u,v), namely, a change of variables $(x,y) \mapsto (u(x,y), v(x,y))$, the first fundamental form would have different coefficients $\tilde{E}(u,v)$, $\tilde{F}(u,v)$, and $\tilde{G}(u,v)$. One basic question in geometry is whether one can make a change of variables such that the coefficients in the new coordinates become simpler. Two concrete such questions are

(a). Whether there is a change of variables, at least locally, such that $\tilde{E}(u,v) \equiv 1$, $\tilde{F}(u,v) \equiv 0$, and $\tilde{G}(u,v) \equiv 1$?

(b). Whether there is a change of variables, at least locally, such that $\tilde{E}(u,v) \equiv \tilde{G}(u,v)$, and $\tilde{F}(u,v) \equiv 0$?

Question (a) amounts to three conditions (PDEs) for two unknowns $u(x,y)$ and $v(x,y)$. These would be an **overdetermined** system of PDEs, and usually do not have solutions, unless certain compatibility condition derived from the system is satisfied. That condition turns out to be the vanishing of the Gaussian curvature of the metric ds^2.

Question (b) amounts to two conditions (PDEs) for two unknowns $u(x,y)$ and $v(x,y)$. The counting of the variables makes this problem to be probably tractable. When such a change of variables exists, then, letting $\Lambda(u,v) = \tilde{E}(u,v) \equiv \tilde{G}(u,v)$, the metric in the new coordinates (u,v) is represented as $\Lambda(u,v)(du^2 + dv^2)$. Such a coordinate would be called an isothermal coordinate. This question can be formulated as the local existence of solution of a system of PDEs, or as the local existence of solution of a single PDE.

More specifically, we ask whether there exist $u(x,y)$, $v(x,y)$, and $\widetilde{\Lambda}(x,y)$ such that such that

$$E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2 = \widetilde{\Lambda}(x,y)(du^2 + dv^2).$$

Substituting $du = u_x dx + u_y dy$, and $dv = v_x dx + v_y dy$ into $\widetilde{\Lambda}(x,y)(du^2 + dv^2)$, and equating the resulting expression to $E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2$, question (b) is equivalent to the existence of functions $\widetilde{\Lambda}(x,y)$, $u(x,y)$, and $v(x,y)$ such that

$$\begin{cases} E(x,y) = \widetilde{\Lambda}(x,y)\left[u_x^2(x,y) + v_x^2(x,y)\right], \\ F(x,y) = \widetilde{\Lambda}(x,y)\left[u_x(x,y)u_y(x,y) + v_x(x,y)v_y(x,y)\right], \\ G(x,y) = \widetilde{\Lambda}(x,y)\left[u_y^2(x,y) + v_y^2(x,y)\right], \\ u_x(x,y)v_y(x,y) - u_y(x,y)v_x(x,y) \neq 0. \end{cases}$$

We also require that $(x,y) \mapsto (u(x,y), v(x,y))$ is locally invertible. This is a system of *nonlinear* equations for the three unknowns $\widetilde{\Lambda}(x,y)$, $u(x,y)$, and $v(x,y)$. It can be reduced to a system of first order *linear* PDEs or a single second order *linear* PDE as follows. First note that

$$\widetilde{\Lambda}(x,y)\left[u_x(x,y)v_y(x,y) - u_y(x,y)v_x(x,y)\right] = \sqrt{E(x,y)G(x,y) - F^2(x,y)}$$

and we may look for solutions such that $\widetilde{\Lambda}(x,y) > 0$. Then the equations can be interpreted as

$$\sqrt{\frac{\widetilde{\Lambda}(x,y)}{E(x,y)}}(u_x(x,y), v_x(x,y)) \text{ and } \sqrt{\frac{\widetilde{\Lambda}(x,y)}{G(x,y)}}(u_y(x,y), v_y(x,y)) \text{ are unit vectors in } \mathbb{R}^2,$$

$$\text{and the angle } \gamma \text{ between them satisfies } \cos\gamma = \frac{F(x,y)}{\sqrt{E(x,y)G(x,y)}}.$$

Thus $\sqrt{\frac{\widetilde{\Lambda}(x,y)}{G(x,y)}}(u_y(x,y), v_y(x,y))$ can be obtained from $\sqrt{\frac{\widetilde{\Lambda}(x,y)}{E(x,y)}}(u_x(x,y), v_x(x,y))$ by a rotation of angle γ. Using the rotation matrix

$$\begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} = \begin{bmatrix} \frac{F(x,y)}{\sqrt{E(x,y)G(x,y)}} & -\frac{\sqrt{E(x,y)G(x,y)-F^2(x,y)}}{\sqrt{E(x,y)G(x,y)}} \\ \frac{\sqrt{E(x,y)G(x,y)-F^2(x,y)}}{\sqrt{E(x,y)G(x,y)}} & \frac{F(x,y)}{\sqrt{E(x,y)G(x,y)}} \end{bmatrix}$$

we have

$$\sqrt{\frac{\widetilde{\Lambda}(x,y)}{G(x,y)}}\begin{bmatrix} u_y(x,y) \\ v_y(x,y) \end{bmatrix} = \sqrt{\frac{\widetilde{\Lambda}(x,y)}{E(x,y)}}\begin{bmatrix} \frac{F(x,y)}{\sqrt{E(x,y)G(x,y)}} & -\frac{\sqrt{E(x,y)G(x,y)-F^2(x,y)}}{\sqrt{E(x,y)G(x,y)}} \\ \frac{\sqrt{E(x,y)G(x,y)-F^2(x,y)}}{\sqrt{E(x,y)G(x,y)}} & \frac{F(x,y)}{\sqrt{E(x,y)G(x,y)}} \end{bmatrix}\begin{bmatrix} u_x(x,y) \\ v_x(x,y) \end{bmatrix},$$

from which it follows that

$$\begin{cases} u_y(x,y) = \dfrac{F(x,y)u_x(x,y) - \sqrt{E(x,y)G(x,y) - F^2(x,y)}\,v_x(x,y)}{E(x,y)}, \\ v_y(x,y) = \dfrac{\sqrt{E(x,y)G(x,y) - F^2(x,y)}\,u_x(x,y) + F(x,y)v_x(x,y)}{E(x,y)}. \end{cases}$$

This is a linear system of first order PDEs for the two unknowns $u(x,y)$ and $v(x,y)$. It further follows from this that

$$\begin{cases} v_x(x,y) = \dfrac{F(x,y)u_x(x,y) - E(x,y)u_y(x,y)}{\sqrt{E(x,y)G(x,y) - F^2(x,y)}}, \\ v_y(x,y) = \dfrac{G(x,y)u_x(x,y) - F(x,y)u_y(x,y)}{\sqrt{E(x,y)G(x,y) - F^2(x,y)}}. \end{cases} \tag{1.32}$$

Thus $u(x,y)$ needs to satisfy

$$\left(\frac{F(x,y)u_x(x,y) - E(x,y)u_y(x,y)}{\sqrt{E(x,y)G(x,y) - F^2(x,y)}}\right)_y - \left(\frac{G(x,y)u_x(x,y) - F(x,y)u_y(x,y)}{\sqrt{E(x,y)G(x,y) - F^2(x,y)}}\right)_x = 0. \tag{1.33}$$

(1.33) is a second order linear PDE for $u(x,y)$, and is called the **Beltrami** equation. Conversely, if $u(x,y)$ is a solution of (1.33), then we can find $v(x,y)$ satisfying (1.32) locally, and reverse the computations above to show that $(x,y) \mapsto (u(x,y), v(x,y))$ is a desired change of variables.

A related question is the question of **isometric imbedding**: whether, for a given metric, namely, a positive definite bilinear form $E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2$, there is a local map $(x,y) \mapsto \vec{r}(x,y) \in \mathbb{R}^n$ such that

$$|d\vec{r}(x,y)|^2 = E(x,y)\,dx^2 + 2F(x,y)\,dxdy + G(x,y)\,dy^2?$$

This amounts to asking $\vec{r}(x,y)$ to satisfy $E(x,y) = \vec{r}_x \cdot \vec{r}_x$, $F(x,y) = \vec{r}_x \cdot \vec{r}_y$, and $G(x,y) = \vec{r}_y \cdot \vec{r}_y$. These are three nonlinear PDEs for the n-components of $\vec{r}(x,y)$. When $n > 3$, this system is **underdetermined**. In the 1950's John Nash provided the first positive answer to a global version of this question and its higher dimensional analogue. For $n = 3$, this is a determined system. In 1894 Darboux reduced this system to a single nonlinear PDE. But the PDE is notoriously difficult to solve; the behavior of the solution depends on the sign of the curvature of the given metric. H. Weyl proposed a related global embedding problem: given a metric on a topological sphere with positive Gaussian curvature, can one find an embedding of the sphere into $\mathbb{R}^3$ whose induced metric equals the given metric? The problem was solved in the early 1950's by Nirenberg and Pogorelov, using different approaches. Hartman and Wintner also solved the local existence of the Darboux equation in the 1950's under the condition that the Gaussian curvature of the given metric does not vanish. There has been renewed interest in this and related problems in the last 30 years, and great progress has been made.

1.5 Additional Problems

Problem 1.5.1. The $-\frac{\partial p(x,t)}{\partial x}$ term in (1.4) comes from $-p(x_2,t)+p(x_1,t)$, which accounts for the pressure force acting on the section $[x_1,x_2]$ at x_2 and x_1, respectively. In higher dimensions, the fluid force acting on a region Ω through contact along $\partial\Omega$ is accounted for by $\int_{\partial\Omega}\vec{F}(x,t)d\sigma(x)$; and one commonly used assumption is that

$$\vec{F}(x,t)=[-p(x,t)I+2\mu S(D\vec{v}(x,t))]\cdot\vec{n}(x),$$

where I is the identity matrix, $\mu\geq 0$ is the viscosity coefficient of the fluid, $S(D\vec{v}(x,t))$ is the strain-rate matrix, whose (i,j) entry is $(\partial_{x_i}v_j(x,t)+\partial_{x_j}v_i(x,t))/2$. To convert the surface flux integrals

$$\int_{\partial\Omega}\rho(x,t)\vec{v}(x,t)\,\vec{v}(x,t)\cdot\vec{n}(x)d\sigma(x),$$

$$\int_{\partial\Omega}\{-p(x,t)\vec{n}(x)+2\mu S(D\vec{v}(x,t))\cdot\vec{n}(x)\}\,d\sigma(x)$$

into volume integrals in Ω, it's more convenient to treat the integrals component wise.

(i) Prove that

$$\begin{aligned}
&\int_{\partial\Omega}\rho(x,t)v_i(x,t)\,\vec{v}(x,t)\cdot\vec{n(x)}d\sigma(x)\\
=&\int_{\Omega}\operatorname{div}\left(\rho(x,t)v_i(x,t)\,\vec{v}(x,t)\right)dx\\
=&\int_{\Omega}\left\{\sum_{j=1}^{n}v_j(x,t)\partial_{x_j}\left(\rho(x,t)v_i(x,t)\right)+\rho(x,t)v_i(x,t)\operatorname{div}\,\vec{v}(x,t)\right\}dx\\
=&\int_{\Omega}\left\{\vec{v}(x,t)\cdot\nabla\left(\rho(x,t)v_i(x,t)\right)+\rho(x,t)v_i(x,t)\operatorname{div}\,\vec{v}(x,t)\right\}dx.
\end{aligned}$$

(ii) Prove that

$$\int_{\partial\Omega}p(x,t)n_i(x)d\sigma(x)=\int_{\Omega}\frac{\partial p(x,t)}{\partial x_i}dx.$$

(iii) Let $S_i(D\vec{v}(x,t))$ denote the vector formed by the ith row of $S(D\vec{v}(x,t))$. Prove that

$$\begin{aligned}
&\int_{\partial\Omega}S_i(D\vec{v}(x,t))\cdot\vec{n}(x)d\sigma(x)\\
=&\int_{\Omega}\frac{1}{2}\sum_{j=1}^{n}\left(\partial^2_{x_jx_i}v_j(x,t)+\partial^2_{x_jx_j}v_i(x,t)\right)dx\\
=&\int_{\Omega}\frac{1}{2}\left(\partial_{x_i}(\operatorname{div}\,\vec{v})+\Delta v_i(x,t)\right)dx
\end{aligned}$$

(iv) Derive the momentum equations

$$\begin{aligned}&\frac{\partial[\rho(x,t)v_i(x,t)]}{\partial t}+\sum_{j=1}^{n}\frac{\partial\left[\rho(x,t)v_i(x,t)v_j(x,t)\right]}{\partial x_j}\\&=-\frac{\partial p(x,t)}{\partial x_i}+\mu\Delta v_i(x,t)+\mu\frac{\partial(\text{div }\vec{v}(x,t))}{\partial x_i}.\end{aligned}\tag{1.34}$$

(v) Derive a similar, but simpler, equation based on the conservation of mass:

$$\frac{\partial\rho(x,t)}{\partial t}+\sum_{j=1}^{n}\frac{\partial\left[\rho(x,t)v_j(x,t)\right]}{\partial x_j}=0.$$

(vi) The equations in (iv) and (v) form part of the Navier-Stokes system. If one assumes that the fluid is incompressible, namely, div $\vec{v}(x,t)\equiv 0$, then, using (v), (1.34) can be simplified into

$$\rho(x,t)\frac{\partial v_i(x,t)}{\partial t}+\rho(x,t)\sum_{j=1}^{n}v_j(x,t)\frac{\partial\left[v_i(x,t)\right]}{\partial x_j}=-\frac{\partial p(x,t)}{\partial x_i}+\mu\Delta v_i(x,t).$$

When $\mu=0$, we obtain the Euler system.

Problem 1.5.2. Consider separable solutions of the form $u(x,t)=e^{-\lambda t}X(x)$ to

$$u_t(x,t)-\left[p(x)u_x(x,t)\right]_x=0,$$

where $p(x)$ is a piecewise constant function defined by

$$p(x)=\begin{cases}p_1>0 & \text{if } x<0,\\ p_2>0 & \text{if } x>0.\end{cases}$$

The equation is to hold for $x\neq 0$ and $t>0$, and we require $u(x,t)$ and $p(x)u_x(x,t)$ to be continuous across $x=0$, which implies $p_1X'(0-)=p_2X'(0+)$.

(i). Derive the equation satisfied by $X(x)$, and prove that for any $\lambda>0$, there exist solutions $X(x)$ on $\mathbb{R}$ which are continuous, piecewise C^1, and bounded—$X'(x)$ would have a discontinuity at $x=0$; and this example shows that when the diffusion coefficient has a jump discontinuity, the diffusion equation has solutions which are only piecewise C^1, in contrast to the behavior of the standard diffusion equation.

(ii). Also verify that, if we impose additionally the boundary conditions $X(-\pi) = X(\pi) = 0$, then the λ's for which there are not-identically-zero solutions X are given by

$$\sqrt{p_1}\sin\left(\sqrt{\frac{\lambda}{p_2}}\pi\right)\cos\left(\sqrt{\frac{\lambda}{p_1}}\pi\right) + \sqrt{p_2}\sin\left(\sqrt{\frac{\lambda}{p_1}}\pi\right)\cos\left(\sqrt{\frac{\lambda}{p_2}}\pi\right) = 0.$$

Problem 1.5.3. Identify all the eigenvalues λ of the following BVP

$$\begin{cases} X^{(4)}(x) + \lambda X(x) = 0, \quad 0 < x < l, \\ X(0) = X''(0) = 0, \\ X(l) = X''(l) = 0. \end{cases}$$

Problem 1.5.4. Identify all the eigenvalues λ of the following BVP

$$\begin{cases} X^{(4)}(x) + \lambda X(x) = 0, \quad 0 < x < l, \\ X(0) = X'(0) = 0, \\ X(l) = X'(l) = 0. \end{cases}$$

Chapter 2

Fourier's Method Applied to the Heat, Wave, and Laplace Equations

We now make some more definite statements about the solutions to the homogeneous case of (1.8) obtained in the previous chapter through separation of variables and superposition principle—both lead to Fourier's method, one to Fourier series, the other to Fourier transforms.

2.1 Convergence Issues in the Fourier Series Solution of the Heat Equation

We first make some comments on Fourier sine series which arise from solving the initial-boundary value problem for the heat equation with homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t - u_{xx} = 0, & \text{for} \quad (x,t) \in (0,l) \times (0,\infty), \\ u(0,t) = u(l,t) = 0, & \text{for} \quad t > 0, \\ u(x,0) = g(x) & \text{for} \quad x \in [0,l]. \end{cases} \tag{2.1}$$

Recall that we were led to propose to use $\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l}) e^{-(\frac{n\pi}{l})^2 t}$ as a solution. The c_n's in (1.24) are determined in terms of $u(x,0) = g(x)$ by

$$c_n = \frac{2}{l} \int_0^l u(x,0) \sin(\frac{n\pi x}{l})\, dx.$$

This relies on the fundamental fact that

$$\int_0^l \sin(\frac{m\pi x}{l}) \sin(\frac{n\pi x}{l})\, dx = 0, \text{if } m \neq n \in \mathbb{N}, \text{ and } = \frac{l}{2} \text{ if } m = n \in \mathbb{N};$$

namely, the functions in the set $\{\sin(\frac{n\pi x}{l}) : n \in \mathbb{N}\}$ are mutually orthogonal to each other over $[0,l]$. Formally one multiplies both sides of (1.24) by $\sin(\frac{m\pi x}{l})$ and integrates over $[0,l]$, then interchanges the order of integration and summation to lead to the formula for c_n, using the orthogonal relations above.

The c_n's are well defined for any $u(x,0)$ which is Riemann integrable on $[0,l]$. It seems natural to investigate whether the partial sums $\sum_{n=1}^{k} c_n \sin(\frac{n\pi x}{l})$ converge to $u(x,0)$ in the integral norm $||u||_{L[0,l]} = \int_0^l |u(x)|\,dx$. But the space of Riemann integrable functions on $[0,l]$ under this norm is not **complete**, namely, there are sequences of Riemann integrable functions on $[0,l]$, $\{u_k(x)\}$, which is a Cauchy sequence in the sense that $||u_k - u_{k'}||_{L[0,l]} \to 0$ as $k, k' \to \infty$, but does not converge to a Riemann integrable function in this norm. This makes it inconvenient for many issues in analysis; we prefer to work with a complete space in which any Cauchy sequence converges. In this context we work with the **completion** of the space of Riemann integrable functions on $[0,l]$ under this norm, which turns out to be the space of Lebesgue integrable functions on $[0,l]$, denoted as $L^1[0,l]$. It was only since the early 20th century that mathematicians began to appreciate the advantages of working with function spaces with completeness in integral norms.

It turns out that it's more convenient to work with $L^2[0,l]$ in this context, and we may also consider $L^p[0,l]$, for $\infty > p \geq 1$, which is the completion of space of Riemann integrable functions on $[0,l]$ under the *norm*

$$||u||_{L^p[0,l]} = (\int_0^l |u(x)|^p\,dx)^{1/p}.$$

A norm on a linear space X is a function $\|\cdot\| : X \mapsto \mathbb{R}^{\geq 0}$ such that

(i) $\|u\| \geq 0$ for any $u \in X$, with $=$ only when $u = 0$,

(ii) $\|u + v\| \leq \|u\| + \|v\|$ for any $u, v \in X$, and

(iii) $\|cu\| = |c|\|u\|$ for any scalar c and any $u \in X$.

The only non-trivial verification of these properties for $||u||_{L^p[0,l]}$ is (ii):

$$||u+v||_{L^p[0,l]} \leq ||u||_{L^p[0,l]} + ||v||_{L^p[0,l]} \quad \text{whenever } ||u||_{L^p[0,l]}, ||v||_{L^p[0,l]} < \infty.$$

This is simple for $p = 1$. The $p = 2$ case is a consequence of the Cauchy-Schwarz inequality, and the $1 < p < \infty$ case is called the Minkowski inequality. Guided proofs will be provided in the exercises.

$L^\infty[0,l]$ is an analogue for the space of bounded integrable functions, but its proper definition needs more background on Lebesgue measure. We almost always deal with solutions that are continuous (or at least piecewise continuous) and bounded, and can use the sup norm, $\sup\{|u(x)| : 0 \le x \le l\}$, instead of the more technical $L^\infty[0,l]$ norm in our contexts.

Three commonly used notions of convergence of sequences/series of functions occurring in the PDE context are: **point-wise convergence, uniform convergence**, and **mean square convergence**. In our context, we need to verify that

(a). The series $u(x,t) = \sum_{n=1}^\infty c_n e^{-(\frac{n\pi}{l})^2 t} \sin(\frac{n\pi x}{l})$ defines a C^2 function at least for $(x,t) \in (0,l) \times (0,\infty)$ and is in $C([0,l] \times [0,\infty))$;

(b). It satisfies the equation in $(0,l)\times(0,\infty)$ and the boundary condition over $\{0,l\}\times(0,\infty)$; and

(c). It satisfies the initial condition in appropriate sense.

Uniform convergence is the most commonly used condition to verify these; but a necessary condition of the uniform convergence of the series here over $[0,l] \times [0,\infty)$, at least over $[0,l] \times [0,T]$ for any finite $T > 0$, is that $g \in C[0,l]$ and $g(0) = g(l) = 0$. This restriction is sometimes considered too restrictive.

A commonly used alternative is that we demand that $u(x,t) \to u(x,0) = g(x)$ in the mean square sense, namely, $||u(x,t) - g(x)||_{L^2[0,l]} \to 0$ as $t \searrow 0$, or that $u(x',t) \to g(x)$ as $x' \to x$ and $t \searrow 0$ only for $0 < x < l$. Note that the latter amounts to $\sum_{n=1}^\infty c_n e^{-(\frac{n\pi}{l})^2 t} \sin(\frac{n\pi x'}{l}) \to g(x)$ as $x' \to x$ and $t \searrow 0$, which is different from the usual point-wise convergence of the Fourier series: $\sum_{n=1}^N c_n \sin(\frac{n\pi x}{l}) \to g(x)$ as $N \to \infty$.

Nonetheless, we first review the notion of convergence of the series $\sum_{n=1}^\infty c_n \sin(\frac{n\pi x}{l})$ to $g(x)$. It can be either in the sense of point-wise convergence and answered via a reduction to the convergence of a classical full Fourier series, or in the mean square sense, treating $\{\sin\left(\frac{n\pi x}{l}\right)\}_{n=1}^\infty$, as a **complete set of orthogonal eigenfunctions** for the boundary value problem (1.23). We will study the latter point of approach later on.

Since students are often more familiar with the convergence of a full Fourier series, we first reduce the convergence of the series $\sum_{n=1}^\infty c_n \sin(\frac{n\pi x}{l})$ to that of a full Fourier series.

If $\sum_{n=1}^\infty c_n \sin(\frac{n\pi x}{l})$ converges for $0 < x < l$, then it would also converge for $-l < x < 0$, in fact, for all $x \in \mathbb{R}$, and the limit function would define an odd, period $2l$ function on $\mathbb{R}$. Taking this as a clue, for a given function $g(x)$ defined on $0 < x < l$, we first construct an odd extension g_{odd} of g to $(-l, l)$, and then construct the full Fourier series expansion of g_{odd} over $(-l, l)$, which would have the $2l$-periodic extension of g_{odd} as its limit in the appropriate sense:

$$g_{\mathrm{odd}}(x) \sim a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right], \quad x \in (-l, l).$$

Here one uses that the functions in the family $\{1, \cos\left(\frac{n\pi x}{l}\right), \sin\left(\frac{n\pi x}{l}\right) : n \in \mathbb{N}\}$ are mutually orthogonal to each other on $(-l, l)$, and

$$\int_{-l}^{l} \cos^2\left(\frac{n\pi x}{l}\right) dx = \int_{-l}^{l} \sin^2\left(\frac{n\pi x}{l}\right) dx = l \quad \text{for } n \in \mathbb{N},$$

to obtain

$$a_0 = \frac{1}{2l} \int_{-l}^{l} g_{\mathrm{odd}}(x)\, dx,$$
$$a_n = \frac{1}{l} \int_{-l}^{l} g_{\mathrm{odd}}(x) \cos\left(\frac{n\pi x}{l}\right) dx,$$
$$b_n = \frac{1}{l} \int_{-l}^{l} g_{\mathrm{odd}}(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

One notes that $a_n = 0$ for $n = 0, 1, \cdots$, due to the oddness of g_{odd}, and b_n's are determined through as

$$b_n = \frac{2}{l} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx.$$

Thus $b_n = c_n$, and

$$g_{\mathrm{odd}}(x) \sim \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right), \quad x \in (-l, l).$$

In particular,

$$g(x) = g_{\mathrm{odd}}(x) \sim \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right), \quad x \in (0, l).$$

Thus we will treat the convergence of $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$ to $g(x)$ as a special case of the convergence of a full Fourier series.

2.2 A Brief Review of Fourier Series

We summarize below a few key facts on the classical Fourier series that are used often. Computations using complex exponentials are often easier than using the classical trigonometric functions, so we will introduce Fourier series in terms of the complex exponentials.

Suppose now that $g(x)$ is (Riemann or Lebesgue) integrable over $(-l, l)$. Using the relations

$$\sin\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i}, \quad \cos\left(\frac{n\pi x}{l}\right) = \frac{e^{\frac{in\pi x}{l}} + e^{-\frac{in\pi x}{l}}}{2},$$

the partial sum $a_0 + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$ can also be written as

$$\begin{aligned} &a_0 + \sum_{n=1}^{N} \left[a_n \frac{e^{\frac{in\pi x}{l}} + e^{\frac{-in\pi x}{l}}}{2} + b_n \frac{e^{\frac{in\pi x}{l}} - e^{-\frac{in\pi x}{l}}}{2i} \right] \\ =&a_0 + \sum_{n=1}^{N} \left[\frac{a_n - b_n i}{2} e^{\frac{in\pi x}{l}} + \frac{a_n + b_n i}{2} e^{-\frac{in\pi x}{l}} \right] \\ =&\sum_{n=-N}^{N} g_n e^{\frac{in\pi x}{l}}, \end{aligned}$$

where

$$g_n = \begin{cases} \frac{a_n - b_n i}{2} & \text{if } n \in \mathbb{N}, \\ a_0 = \frac{1}{2l}\int_{-l}^{l} g(x)\,dx & \text{if } n = 0, \\ \frac{a_{|n|} + b_{|n|} i}{2} & \text{if } -n \in \mathbb{N}. \end{cases}$$

Since

$$\frac{a_n - b_n i}{2} = \frac{1}{2l}\int_{-l}^{l} g(x)\left[\cos\left(\frac{n\pi x}{l}\right) - i\sin\left(\frac{n\pi x}{l}\right)\right] dx = \frac{1}{2l}\int_{-l}^{l} g(x) e^{-\frac{in\pi x}{l}}\,dx,$$

$$\frac{a_{|n|} + b_{|n|} i}{2} = \frac{1}{2l}\int_{-l}^{l} g(x)\left[\cos\left(\frac{|n|\pi x}{l}\right) + i\sin\left(\frac{|n|\pi x}{l}\right)\right] dx = \frac{1}{2l}\int_{-l}^{l} g(x) e^{\frac{i|n|\pi x}{l}}\,dx,$$

we have

$$g_n = \frac{1}{2l}\int_{-l}^{l} g(x) e^{-\frac{in\pi x}{l}}\,dx \quad \text{for all } n \in \mathbb{Z}.$$

The functions in the family $\{e^{\frac{in\pi x}{l}}\}_{n\in\mathbb{Z}}$ are orthogonal to each other over $(-l, l)$ in the sense that

$$(e^{\frac{im\pi x}{l}}, e^{\frac{in\pi x}{l}}) := \int_{-l}^{l} e^{\frac{im\pi x}{l}} \overline{e^{\frac{in\pi x}{l}}}\,dx = \int_{-l}^{l} e^{\frac{im\pi x}{l}} e^{\frac{-in\pi x}{l}}\,dx = 0$$

for $m \neq n$ in $\mathbb{Z}$—the inner product between complex valued functions f and g should be defined as $\int_{-l}^{l} f(x)\overline{g(x)}\,dx$. It's often more convenient to work with Fourier series in this "basis" than in the traditional basis $\{1, \cos\left(\frac{n\pi x}{l}\right), \sin\left(\frac{n\pi x}{l}\right)\}_{n\in\mathbb{N}}$.

Theorem 2.1. *For any g integrable over $(-l,l)$, define its Fourier series as above. Then*

(i). $g_n \to 0$ *as* $n \to \infty$ *(Riemann-Lebesgue Lemma).*

(ii). *If* $g(x) \in L^2[-l,l]$, *then* $\sum_{n=-\infty}^{\infty} g_n e^{\frac{in\pi x}{l}}$ *converges to* g *in the sense of* $L^2[-l,l]$, *namely*

$$\left\| \sum_{n=-N}^{N} g_n e^{\frac{in\pi x}{l}} - g(x) \right\|_{L^2[-l,l]} \to 0 \text{ as } N \to \infty.$$

In addition, $\int_{-l}^{l} |g(x)|^2\,dx = 2l \sum_{n=-\infty}^{\infty} |g_n|^2$ *(Parseval identity). Using the form* $a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$, *this relation is expressed as*

$$\int_{-l}^{l} |g(x)|^2\,dx = l\left[2|a_0|^2 + \sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right)\right].$$

(iii). *The point-wise convergence of the Fourier series at some* $x_0 \in (-l,l)$ *depends on the local behavior of* $g(x)$ *near* x_0, *namely, if* $g(x) = h(x)$ *in a neighborhood of* x_0, *then the Fourier series of* $g(x)$ *converges at* x_0 *if and only if the Fourier series of* $h(x)$ *converges at* x_0. *If* $g(x)$ *is* piecewise locally Lipschitz continuous at x_0*, *then it converges to* $[g(x_0+) + g(x_0-)]/2$, *where* $g(x_0\pm) := \lim_{x\to x_0\pm} g(x)$.

(iv). *If* $\sum_{n=-\infty}^{\infty} |g_n| < \infty$, *then* $\sum_{n=-\infty}^{\infty} g_n e^{\frac{in\pi x}{l}}$ *converges absolutely and uniformly to* $g(x)$ *over* $\mathbb{R}$, *therefore defines a continuous* $2l$*-periodic function over* $\mathbb{R}$.

(v). *If* $g(x)$ *is continuous over* $[-l,l]$ *with* $g(-l) = g(l)$ *and* $g(x)$ *is piecewise* C^1 *over* $[-l,l]$ *(this can be replaced by* $g(x)$ *be absolutely continuous over* $[-l,l]$ *with* $g'(x) \in L^2[0,l]$*), then the corresponding* $\{g_n\}$ *satisfies* $\sum_{n=-\infty}^{\infty} |g_n| < \infty$, *therefore the Fourier series converges to* $g(x)$ *uniformly for* $x \in [-l,l]$.

Remark 2.1. It is known that if $g(x)$ is only continuous over $[-l,l]$, then its Fourier series may not converge to $g(x)$ point-wise. However, if $g(x)$ is continuous over $[-l,l]$ with $g(-l) = g(l)$, then one can construct trigonometric polynomials $p_N(x)$ of the form $\sum_{n=-N}^{N} c_{N,n} e^{\frac{in\pi x}{l}}$ such that $p_N(x) \to g(x)$ uniformly over $[-l,l]$ as $N \to \infty$. One simple construction was due to Fejér, where one takes $c_{N,n} = \left(1 - \frac{|n|}{N}\right) g_n$ so that

$$\sum_{n=-N}^{N} c_{N,n} e^{\frac{in\pi x}{l}} = N^{-1} \sum_{n=0}^{N-1} \left(\sum_{-n}^{n} g_k e^{\frac{ik\pi x}{l}} \right)$$

*this means that the limits $\lim_{x\to x_0 \pm 0} g(x) = g(x_0 \pm 0)$ exist, and there exist $a < x_0 < b$ and constant $L > 0$, such that $|g(x) - g(x_0-)| \le L|x - x_0|$ for $x \in [a, x_0)$, and $|g(x) - g(x_0+)| \le L|x - x_0|$ for $x \in (x_0, b]$. The conclusion here holds under the more general condition that g has bounded variation.

is the arithmetic mean of the partial sums of the Fourier series $\sum_{-n}^{n} g_k e^{\frac{ik\pi x}{l}}$ for $n = 0, \cdots, N-1$. The main reason for the difference between the behavior of the Fourier series and the modification due to Fejér is that the partial sums

$$\sum_{n=-N}^{N} g_n e^{\frac{in\pi x}{l}} = \int_{-l}^{l} g(y) \frac{\sin\left((2N+1)\frac{\pi(x-y)}{2l}\right)}{2l \sin\frac{\pi(x-y)}{2l}}\, dy, \tag{2.2}$$

and

$$N^{-1}\sum_{n=0}^{N-1}\left(\sum_{-n}^{n} g_k e^{\frac{ik\pi x}{l}}\right) = \int_{-l}^{l} g(y) \frac{\sin^2\left((N+1)\frac{\pi(x-y)}{2l}\right)}{2l(N+1)\sin^2\frac{\pi(x-y)}{2l}}\, dy \tag{2.3}$$

have different behavior in their kernel functions:

$$\int_{-l}^{l} \frac{\sin^2\left((N+1)\frac{\pi(x-y)}{2l}\right)}{2l(N+1)\sin^2\frac{\pi(x-y)}{2l}}\, dy = 1, \text{ but } \int_{-l}^{l} \left|\frac{\sin\left((2N+1)\frac{\pi(x-y)}{2l}\right)}{2l \sin\frac{\pi(x-y)}{2l}}\right| dy \to \infty \text{ as } N \to \infty. \tag{2.4}$$

Remark 2.2. In (v) the everywhere continuity of $g(x)$ in $[-l, l]$ and the condition $g(-l) = g(l)$ are necessary (these conditions would make the $2l$-periodic extension of the odd extension of g to $(-l, l)$ continuous on $\mathbb{R}$); the conclusion may not hold if $g(x)$ is only piecewise C^1 on $[-l, l]$, but fails to be continuous everywhere or fails the condition $g(-l) = g(l)$. These conditions are used to derive appropriate decay rate of g_n as $n \to \infty$.

A subtle issue when $g(x)$ is only piecewise C^1 on $[-l, l]$ is the relation between the Fourier series of g over $(-l, l)$ and that of $g'(x)$: if $g(x) \sim \sum_{n=-\infty}^{\infty} g_n e^{\frac{in\pi x}{l}}$, does it follow that $g'(x) \sim \sum_{n=-\infty}^{\infty} g_n \left(\frac{in\pi}{l}\right) e^{\frac{in\pi x}{l}}$, namely, by term-wise differentiation? This holds true if g continuous everywhere over $[-l, l]$ and $g(-l) = g(l)$, for, $g'(x)$ has its own Fourier series $\sum_{n=-\infty}^{\infty} g'_n e^{\frac{in\pi x}{l}}$, where

$$g'_n = \frac{1}{2l}\int_{-l}^{l} g'(x) e^{-\frac{in\pi x}{l}}\, dx.$$

When g is continuous everywhere over $[-l, l]$ and $g(-l) = g(l)$, we can carry out integration by parts as follows and derive

$$g'_n = \frac{1}{2l}\left[g(x) e^{-\frac{in\pi x}{l}}\Big|_{x=-l}^{x=l} + \frac{in\pi}{l}\int_{-l}^{l} g(x) e^{-\frac{in\pi x}{l}}\, dx\right] = \left(\frac{in\pi}{l}\right) g_n.$$

Note that if either of these two condition fails, then g'_n may not be equal to $\left(\frac{in\pi}{l}\right) g_n$: the jump discontinuities of g would create a δ-function type singularity in

g', and the series $\sum_{n=-\infty}^{\infty} g_n \left(\frac{in\pi}{l}\right) e^{\frac{in\pi x}{l}}$ would reflect such singular behavior—it often does not converge point-wise due to the contribution from the boundary terms above not decaying in n, so this series does not necessarily represent the Fourier series of the function which is defined point-wise, except at the points of jump discontinuity, by $g'(x)$. The simplest such an example is a step function.

When g is continuous everywhere over $[-l,l]$, but fails to satisfy $g(-l)=g(l)$, its $2l$-periodic extension to $\mathbb{R}$ would have jump discontinuities at $-l$, l, and at the translates of these points by integer multiples of $2l$, which also cause the failure of (v) as a jump discontinuity at an interior point of $(-l,l)$ would do; in such cases the Fourier series of the point-wise derivative g' over $(-l,l)$ would not agree with the one obtained through term-wise differentiation of the Fourier series of g over $(-l,l)$.

On the other hand, term-wise integration of Fourier series holds under very relaxed conditions, say, $g \in L^2[-l,l]$: if $g(x) \sim \sum_{n=-\infty}^{\infty} g_n e^{\frac{in\pi x}{l}}$, then for any $a<b$, $\int_a^b g(x)\,dx = \sum_{n=-\infty}^{\infty} g_n \int_a^b e^{\frac{in\pi x}{l}}\,dx$.

In the context of Fourier series construction for a solution of IBVP, the conditions in (v) translate into $u(x,0)$ being continuous over $[0,l]$, $u(0,0)=u(l,0)=0$ (so that its odd extension satisfies the conditions specified in (v)), and $u(x,0)$ being piecewise C^1 over $[0,l]$. Note that in such a situation, the convergence of $\sum_{n=-\infty}^{\infty}|c_n|$ also implies the uniform convergence of $\sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi y}{l}) \to g(x)$ as $y \to x$ and $t \searrow 0$.

This is seen by using a divide and conquer strategy. We first split $\sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi y}{l})$ as $\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}$, and for any $\epsilon>0$, we know that for all sufficiently large N,

$$\left|\sum_{n=N+1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi y}{l})\right| \le \sum_{n=N+1}^{\infty}|c_n| < \epsilon/3;$$

we can further guarantee, under our assumption on g, that for all sufficiently large N,

$$\left|\sum_{n=1}^{N} c_n \sin(\frac{n\pi x}{l}) - g(x)\right| < \epsilon/3;$$

and finally using

$$\begin{aligned}
&\left|\sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi y}{l}) - g(x)\right| \\
\le &\left|\sum_{n=1}^{N} c_n \left[e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi y}{l}) - \sin(\frac{n\pi x}{l})\right]\right| \\
&+\left|\sum_{n=N+1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi y}{l})\right| + \left|\sum_{n=1}^{N} c_n \sin(\frac{n\pi x}{l}) - g(x)\right|,
\end{aligned}$$

we see that we find some $\delta > 0$ such that, whenever $|y - x| < \delta$ and $0 < t < \delta$, we have

$$\left| \sum_{n=1}^{N} c_n \left[e^{-(\frac{n\pi}{l})^2 t} \sin(\frac{n\pi y}{l}) - \sin(\frac{n\pi x}{l}) \right] \right| < \epsilon/3.$$

The family $\left\{\sin\left(\frac{n\pi x}{l}\right)\right\}_{n=1}^{\infty}$ arises as a complete set of orthogonal eigenfunctions for (1.23). In fact, statements analogous to (i)–(ii) hold for expansions in a complete set of orthogonal eigenfunctions for many boundary value problems similar to (1.23)—they are called **Sturm-Liouville** (eigenvalue) problems (first studied in the mid-1830's). Fourier cosine series in $\left\{\cos\left(\frac{n\pi x}{l}\right)\right\}_{n=0}^{\infty}$ is another such example— they arise as a complete set of orthogonal eigenfunctions for a boundary value problem similar to (1.23), replacing the boundary conditions $X(0) = X(l) = 0$ there by $X'(0) = X'(l) = 0$. We will say more about the Sturm-Liouville problems later on, noting in particular their similarities with the eigenvalue problems for real symmetric (or complex Hermitian) matrices.

When $u \in L^2[0, l]$, we write $u(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$, where the equality is interpreted as equality in $L^2[0, l]$ by (ii) above, not necessarily in the point-wise sense. When we do not necessarily know or care whether the equality holds in a specific sense, we may simply write $u(x) \sim \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$.

Remark 2.3. It's often easier to derive properties of a series with direct knowledge of its coefficients—usually a fast enough decay condition would help, such as the one in (iv) of the theorem above; one then needs to study conditions on the original function which would guarantee that the coefficients of its series expansion has the desired decay rate ((v) is such an example). This approach is usually good for getting first results; it usually imposes unnecessarily strong conditions than needed on the original function for answering questions about solutions to our PDE problems.

One sufficient condition to establish the continuous differentiability of a function $u(x)$ via its Fourier coefficients c_n is the condition that $\sum_{n=1}^{\infty} n|c_n| < \infty$: under this condition, we would have that $\sum_{n=0}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$ converges to $u(x)$ uniformly, and

$$\left[\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)\right]' = \sum_{n=1}^{\infty} \frac{n c_n \pi}{l} \cos\left(\frac{n\pi x}{l}\right) \quad \text{for } x \in [0, l]. \tag{2.5}$$

See Lemma A.1 in the Appendix.

It follows from this discussion that, for any $g \in L^2[0, l]$, $\sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t} \sin(\frac{n\pi x}{l})$ is continuously differentiable in x for $t > 0$, for $\sum_{n=1}^{\infty} n|c_n| e^{-(\frac{n\pi}{l})^2 t} < \infty$ for $t > 0$ by the Cauchy-Schwarz* inequality

$$\begin{aligned}&\sum_{n=1}^{\infty} n|c_n|e^{-(\frac{n\pi}{l})^2 t}\\ \le&\left(\sum_{n=1}^{\infty} c_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty} n^2 e^{-2(\frac{n\pi}{l})^2 t}\right)^{1/2}\\ <&\infty\end{aligned}$$

and the Parseval relation $\sum_{n=1}^{\infty} c_n^2 = \frac{2}{\pi}\int_0^{\pi} g^2(x)\,dx < \infty$. A similar discussion shows that $\sum_{n=1}^{\infty} c_n e^{-(\frac{n\pi}{l})^2 t}\sin(\frac{n\pi x}{l})$ is infinitely continuously differentiable in x and t for $t > 0$. We will provide more details in the next section.

Exercises

Exercise 2.2.1. Using the property that if $||S_N(x) - g(x)||_{L^2[0,l]} \to 0$ as $N \to \infty$, then for any $h \in L^2[0,l]$, there holds $\int_0^l S_N(x)h(x)\,dx \to \int_0^l g(x)h(x)\,dx$ to prove that if $\sum_{n=1}^{N} c_n \sin(\frac{n\pi x}{l}) \to g(x)$ in $L^2[0,l]$, then

$$c_n = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})\,dx.$$

Exercise 2.2.2. Compute the Fourier sine expansion $\sum_{n=1}^{\infty} c_n \sin(nx)$ of $u(x) = x$ over $[0,\pi]$, and study the convergence of $\sum_{n=1}^{\infty} |c_n|$. Does the term-wise differentiation of $\sum_{n=1}^{\infty} c_n \sin(nx)$ give the Fourier cosine expansion of $u'(x) = 1$ for $0 < x < \pi$?

Exercise 2.2.3. Verify (2.2), (2.3) and (2.4).

Exercise 2.2.4. For $1 < p < \infty$, let p' be defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. One way the convexity of the function $u \mapsto |u|^p$ manifests itself is the following inequality

$$\text{For any } u, v,\ |uv| \le \frac{|u|^p}{p} + \frac{|v|^{p'}}{p'}. \tag{2.6}$$

Prove this inequality.

Exercise 2.2.5. Assume that $\sum_{n=1}^{\infty} |A_n|^p = 1$ and $\sum_{n=1}^{\infty} |B_n|^{p'} = 1$. Use (2.6) to show that $\sum_{n=1}^{\infty} |A_n B_n| \le 1$. Then assume $\sum_{n=1}^{\infty} |a_n|^p < \infty$ and $\sum_{n=1}^{\infty} |b_n|^{p'} < \infty$, and prove that $\sum_{n=1}^{\infty} |a_n b_n| \le \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}\left(\sum_{n=1}^{\infty} |b_n|^{p'}\right)^{1/p'}$. Note that the $p = 2$ case is the discrete version of the Cauchy-Schwarz inequality $\sum_{n=1}^{\infty} |a_n b_n| \le \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty} |b_n|^2\right)^{1/2}$. (Hint: Set $A_n = a_n/\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$ and $B_n = b_n/\left(\sum_{n=1}^{\infty} |b_n|^{p'}\right)^{1/p'}$.)

*We are using here the discrete version $\sum_{n=1}^{\infty} |a_n b_n| \le \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}\left(\sum_{n=1}^{\infty} |b_n|^2\right)^{1/2}$.

Exercise 2.2.6. Assume that $\int_0^l |F(x)|^p\, dx = 1$ and $\int_0^l |G(x)|^{p'}\, dx = 1$. Show that $\int_0^l |F(x)G(x)|\, dx \leq 1$. Then assume that $||f||_p := \left(\int_0^l |f(x)|^p\, dx\right)^{1p} < \infty$ and $||g||_{p'} := \left(\int_0^l |g(x)|^{p'}\, dx\right)^{1p'} < \infty$ and show that $\int_0^l |f(x)g(x)|\, dx \leq ||f||_p ||g||_{p'}$. This is called Hölder's inequality.

Exercise 2.2.7. Assume that $||u||_{L^2[0,l]}, ||v||_{L^2[0,l]} < \infty$, and show that $||u+v||_{L^2[0,l]} \leq ||u||_{L^2[0,l]} + ||v||_{L^2[0,l]}$. (Hint: Expand out $||u+v||^2_{L^2[0,l]}$ and apply the Cauchy-Schwarz inequality.)

Exercise 2.2.8. Assume that $||u||_{L^p[0,l]}, ||v||_{L^p[0,l]} < \infty$, and show that $||u+v||_{L^p[0,l]} \leq ||u||_{L^p[0,l]} + ||v||_{L^p[0,l]}$. (Hint: Use $||u+v||^p_{L^p[0,l]} \leq \int_0^l |u(x)+v(x)|^{p-1}|u(x)|\, dx + \int_0^l |u(x)+v(x)|^{p-1}|v(x)|\, dx$ and apply the Hölder inequality to each integral on the right.)

Exercise 2.2.9. Assume $\int_0^l ||u(\cdot, y)||_{L^p[0,l]}\, dy := \int_0^l \left(\int_0^l |u(x,y)|^p\, dx\right)^{1/p} dy < \infty$. Prove the integral version of the Minkowski inequality (the L^p norm of the average/integral is less than or equal to the average/integral of the L^p norm):

$$||\int_0^l |u(x,y)|\, dy||_{L^p[0,l]} \leq \int_0^l ||u(\cdot, y)||_{L^p[0,l]}\, dy.$$

(Hint: Write

$$\begin{aligned} ||\int_0^l |u(x,y)|\, dy||^p_{L^p[0,l]} &= \int_0^l \left|\int_0^l |u(x,y)|\, dy\right|^{p-1} \int_0^l |u(x,z)|\, dz\, dx \\ &= \int_0^l \left(\int_0^l \left[\left|\int_0^l |u(x,y)|\, dy\right|^{p-1} \int_0^l |u(x,z)|\right] dx\right) dz, \end{aligned}$$

and apply Hölder's inequality to $\int_0^l \left[\left|\int_0^l |u(x,y)|\, dy\right|^{p-1} \int_0^l |u(x,z)|\right] dx$.)

2.3 Fourier Series Solution of the One Dimensional Initial-Boundary Value Problem for the Heat Equation

We now apply our knowledge of Fourier series to obtain a solution of our IBVP (2.1).

Theorem 2.2. *For any $g \in L^2[0,l]$, there is a unique solution $u \in C^\infty([0,l]\times(0,\infty)) \cap C([0,\infty), L^2[0,l])$ to (2.1). Here $u \in C([0,\infty), L^2[0,l])$ means $u(\cdot,t)$ is continuous as an element of $L^2[0,l]$, namely, $||u(\cdot,t) - u(\cdot,s)||_{L^2[0,l]} \to 0$ as $t \to s$.*

Furthermore, if $g(x)$ is continuous over $[0, l]$ with $g(0) = g(l) = 0$ and $g'(x) \in L^2[0, l]$, then the solution $u(x,t) \in C([0,l] \times [0,\infty))$.

Remark 2.4. The reason we choose to work with solution in the space $C([0,\infty), L^2[0,l])$ is partly influenced by the energy estimates as in Corollary 1.2, which make $L^2[0,l])$ a natural space in which to consider solutions. As the statement of the above theorem (and its proof below) shows, construction of a solution in $C([0,\infty), L^2[0,l])$ only requires the initial data $g \in L^2[0,l]$, while construction of a solution in $C([0,l] \times [0,\infty))$ would require some additional conditions on g. The condition $g'(x) \in L^2[0,l]$ can be removed using the maximum principle to be developed later on.

Proof. The uniqueness in the class of solutions that are $C^2([0,l] \times (0,\infty)) \cap C([0,l] \times [0,\infty))$ was addressed by the energy method in Corollary 1.2. The same proof works in the class $C^{2,1}_{x,t}([0,l] \times (0,\infty)) \cap C([0,\infty), L^2[0,l])$. More advanced theory will show that any solution of (2.1) in the class $C^{2,1}_{x,t}((0,l) \times (0,\infty)) \cap C([0,l] \times [0,\infty))$ is automatically in $C^\infty([0,l] \times (0,\infty)) \cap C([0,\infty), L^2[0,l])$. Therefore uniqueness holds in the class $C^{2,1}_{x,t}((0,l) \times (0,\infty)) \cap C([0,l] \times [0,\infty))$. We will also prove the uniqueness in this class directly using the maximum principle in later sections.

For the existence and regularity part, we simply take c_n to be the Fourier coefficients of g, and construct

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}.$$

The proof for $u \in C^\infty([0,l] \times (0,\infty))$ relies on a standard property taught in advance calculus, see Lemma A.1 in the Appendix. Below is a sketch of the argument.

To prove $u \in C^\infty([0,l] \times (0,\infty))$, it suffices to prove that for any $\tau > 0$, $u \in C^\infty([0,l] \times [\tau,\infty))$. We take $a_n(x,t) = c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$ on $[0,l] \times [\tau,\infty)$.

$$\sum_{n=1}^{\infty} |\partial_x a_n(x,t)| = \sum_{n=1}^{\infty} \left| \frac{n\pi c_n}{l} \cos\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t} \right| \le \frac{\pi}{l} \sum_{n=1}^{\infty} \left| n c_n e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \right|,$$

for $t \ge \tau$, and the series on the R.H.S converges, using $\sum_{n=1}^{\infty} |c_n|^2 < \infty$ and the exponentially fast decay of $e^{-\left(\frac{n\pi}{l}\right)^2 \tau}$ in n:

$$\sum_{n=1}^{\infty} \left| n c_n e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \right| \le \left(\sum_{n=1}^{\infty} |c_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=1}^{\infty} n^2 e^{-2\left(\frac{n\pi}{l}\right)^2 \tau} \right)^{\frac{1}{2}} < \infty.$$

So $\sum_{n=1}^{\infty} \frac{n\pi c_n}{l} \cos\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$ converges uniformly over $(x,t) \in [0,l] \times [\tau,\infty)$, and by the Lemma A.1, $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$ is continuously differentiable in x for (x,t)

over $[0,l]\times[\tau,\infty)$. A similar argument shows that it is continuously differentiable in t for (x,t) over $[0,l]\times[\tau,\infty)$. Then an induction argument, using the exponentially fast decay of $e^{-\left(\frac{n\pi}{l}\right)^2\tau}$ in n shows that $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$ is infinitely continuously differentiable in both x and t for (x,t) over $[0,l]\times[\tau,\infty)$. Since $\tau>0$ is arbitrary, we conclude that $u\in C^{\infty}([0,l]\times(0,\infty))$.

Then, the remaining issue is the continuity of $u(\cdot,t)$ in $L^2[0,l]$, which follows as a consequence of the Parseval identity for the Fourier coefficients: for any $t,t'\geq 0$,

$$||u(\cdot,t)-u(\cdot,t')||_{L^2[0,l]}^2=\frac{2}{l}\sum_{n=1}^{\infty} c_n^2\left|e^{-\left(\frac{n\pi}{l}\right)^2 t}-e^{-\left(\frac{n\pi}{l}\right)^2 t'}\right|^2.$$

Now for any given $\epsilon>0$, we can find N such that $\sum_{n=N+1}^{\infty} c_n^2<\frac{\epsilon}{8}$, which then leads to

$$\sum_{n=N+1}^{\infty} c_n^2\left|e^{-\left(\frac{n\pi}{l}\right)^2 t}-e^{-\left(\frac{n\pi}{l}\right)^2 t'}\right|^2\leq 4\sum_{n=N+1}^{\infty} c_n^2\leq\frac{\epsilon}{2}\ \text{ for } t,t'\geq 0;$$

on the other hand, there exists $\delta>0$ such that, when $|t-t'|<\delta$,

$$\sum_{n=1}^{N} c_n^2\left|e^{-\left(\frac{n\pi}{l}\right)^2 t}-e^{-\left(\frac{n\pi}{l}\right)^2 t'}\right|^2\leq\frac{\epsilon}{2},$$

which proves that $u(\cdot,t)$ is (uniformly) continuous in $L^2[0,l]$. (Those familiar with Lebesgue's integral should recognize a proof using Lebesgue's Dominated Convergence Theorem.)

When g satisfies that $g(x)$ is continuous over $[0,l]$ with $g(0)=g(l)=0$ and $g'(x)\in L^2[0,l]$,

$$c_n=\frac{2}{l}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx=\frac{2}{n\pi}\int_0^l g'(x)\cos\left(\frac{n\pi x}{l}\right)dx.$$

The Fourier coefficients of $g'(x)$ is l^2- summable by Parseval relation. So by the Cauchy-Schwarz inequality

$$\sum_{n=1}^{\infty}|c_n|\leq\frac{2}{\pi}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left[\int_0^l g'(x)\cos\left(\frac{n\pi x}{l}\right)dx\right]^2\right)^{1/2}<\infty.$$

Therefore, $\sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi x}{l}\right)e^{-\left(\frac{n\pi}{l}\right)^2 t}$ converges to $u(x,t)$ uniformly over $(x,t)\in[0,l]\times[0,\infty)$, and $u(x,t)$ is continuous there. A deeper analysis using the heat kernel or other tools—to be introduced later—can give the same result but remove the condition that $g'(x)\in L^2[0,l]$.

$\square$

Note that the solution $u(x,t)$ constructed above decays in t exponentially, and the rate of exponential decay is determined by the first term $e^{-\left(\frac{\pi}{l}\right)^2 t}$ (when $c_1 \neq 0$), as

$$u(x,t) = e^{-\left(\frac{\pi}{l}\right)^2 t}\left\{c_1 \sin\left(\frac{\pi x}{l}\right) + \sum_{n=2}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{(n^2-1)\pi^2}{l^2}t}\right\}.$$

and $\sum_{n=2}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{(n^2-1)\pi^2}{l^2}t}$ decays exponentially in t, and can be considered as a perturbation of $c_1 \sin\left(\frac{\pi x}{l}\right)$.

Note also that, unless further assumptions on g as given in the second part of the above Theorem are made, the solution $u(x,t)$ may not be continuous at $(0,0)$ and $(l,0)$. At issue here is the compatibility of the given initial and boundary data at these two corner points: the boundary conditions $u(0,t) = u(l,t) = 0$ for $t > 0$ dictates that $u(0,0) = u(l,0) = 0$, if u is continuous on the closure of the region, it would require $g(0) = u(0,0) = 0$, and $g(l) = u(l,0) = 0$; but the conclusions in the first paragraph of the above theorem does not require any such assumption.

Remark 2.5. The assumption that $g' \in L^2[0,l]$ in the second part of the above Theorem can be removed, *i.e.*, the second part of the above Theorem can be established under only the condition that $g \in C[0,l]$ and $g(0) = g(l) = 0$. Recall that for any point $x_0 \in [0,l]$, there exists a continuous function g on $[0,l]$ whose Fourier (sine) series $\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x_0}{l})$ does not converge to $g(x_0)$. We will prove later that even though the Fourier (sine) series $\sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l})$ of g may not converge point wise to $g(x)$ for $x \in [0,l]$, $\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t} \in C([0,l]\times[0,\infty))$, and $\to g(z) = u(z,0)$, as $(x,t) \to (z,0)$ with $t > 0$, as long as $g \in C[0,l]$ and $g(0) = g(l) = 0$.

One ingredient is to examine the convergence of a sequence of solutions $u^{(N)}(x,t) = \sum_{n=1}^{N} c^N(n) \sin(\frac{n\pi x}{l}) e^{-(\frac{n\pi}{l})^2 t}$, where $c^N(n)$ are chosen such that $\sum_{n=1}^{N} c^N(n) \sin(\frac{n\pi x}{l}) \to g(x)$ (uniformly) as $N \to \infty$ (Look up Fejér Theorem). Other ingredients include the Maximum Principle and gradient estimates for solutions to (2.1).

A general theme in our course is to understand the question: **if an IBVP or IVP has solutions for a sequence of data which is converging in an appropriate norm, would it imply that the sequence of solutions converge in an appropriate norm, or at least a subsequence would converge?**

Answers to such questions often amount to getting estimates of solutions in terms of data in appropriate norms. In the context of (2.1), if we are interested in establishing solvability of (2.1) for initial data in $X = \{g \in C[0,l] : g(0) = g(l) = 0\}$, the following three components would suffice:

(a). Establish solvability for a *dense* set of data in X;

(b). Prove that for any $T > 0$, there exists $C > 0$, which may depend on T, such that for any solution $u(x,t)$ to (1.9) (with $u(\cdot,0) \in X$)), we have

$$\max\{|u(x,t)| : (x,t) \in [0,l] \times [0,T]\} \le C \max\{|u(x,0)| : x \in [0,l]\}. \tag{2.7}$$

(c). Prove that for any $T > \tau > 0$, there exists $C' > 0$, which may depend on T and τ, such that for any solution $u(x,t)$ to (2.1), we have

$$\begin{aligned} &\max\{|u(x,t)|, |u_t(x,t)|, |u_x(x,t)|, |u_{xx}(x,t)| : (x,t) \in [0,l] \times [\tau,T]\} \\ \le &C' \max\{|u(x,0)| : x \in [0,l]\}. \end{aligned} \tag{2.8}$$

For, then, if given $g \in X$, we can find a sequence of solutions $u_k(x,t)$ of (2.1), with $u_k(x,0) \in X$ and $u_k(x,0) \to g$ (uniformly) in X , and if we apply (2.7) to $u_k(x,t) - u_l(x,t)$, we would conclude that $\{u_k(x,t)\}$ is a Cauchy sequence in $C([0,l] \times [0,T])$, therefore converges to a function u in $C([0,l] \times [0,T])$; furthermore, (2.8) applied to $u_k(x,t) - u_l(x,t)$ would imply that, on any subregion $[0,l] \times [\tau,T]$, $\{u_t(x,t)\}$, $\{u_x(x,t)\}$, and $\{u_{xx}(x,t)\}$ are also Cauchy in $C([0,l] \times [\tau,T])$, therefore each has a limit in this space; this then implies that the limit $u(x,t) \in C^{2,1}_{x,t}([0,l] \times [\tau,T])$; and, since $0 < \tau < T$ is arbitrary, we conclude that u is a solution of (2.1) in $C^{2,1}_{x,t}([0,l] \times (0,T]) \cap C([0,l] \times [0,T])$, with $u(x,0) = g(x)$.

We may not need to put in the restrictions on T or τ for the estimates in other context. In fact, we could allow $T = \infty$ in this context; but the restriction on $[0,l] \times [\tau,T]$ is needed for (2.8) in this context, as the right hand side of (2.8) is in terms of $\max\{|u(x,0)| : x \in [0,l]\}$, and there is no way that we can control derivatives of $u(x,t)$ in $[0,l] \times [0,T]$ directly in terms of $\max\{|u(x,0)| : x \in [0,l]\}$.

It actually suffices to establish (2.7) and (2.8) for the set of solutions which correspond to data in the dense subset referred to in (a) (Think through this!). Thus it suffices to establish (2.7) and (2.8) for all *finite* series solutions $u(x,t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\left(\frac{n\pi}{l}\right)^2 t}$. (2.7) will be established later by the Maximum Principle, in fact with $C = 1$.

Finally, we show how to establish one of the estimates in (2.8) for solutions represented in finite series (the argument actually works for solutions in infinite series)— **our discussion above already shows the usefulness of (a)–(c); we are deriving them here using explicit representation for these sample solutions, but will develop methods to derive such estimates even when no explicit representation of solutions is readily available**.

We will estimate $\max\{|u_t(x,t)| : (x,t) \in [0,l] \times [\tau, T]\}$ in terms of $\max\{|u(x,0)| : x \in [0,l]\}$. For $\tau \leq t \leq T$,

$$\begin{aligned}
|u_t(x,t)| \leq & \left(\frac{\pi}{l}\right)^2 \sum_{n=1}^{\infty} |c_n| n^2 e^{-\left(\frac{n\pi}{l}\right)^2 \tau} \\
\leq & \left(\frac{\pi}{l}\right)^2 \left(\sum_{n=1}^{\infty} |c_n|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} n^4 e^{-\left(\frac{n\pi}{l}\right)^2 2\tau}\right)^{1/2} \\
= & C' ||u(\cdot, 0)||_{L^2[0,l]} \\
\leq & C' \sqrt{l} \max\{|u(x,0)| : x \in [0,l]\},
\end{aligned}$$

where $C' = \left(\frac{\pi}{l}\right)^2 \left(\sum_{n=1}^{\infty} n^4 e^{-\left(\frac{n\pi}{l}\right)^2 2\tau}\right)^{1/2}$. One can see that this C' tends to ∞ if we allow $\tau \to 0$.

Question: Can the method be adapted to solutions of the modified equation such as $u_t - u_{xx} + bu_x + cu = 0$ to draw the same or similar conclusions? Which parts of the argument can be adapted to construct a solution of a higher dimensional problem such as the following?

$$\begin{cases}
\partial_t^2 u(\mathbf{x}, t) - c^2 \Delta u(\mathbf{x}, t) = 0, & \mathbf{x} \in D, t > 0, \\
u(\mathbf{x}, t) = 0, & \mathbf{x} \in \partial D, t > 0, \\
u(\mathbf{x}, 0) = g(\mathbf{x}), & \mathbf{x} \in D.
\end{cases}$$

What additional information is needed to implement this approach? Try out the case when D is a simple domain such as a square or a round disc in $\mathbb{R}^2$.

Exercises

Exercise 2.3.1. Suppose u solves $u_t - u_{xx} = 4u$ on the interval $(0, \pi)$, with the homogeneous Neumann condition $u_x(x,t) = 0$ at $x = 0, \pi$. Characterize the initial data $u_0(x) = u(x,0)$ for which $u(x,t)$ stays bounded as $t \to \infty$; then characterize $\lim_{t\to\infty} u(x,t)$ for such solutions.

Exercise 2.3.2. Prove that for any $g \in L^2[0,l]$ there is a solution $u \in C^\infty([0,l] \times (0,\infty)) \cap C([0,\infty), L^2[0,l])$ to the homogeneous heat equation $u_t(x,t) - u_{xx}(x,t) = 0$ over $(0,l) \times (0,\infty)$ satisfying the Neumann boundary condition $u_x(0,t) = u_x(l,t) = 0$ for all $t > 0$, and the initial data $u(x,0) = g(x)$ in the L^2 sense. If, furthermore, $g(x)$ is differentiable over $[0,l]$ and $g'(x) \in L^2[0,l]$, then $u \in C([0,l] \times [0,\infty))$. (However, $u_x(x,t)$ may not be continuous over $[0,l] \times [0,\infty)$); formulate a sufficient condition which would guarantee that $u_x(x,t)$ is continuous over $[0,l] \times [0,\infty)$.)

Exercise 2.3.3. This exercise considers the regularity behavior near $t = 0$ of the solution

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 t} \sin(nx) \tag{2.9}$$

to (2.1) with $l = \pi$, $c_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx$. Recall that the function $u(x,t)$ defined by (2.9) is C^∞ in $(x,t) \in [0,\pi] \times (0,\infty)$.

(i). Assume now that $g(x)$ has bounded derivative, with $|g'(x)| \le M$ for some constant M. Assume further that g satisfies the boundary condition, i.e. $g(0) = g(\pi) = 0$.

- (a). Prove that $\sum_{n=1}^{\infty} |c_n| < \infty$.
- (b). Show that as t decreases to 0, the function $u(x,t)$ defined by (2.9) converges uniformly to $g(x)$.

(ii). Suppose now that, in addition to the assumptions in (i), $g(x)$ has three bounded derivatives ($|g'''(x)| \le M$), and $g''(0) = g''(\pi) = 0$.

- (a). Show that as t decreases to 0, $u_{xx}(x,t)$ converges uniformly to $g''(x)$.
- (b). If g is smooth in $[0,\pi]$ but $g''(0) \neq 0$ or $g''(\pi) \neq 0$, is it possible that the conclusion of part (a) still holds?

(iii). The solution formula (2.9) makes sense even when g doesn't vanish at the end-points; assume below that $g(x) \equiv 1$ over $(0,\pi)$.

- (a). Does $u(x,t)$ satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$ for $t > 0$?
- (b). Discuss the sense in which $u(x,t)$ approaches g as $t \searrow 0$ in this case.
- (c). Verify that $t \mapsto \int_0^\pi |u_x(x,t)|^2\,dx \to \infty$ as $t \searrow 0$, even though $g'(x) = 0$ over $(0,\pi)$.

(iv). Verify that, for any $g \in L^2[0,l]$, the corresponding solution $u(x,t)$ satisfies that $t \mapsto \int_0^\pi |u(x,t)|^2\,dx$ is in $C[0,\infty)$, and that $t \mapsto \int_0^\pi |u_x(x,t)|^2\,dx$ is in $L^1[0,\infty)$; in fact,

$$\int_0^\infty \int_0^\pi |u_x(x,t)|^2\,dxdt \le \int_0^\pi |g(x)|^2\,dx.$$

Exercise 2.3.4. Does the separation of variables method, as presented, work in constructing solutions to the following IBVP?

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + bu_x(x,t) = 0, \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = g(x), \quad 0 < x < \pi, \end{cases}$$

where b is some non-zero constant. HINT: If you are having some difficulty directly adapting the separation of variables method to this setting, you may need to consider a change of variables of the form $v(x,t) = e^{\delta x}u(x,t)$ to obtain a problem for $v(x,t)$ to which the separation of variables method applies readily.

2.4 Fourier's Method Applied to a Cauchy Problem of the One Dimensional Wave Equation

We now apply Fourier's method to find solutions to

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad \text{in} \quad \mathbb{R} \times [0,\infty), \\ u(x,0) = g(x), \\ u_t(x,0) = h(x). \end{cases} \tag{2.10}$$

We first look for solutions to the first equation in (2.10) of the form $e^{ix\xi}T(t)$. This leads to $T''(t) + c^2|\xi|^2 T(t) = 0$. So $T(t) = A\cos(c\xi t) + B\sin(c\xi t)$ for some constants A and B. It is often more convenient to use the solutions in their complex form $T(t) = Ae^{ic\xi t} + Be^{-ic\xi t}$. Thus for each parameter ξ, we have the solution

$$\left[Ae^{ic\xi t} + Be^{-ic\xi t}\right] e^{ix\xi} = Ae^{i\xi(x+ct)} + Be^{i\xi(x-ct)}.$$

$Ae^{i\xi(x+ct)}$ represents a traveling wave which remains a constant along each line $x+ct =$ a constant, and its form remains unchanged, moving to the left at a speed of c; while $Be^{i\xi(x-ct)}$ represents a traveling wave which remains a constant along each line $x-ct =$ a constant, and its form remains unchanged, moving to the right at a speed of c. This structure remains when we superpose such solutions:

$$\int \left[A(\xi)e^{i\xi(x+ct)} + B(\xi)e^{i\xi(x-ct)}\right] d\xi = \int A(\xi)e^{i\xi(x+ct)}d\xi + \int B(\xi)e^{i\xi(x-ct)}d\xi.$$

Denote the first term as $G(x+ct)$ and the second term as $H(x-ct)$. We are led to solutions of the form $G(x+ct) + H(x-ct)$. It is now routine to check that for any $G, H \in C^2(\mathbb{R})$, both $G(x+ct)$ and $H(x-ct)$ are solutions to the first equation in (2.10), thus so is $G(x+ct) + H(x-ct)$.

We abandon (for now) our initial attempt to construct solutions to (2.10) by superposing $e^{i\xi(x\pm ct)}$, but explore whether we can choose G and H so that $u(x,t) = G(x+ct) + H(x-ct)$ satisfies the initial conditions in (2.10). We need

$$\begin{cases} G(x) + H(x) = g(x), \\ cG'(x) - cH'(x) = h(x). \end{cases}$$

From the second equation, we find $G(x) - H(x) = c^{-1}\int_0^x h(\xi)d\xi + d$ for some constant d. Combining with the first equation, we find

$$\begin{cases} G(x) = \dfrac{g(x)+d}{2} + \dfrac{1}{2c}\displaystyle\int_0^x h(\xi)d\xi, \\ H(x) = \dfrac{g(x)-d}{2} - \dfrac{1}{2c}\displaystyle\int_0^x h(\xi)d\xi. \end{cases}$$

Thus

$$\begin{aligned} u(x,t) &= G(x+ct) + H(x-ct) \\ &= \frac{g(x+ct)+d}{2} + \frac{1}{2c}\int_0^{x+ct} h(\xi)d\xi + \frac{g(x-ct)-d}{2} - \frac{1}{2c}\int_0^{x-ct} h(\xi)d\xi \\ &= \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy. \end{aligned}$$

This formula is called **d'Alembert's formula** (discovered in the mid-1700's).

Theorem 2.3. *For any $g \in C^2(\mathbb{R})$, and $h \in C^1(\mathbb{R})$, there is a unique solution $u(x,t) \in C^2(\mathbb{R} \times [0,\infty))$ solving* (2.10)

Remark 2.6. (i). The uniqueness is not addressed by the method here, and will be taken up in the following sections.

(ii). The d'Alembert's formula exhibits that the solution at (x,t) depends on its initial data only in the range $[x-ct, x+ct]$, which is called the interval of dependence of (x,t) on $t=0$. Viewed from a different perspective, this means that signals travel with a finite speed, which in this case is c. Recall that the solution of the heat equation as provided by (2.18) shows that the value of $u(x,t)$ is influenced by its initial value at every location, in other words, there is an infinite speed of propagation for the heat equation.

(iii). The derivation process here showed that the general solution of the homogeneous one dimensional wave equation is $G(x-ct) + H(x+ct)$ for some C^2 functions G and H. This exhibits the wave character of the solutions. In fact, even if G

or H are not necessarily C^2, but are only continuous, with piecewise continuous derivatives, $G(x-ct)+H(x+ct)$ should still be regarded as a solution of the wave equation, although it may not satisfy its differential form (1.10) everywhere. It still satisfies (1.10) at (x,t) whenever G is C^2 at $x-ct$ and H is C^2 at $x+ct$, and satisfies the integral form (1.11) whenever G' and H' are continuous at (x_1,t) and (x_2,t).

In fact, there is another integral formulation of (1.10) which dispenses with any explicit reference to the derivative of the solution. We will say more about this at the end of this section.

(iv). For the Cauchy problem to the wave equation in higher dimension, we can still easily find solutions of the form $\left[Ae^{ic|\xi|t}+Be^{-ic|\xi|t}\right]e^{ix\cdot\xi}$, and use them to construct the general solution of the wave equation, but it is not as elementary to obtain as simple a representation in terms of the initial data. The phenomenon of finite speed of propagation is still valid for higher dimensional wave equation, but it will take additional efforts to show that using the Fourier representation. For each $\xi\neq 0\in\mathbb{R}^n$, $e^{i(x\cdot\xi\pm c|\xi|t)}$ remains a constant along the planes $x\cdot\xi\pm c|\xi|t=a$, which are perpendicular to ξ and are moving at a speed of c. These are called plane wave solutions. One new ingredient in higher dimensions is that we now have a continuum parameter family of directions in which the plane wave solutions move.

(v). If (1.10) is modified slightly into $u_{tt}-c^2u_{xx}+\alpha u=0$, then the same approach can easily give the formal solution $\int_{\mathbb{R}}\left[A(\xi)e^{i(\xi x+\sqrt{c^2\xi^2+\alpha}t)}+B(\xi)e^{i(\xi x-\sqrt{c^2\xi^2+\alpha}t)}\right]d\xi$, but it would take additional effort to find a more explicit representation of such a solution in terms of the initial data. Note that for each parameter ξ, $e^{i(\xi x\pm\sqrt{c^2\xi^2+\alpha}t)}$ still represents a finite speed plane wave, but its wave speed, $\pm\sqrt{c^2\xi^2+\alpha}/\xi$ for $\xi\neq 0$, depends on ξ now.

Question. Is there a difference of behavior between the cases of $\alpha>0$ and $\alpha<0$?

Exercises

Exercise 2.4.1. Using separation of variables (or Fourier transforms) to verify that a (formal) solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} + \alpha u = 0, \quad \text{in} \quad \mathbb{R} \times [0, \infty), \\ u(x,0) = g(x), \\ u_t(x,0) = h(x), \end{cases}$$

with $\alpha \geq 0$, takes the form of

$$u(x,t) = \int_{\mathbb{R}} \left[\widehat{g}(\xi) \cos(\sqrt{c^2\xi^2 + \alpha}t) + \widehat{h}(\xi) \frac{\sin(\sqrt{c^2\xi^2 + \alpha}t)}{\sqrt{c^2\xi^2 + \alpha}} \right] e^{i\xi x} d\xi,$$

where $\widehat{g}(\xi)$ and $\widehat{h}(\xi)$ are the Fourier transforms of g and h, respectively, defined through $g(x) = \int_{\mathbb{R}} \widehat{g}(\xi) e^{ix\xi} d\xi$, and $h(x) = \int_{\mathbb{R}} \widehat{h}(\xi) e^{ix\xi} d\xi$. Note that the property of finite speed of propagation of solutions to (2.10) (the $\alpha = 0$ case) can't be read off directly from this representation.

2.5 Fourier Series Solution of the One Dimensional Initial-Boundary Value Problem for the Wave Equation

When we apply separation of variables method to the one dimensional initial-boundary value problem for the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & \text{on } (x,t) \in [0,l] \times \mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l], \end{cases} \tag{2.11}$$

we find that a separable solution of the form $X(x)T(t)$ to the first two equations leads to the same Sturm-Liouville eigenvalue problem (1.23) for $X(x)$, from which we know that for each $n \in \mathbb{N}$ we can take $X(x) = \sin(\frac{n\pi}{l}x)$ and the accompanying $T(t)$ must solve $T''(t) + (\frac{cn\pi}{l})^2 T(t) = 0$. This leads us to separable solutions of the form $\left[A\cos(\frac{cn\pi}{l}t) + B\sin(\frac{cn\pi}{l}t)\right]\sin(\frac{n\pi}{l}x)$, which satisfy the first two sets of equations in (2.11). Thus we can look for solutions to the initial-boundary value problem (2.11) in the form

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos(\frac{cn\pi}{l}t) + B_n \sin(\frac{cn\pi}{l}t) \right] \sin(\frac{n\pi}{l}x).$$

The A_n, B_n are chosen based on the formal relations

$$g(x) = u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi}{l}x),$$

and

$$h(x) = u_t(x,0) = \sum_{n=1}^{\infty} \frac{cn\pi}{l} B_n \sin(\frac{n\pi}{l}x).$$

Thus

$$A_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi}{l}x) g(x)\, dx,$$

and

$$\frac{cn\pi}{l} B_n = \frac{2}{l}\int_0^l \sin(\frac{n\pi}{l}x) h(x)\, dx.$$

Unlike the case for the heat equation, where the constructed solution series decays in n exponentially for $t > 0$, the series solution for (2.11) does not have exponential decay in n, unless the initial data $g(x)$ and $h(x)$ make A_n and B_n decay exponentially in n, or at least fast enough to make the series converges. The solution does not show decay in t either. This is consistent with conservation of energy to be discussed later. The wave character of the solution can be seen by the trigonometric identities:

$$\cos\left(\frac{cn\pi}{l}t\right)\sin\left(\frac{n\pi}{l}x\right) = \frac{1}{2}\left[\sin\left(\frac{n\pi}{l}(x-ct)\right) + \sin\left(\frac{n\pi}{l}(x+ct)\right)\right],$$
$$\sin\left(\frac{cn\pi}{l}t\right)\sin\left(\frac{n\pi}{l}x\right) = \frac{1}{2}\left[\cos\left(\frac{n\pi}{l}(x-ct)\right) - \cos\left(\frac{n\pi}{l}(x+ct)\right)\right].$$

The terms $\sin\left(\frac{n\pi}{l}(x-ct)\right)$ and $\cos\left(\frac{n\pi}{l}(x-ct)\right)$ represent waves moving to the right with a speed of c, while the terms $\sin\left(\frac{n\pi}{l}(x+ct)\right)$ and $\cos\left(\frac{n\pi}{l}(x+ct)\right)$ represent waves moving to the left with a speed of c. The above shows that certain superposition of two traveling waves gives rise to a standing wave, as a result, the constructed solution is really the superposition of two traveling waves with velocity $\pm c$.

The following theorem provides some sufficient conditions to make sense of the (uniform) convergence of the series solution; as we will see in a later section, there is a more robust method to prove the convergence of the series solution in an L^2 sense.

Theorem 2.4. *Suppose that $g \in C^3[0,l]$ with $g(0) = g(l) = 0$ and $g''(0) = g''(l) = 0$, and that $h \in C^2[0,l]$ with $h(0) = h(l) = 0$. Then the series solution $u(x,t)$ constructed above is $C^2([0,l]\times[0,\infty))$ and satisfies* (2.11).

Proof. We will use Lemma (A.1) in the Appendix to prove that the series solution $u(x,t)$ constructed above is $C^2([0,l]\times[0,\infty))$ and

$$u_{tt}(x,t) = -\sum_{n=1}^{\infty}\left(\frac{cn\pi}{l}\right)^2\left[A_n\cos(\frac{cn\pi}{l}t) + B_n\sin(\frac{cn\pi}{l}t)\right]\sin(\frac{n\pi}{l}x),$$

and

$$u_{xx}(x,t) = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^2 \left[A_n \cos(\frac{cn\pi}{l}t) + B_n \sin(\frac{cn\pi}{l}t)\right] \sin(\frac{n\pi}{l}x),$$

therefore $u(x,t)$ satisfies the homogeneous equation in (2.11).

The proof also shows that $u(0,t)$, $u(l,t)$, $u(x,0)$ and $u_t(x,0)$ can be evaluated by the series, and therefore satisfies the remaining equations in (2.11).

According to Lemma (A.1) in the Appendix, it suffices to check that the series on the right hand sides of the equations for $u_{tt}(x,t)$ and $u_{xx}(x,t)$ are uniformly and absolutely convergent on $[0,l]\times[0,T]$ for any $T>0$, and for that it suffices to check that $\sum_{n=1}^{\infty} n^2(|A_n|+|B_n|) < \infty$.

Using the regularity and boundary conditions on g and h, we can do integration by parts as follows to find

$$\begin{aligned}
A_n &= \frac{2}{l}\int_0^l \sin(\frac{n\pi}{l}x) g(x)\,dx \\
&= \frac{2}{n\pi}\left[-g(x)\cos(\frac{n\pi}{l}x)\Big|_{x=0}^{l} + \int_0^l g'(x)\cos(\frac{n\pi}{l}x)\,dx\right] \\
&= \frac{2l}{(n\pi)^2}\left[g'(x)\sin(\frac{n\pi}{l}x)\Big|_{x=0}^{l} - \int_0^l g''(x)\sin(\frac{n\pi}{l}x)\,dx\right] \\
&= \frac{2l^2}{(n\pi)^3}\left[g''(x)\cos(\frac{n\pi}{l}x)\Big|_{x=0}^{l} + \int_0^l g'''(x)\cos(\frac{n\pi}{l}x)\,dx\right] \\
&= \frac{2l^2}{(n\pi)^3}\left[\int_0^l g'''(x)\cos(\frac{n\pi}{l}x)\,dx\right].
\end{aligned}$$

It now follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2|A_n| \le& \sum_{n=1}^{\infty} \frac{2l^2}{n\pi^3}\left|\int_0^l g'''(x)\cos(\frac{n\pi}{l}x)\,dx\right| \\
\le& \frac{2l^2}{\pi^3}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left[\int_0^l g'''(x)\cos(\frac{n\pi}{l}x)\,dx\right]^2\right)^{1/2} < \infty,
\end{aligned}$$

here we used $\sum_{n=1}^{\infty}\left[\int_0^l g'''(x)\cos(\frac{n\pi}{l}x)\,dx\right]^2 = \frac{2}{l}\int_0^l |g'''(x)|^2\,dx < \infty$ by Parseval identity.

Similarly,

$$\begin{aligned}
B_n &= \frac{2}{cn\pi}\int_0^l \sin(\frac{n\pi}{l}x)h(x)\,dx \\
&= \frac{2l}{c(n\pi)^2}\left[-h(x)\cos(\frac{n\pi}{l}x)\Big|_0^l + \int_0^l \cos(\frac{n\pi}{l}x)h'(x)\,dx\right] \\
&= \frac{2l^2}{c(n\pi)^3}\left[h'(x)\sin(\frac{n\pi}{l}x)\Big|_0^l - \int_0^l \sin(\frac{n\pi}{l}x)h''(x)\,dx\right] \\
&= -\frac{2l^2}{c(n\pi)^3}\int_0^l \sin(\frac{n\pi}{l}x)h''(x)\,dx,
\end{aligned}$$

so we also have

$$\begin{aligned}
\sum_{n=1}^{\infty} n^2|B_n| \le& \sum_{n=1}^{\infty}\frac{2l^2}{cn\pi^3}\left|\int_0^l \sin(\frac{n\pi}{l}x)h''(x)\,dx\right| \\
\le& \frac{2l^2}{\pi^3}\left(\sum_{n=1}^{\infty}\frac{1}{(cn)^2}\right)^{1/2}\left(\sum_{n=1}^{\infty}\left|\int_0^l \sin(\frac{n\pi}{l}x)h''(x)\,dx\right|^2\right)^{1/2} < \infty.
\end{aligned}$$

□

Remark 2.7. In the above theorem, we look for a solution in $C^2([0,l]\times[0,\infty))$, and expect the first equation in (2.11) to also hold on the boundary of $[0,l]\times[0,\infty)$. Along $x=0$ or l, since we expect $u(0 \text{ or } l,t)=0$ for all $t>0$, this leads to $u_{tt}(0 \text{ or } l,t)=0$. Thus, applying the equation at $(0,0)$ and $(l,0)$ to a solution in $C^2([0,l]\times[0,\infty))$, we expect $u_{xx}(0,0)=g''(0)=0$, and $u_{xx}(l,0)=g''(l)=0$. The other conditions, $g(0)=g(l)=0$ and $h(0)=h(l)=0$ come from similar consideration. Note that the differentiability assumptions on the initial data are one order higher than the differentiability of the solution. This is an artifact of the method. We will discuss how to address this issue shortly later in this section.

Note also that if the compatibility conditions, $g(0)=g(l)=0$, $h(0)=h(l)=0$, or $g''(0)=g''(l)=0$ do not hold, then the series will not produce a function in $C^2([0,l]\times[0,\infty))$ or one in $C^2((0,l)\times(0,\infty))$, and that any discontinuities of $u(x,t)$ or its first or second derivatives at the corner points $(0,0)$ or $(l,0)$ would be propagated along $x-ct=0$, $x+ct=l$, and their iterated reflections in the vertical sides of the rectangle $\{(x,t): 0\le x\le l, t\ge 0\}$. This is a feature of the wave equation: any singularity of the initial or boundary data gets propagated into the interior region where the homogeneous wave equation holds—this is in contrast with the behavior of solutions to the heat equation. Such non-$C^2((0,l)\times(0,\infty))$ solutions should be regarded as generalized solutions to the wave equation.

In fact, the above features can be seen as follows.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} \left[A_n \cos(\frac{cn\pi}{l}t) + B_n \sin(\frac{cn\pi}{l}t)\right] \sin(\frac{n\pi}{l}x) \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \left\{ A_n \left[\sin\left(\frac{n\pi}{l}(x-ct)\right) + \sin\left(\frac{n\pi}{l}(x+ct)\right)\right] \right. \\ &\qquad \left. + B_n \left[\cos\left(\frac{n\pi}{l}(x-ct)\right) - \cos\left(\frac{n\pi}{l}(x+ct)\right)\right]\right\} \\ &= \frac{1}{2}\left\{\tilde{g}(x-ct) + \tilde{g}(x+ct) + k(x-ct) - k(x+ct)\right\}, \end{aligned}$$

where

$$\begin{aligned} \tilde{g}(y) &= \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}y\right), \\ k(y) &= \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi}{l}y\right). \end{aligned}$$

We recognize that $\tilde{g}(y)$ is the Fourier sine series expansion of g over $(0,l)$, therefore, represents the $2l$-periodic extension of the odd extension of g to $(-l,l)$. Since $h(x) = u_t(x,0) = \sum_{n=1}^{\infty}(\frac{cn\pi}{l})B_n \sin(\frac{n\pi}{l}x)$, for $0 < x < l$, we see that

$$\begin{aligned} \int_0^y h(z)\,dz &= \int_0^y \left(\sum_{n=1}^{\infty}(\frac{cn\pi}{l})B_n \sin(\frac{n\pi}{l}z)\right) dz \\ &= \sum_{n=1}^{\infty}\left(\frac{cn\pi}{l}\right) B_n \int_0^y \sin(\frac{n\pi}{l}z)dz \\ &= -c\sum_{n=1}^{\infty} B_n \left(\cos(\frac{n\pi}{l}y) - 1\right) \\ &= -c\left[k(y) - k(0)\right], \end{aligned}$$

for $0 < y < l$. If we define $\tilde{h}(z) = \sum_{n=1}^{\infty}\left(\frac{2}{l}\int_0^l \sin(\frac{n\pi}{l}x)h(x)\,dx\right)\sin(\frac{n\pi}{l}z)$ as the Fourier sine series expansion of h over $(0,l)$, which represents the $2l$-periodic extension of the odd extension of h to $(-l,l)$, then $k(y) = k(0) - c^{-1}\int_0^y \tilde{h}(z)\,dz$ for all y. Thus, we find

$$u(x,t) = \frac{1}{2}\left[\tilde{g}(x-ct) + \tilde{g}(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(z)\,dz,$$

which is the same d'Alembert solution to an IVP for (1.10). From this representation, it's clear that if $\tilde{g}(y) \in C^2(\mathbb{R})$ and $\tilde{h}(y) \in C^1(\mathbb{R})$, we would get a solution which is in $C^2(\mathbb{R} \times [0, \infty))$. But these conditions on $\tilde{g}(y)$ and $\tilde{h}(y)$ are equivalent to the compatibility conditions at the corner points on g and h respectively.

Note that this approach has lowered the differentiability assumptions on g and h. On the other hand, if one or more of the compatibility conditions fails at a corner point, then $\tilde{g}(y)$ or $\tilde{h}(y)$, or both, would lose some differentiability at the corresponding corner points, and that behavior is propagated along $x \pm ct = 0$, or l, or ml for $m \in \mathbb{Z}$—when examining the behavior of the solution in $(0, l) \times [0, \infty)$, these lines look like the reflections of $x \pm ct = 0$, or l at the vertical lines $x = 0$ or l.

In contrast, for the IBVP of the heat equation (2.1), even if its initial data g has discontinuities, they *are not* propagated into the interior region where the homogeneous heat equation holds. In other words, the (homogeneous) heat equation smoothes out any discontinuities of its initial (and boundary) data. Of course, if the coefficients in a diffusion equation have discontinuities, such as in **Exercise** 1.5.2, its solution may not be infinitely times differentiable. In the case of **Exercise** 1.5.2, the solutions there are continuous with a jump discontinuity of their first derivative at $x = 0$, where the diffusion coefficient has a jump discontinuity.

We now summarize our above discussion on the improved Theorem 2.4 as

Theorem 2.5. *Suppose that $g \in C^2[0, l]$ with $g(0) = g(l) = 0$ and $g''(0) = g''(l) = 0$, and that $h \in C^1[0, l]$ with $h(0) = h(l) = 0$. Then* (2.11) *has a solution in $C^2([0, l] \times [0, \infty))$; in fact, the series solution $u(x, t)$ constructed for Theorem 2.4 is $C^2([0, l] \times [0, \infty))$ and satisfies* (2.11).

Proof. Under the assumptions on $g(x)$ and $h(x)$ here, we can first extend $g(x)$ to be an odd function on $[-l, l]$, and then extend it by period $2l$ to $\mathbb{R}$, call it $\tilde{g}(x)$; and do the same for $h(x)$, and call the extended function $\tilde{h}(x)$. Then $\tilde{g}(x) \in C^2(\mathbb{R})$, and $\tilde{h}(x) \in C^1(\mathbb{R})$. Let $\tilde{u}(x, t)$ denote the solution provided by d'Alembert's formula. Then $\tilde{u}(x, t)$ satisfies the initial conditions in (2.11) on $(0, l)$, and

$$\tilde{u}(0, t) = \frac{\tilde{g}(ct) + \tilde{g}(-ct)}{2} + \frac{1}{2c} \int_{-ct}^{+ct} \tilde{h}(y) \, dy = 0;$$

in addition,

$$
\begin{aligned}
&\tilde{u}(l,t)\\
=&\frac{\tilde{g}(l+ct)+\tilde{g}(l-ct)}{2}+\frac{1}{2c}\int_{l-ct}^{l+ct}\tilde{h}(y)\,dy\\
=&\frac{\tilde{g}(-l+ct)+\tilde{g}(l-ct)}{2}+\frac{1}{2c}\left(\int_{l-ct}^{l}+\int_{l}^{l+ct}\right)\tilde{h}(y)\,dy \quad (\text{using } \tilde{g}(l+ct)=\tilde{g}(-l+ct))\\
=&\frac{1}{2c}\left(\int_{l-ct}^{l}+\int_{-l}^{-l+ct}\right)\tilde{h}(y)\,dy \quad (\text{using } \tilde{g}(-l+ct)+\tilde{g}(l-ct)=0 \text{ and } \tilde{h}(y-2l)=\tilde{h}(y))\\
=&0 \quad (\text{using } \tilde{h}(y)=-\tilde{h}(-y)).
\end{aligned}
$$

Thus this $\tilde{u}$ provides a solution for (2.11). After uniqueness property is established, we know that this $\tilde{u}$ is identical to the series solution $u(x,t)$ constructed for Theorem Theorem 2.4, therefore it is $C^2([0,l]\times[0,\infty))$. □

Question: Can the method be adapted to solutions of the modified equation such as $u_{tt}-c^2u_{xx}+au_t+bu_x+du=0$ to draw the same or similar conclusions? Which parts of the argument can be adapted to construct a solution of a higher dimensional problem such as the following?

$$
\begin{cases}
\partial_{tt}^2 u(\mathbf{x},t)-c^2\Delta u(\mathbf{x},t)=0, & \mathbf{x}\in D, t>0,\\
u(\mathbf{x},t)=0, & \mathbf{x}\in\partial D, t>0,\\
u(\mathbf{x},0)=g(\mathbf{x}), & \mathbf{x}\in D,\\
\partial_t u(\mathbf{x},0)=h(\mathbf{x}), & \mathbf{x}\in D.
\end{cases}
$$

What additional information is needed to implement this approach? Try out the case when D is a simple domain such as a square or a round disc in $\mathbb{R}^2$.

Exercises

Exercise 2.5.1. Construct a Fourier series solution $u(x,t)$ to (2.11) with $g(x)=\cos(\frac{\pi x}{l})$ and $h(x)=0$ for $0<x<l$, and prove that it is discontinuous along $x=ct$ and $x=-ct+l$. (Hint: Study the discontinuities of the Fourier sine series of g over $(0,l)$, and use

$$
\cos(\frac{cn\pi}{l}t)\sin(\frac{n\pi}{l}x)=\frac{1}{2}\left[\sin\left(\frac{n\pi}{l}(x-ct)\right)+\sin\left(\frac{n\pi}{l}(x+ct)\right)\right]
$$

in studying the behavior of $\sum_{n=1}^{\infty}A_n\cos(\frac{cn\pi}{l}t)\sin(\frac{n\pi}{l}x)$.)

Exercise 2.5.2. Prove that if $g(x) = 0$ and $h(x) = 1$ for $0 < x < l$, then the series solution $u(x,t) \in C([0,l] \times [0,\infty))$, but fails to be in $C^1([0,l] \times [0,\infty))$. Study the set where its first derivatives have discontinuities.

Exercise 2.5.3. Use the d'Alembert representation to redo the above two exercises.

Exercise 2.5.4. Construct a Fourier series solution $u(x,t)$ to the following problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + \alpha u = 0, & \text{on } (x,t) \in [0,l] \times \mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l], \end{cases}$$

where α is a positive parameter. Discuss the effect of α on the behavior of the solution. Recall that the constructed solutions to (2.11) are all time periodic with period $2l/c$. Are the solutions here also a common time-period?

Exercise 2.5.5. Construct a Fourier series solution $u(x,t)$ to the following problem

$$\begin{cases} u_{tt} - c^2 u_{xx} + \beta u_t = 0, & \text{on } (x,t) \in [0,l] \times \mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l], \end{cases}$$

where β is a positive parameter. Discuss the effect of β on the behavior of the solution; in particular, discuss the cases $0 < \beta < \frac{2c\pi}{l}$, $\beta = \frac{2c\pi}{l}$, and $\frac{2c\pi}{l} < \beta < \frac{4c\pi}{l}$.

Exercise 2.5.6. Does the separation of variables method work in constructing a solution to the following IBVP?

$$\begin{cases} u_{tt} - c^2 u_{xx} + \gamma u_x = 0, & \text{on } (x,t) \in [0,l] \times \mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l], \end{cases}$$

where γ is some non-zero constant.

2.6 Separation of Variables in Polar Coordinates and Fourier Series Solution of the Dirichlet Problem of the Laplace Equation on a Round Disk

We now apply similar methods to find a solution formula for the Laplace equation on a round disk in $\mathbb{R}^2$. Based on knowledge of complex analytic functions, we know directly that for any $n \geq 0$, $z^n = r^n e^{in\theta}$ and $\bar{z}^n = r^n e^{-in\theta}$ are harmonic functions on $B_1(0) \subset \mathbb{R}^2$.

We now examine how the separation of variables method produces the same family of harmonic functions. We first write the Laplace operator in polar coordinates $\Delta u = u_{rr} + u_r/r + u_{\theta\theta}/r^2$, and apply separation of variables to look for solutions of the form $u = R(r)\Theta(\theta)$. Then $R(r)$ and $\Theta(\theta)$ must satisfy

$$\left[R''(r) + \frac{R'(r)}{r}\right]\Theta(\theta) + \frac{R(r)}{r^2}\Theta''(\theta) = 0,$$

which can be rewritten as

$$r^2 R^{-1}(r)\left[R''(r) + \frac{R'(r)}{r}\right] + \frac{\Theta''(\theta)}{\Theta(\theta)} = 0,$$

whenever $R(r) \neq 0$ and $\Theta(\theta) \neq 0$. It now follows that $\Theta''(\theta) + \lambda\Theta(\theta) = 0$ for some constant λ and all $\theta \in [0, 2\pi]$, and that

$$R''(r) + \frac{R'(r)}{r} - \frac{\lambda}{r^2}R(r) = 0 \quad \text{for } 0 < r < 1. \tag{2.12}$$

It's natural to impose the condition that $u(r, 0) = u(r, 2\pi)$ and $u_\theta(r, 0) = u_\theta(r, 2\pi)$ for all $0 < r < 1$, which leads to the condition that $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$. Thus we must look for those constants λ for which the problem

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \quad \text{for all } \theta \in [0, 2\pi], \\ \Theta(0) = \Theta(2\pi) \text{ and } \Theta'(0) = \Theta'(2\pi) \end{cases} \tag{2.13}$$

has non-trivial solutions.

One can check that this problem has non-trivial solutions iff $\lambda = n^2$ for $n \in \mathbb{Z}_{\geq 0}$; and for $n \in \mathbb{N}$, the solution space to (2.13) is spanned by $\{\sin(n\theta), \cos(n\theta)\}$; while for $n = 0$, the solution space to (2.13) is spanned by $\{1\}$.

For each $\lambda = n^2$, we look for solutions $R(r)$ to (2.12) in the form $R(r) = r^\alpha$ and find that α must satisfy $\alpha^2 - n^2 = 0$. Thus, for each $n \in \mathbb{N}$, the solution space to (2.12) is spanned by $\{r^n, r^{-n}\}$, and for $n = 0$, the solution space to (2.12) is spanned by $\{1, \ln r\}$. To conclude, for each $n \in \mathbb{Z} \setminus \{0\}$, $r^{\pm n} e^{in\theta}$ is a harmonic function of $(x, y) = (r\cos\theta, r\sin\theta)$, at least in the domain where $r > 0$.

(2.12) is an ODE with a singularity at $r = 0$, which has solutions that may be singular at $r = 0$. The singular coefficients in (2.12) is an artifact of the degeneracy in the change of coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ at $r = 0$, as (2.12) comes from the Laplace equation, and the Laplace equation does not have any singularity at $(x, y) = (0, 0)$. Since we are looking for solutions which are regular in a region that contains $(x, y) = (0, 0)$, this consideration implies that we should limit our consideration of solutions $R(r)$ to (2.12) to those which are bounded near $r = 0$.

Thus, to construct harmonic functions that are smooth in $B_1(0)$, we drop the choices $r^{-n}\cos(n\theta)$, $r^{-n}\sin(n\theta)$ for $n \in \mathbb{N}$, as well as $\ln r$. So

$$\sum_{n \in \text{a finite set in } \mathbb{Z}_{\geq 0}} \left\{a_n r^n e^{in\theta} + b_n r^n e^{-in\theta}\right\}$$

provides a harmonic function on $B_1(0)$ with boundary value

$$\sum_{n \in \text{a finite set in } \mathbb{Z}_{\geq 0}} \left\{a_n e^{in\theta} + b_n e^{-in\theta}\right\},$$

which is a trigonometric polynomial in θ.

In fact, we now have a solution of the Dirichlet problem on $B_1(0)$ with any trigonometric polynomial as boundary value. To deal with the problem of arbitrary boundary data (continuous, say), we form the infinite sum

$$u = \sum_{n=0}^{\infty} \left\{a_n r^n e^{in\theta} + b_n r^n e^{-in\theta}\right\}$$

and hope to be able to choose a_n and b_n to obtain arbitrary boundary data.

For the Dirichlet problem

$$\begin{cases} \Delta u(re^{i\theta}) = 0 & \text{for } 0 \leq r < 1, \\ u(e^{i\theta}) = g(e^{i\theta}) & \text{for } 0 \leq \theta \leq 2\pi, \end{cases} \tag{2.14}$$

where $g(e^{i\theta})$ for $0 \leq \theta \leq 2\pi$ is the given boundary value, we have formally

$$g(e^{i\theta}) = u(e^{i\theta}) = \sum_{n=0}^{\infty} \left\{a_n e^{in\theta} + b_n e^{-in\theta}\right\},$$

which leads us to take

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\phi})e^{-in\phi}d\phi, \qquad \text{for } n \geq 0,$$
$$b_n = \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\phi})e^{in\phi}d\phi, \qquad \text{for } n > 0, \quad \text{and} \quad b_0 = 0.$$

Then the series solution is

$$\begin{aligned} u(re^{i\theta}) &= \frac{1}{2\pi}\left[\int_0^{2\pi} g(e^{i\phi})d\phi + \sum_{n=1}^{\infty}\int_0^{2\pi} g(e^{i\phi})r^n\left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right)d\phi\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\phi})\left[1 + \sum_{n=1}^{\infty} r^n\left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right)\right]d\phi \\ &= \int_0^{2\pi} g(e^{i\phi})\frac{1-r^2}{2\pi\left(1+r^2-2r\cos(\theta-\phi)\right)}d\phi. \end{aligned} \tag{2.15}$$

Here we have used the summation of the two geometric series

$$\sum_{n=1}^{\infty} r^n e^{in(\theta-\phi)} = \frac{re^{i(\theta-\phi)}}{1-re^{i(\theta-\phi)}} \quad \text{and} \quad \sum_{n=1}^{\infty} r^n e^{-in(\theta-\phi)} = \frac{re^{-i(\theta-\phi)}}{1-re^{-i(\theta-\phi)}}.$$

Let

$$p(\theta, r) = \frac{1-r^2}{2\pi\left(1+r^2-2r\cos\theta\right)}.$$

Then we have

$$u(re^{i\theta}) = \int_0^{2\pi} g(e^{i\phi})p(\theta-\phi, r)\, d\phi.$$

(2.15) is called the Poisson formula for the Laplace operator on the unit disk. In terms of rectangular coordinates $X = (x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{D}$, $Y = (\cos\phi, \sin\phi) \in \partial\mathbb{D}$, $p(\theta-\phi, r) = \frac{1-|X|^2}{2\pi|X-Y|^2}$, which we denote as $P(X, Y)$. $P(X, Y)$ (and often $p(\theta-\phi, r)$) is called **Poisson kernel** for the Laplace operator on the unit disk.

Remark 2.8. Note that we initially aim for an infinite series solution using Fourier expansion, but find out that the resulting Fourier expansion can be expressed directly as an integral in terms of the boundary value and the Poisson kernel. At this point, we need not necessarily spend effort to provide verification using the infinite series; in fact it's a lot harder to use the Fourier series directly to verify that the constructed solution is continuous on the closed unit disc when g is continuous on $\mathbb{S}^1 = \partial\mathbb{D}$. We will use the integral representation directly to verify that (2.15) provides a solution in $C(\overline{\mathbb{D}}) \cap C^2(\mathbb{D})$. Such approaches of starting with looking for a formal solution and then using the formal solution to find more useful representations or structures of the solution is very common in the study of PDEs.

Theorem 2.6. *For any $g \in C(\partial\mathbb{D})$, there is a unique $u \in C(\overline{\mathbb{D}}) \cap C^2(\mathbb{D})$ solving* (2.14).

Proof. Again the uniqueness part will be addressed by the maximum principle in a later section. For the existence part, our proof will use the following properties of the Poisson kernel $P(X,Y) = \frac{1-|X|^2}{2\pi|X-Y|^2}$ for $X = (x,y) \in \mathbb{D}$ and $Y = (\cos\phi, \sin\phi) \in \partial\mathbb{D}$

$$P(X,Y) \in C^\infty(\mathbb{D} \times \partial\mathbb{D}), \text{ and } \Delta_X P(X,Y) = 0 \text{ for any } (X,Y) \in \mathbb{D} \times \partial\mathbb{D}.$$

(AP1). $\int_{\partial\mathbb{D}} P(X,Y)\, d\sigma(Y) = 1$, for any $X \in \mathbb{D}$.

(AP2). For any $Y_0 \in \partial\mathbb{D}$ and any $\delta > 0$, $\int_{Y\in\partial\mathbb{D}, |Y_0-Y|>\delta} P(X,Y)\, d\sigma(Y) \to 0$ as $X \to Y_0$ in $\mathbb{D}$.

(AP3). $P(X,Y) \geq 0$ for all $(X,Y) \in \mathbb{D} \times \partial\mathbb{D}$.

The first item can be verified directly, or can be seen from the relation that

$$P(X,Y) = (2\pi)^{-1}\left[1 + \sum_{n=1}^{\infty} r^n \left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right)\right],$$

which can be regarded as the sum of two power series both having their radii of convergence equal to 1, therefore, term-wise differentiation can be carried out inside $\mathbb{D}$, and noting $\Delta_X r^n e^{\pm in(\theta-\phi)} = 0$, we have $\Delta_X P(X,Y) = 0$ for any $(X,Y) \in \mathbb{D} \times \partial\mathbb{D}$.

(AP1) can also be verified directly, or can be seen from the uniform convergence in $Y \in \partial\mathbb{D}$ of $P(X,Y) = 1 + \sum_{n=1}^{\infty} r^n \left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right)$ for any fixed $X \in \mathbb{D}$, and $\int_0^{2\pi} e^{\pm in(\theta-\phi)} d\phi = 0$ for any $n \in \mathbb{N}$.

(AP3) is obvious from the form for $P(X,Y)$, while (AP2) is established by noting that, as a function of $Y \in \partial\mathbb{D}$, the family of functions $\{P(X,Y)\}_{X\in\mathbb{D}}$ becomes concentrated near $Y_0 \in \partial\mathbb{D}$ as $X \to Y_0$. More precisely, for any fixed $\delta > 0$, when $|X - Y_0| \leq \delta/2$ and $|Y - Y_0| \geq \delta$, $|X - Y| \geq |Y - Y_0| - |X - Y_0| \geq \delta/2$, so $P(X,Y) \leq \frac{1-|X|^2}{2\pi(\delta/2)^n}$, and for any given $\epsilon > 0$, there exists $0 < \delta' < \delta/2$ such that when $|X - Y_0| \leq \delta'$, $P(X,Y) \leq \epsilon$.

Now for any $X \in \mathbb{D}$, there exist $\delta > 0$ and constant $C = C(\delta) > 0$ such that $|\nabla_Z^\alpha P(Z,Y)| \leq C$ for all $y \in \partial\mathbb{D}$, X with $|Z - X| \leq \delta$, and $|\alpha| \leq 2$, so we can apply Lemma A.1 in the Appendix to conclude that

$$\Delta_X u(X) = \int_{Y\in\partial\mathbb{D}} g(Y)\Delta_X P(X,Y)\, d\sigma(Y) = 0.$$

To prove $u(X) \to g(Y_0)$ as $X \to Y_0 \in \partial\mathbb{D}$, for a given $\epsilon > 0$, we first use the continuity of g to find $\delta > 0$ such that $|g(Y) - g(Y_0)| < \epsilon$ when $|Y - Y_0| < \delta$. Then we write, using (AP1),

$$u(X) - g(Y_0) = \int_{Y \in \partial\mathbb{D}} [g(Y) - g(Y_0)]\, P(X,Y)\, d\sigma(Y),$$

and estimate

$$|u(X) - g(Y_0)| \leq \left(\int_{Y \in \partial\mathbb{D}, |Y-Y_0| \leq \delta} + \int_{Y \in \partial\mathbb{D}, |Y-Y_0| \geq \delta} \right) |g(Y) - g(Y_0)|\, P(X,Y)\, d\sigma(Y).$$

The first integral is bounded above by $\epsilon \int_{Y \in \partial\mathbb{D}} P(X,Y)\, d\sigma(Y) = \epsilon$, while the second integral is bounded above by $2 \max |g(Y)| \int_{Y \in \partial\mathbb{D}, |Y-Y_0| \geq \delta} P(X,Y)\, d\sigma(Y)$, which can be made smaller than ϵ as $X \to Y_0$ by using (AP2).

$\square$

A family of functions $\{P(X,Y)\}_X$, treated as functions of $Y \in \partial\mathbb{D}$ and parametrized by $X \in \mathbb{D}$, satisfying (AP1)–(AP3) above is called an *approximation to identity* (on $\partial\mathbb{D}$). As $X \to Z \in \partial\mathbb{D}$, this family of functions of Y becomes concentrated at Z and behaves like Dirac's delta function of Y at Z. Families of functions having this property will arise in other contexts.

Question: Can the method be adapted to solutions of the modified equation such as $\Delta u + cu = 0$ to draw the same or similar conclusions?

Exercise 2.6.1. (a). Prove that for any bounded sequence $\{a_n\}$ and $\{b_n\}$, the series defining u above converges uniformly on any smaller disk $\{r \leq r_0 < 1\}$ and gives rise to a smooth harmonic function in $B_1(0)$.

(b). Verify that

$$p(\theta-\phi, r) := \frac{1}{2\pi} \left[1 + \sum_{n=1}^{\infty} r^n \left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)} \right) \right] = \frac{1 - r^2}{2\pi \left(1 + r^2 - 2r\cos(\theta - \phi)\right)}.$$

(c). Verify that if we write $X = (r\cos\theta, r\sin\theta)$ and $Y = (\cos\phi, \sin\phi)$, then

$$p(\theta - \phi, r) = \frac{1 - |X|^2}{2\pi |X - Y|^2} = P(X,Y).$$

(d). Verify that for $X \in \mathbb{D}$, $\int_{\partial\mathbb{D}} P(X,Y) d\sigma(Y) = 1$.

Exercise 2.6.2. Apply separation of variables to construct solutions to the following BVP on the sector $\Sigma_{\theta_0} := \{(x,y) = (r\cos\theta, r\sin\theta) : 0 \le r < 1, 0 < \theta < \theta_0\}$:

$$\begin{cases} \Delta u(x,y) = 0 & \text{for } (x,y) \in \Sigma_{\theta_0}. \\ u(r\cos\theta, r\sin\theta) = 0 & \text{for } \theta = 0 \text{ or } \theta_0 \text{ and } 0 \le r \le 1, \\ u(\cos\theta, \sin\theta) = g(\theta) & \text{for } 0 < \theta < \theta_0, \end{cases}$$

where $g(\theta)$ is a given continuous function on $[0, \theta_0]$. Then change one of the boundary conditions, say, using $u_\theta(r\cos\theta_0, r\sin\theta_0) = 0$ to replace $u(r\cos\theta_0, r\sin\theta_0) = 0$ for $0 \le r \le 1$, and find a solution of this modified BVP.

Exercise 2.6.3. Model on the proof of Theorem 2.6 to show that if g is Riemann integrable on $[-l, l]$ and continuous at some $x \in (-l, l)$, then

$$N^{-1} \sum_{n=0}^{N-1} \left(\sum_{-n}^{n} g_k e^{\frac{ik\pi x}{l}} \right) \to g(x) \text{ as } N \to \infty.$$

Furthermore, if $g \in C[-l, l]$ with $g(-l) = g(l)$, then the convergence is uniform over $x \in [-l, l]$. (HINT: (2.3) and (2.4) will play a role.)

2.7 Solution of the Cauchy Problem for the Heat Equation

We next make sense of the formal solution $\int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi$ to the homogeneous heat equation which we have found in the last chapter.

Theorem 2.7. *For any $c(\xi) \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi$ defines a smooth function for $x \in \mathbb{R}$ and $t > 0$, which is in $C(\mathbb{R} \times [0, \infty))$ and satisfies $\frac{\partial u}{\partial t} = \Delta u$.*

Proof. We first sketch a proof that $(x,t) \in \mathbb{R} \times \mathbb{R}^+ \mapsto \int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi$ is a smooth function by first proving that

$$\frac{\partial}{\partial x} \left(\int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi \right) = \int_{\mathbb{R}} c(\xi) i\xi e^{ix\xi - \xi^2 t} d\xi, \tag{2.16}$$

and

$$\frac{\partial}{\partial t} \left(\int_{\mathbb{R}} c(\xi) e^{ix\xi - \xi^2 t} d\xi \right) = \int_{\mathbb{R}} c(\xi)(-\xi^2) e^{ix\xi - \xi^2 t} d\xi, \tag{2.17}$$

for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, then using similar arguments to prove the continuous differentiability of any order of derivatives.

It suffices to prove the continuous differentiability for $(x,t) \in \mathbb{R} \times [\tau, \infty)$, where $\tau > 0$ is arbitrarily given; and it suffices to check the conditions in Lemma A.1 in the Appendix for $(x,t;\xi)$ over $\mathbb{R} \times (\tau,\infty) \times \mathbb{R}$, and with $s(x,t;\xi) = c(\xi)e^{ix\xi-\xi^2 t}$. This follows easily because

$$\left|s_x(x,t;\xi)\right| = \left|i\xi c(\xi)e^{ix\xi-\xi^2 t}\right| \le |\xi||c(\xi)|e^{-\tau\xi^2},$$

and

$$\left|s_t(x,t;\xi)\right| = \left|(-\xi^2)c(\xi)e^{ix\xi-\xi^2 t}\right| \le |\xi|^2|c(\xi)|e^{-\tau\xi^2}.$$

Note that each of the bounds above on the right is integrable over $\xi \in \mathbb{R}$, so we can apply Lemma A.1 in the Appendix.

Version of (2.16) for second order differentiation in x gives

$$\frac{\partial^2}{\partial x^2}\left(\int_{\mathbb{R}} c(\xi)e^{ix\xi-\xi^2 t}d\xi\right) = \int_{\mathbb{R}} c(\xi)(-\xi^2)e^{ix\xi-\xi^2 t}d\xi,$$

which, together with (2.17), proves that $\frac{\partial u}{\partial t} = \Delta u$. The continuity of $\int_{\mathbb{R}} c(\xi)e^{ix\xi-\xi^2 t}d\xi$ in $\mathbb{R} \times [0,\infty)$ follows also from Lemma A.1 in the Appendix, or alternatively, Lebesgue's Dominated Convergence Theorem, using $c(\xi) \in L(\mathbb{R})$.

□

The main issue next is to verify that, if we choose $c(\xi)$ appropriately, namely, $c(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} u(y,0)e^{-iy\xi}\,dy$*, then the $u(x,t)$ defined above takes on the initial value $u(x,0)$ in appropriate sense as $t \to 0$.

One situation where this can be checked easily is when we assume $u(x,0)$ can be represented as in (1.22) with $c(\xi) \in L^1(\mathbb{R})$. Then $u(x,t) \to u(x_0,0)$ as $(x,t) \to (x_0,0)$ by Lebesgue's dominated convergence theorem (or Lemma A.1). But this assumption on $u(x,0)$ is through $c(\xi)$, its Fourier transform, and may not be as easy to verify.

It turns out we can verify the continuity of $u(x,t)$ up to $t=0$ under much weaker condition on $u(x,0)$. We proceed by deducing a further representation formula for

*A more detailed discussion of this formula will be given in a later section on Fourier transform.

$u(x,t)$ as follows.

$$
\begin{aligned}
&u(x,t)\\
=&\frac{1}{2\pi}\int_{\mathbb{R}}\left\{\int_{\mathbb{R}} u(y,0)e^{-iy\xi}\,dy\right\}e^{ix\xi-\xi^2 t}d\xi\\
=&\frac{1}{2\pi}\int_{\mathbb{R}} u(y,0)\left\{\int_{\mathbb{R}} e^{i(x-y)\xi-\xi^2 t}d\xi\right\}dy \quad \text{(interchange the order of integration)}\\
=&\frac{1}{2\pi}\int_{\mathbb{R}} u(y,0)e^{-\frac{|x-y|^2}{4t}}\sqrt{\frac{\pi}{t}}\,dy \quad \left(\text{using } \int_{\mathbb{R}} e^{-u^2+iau}du=\sqrt{\pi}e^{-\frac{a^2}{4}}\right)\\
=&\int_{\mathbb{R}} u(y,0)\frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}}\,dy,
\end{aligned}
\tag{2.18}
$$

here, to justify the interchange of integrals, we could assume $u(y,0)$ to have compact support; but the above representation can be used under much relaxed condition on $u(y,0)$.

The integral kernel

$$
K(x-y,t)=\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi-\xi^2 t}d\xi=\frac{1}{\sqrt{4\pi t}}e^{-\frac{|x-y|^2}{4t}},
$$

is called the **heat kernel** or **fundamental solution** to the heat equation and enjoys the following properties:

(HK). For any $y\in\mathbb{R}$, $(\partial_t-\partial_x^2)K(x-y,t)=0$, for any x and $t>0$.

(AP1). $\int_{\mathbb{R}} K(x-y,t)\,dy=1$, for any x and $t>0$.

(AP2). For any $\delta>0$, $\int_{|x-y|>\delta} K(x-y,t)\,dy\to 0$ as $t\to 0$.

(AP3). $K(x-y,t)\geq 0$.

(HK) follows from almost identical arguments as those used in the proof of the previous Theorem, using the integral representation for $K(x-y,t)$, or can be verified by direct differentiation. (AP1)–(AP3) above are similar to the (AP1)–(AP3) satisfied by the Poisson kernel on the unit disk. This family of functions $\{K(x-y,t)\}_{t>0}$ also forms an *approximation to identity* (on $\mathbb{R}$). With these properties above, we will prove

Theorem 2.8. *Define $u(x,t)$ in terms of $u(x,0)$ through (2.18), where $u(x,0)\in L^1(\mathbb{R})$ (or is bounded and Lebesgue measurable or Riemann integrable over any bounded interval in $\mathbb{R}$). Then* *(i).* $u(x,t)$ *is a* C^∞ *solution of* $u_t - u_{xx} = 0$ *on* $(x,t) \in \mathbb{R} \times (0,\infty)$.

(ii). $||u(x,t) - u(x,0)||_{L^1(\mathbb{R})} \to 0$, *as* $t \to 0$, *if* $u(x,0) \in L^1(\mathbb{R})$.

(iii). $u(x,t)$ *is continuous at* $u(x_0,0)$, *if* $u(x,0)$ *is continuous at* x_0.

(iv). $||u(\cdot,t)||_{L^1(\mathbb{R})} \le ||u(\cdot,0)||_{L^1(\mathbb{R})}$.

Remark 2.9. Notice also that even if the initial data has discontinuity, the solution given by (2.18) instantly becomes a smooth function of (x,t) for $t > 0$. In addition, the value of $u(x,t)$ depends on the values of $u(y,0)$ for all $y \in \mathbb{R}$, which indicates that solutions to the heat equation have an *infinite speed* of "propagation".

One advantage of the above representation (2.18) is that it provides a solution (possibly on a finite time interval) even for initial data $u(x,0)$ that has some growth, as long as it is slower than $e^{a|x|^2}$ for some $a > 0$—even though the formula was derived under much stricter requirement on $u(x,0)$; Chapter 4 will pick up the technical discussion on this again. In fact, this kind of approach will be used repeatedly: on a first try, one does not pay too close attention to justify every step, only after one obtains some useful results, does one go back to check every step, or verify the conclusion by other means.

Uniqueness is not settled by this approach.

Remark 2.10. Even though (2.18) and (1.21) are both established for one space dimension times the time axis — the latter was discovered through separation of variables in one space dimension times the time axis and led to (2.18), their forms and the content of the above Theorem remain the same when $x \in \mathbb{R}^n$: one simply takes $\xi \in \mathbb{R}^n$, replaces $x\xi$ by $\mathbf{x} \cdot \boldsymbol{\xi}$, the dot product between $\mathbf{x}$ and $\boldsymbol{\xi}$, replaces ξ^2 in (1.21) by $|\boldsymbol{\xi}|^2$, replaces $\sqrt{4\pi t}$ in $K(x-y,t)$ by $(4\pi t)^{n/2}$, and interprets $|\mathbf{x}-\mathbf{y}|$ in $K(\mathbf{x}-\mathbf{y},t)$ as $\sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$. This is due to the simple reason that $K(\mathbf{x}-\mathbf{y},t) = K(x_1-y_1,t)\cdots K(x_n-y_n,t)$ has a product structure, and

$$\begin{aligned}
&(\partial_t - \Delta_x)\, K(x-y,t) \\
=&\sum_{k=1}^{n} K(x_1-y_1,t)\cdots[K_t(x_k-y_k,t) - K_{xx}(x_k-y_k,t)]\cdots K(x_n-y_n,t) \\
=&0.
\end{aligned}$$

Alternatively,

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\xi} - |\boldsymbol{\xi}|^2 t} d\boldsymbol{\xi} = \frac{1}{(2\pi)^n}\prod_{j=1}^{n}\int_{\mathbb{R}} e^{i(x_j-y_j)\xi_j - \xi_j^2 t} d\xi_j = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}}.$$

Our development here did not depend logically on any systematic theory of Fourier transforms, other than using the inversion formula

$$c(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}} u(x,0)e^{-ix\xi}\,dx$$

(for one space dimension) to (1.22) in formally deriving (2.18). Once we came up with (2.18), we justified its validity without any direct usage of Fourier transforms.

It is often efficient to use this inversion relation to explore how certain prototype solutions (to constant coefficient PDEs) behave. For exploring solutions to the Cauchy problem for constant coefficient PDEs, we often make a variation of the separation of variables method and look for solutions of the form $e^{i\mathbf{x}\cdot\boldsymbol{\xi}}T(t)$ and try to understand how the "Fourier mode" $e^{i\mathbf{x}\cdot\boldsymbol{\xi}}$ would evolve under the PDE, as the gist of Fourier transform says that reasonable functions can be synthesized as superpositions of various Fourier modes $e^{i\mathbf{x}\cdot\boldsymbol{\xi}}$.

Proof of Theorem 2.8. This proof is a bit lengthy. The proof for (i) is very similar to our proof for Theorem 2.6; students on a first reading can first focus on the proof for (ii) and/or (iii) below.

We will apply Lemma A.1 in the Appendix to prove (i).

When $u(y,0) \in L^1(\mathbb{R})$, we set $s(x,t;y) = K(x-y,t)u(y,0)$, then for any $y \in \mathbb{R}$, $(x,t) \in \mathbb{R}\times\mathbb{R}^+ \mapsto s(x,t;y)$ is smooth in (x,t). Fix any $\tau > 0$. Now for any $(x,t) \in \mathbb{R}\times(\tau,\infty)$, we have

$$\begin{aligned}|\partial_t s(x,t;y)| &= \left|\frac{|x-y|^2}{4t} - \frac{1}{2}\right| \frac{1}{\sqrt{4\pi t^3}} e^{-\frac{|x-y|^2}{4t}}|u(y,0)| \\ &\le \left(\frac{|x-y|^2}{4t} + \frac{1}{2}\right)\frac{1}{\sqrt{4\pi t^3}} e^{-\frac{|x-y|^2}{4t}}|u(y,0)| \\ &\le \frac{C}{\sqrt{4\pi\tau^3}}|u(y,0)|,\end{aligned}$$

using $\left(\frac{|x-y|^2}{4t}+\frac{1}{2}\right)e^{-\frac{|x-y|^2}{4t}} \le C$. Now the bound on the right hand side is integrable in $\mathbb{R}$ so we can appeal to Lemma A.1 in the Appendix to conclude that

$$\partial_t\left(\int_{\mathbb{R}} K(x-y,t)u(y,0)\,dy\right) = \int_{\mathbb{R}} \partial_t K(x-y,t)u(y,0)\,dy.$$

Differentiation in x, and higher order (mixed) differentiation, is proved in a similar way, using the following pattern proved by induction: there exists a polynomial $p_{k,l}(\frac{x}{\sqrt{t}})$ of degree $k+2l$ in $\frac{x}{\sqrt{t}}$ such that

$$\left|\partial_x^k\partial_t^l K(x,t)\right| \le \frac{|p_{k,l}(\frac{x}{\sqrt{t}})|}{t^{k/2+l}}K(x,t), \text{for all } (x,t)\in\mathbb{R}\times\mathbb{R}^+. \tag{2.19}$$

When $u(y,0)$ is bounded, for any $(x,t) \in \mathbb{R} \times (0,\infty)$, we will verify that

$$\hat{s}(\tau; y) = s(x, t+\tau; y) = K(x-y, t+\tau)u(y,0)$$

satisfies the hypothesis for Lemma A.1 in the Appendix for $|\tau| < t/2$, therefore it is differentiable at $\tau = 0$, and

$$\begin{aligned}\partial_t \left(\int_{\mathbb{R}} K(x-y,t)u(y,0)\,dy \right) &= \frac{d}{d\tau}\bigg|_{\tau=0} \left(\int_{\mathbb{R}} K(x-y,t+\tau)u(y,0)\,dy \right) \\ &= \int_{\mathbb{R}} \partial_t K(x-y,t)u(y,0)\,dy.\end{aligned}$$

When $|\tau| < t/2$, we have the bound

$$\begin{aligned}|\partial_\tau \hat{s}(\tau;y)| &\le \left(\frac{|x-y|^2}{4(t+\tau)} + \frac{1}{2} \right) \frac{1}{\sqrt{4\pi(t+\tau)^3}} e^{-\frac{|x-y|^2}{4(t+\tau)}} |u(y,0)| \\ &\le \left(\frac{|x-y|^2}{2t} + \frac{1}{2} \right) \frac{1}{\sqrt{\pi t^3/2}} e^{-\frac{|x-y|^2}{8t}} |u(y,0)|\end{aligned}$$

The bound on the right hand side above is in $L^1(\mathbb{R})$ considered as a function of y:

$$\begin{aligned}&\int_{\mathbb{R}} |u(y,0)| \left(\frac{|x-y|^2}{2t} + \frac{1}{2} \right) \frac{1}{\sqrt{\pi t^3/2}} e^{-\frac{|x-y|^2}{8t}}\,dy \\ =&\frac{||u(\cdot,0)||_{L^\infty(\mathbb{R})}}{\sqrt{\pi/2}t} \int_{\mathbb{R}} \frac{|z|^2+1}{2} e^{-\frac{|z|^2}{8}}\,dz < \infty.\end{aligned}$$

We are now ready to appeal to Lemma A.1 in the Appendix (or Lebesgue's dominated convergence theorem) to conclude that

$$u_t(x,t) = \frac{d}{d\tau}\bigg|_{\tau=0} \left(\int_{\mathbb{R}} \hat{s}(\tau;y)\,dy \right) = \int_{\mathbb{R}} u(y,0)\partial_t K(x-y,t)\,dy.$$

The additional differentiability of u follows in a similar fashion. In particular

$$u_{xx}(x,t) = \int_{\mathbb{R}} u(y,0)\partial_x^2 K(x-y,t)\,dy.$$

Finally $u_t(x,t) - u_{xx}(x,t) = 0$ because $K_t(x-y,t) - K_{xx}(x-y,t) = 0$.

For (ii), using (AP1), we have

$$u(x,t)-u(x,0) = \int_{\mathbb{R}} u(y,0)K(x-y,t)\,dy - u(x,0) = \int_{\mathbb{R}} \left[u(x-z,0) - u(x,0)\right] K(z,t)\,dz.$$

Since $K(z,t) > 0$ and $\int_{\mathbb{R}} K(z,t)\,dz = 1$, the above integral can be thought of as a convex linear combination of the family of functions $u(\cdot - z, 0) - u(\cdot, 0)$. Thus from the convexity of L^1 norm, we get

$$\begin{aligned}
&||u(x,t) - u(x,0)||_{L^1(\mathbb{R})} \\
&= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} [u(x-z,0) - u(x,0)]\, K(z,t)\, dz \right| dx \\
&\le \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |u(x-z,0) - u(x,0)|\, dx \right) K(z,t)\, dz \\
&\le \int_{|z|\le\delta} ||u(x-z,0) - u(x,0)||_{L^1(\mathbb{R})} K(z,t)\, dz + \int_{|z|>\delta} ||u(x-z,0) - u(x,0)||_{L^1(\mathbb{R})} K(z,t)\, dz.
\end{aligned}$$

For any $\epsilon > 0$, using the continuity of translation in $L^1(\mathbb{R})$, there exists $\delta > 0$ such that for $|z| \le \delta$, we have

$$||u(x-z,0) - u(x,0)||_{L^1(\mathbb{R})} < \epsilon,$$

which implies

$$\int_{|z|\le\delta} ||u(x-z,0) - u(x,0)||_{L^1(\mathbb{R})} K(z,t)\, dz < \epsilon \int_{|z|\le\delta} K(z,t)\, dz < \epsilon,$$

using (AP1) and (AP3). Using (AP2) and (AP3),

$$\int_{|z|>\delta} ||u(x-z,0) - u(x,0)||_{L^1(\mathbb{R})} K(z,t)\, dz \le 2||u(x,0)||_{L^1(\mathbb{R})} \int_{|z|>\delta} K(z,t)\, dz < \epsilon$$

when $t > 0$ is sufficiently small.

For (iii), we again express

$$u(x,t) - u(x_0,0) = \int_{\mathbb{R}} u(y,0) K(x-y,t)\, dy - u(x_0,0) = \int_{\mathbb{R}} [u(x-z,0) - u(x_0,0)]\, K(z,t)\, dz.$$

Since $u(y,0)$ is assumed to be continuous at $y = x_0$, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|u(y,0) - u(x_0,0)| \le \epsilon \quad \text{for } |y - x_0| \le 2\delta.$$

Then for $|x - x_0| \le \delta$, when $|z| \le \delta$, we have $|x - z - x_0| \le |x - x_0| + |z| \le 2\delta$, so $|u(x-z,0) - u(x_0,0)| < \epsilon$, and

$$\int_{|z|\le\delta} |u(x-z,0) - u(x_0,0)| K(z,t)\, dz \le \epsilon.$$

Let's assume $u \in L^\infty(\mathbb{R})$ here, then

$$\int_{|z|\ge\delta} |u(x-z,0) - u(x_0,0)| K(z,t)\, dz \le 2||u(\cdot,0)||_{L^\infty(\mathbb{R})} \int_{|z|\ge\delta} K(z,t)\, dz.$$

But by (AP2), there exists $h > 0$ such that when $0 < t < h$, $2||u(\cdot,0)||_{L^\infty(\mathbb{R})} \int_{|z|\geq\delta} K(z,t)\,dz \leq \epsilon$. Putting these together, we find that, when $0 < t < h$, $|x - x_0| \leq \delta$,

$$
\begin{aligned}
&|u(x,t) - u(x_0,0)| \\
\leq &\int_{|z|\leq\delta} |u(x-z,0) - u(x_0,0)|K(z,t)\,dz + \int_{|z|\geq\delta} |u(x-z,0) - u(x_0,0)|K(z,t)\,dz \\
\leq &2\epsilon.
\end{aligned}
$$

(iv) is proved using Fubini Theorem to interchange the integrals

$$
u(x,t) = \int_{\mathbb{R}} u(x-z,0)K(z,t)\,dz,
$$

so that

$$
||u(\cdot,t)||_{L^1(\mathbb{R})} \leq \int_{\mathbb{R}} ||u(\cdot - z,0)||_{L^1(\mathbb{R})} K(z,t)\,dz \leq ||u(\cdot,0)||_{L^1(\mathbb{R})} \int_{\mathbb{R}} K(z,t)\,dz = ||u(\cdot,0)||_{L^1(\mathbb{R})}.
$$

□

Exercise 2.7.1. Let $u(x,t)$ be given by (2.18) and $|u(\cdot,0)|$ be bounded by M. Prove that there exists a constant $C > 0$ such that

$$
|u_x(x,t)| \leq \frac{CM}{\sqrt{t}}, \quad |u_t(x,t)| + |u_{xx}(x,t)| \leq \frac{CM}{t}.
$$

Exercise 2.7.2. (a). Apply separation of variables/Fourier transforms to construct solutions to the Laplace equation $\Delta u = 0$ on the half plane $\mathbb{R}^2_+$:

$$
\begin{cases}
\Delta u = 0, & \text{on } (x,y) \in \mathbb{R}^2_+, \\
u(x,0) = g(x), & \text{on } (x,0) \in \partial R^2_+.
\end{cases}
\tag{2.20}
$$

Here we are implicitly assuming that u remains bounded as (x,y) approaches ∞.

(b). Rewrite the solution in the last part directly as a convolution with g. HINT: the solution should have the form

$$
u(x,y) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y g(u)}{|x-u|^2 + y^2} du.
$$

(c). Prove if $g \in L^1(\mathbb{R})$, then the $u(x,y)$ above is smooth in $(x,y) \in \mathbb{R}^2_+$ and satisfies $\Delta u = 0$ there.

(d). Prove that if $g \in L^1(\mathbb{R})$, then $u(x,y) \to g$ in $L^1(\mathbb{R})$ as $y \searrow 0$.

(e). Prove that if g is bounded over $\mathbb{R}$ and that g is continuous at x_0, then $u(x,y)$ is continuous at $(x_0, 0)$.

Exercise 2.7.3. Follow the procedure below to weaken the hypotheses in Theorem 2.2, namely, replace the assumptions on $g(x)$ there by assuming only $g \in C[0,l]$ with $g(0) = g(l) = 0$, and prove that there is a solution $u(x,t) \in C^2([0,l] \times (0,\infty)) \cap C([0,l] \times [0,\infty))$ that satisfies (2.1). First extend g to be an odd function to $[-l,l]$, then extend it to a $2l$-periodic function on $\mathbb{R}$. Call it $\tilde{g}$. Show that $u(x,t) := \int_{\mathbf{R}} K(x-y,t)\tilde{g}(y)dy$ is a solution of (2.1) in $C^2([0,l] \times (0,\infty)) \cap C([0,l] \times [0,\infty))$. Show that there is also a version of Theorem 2.2 under a weaker assumption on g: if $g \in L^p[0,l]$ for some $1 \le p \le \infty$, and g is continuous at some $0 < x_0 < l$, then the solution $u(x,t)$ defined above is continuous at x_0.

Exercise 2.7.4. Construct a solution to the Cauchy problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) + \gamma u_x(x,t) = 0, & x \in \mathbb{R}, t > 0, \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

where γ is some real constant. If you run into difficulty in applying separation of variables or Fourier transforms, you may consider a new variable $v(x,t) = u(x+\gamma t, t)$ (or $v(x,t) = e^{\delta x}u(x,t)$ for some δ) and solve for a Cauchy problem for $v(x,t)$.

2.8 A Notion of Generalized Solution to the Wave Equation

We now discuss a notion of generalized solution using an integral form. It is based on the following divergence structure for two C^2 functions $u(x,t)$ and $\eta(x,t)$:

$$\begin{aligned} &\left[u_{tt}(x,t) - c^2 u_{xx}(x,t)\right]\eta(x,t) - u(x,t)\left[\eta_{tt}(x,t) - c^2\eta_{xx}(x,t)\right] \\ &= \left[u_t(x,t)\eta(x,t) - u(x,t)\eta_t(x,t)\right]_t - c^2\left[u_x(x,t)\eta(x,t) - \eta_x(x,t)u(x,t)\right]_x. \end{aligned}$$

If $u(x,t)$ is $C^2(\mathbb{R}^2)$ solution of (1.10), and $\eta \in C_c^2(\mathbb{R}^2)$, then integrating the above over $\mathbb{R}^2$, we see that

$$\iint_{\mathbb{R}^2} u(x,t)\left[\eta_{tt}(x,t) - c^2\eta_{xx}(x,t)\right] dxdt = 0. \tag{2.21}$$

If one would like to take into account of the initial data $u(x,0)$ and $u_t(x,0)$ of $u(x,t)$, then integrating over $\mathbb{R}\times\mathbb{R}^+$, we obtain

$$\int_0^\infty\int_{\mathbb{R}} u(x,t)\left[\eta_{tt}(x,t)-c^2\eta_{xx}(x,t)\right]\,dxdt=\int_{\mathbb{R}}\left[h(x)\eta(x,0)-g(x)\eta_t(x,0)\right]dx. \tag{2.22}$$

Both (2.21) and (2.22) make sense if $u(x,t)$ is locally integrable in $\mathbb{R}^2$.

We define $u(x,t)$ to be a **generalized solution** to (2.10) if it is locally integrable in $\mathbb{R}^2$, and (2.22) holds for all $\eta\in C_c^2(\mathbb{R}^2)$. If consideration of initial data is not relevant, we define $u(x,t)$ to be a **generalized solution** to (1.10) if it is locally integrable in $\mathbb{R}^2$, and (2.21) holds for all $\eta\in C_c^2(\mathbb{R}^2)$.

One advantage of this definition is that uniform limits of C^2 solutions are generalized solutions (although they may not known to be differentiable).

One can also construct a generalized solution easily when $g\in C(\mathbb{R})$ and $h\in L^1_{\text{local}}(\mathbb{R})$, as one can choose $g_k\in C^2(\mathbb{R})$ and $h_k\in C^1(\mathbb{R})$ such that $g_k\to g$ uniformly over any compact interval of $\mathbb{R}$, and $h_k\to h$ in $L^1(I)$ over any compact interval I of $\mathbb{R}$, and use the d'Alembert's formula to construct a solution $u_k(x,t)$ to (2.10) with $u_k(x,0)=g_k(x)$ and $u_t(x,0)=h_k(x)$. Then d'Alembert's formula shows that, on any compact subset of $\mathbb{R}^2$, $u_k(x,t)$ has a uniform limit. This limit defines a generalized solution of the Cauchy problem for (2.10)—we could use the d'Alembert's formula to define a notion of generalized solution, but the notion defined here can easily extend to situations where an explicit representation is either unavailable or difficult to obtain.

In general, if G and H are continuous, then $u(x,t)=G(x-ct)+H(x+ct)$ is a generalized solution of (1.10).

Note also that, suppose that G is C^2 except at a finite number of points, say, $\{x_1,\cdots,x_N\}$, then $G(x-ct)$ is a smooth solution of the wave equation except along the lines $x-ct=x_i$, $1\le i\le N$, along which the discontinuity of the derivative persists. This is different from the behavior of solutions to the heat equation, which smoothes out any discontinuity instantly. A similar relation holds for $H(x+ct)$.

If we choose $H(x)=-G(x)$, then $u(x,t)=G(x-ct)+H(x+ct)$ satisfies $u(x,0)=0$, and $u_t(x,0)=-2cG'(x)$ if $G'(x)$ is defined. If we choose $G(x)$ such that $-2cG'(x)=\delta(x)$, e.g., $G(x)$ can be taken to be the limit of $G_k(x)$, with

$$-2cG_k'(x)=\begin{cases}\frac{k}{2} & \text{for } -\frac{1}{k}\le x\le\frac{1}{k},\\ 0 & \text{otherwise,}\end{cases}$$

and $G_k(x)$ normalized to be equal to 0 for $x \geq \frac{1}{k}$. This leads to

$$G_k(x) = \begin{cases} \frac{1}{2c} & \text{for } x \leq -\frac{1}{k}, \\ -\frac{k}{4c}(x - \frac{1}{k}) & \text{for } -\frac{1}{k} \leq x \leq \frac{1}{k}, \\ 0 & \text{for } x \geq \frac{1}{k}. \end{cases}$$

Thus

$$G(x) = \begin{cases} \frac{1}{2c} & \text{if } x \leq 0, \\ 0 & \text{if } x > 0; \end{cases}$$

and

$$E(x,t) = \begin{cases} G(x-ct) - G(x+ct) = \frac{1}{2c} & \text{if } t > 0, -ct \leq x \leq ct, \\ 0 \quad \text{elsewhere.} \end{cases}$$

In the above we see that for any $x \neq 0$, $G_k(x) \to G(x)$ as $k \to \infty$, but a more proper view is the convergence in the sense of distribution defined by

$$\int_{\mathbb{R}} G_k(x)\eta(x)\, dx \to \int_{\mathbb{R}} G(x)\eta(x) dx \quad \text{for any } \eta \in C_c^\infty(\mathbb{R}) \text{ as } k \to \infty.$$

E(x,t) satisfies

$$\left[\partial_t^2 - c^2\partial_x^2\right] E(x,t) = \delta(0,0) \quad \text{in the sense of distribution,}$$

and is called a fundamental solution to the wave equation. Another view is that this $E(x,t)$ solves the wave equation with a single point source at $(0,0)$:

$$\begin{cases} \left[\partial_t^2 - c^2\partial_x^2\right] E(x,t) = 0, \text{ for } t > 0, \\ E(x,0) = 0, E_t(x,0) = \delta(x), \end{cases}$$

This will be discussed in more detail in a later chapter.

In order for this notion of generalized solution of be a useful one, there is a need to establish uniqueness of solution among this class of solutions to the appropriately formulated IVP. This can be done following an approach by Holmgren, using the existence of solutions to the non-homogeneous wave equation. We will supply a proof after establishing the existence of solutions to these equations in the first section of next Chapter.

In addition to having appropriate limits of smooth solutions as candidates for generalized solutions, another class of potential candidates for generalized solutions are piecewise smooth solutions. We will first present a simple example to illustrate that this can't be done arbitrarily, and that the integral formulation for a generalized solution may provide some requirement on the behavior of the solution across potential interface of discontinuity.

Example 2.1. All (C^2) solutions to $u''(x) = 0$ on $\mathbb{R}$ are of the form $u(x) = ax + b$ for some constants a and b. An integral formulation for a generalize solution of $u''(x) = 0$ is $\int u(x)\eta''(x)dx = 0$ for all $\eta \in C_c^2$. Suppose that $u(x)$ is piecewise linear on either side of $x = 0$, then the condition $\int u(x)\eta''(x)dx = 0$ implies that

$$\begin{aligned} 0 &= \int_{-\infty}^{0} u(x)\eta''(x)dx + \int_{0}^{\infty} u(x)\eta''(x)dx \\ &= u(0-)\eta'(0) - \int_{-\infty}^{0} u'(x)\eta'(x)dx - u(0+)\eta'(0) - \int_{0}^{\infty} u'(x)\eta'(x)dx \\ &= [u(0-) - u(0+)]\eta'(0) - [u'(0-) - u'(0+)]\eta(0). \end{aligned}$$

Since we have the freedom to choose $\eta(0)$ and $\eta'(0)$ arbitrarily, it follows that $u(0-) - u(0+) = 0$, and $u'(0-) - u'(0+) = 0$. Thus a piecewise solution satisfying the integral formulation must in fact be smooth across the interface point; namely, one can't arbitrarily use piecewise solutions to construct a generalized solution.

This simple example is inserted here to illustrate that a generalized solution requires more than asking for the equation to be satisfied almost everywhere, and how the integral formulation is used to extract information on the behavior of a generalized solution. More general results and proofs on generalized solutions to equations like the Laplace equation will be presented later on. For equations like the one in this example, a continuous, or just locally integrable, generalized solution will in fact be C^∞ smooth.

Remark 2.11. Another point which should be pointed out is that a jump discontinuity of second derivatives to solutions to (1.10) can only occur along $x \pm ct =$ constant, the so called characteristic lines of (1.10).

More precisely, suppose that $x = \xi(t)$ defines a C^1 curve, that $u^\pm$ are C^2 functions of (x, t) in $\{(x, t) : x \geq (\leq) \xi(t)\}$ (respectively), that $u^\pm$ are solution of (1.10) in their corresponding region, and that $u^\pm$, $u_t^\pm$, and $u_x^\pm$ agree along $x = \xi(t)$, but some of the second derivatives of $u^\pm$ do not agree along $x = \xi(t)$. Then $\xi(t)$ must satisfy $\xi'(t) = \pm c$, namely, $x = \xi(t) = \pm ct$+constant.

This is seen as follows. Define $\chi(t) := u^\pm(\xi(t), t)$, $\phi(t) := u_x^\pm(\xi(t), t)$, and $\psi(t) := u_t^\pm(\xi(t), t)$. Then $\chi(t)$, $\phi(t)$, $\psi(t)$ are C^1 functions of t, and

$$\phi'(t) = u_{xx}^\pm(\xi(t), t)\xi'(t) + u_{xt}^\pm(\xi(t), t), \tag{2.23}$$

$$\psi'(t) = u_{tx}^\pm(\xi(t), t)\xi'(t) + u_{tt}^\pm(\xi(t), t). \tag{2.24}$$

Together with $u_{tt}^\pm(\xi(t), t) - c^2 u_{xx}^\pm(\xi(t), t) = 0$, these form 3 linear equations in $u_{xx}^\pm(\xi(t), t)$, $u_{tx}^\pm(\xi(t), t)$, and $u_{tt}^\pm(\xi(t), t)$.

A jump discontinuity along $x = \xi(t)$ in one of these second derivatives is interpreted as this 3×3 linear system not determining a unique solution. Thus the determinant of its coefficient matrix must be singular:

$$\det \begin{vmatrix} \xi'(t) & 1 & 0 \\ 0 & \xi'(t) & 1 \\ -c^2 & 0 & 1 \end{vmatrix} = 0.$$

This leads to $|\xi'(t)|^2 - c^2 = 0$. Therefore, $\xi'(t) = \pm c$.

The same holds for solutions defined in terms of the integral formulation (2.21). In fact, a stronger conclusion holds: if $u^\pm$ are C^2 functions of (x,t) in $\{(x,t) : x \geq (\leq)\xi(t)\}$ (respectively), that $u^\pm$ are solution of (1.10) in their corresponding region, and that (2.21) holds in a domain containing the curve $x = \xi(t)$, then, *unless this curve is a characteristic curve*, namely, $x = \pm ct + x_0$ for some x_0, $u^\pm$, together with their derivatives up to order two, must agree along $x = \xi(t)$. This indicates that (2.21) places restrictions on the behavior on piecewise defined solutions: one can't simply piece together two solutions to (1.10) along a non-characteristic curve and expect the resulting function to be a generalized solution.

One way to prove this conclusion is to use what we have done earlier: carrying out integration by parts of the integrals in (2.21) in each region on either side of the curve $x = \xi(t)$, resulting in some integrals along this curve, then examining the implications of the vanishing of the integral along this curve. The details will be omitted here.

Exercises

Exercise 2.8.1. Define

$$E_k(x,t) = \begin{cases} G_k(x-ct) - G_k(x+ct) & \text{if } t > 0, \\ 0 & \text{elsewhere,} \end{cases}$$

where $G_k(x,t)$ is given earlier in the section. Prove that, for any $\eta \in C_c^2(\mathbb{R}^2)$,

$$\begin{aligned} \iint_{\mathbb{R}^2} E_k(x,t) \left[\eta_{tt}(x,t) - c^2\eta_{xx}(x,t)\right] dxdt &= \int_0^\infty \int_{\mathbb{R}} E_k(x,t) \left[\eta_{tt}(x,t) - c^2\eta_{xx}(x,t)\right] dxdt \\ &= \int_{\mathbb{R}} -2cG_k'(x)\eta(x,0)dx, \end{aligned}$$

and that

$$\iint_{\mathbb{R}} E(x,t) \left[\eta_{tt}(x,t) - c^2\eta_{xx}(x,t)\right] dxdt = \eta(0,0),$$

where $E(x,t)$ is the fundamental solution of the wave equation defined earlier in the section.

Exercise 2.8.2. Define

$$E(\cdot,t)*h(x)=\int_{\mathbb{R}}h(x-y)E(y,t)\ dy,$$

and

$$E(\cdot,\cdot)*f(x,t)=\iint E(x-y,t-s)f(y,s)dyds.$$

Verify that

$$E(\cdot,t)*h(x)=\frac{1}{2c}\int_{x-ct}^{x+ct}h(y)dy,$$

and

$$E(\cdot,\cdot)*f(x,t)=\frac{1}{2c}\int_0^t\int_{x-c(t-s)}^{x+c(t-s)}f(y,s)dyds.$$

Exercise 2.8.3. Here is another way to analyze (1.10) and its integral formulation.

(i) Consider the change of variables $(x,t)\mapsto(y,s)$, where $y=x-ct$, $s=x+ct$, and $v(y,s)=u(x,t)$. Verify that $u_{tt}(x,t)-c^2u_{xx}(x,t)=-4c^2v_{ys}(y,s)$.

(ii) Prove that any C^2 solution of $v_{ys}(y,s)=0$ in a rectangle $\mathcal{R}$: $y_0<y<y_1$ and $s_0<s<s_1$ must be of the form of $g(y)+h(s)$ for some C^2 function g and h.

(iii) A continuous function $v(y,s)$ is called a generalized solution of $v_{ys}(y,s)=0$ in $\mathcal{R}$ if $\iint_{\mathcal{R}}v(y,s)\eta_{ys}(y,s)dyds=0$ for any $C_c^2(\mathcal{R})$. For $s_0<s_*<s_1$, let $\mathcal{R}_0=\{(y,s):y_0<y<y_1,s_0<s<s_*\}$, and $\mathcal{R}_1=\{(y,s):y_0<y<y_1,s_*<s<s_1\}$. Suppose that $v(y,s)\in C^2(\overline{\mathcal{R}_0})$ satisfies $v_{ys}(y,s)=0$ in $\overline{\mathcal{R}_0}$, and $v(y,s)\in C^2(\overline{\mathcal{R}_1})$ satisfies $v_{ys}(y,s)=0$ in $\overline{\mathcal{R}_1}$. Denote $v(y,s_*\pm)=\lim_{s\to s_*\pm}v(y,s)$. Prove that $v(y,s)$ is a generalized solution of $v_{ys}(y,s)=0$ in $\mathcal{R}$, iff $v(y,s_*+)-v(y,s_*-)$ is a constant.

(iv) Suppose that $s=\phi(y)$ is a C^1 curve in $\mathcal{R}$ and that $v(y,s)$ is a C^2 solution of $v_{ys}(y,s)=0$ in $\{(y,s)\in\mathcal{R}:s\geq\phi(y)\}$ and in $\{(y,s)\in\mathcal{R}:s\leq\phi(y)\}$ separately, and is a generalized solution of $v_{ys}(y,s)=0$ in $\mathcal{R}$. Prove that at any $(y,s)\in\mathcal{R}$, if $\phi'(s)\neq0$, then $\lim_{h\to0\pm}v(y,s+h)$, $\lim_{h\to0\pm}v_y(y,s+h)$, $\lim_{h\to0\pm}v_s(y,s+h)$ are all respectively equal. Finally prove that $v(y,s)$ is C^2 near (y,s). Translated back to (1.10), conclude that a generalized solution of (1.10) which is piecewise C^2 across a C^1 curve must be C^2 across it unless the curve is of the form $x\pm ct=x_0$ for some constant x_0, in which case the solution can have a certain jump discontinuity.

2.9 Basic Properties of Fourier Transforms*

We first give an argument for the Fourier inversion formula in the case that g is a C^1 function with compact support on $\mathbb{R}$, namely,

$$g(x) = \int_{\mathbb{R}} \hat{g}(\xi) e^{ix\xi}\, d\xi, \text{ with } \hat{g}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}} g(x) e^{-ix\xi}\, dx. \tag{2.25}$$

Let l be large enough that the support of g is contained in $(-l, l)$. Then we know that the full Fourier series of g on $(-l, l)$ converges to $g(x)$ uniformly over $[-l, l]$. The full Fourier series of g on $(-l, l)$ is given by

$$\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}},$$

with $c_n = (2l)^{-1} \int_{-l}^{l} g(y) e^{-\frac{in\pi y}{l}}\, dy = (2l)^{-1} \int_{\mathbb{R}} g(y) e^{-\frac{in\pi y}{l}}\, dy$. This relation holds for any such large l. Since g has compact support,

$$\hat{g}(\xi) = (2\pi)^{-1} \int_{-\infty}^{\infty} g(y) e^{-i\xi y}\, dy = (2\pi)^{-1} \int_{-l}^{l} g(y) e^{-i\xi y}\, dy$$

is defined for all $\xi \in \mathbb{R}$, and $c_n = \frac{\pi}{l} \hat{g}(\xi_n)$, where $\{\xi_n = \frac{n\pi}{l}\}$ are the grid points of the partition of $\mathbb{R}$ into intervals of length $\frac{\pi}{l}$. Then for any x, $-l < x < l$,

$$g(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} = \sum_{n=-\infty}^{\infty} \hat{g}(\xi_n) e^{i\xi_n x} \frac{\pi}{l},$$

which is a Riemann sum for the (improper) integral $\int_{\mathbb{R}} \hat{g}(\xi) e^{i\xi x} d\xi$. We will see momentarily that, under our conditions on g, this improper integral converges absolutely, so in the limit $l \to \infty$, we obtain

$$g(x) = \int_{\mathbb{R}} \hat{g}(\xi) e^{i\xi x} d\xi. \tag{2.26}$$

The convergence of the integral $\int_{\mathbb{R}} |\hat{g}(\xi)|\, d\xi$ uses the Parseval relation for g over $(-l, l)$

$$\int_{-l}^{l} |g(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 2l = \sum_{n=-\infty}^{\infty} \frac{2\pi^2 |\hat{g}(\xi_n)|^2}{l}.$$

The R.H.S. is a Riemann sum of the integral $2\pi \int_{\mathbb{R}} |\hat{g}(\xi)|^2 d\xi$. We justify the integrability of this integral by noting that

$$\hat{g}(\xi_n + t) = (2\pi)^{-1} \int_{-\infty}^{\infty} g(y) e^{-iyt} e^{-i\xi_n y}\, dy = \widehat{g(x)e^{-ixt}}(\xi_n),$$

*This section is included for completeness. Most of the material is not used in a substantial way in the remaining notes; a student may look up the material of this section only when needed.

so the same Parseval relation for $g(x)e^{-ixt}$ over $(-l, l)$ implies that for any t

$$\int_{-l}^{l} |g(x)|^2 \, dx = \int_{-l}^{l} |g(x)e^{-ixt}|^2 \, dx = \sum_{n=-\infty}^{\infty} \frac{2\pi^2 |\hat{g}(\xi_n + t)|^2}{l}.$$

Integrating both sides over $t \in [0, \frac{\pi}{l}]$, we obtain

$$\frac{\pi}{l} \int_{\mathbb{R}} |g(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} \frac{2\pi^2}{l} \int_0^{\frac{\pi}{l}} |\hat{g}(\xi_n + t)|^2 \, dt = \frac{2\pi^2}{l} \int_{\mathbb{R}} |\hat{g}(\xi)|^2 \, d\xi.$$

Eliminating the factors $\frac{\pi}{l}$ on both sides gives us

$$\int_{\mathbb{R}} |g(x)|^2 \, dx = 2\pi \int_{\mathbb{R}} |\hat{g}(\xi)|^2 d\xi \tag{2.27}$$

for any C^1 function g with compact support on $\mathbb{R}$. This is called the Plancherel identity, and it is based on this identity that Fourier transform is extended to all $L^2(\mathbb{R})$ functions.

Now, for $\xi \neq 0$, integrating-by-parts gives us

$$\hat{g}(\xi) = -i\xi^{-1}\widehat{g'}(\xi).$$

The Plancherel relation for g' implies that $\int_{|\xi|>1} |\widehat{g'}(\xi)|^2 \, d\xi$ is convergent. Thus by the Cauchy-Schwarz inequality we have

$$\int_{|\xi|>1} |\hat{g}(\xi)| \, d\xi \leq \left(\int_{|\xi|>1} |\xi|^{-2} \, d\xi \right)^{1/2} \left(\int_{|\xi|>1} |\widehat{g'}(\xi)|^2 \, d\xi \right)^{1/2} < \infty,$$

which justifies the absolute integrability of $\hat{g}(\xi)$ over $\mathbb{R}$.

We have so far used the Parseval relation in its quadratic form; it has a bilinear form:

$$\int_{-l}^{l} g(x)\overline{h(x)} \, dx = 2l \sum_{n=-\infty}^{\infty} c_n \overline{d_n},$$

where $d_n = (2l)^{-1} \int_{-l}^{l} h(x) e^{-\frac{inx}{l}} \, dx$ are the Fourier coefficients of $h(x)$ over $(-l, l)$. This then leads to a bilinear form of the Plancherel identity, first for $C^1(\mathbb{R})$ functions g, h with compact support, then for $g, h \in L^2(\mathbb{R})$ by a density argument:

$$\int_{\mathbb{R}} g(x)\overline{h(x)} \, dx = 2\pi \int_{\mathbb{R}} \hat{g}(\xi)\overline{\hat{h}(\xi)} \, d\xi. \tag{2.28}$$

There are various conventions in defining the Fourier transform. Another commonly used convention is to define

$$\tilde{g}(\xi) = \int_{-\infty}^{\infty} g(y) e^{-i\xi y} \, dy = 2\pi \hat{g}(\xi), \tag{2.29}$$

then (2.26) turns into

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \tilde{g}(\xi)e^{i\xi x}d\xi, \tag{2.30}$$

and the Plancherel identity would take the form of

$$\int_{\mathbb{R}} |g(x)|^2\,dx = (2\pi)^{-1}\int_{\mathbb{R}} |\tilde{g}(\xi)|^2 d\xi. \tag{2.31}$$

Our definition of $\hat{g}(\xi)$ follows closely the convention for the Fourier coefficients of the Fourier series of a function, as reflected in (2.26).

A further commonly used convention is to define

$$\mathcal{F}g(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(y)e^{-i\xi y}\,dy = \sqrt{2\pi}\hat{g}(\xi), \tag{2.32}$$

then (2.26) takes on a symmetric form

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathcal{F}g(\xi)e^{i\xi x}d\xi = \left[\mathcal{F}\circ\mathcal{F}g\right](-x), \tag{2.33}$$

and the Plancherel identity would take the form of

$$\int_{\mathbb{R}} |g(x)|^2\,dx = \int_{\mathbb{R}} |\mathcal{F}g(\xi)|^2 d\xi, \tag{2.34}$$

which shows that $\mathcal{F}$ is an isometry on $L^2(\mathbb{R})$.

However, a property of the Fourier transform involving convolution, to be introduced in Theorem 2.10 below, would carry a power of 2π as a factor when formulated in terms of $\hat{g}$ or $\mathcal{F}$. It turns out that if we define

$$\mathscr{F}g(k) = \int_{-\infty}^{\infty} g(y)e^{-i2\pi ky}\,dy = \tilde{g}(2\pi k) = 2\pi\hat{g}(2\pi k) = \sqrt{2\pi}\mathcal{F}(2\pi k), \tag{2.35}$$

then we still have

$$g(x) = \int_{\mathbb{R}} \mathscr{F}g(k)e^{i2\pi kx}\,dk, \quad \text{and} \quad \int_{\mathbb{R}} |g(x)|^2\,dx = \int_{\mathbb{R}} |\mathscr{F}g(\xi)|^2 d\xi. \tag{2.36}$$

In physics context k is called the wave number while $\xi = 2\pi k$ is called the angular wave number; k^{-1} is called the wave length (the distance between two neighboring peaks).

We have done our discussion so far in $\mathbb{R}$. We extend the definitions to $\mathbb{R}^n$ and summarize the main properties.

For any $g \in L(\mathbb{R}^n)$, we define

$$\hat{g}(\boldsymbol{\xi}) = (2\pi)^{-n}\int_{\mathbb{R}^n} g(\mathbf{x})e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\,d\mathbf{x} \text{ for } \boldsymbol{\xi} \in \mathbb{R}^n, \tag{2.37}$$

and

$$\mathscr{F}g(\mathbf{k}) = (2\pi)^n\hat{g}(2\pi\mathbf{k}) = \int_{\mathbb{R}^n} g(\mathbf{x})e^{-i2\pi\mathbf{x}\cdot\mathbf{k}}\,d\mathbf{x} \text{ for } \mathbf{k} \in \mathbb{R}^n. \tag{2.38}$$

Theorem 2.9. *The following holds for any g, h in $C_c^m(\mathbb{R}^n)$, the space of m-times continuously differentiable functions with compact support, when $m > n/2$.*

(i). $|\boldsymbol{\xi}|^m |\hat{g}(\boldsymbol{\xi})| \to 0$ *as* $\boldsymbol{\xi} \to \infty$.

(ii). $g(\mathbf{x}) = \int_{\mathbb{R}^n} \hat{g}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^n} \mathscr{F} g(\mathbf{k}) e^{i2\pi\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}$.

(iii). $\mathscr{F} \circ \mathscr{F}(g)(\mathbf{x}) = g(-\mathbf{x})$.

(iv). $\int_{\mathbb{R}^n} g(\mathbf{x})\overline{h(\mathbf{x})}\, d\mathbf{x} = (2\pi)^n \int_{\mathbb{R}^n} \hat{g}(\boldsymbol{\xi})\overline{\hat{h}(\boldsymbol{\xi})}\, d\boldsymbol{\xi} = \int_{\mathbb{R}^n} \mathscr{F}(g)(\mathbf{k})\overline{\mathscr{F}(h)(\mathbf{k})}\, d\mathbf{k}$.

(v). $\int_{\mathbb{R}^n} |g(\mathbf{x})|^2\, d\mathbf{x} = (2\pi)^n \int_{\mathbb{R}^n} |\hat{g}(\boldsymbol{\xi})|^2\, d\boldsymbol{\xi} = \int_{\mathbb{R}^n} |\mathscr{F}(g)(\mathbf{k})|^2\, d\mathbf{k}$.

(vi). $g\widehat{(\mathbf{x}+\mathbf{y})}(\boldsymbol{\xi}) = e^{i\boldsymbol{\xi}\cdot\mathbf{y}}\hat{g}(\boldsymbol{\xi})$.

(vii). $g\widehat{(\mathbf{x})e^{i\boldsymbol{\xi}'\cdot\mathbf{x}}}(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi} - \boldsymbol{\xi}')$.

(viii). $\widehat{D_{x_j} g(\mathbf{x})}(\boldsymbol{\xi}) = i\xi_j \hat{g}(\boldsymbol{\xi}); \mathscr{F}\left(D_{x_j} g(\mathbf{x})\right)(\mathbf{k}) = i2\pi k_j \mathscr{F} g(\mathbf{k})$.

These properties, except for the first and last one, also hold for $g, h \in L^2(\mathbb{R}^n)$ by a density argument.

Even for $g \in L(\mathbb{R}^n)$ one can show that $\hat{g}(\mathbf{y}) \to 0$ as $\mathbf{y} \to \infty$. The condition $m > n/2$ is imposed to make sure that $\hat{g}(\mathbf{y})$ has sufficiently fast decay to be integrable in $\mathbb{R}^n$ so the integrals in (ii) converge absolutely; relaxing this condition would necessitate the interpretation of (ii) as elements of $L^2(\mathbb{R}^n)$.

As a consequence of (iii) and (iv), $\mathscr{F}$ is invertible on $L^2(\mathbb{R}^n)$* and in fact is an isometry on it. (iii) also implies that $\mathscr{F}^4$ is the identity map, so as a consequence, if $\mathscr{F}$ has any eigenvalue λ, it must be a 4th root of 1.

The convolution plays an important role in studying Fourier transforms. The convolution between ϕ, $\eta \in C_c(\mathbb{R}^n)$ is defined as

$$\phi * \eta(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(\mathbf{x}-\mathbf{y})\eta(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^n} \phi(\mathbf{y})\eta(\mathbf{x}-\mathbf{y})\, d\mathbf{y}.$$

Heuristically it is an average of ϕ weighted by η (or average of η weighted by ϕ).

Here are some basic properties of convolution.

Theorem 2.10. *Suppose that $\phi, \psi, \eta \in C_c(\mathbb{R}^n)$ and $p, q, r \geq 1$ satisfy $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, then*

*Note that $\mathscr{F}$ does not map $C_c^m(\mathbb{R}^n)$ into $C_c^m(\mathbb{R}^n)$ so does not define an invertible map on $C_c^m(\mathbb{R}^n)$; our condition on m guarantees that $\mathscr{F} : C_c^m(\mathbb{R}^n) \mapsto L(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is well defined so the integrals in $\mathscr{F} \circ \mathscr{F}$ converge absolutely when acting on $C_c^m(\mathbb{R}^n)$.

(a). $\phi * \eta(\mathbf{x}) = \eta * \phi(\mathbf{x})$.

(b). $(\phi * \psi) * \eta = \phi * (\psi * \eta)$.

(c). $\mathscr{F}(\phi * \eta) = \mathscr{F}(\phi)\mathscr{F}(\eta)$ *and* $\widehat{\phi * \eta} = (2\pi)^n \hat{\phi}\hat{\eta}$.

(d). $\|\phi * \eta\|_{L^r(\mathbb{R}^n)} \leq \|\phi\|_{L^p(\mathbb{R}^n)} \|\eta\|_{L^q(\mathbb{R}^n)}$.

(d) allows to extend $\phi * \eta$ *as an element in* $L^r(\mathbb{R}^n)$ *when* $\phi \in L^p(\mathbb{R}^n)$ *and* $\eta \in L^q(\mathbb{R}^n)$.

We will only provide a proof for (c) and the $q = 1$ special case of (d).

Proof of (c) and $q = 1$ *case of (d) of Theorem 2.10.*

$$\begin{aligned}
\mathscr{F}(\phi * \eta)(\mathbf{k}) &= \int_{\mathbb{R}^n} \phi * \eta(\mathbf{x}) e^{-i2\pi \mathbf{x}\cdot\mathbf{k}}\, d\mathbf{x} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(\mathbf{x}-\mathbf{y})\eta(\mathbf{y}) e^{-i2\pi \mathbf{x}\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \phi(\mathbf{x})\eta(\mathbf{y}) e^{-i2\pi (\mathbf{x}+\mathbf{y})\cdot\mathbf{k}}\, d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \phi(\mathbf{x}) e^{-i2\pi \mathbf{x}\cdot\mathbf{k}}\, d\mathbf{x} \right) \eta(\mathbf{y}) e^{-i2\pi \mathbf{y}\cdot\mathbf{k}}\, d\mathbf{y} \\
&= \mathscr{F}(\phi)(\mathbf{k})\mathscr{F}(\eta)(\mathbf{k}).
\end{aligned}$$

The $q = 1$ case of (d) is an integral form of the Minkowski inequality with $|\eta(\mathbf{y})|\, d\mathbf{y}$ as weight, as in this case $r = p$, so

$$\begin{aligned}
\|\phi * \eta\|_{L^p(\mathbb{R}^n)} &\leq \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \phi(\mathbf{x}-\mathbf{y})\eta(\mathbf{y})\, d\mathbf{y} \right|^p d\mathbf{x} \right)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\phi(\mathbf{x}-\mathbf{y})|^p\, d\mathbf{x} \right)^{1/p} |\eta(\mathbf{y})|\, d\mathbf{y} \\
&= \|\phi\|_{L^p(\mathbb{R}^n)} \|\eta\|_{L^1(\mathbb{R}^n)}.
\end{aligned}$$

□

Here is a typical example of how Fourier transform is used in constructing solutions of a linear PDE with constant coefficients. To construct a solution to

$$u_{tt} - c^2 u_{xx} + \alpha u_x + \beta u = f(x,t), \quad \text{in} \quad \mathbb{R} \times [0, \infty),$$

for each t one treats both sides of the above equation as functions of x and takes Fourier transform on both sides—assuming that Fourier transform of each of the terms is well defined. Using the linearity and the above properties and denoting

$\hat{u}(\xi, t)$ and $\hat{f}(\xi, t)$ as the respective Fourier transform of $u(x,t)$ and $f(x,t)$ in x, and assuming $\widehat{u(\cdot,t)_{tt}}(\xi) = \hat{u}(\xi,t)_{tt}$, we have

$$\hat{u}(\xi,t)_{tt} + c^2\xi^2\hat{u}(\xi,t) + i\alpha\xi\hat{u}(\xi,t) + \beta\hat{u}(\xi,t) = \hat{f}(\xi,t),$$

which produces an ODE in $\hat{u}(\xi,t)$ in the t variable, with ξ as a parameter. For simplicity, let's take $\alpha = 0$, $\beta > 0$, and $f \equiv 0$, then

$$\hat{u}(\xi,t) = A(\xi)\cos\left(\sqrt{c^2\xi^2+\beta}t\right) + B(\xi)\sin\left(\sqrt{c^2\xi^2+\beta}t\right)$$

for some constants $A(\xi), B(\xi)$. Suppose that $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$, then setting $t = 0$ gives us

$$A(\xi) = \widehat{u_0}(\xi), \quad \sqrt{c^2\xi^2+\beta}B(\xi) = \widehat{u_1}(\xi).$$

In the final construction of $u(x,t)$, if we are able to identify an appropriate $S(x,t)$ such that $\hat{S}(\xi,t) = \sin\left(\sqrt{c^2\xi^2+\beta}t\right)/\sqrt{c^2\xi^2+\beta}$, then we would have the relation

$$\hat{u}(\xi,t) = \widehat{u_0}(\xi)\partial_t\hat{S}(\xi,t) + \widehat{u_1}(\xi)\hat{S}(\xi,t).$$

We would then be able to use (c) to identify $u(x,t)$ as

$$(2\pi)^{-1}\left(u_0 * \partial_t S(x,t) + u_1 * S(x,t)\right).$$

Unfortunately it is not a particularly simple task to identify $S(x,t)$ using the definition and properties of Fourier transforms—in fact, very rarely can we compute the Fourier transform of a function explicitly, or identify a function directly as an explicit (elementary) function via its Fourier transform; but there are tools to read-off information about a function through its Fourier transform. Note that $\hat{S}(\xi,t) \in L^2(\mathbb{R})$, but not in $L(\mathbb{R})$, so one can't directly use the integral $\int_{\mathbb{R}} \hat{S}(\xi,t)e^{i\xi x}\,d\xi$ to identify $S(x,t)$.

To make this procedure rigorous, one needs to assume that all the terms in the PDE are in appropriate function spaces for which the properties of the Fourier transform in Theorem 2.9 are valid, and also has to justify that one can exchange the order of differentiation in t and Fourier transform, namely, $\widehat{u(x,t)_{tt}} = \widehat{u(x,t)}_{tt}$. Usually one does not expect to settle the existence and uniqueness of the solution at the same time, so needs not justify each step in constructing a solution candidate. One can use the heuristic idea that $u(x,t)$ is the superposition of its "Fourier modes" $\hat{u}(\xi,t)e^{i\xi x}$, and each of this Fourier mode should satisfy the same equation. Plugging this into the equation (and replacing $f(x,t)$ by its Fourier mode $\hat{f}(\xi,t)$ at "frequency" ξ) would produce an equation for $\hat{u}(\xi, t)$—the same equation as obtained by using the properties of the Fourier transform and one does not need to memorize the formal properties in Theorem 2.9 in carrying out this procedure.

The Gaussian function, e^{-ax^2}, $a > 0$, is one of the rare functions whose Fourier transform can be computed explicitly, and it is used in many contexts. In fact we did the computation in deriving (2.18). The derivation contained the following computation for $t > 0$

$$\int_{\mathbb{R}} e^{ix\xi-\xi^2 t} d\xi = \sqrt{\frac{\pi}{t}} e^{-\frac{x^2}{4t}},$$

which implies that

$$\widehat{e^{-tx^2}} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{\xi^2}{4t}}; \quad \widehat{\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}} = \frac{1}{2\pi} e^{-t\xi^2}; \quad \mathscr{F}(e^{-tx^2}) = \sqrt{\frac{\pi}{t}} e^{-\frac{\pi^2\xi^2}{t}}.$$

Our earlier derivation for solutions of (1.20) amounts to

$$\widehat{u(\cdot, t)}(\xi) = \widehat{u(\cdot, 0)}(\xi) e^{-\xi^2 t}.$$

By property (c) of Theorem 2.10, we also have

$$\widehat{u(\cdot, 0) * \frac{1}{\sqrt{4\pi t}} e^{-\frac{|\cdot|^2}{4t}}}(\xi) = \widehat{u(\cdot, 0)}(\xi) e^{-\xi^2 t},$$

from which we conclude that

$$u(x, t) = u(x, 0) * \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} = \int_{\mathbb{R}} u(x-y, 0) \frac{1}{\sqrt{4\pi t}} e^{-\frac{y^2}{4t}}\, dy.$$

The same technique can be applied to any function $f \in L^p(\mathbb{R}^n)$ (for $p = 1$ or 2 now) to give

$$\widehat{f * \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\cdot|^2}{4t}}}(\boldsymbol{\xi}) = \widehat{f}(\boldsymbol{\xi}) e^{-|\boldsymbol{\xi}|^2 t}.$$

Since the R.H.S. above decays sufficiently fast in ξ for $t > 0$, we have

$$f * \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\cdot|^2}{4t}}(\mathbf{x}) = \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{-|\boldsymbol{\xi}|^2 t + i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}.$$

Recall that the family $\left\{K(\mathbf{x}, t) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\mathbf{x}|^2}{4t}}\right\}_{t>0}$ forms an approximation of identity as $t \searrow 0$, namely, they satisfy (AP1)–(AP3), which were established for the $n = 1$ case right after (2.18). We have

Theorem 2.11. *For any $u \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$,*

$$u * \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\cdot|^2}{4t}}(\mathbf{x}) \to u \text{ in } L^p(\mathbb{R}^n).$$

Using this theorem we conclude that, for $p = 1$ or 2,

$$f(\mathbf{x}) = \lim_{t\searrow 0} f * \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\cdot|^2}{4t}}(\mathbf{x}) = \lim_{t\searrow 0} \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{-|\boldsymbol{\xi}|^2 t + i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}$$

in the sense of $L^p(\mathbb{R}^n)$.

This is a generalization of the Fourier inversion formula. When it is known that $\widehat{f} \in L(\mathbb{R}^n)$, the limit on the R.H.S exists point-wise and equals $\int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}$. Thus when both f and $\widehat{f}$ are in $L(\mathbb{R}^n)$, we have established the point-wise inversion relation:

$$f(\mathbf{x}) = \int_{\mathbb{R}^n} \widehat{f}(\boldsymbol{\xi}) e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}.$$

We will later extend Fourier transform to a much class of objects, called *tempered distributions*, to make this tool more widely applicable.

Proof of Theorem 2.11. Set

$$u(\mathbf{x}; t) = \int_{\mathbb{R}^n} u(\mathbf{y}) K(\mathbf{x} - \mathbf{y}, t)\, d\mathbf{y}.$$

Using (AP1) we have

$$u(\mathbf{x}; t) - u(\mathbf{x}) = \int_{\mathbb{R}^n} u(\mathbf{y}) K(\mathbf{x} - \mathbf{y}, t)\, dy - u(\mathbf{x}) = \int_{\mathbb{R}^n} \left[u(\mathbf{x} - \mathbf{z}) - u(\mathbf{x})\right] K(\mathbf{z}, t)\, d\mathbf{z},$$

We now apply the integral version of Minkowski* inequality to get

$$\begin{aligned} &||u(\mathbf{x}; t) - u(\mathbf{x})||_{L^p(\mathbb{R}^n)} \\ =&|| \int_{\mathbb{R}^n} \left[u(\mathbf{x} - \mathbf{z}) - u(\mathbf{x})\right] K(\mathbf{z}, t)\, d\mathbf{z}||_{L^p(\mathbb{R}^n)} \\ \le& \int_{\mathbb{R}^n} ||u(\mathbf{x} - \mathbf{z}) - u(\mathbf{x})||_{L^p(\mathbb{R}^n)} K(\mathbf{z}, t)\, d\mathbf{z} \\ \le& \int_{|\mathbf{z}|\le\delta} ||u(\mathbf{x} - \mathbf{z}) - u(\mathbf{x})||_{L^p(\mathbb{R}^n)} K(\mathbf{z}, t)\, d\mathbf{z} + \int_{|\mathbf{z}|>\delta} ||u(\mathbf{x} - \mathbf{z}) - u(\mathbf{x})||_{L^p(\mathbb{R}^n)} K(\mathbf{z}, t)\, d\mathbf{z}. \end{aligned}$$

Since $p < \infty$, we will use the continuity property of translation in $L^p(\mathbb{R}^n)$, namely, for any $\epsilon > 0$, there exists $\delta > 0$ such that for $|\mathbf{z}| \le \delta$, we have

$$||u(\mathbf{x} - \mathbf{z}) - u(\mathbf{x})||_{L^p(\mathbb{R}^n)} < \epsilon,$$

*Minkowski inequality and its integral version are all reflections of the convexity property of the L^p norm, which can be schematically expressed as the L^p norm of a convex combination of a family of functions is $\le$ the convex combination of the L^p norms of this family of functions.

which implies

$$\int_{|\mathbf{z}|\leq\delta} ||u(\mathbf{x}-\mathbf{z})-u(\mathbf{x})||_{L^p(\mathbb{R}^n)} K(\mathbf{z},t)\,d\mathbf{z} < \epsilon \int_{|\mathbf{z}|\leq\delta} K(\mathbf{z},t)\,d\mathbf{z} < \epsilon,$$

using (AP1) and (AP3). Using (AP2) and (AP3),

$$\int_{|\mathbf{z}|>\delta} ||u(\mathbf{x}-\mathbf{z})-u(\mathbf{x})||_{L^p(\mathbb{R}^n)} K(\mathbf{z},t)\,d\mathbf{z} \leq 2||u(\mathbf{x})||_{L^p(\mathbb{R}^n)} \int_{|\mathbf{z}|>\delta} K(\mathbf{z},t)\,d\mathbf{z} < \epsilon$$

when $t > 0$ is sufficiently small.

□

Exercises

Exercise 2.9.1. Find $\widehat{\chi_{[-a,a]}}$, where $\chi_{[-a,a]}$ is the characteristic function of the interval $[-a,a]$. Can you find the Fourier transform of $\widehat{\chi_{[-a,a]}}$ using the definition directly?

Exercise 2.9.2. Use Fourier transform to construct a solution to

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0, & (x,t) \in \mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = g(x), & x \in \mathbb{R} \\ u_t(x,0) = h(x), & x \in \mathbb{R}. \end{cases}$$

What modifications are needed when $x \in \mathbb{R}$ is replaced by $\mathbf{x} \in \mathbb{R}^n$?

Exercise 2.9.3. Find $\widehat{e^{-|x|}}$ and $\widehat{e^{-a|x|}}$ in $\mathbb{R}$, where $a > 0$.

Exercise 2.9.4. Construct a solution to

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0, & (x,y) \in \mathbb{R}\times\mathbb{R}^+ \\ u(x,0) = g(x), & x \in \mathbb{R}. \end{cases}$$

Can you extend your approach to the higher dimensional case on $\mathbb{R}^n \times \mathbb{R}^+$ for $n > 1$?

Exercise 2.9.5. Prove that for $f, g \in L(\mathbb{R}^n)$ we have $\int_{\mathbb{R}^n} \hat{f}(\mathbf{x})g(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^n} f(\mathbf{x})\hat{g}(\mathbf{x})\,d\mathbf{x}$.

2.10 $H^s(\mathbb{R}^n)$ and Sobolev's inequality

For any $k \in \mathbb{N}$, define $H^k(\mathbb{R}^n)$ to be the completion of $C_c^\infty(\mathbb{R}^n)$ in the norm

$$||u||_{H^k(\mathbb{R}^n)} := \left(\sum_{|\alpha|=0}^{k} ||\partial_{\mathbf{x}}^\alpha u||^2_{L^2(\mathbb{R}^n)} \right)^{1/2}.$$

Thus for any $u \in H^k(\mathbb{R}^n)$, there exists a sequence $\{u_j\} \subset C_c^\infty(\mathbb{R}^n)$ such that $u_j - u \to 0$ in $L^2(\mathbb{R}^n)$, and for any α with $|\alpha| \le k$, $\{\partial_{\mathbf{x}}^\alpha u_j\}$ is Cauchy in $L^2(\mathbb{R}^n)$. The $L^2(\mathbb{R}^n)$ limit of $\{\partial_{\mathbf{x}}^\alpha u_j\}$ is then the $L^2(\mathbb{R}^n)$ $\partial_{\mathbf{x}}^\alpha$-derivative of u, as for any $\phi \in C_c^\infty(\mathbb{R}^n)$, the relation

$$\int_{\mathbb{R}^n} \partial_{\mathbf{x}}^\alpha u_j(\mathbf{x})\phi(\mathbf{x})\,dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u_j(\mathbf{x})\partial_{\mathbf{x}}^\alpha \phi(\mathbf{x})\,d\mathbf{x}$$

turns in the limit into

$$\int_{\mathbb{R}^n} \partial_{\mathbf{x}}^\alpha u(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} = (-1)^{|\alpha|} \int_{\mathbb{R}^n} u(\mathbf{x})\partial_{\mathbf{x}}^\alpha \phi(\mathbf{x})\,d\mathbf{x},$$

with $u, \partial_{\mathbf{x}}^\alpha u(\mathbf{x}) \in L^2(\mathbb{R}^n)$. $\partial_{\mathbf{x}}^\alpha u(\mathbf{x})$ is called the $L^2(\mathbb{R}^n)$ $\partial_{\mathbf{x}}^\alpha$ (weak) derivative of u.

The above formulation of weak derivative can be extended to more general contexts.

Definition. Let D be an open domain in $\mathbb{R}^n$ and $1 < p < \infty$. A function $u \in L^p(D)$ is said to have weak derivatives in $L^p(D)$ of order k, if for every multi-index α with $|\alpha| \le k$, there exists a function $u_\alpha \in L^p(D)$ such that

$$\int_D u_\alpha(\mathbf{x})\phi(\mathbf{x})\,d\mathbf{x} = (-1)^{|\alpha|} \int_D u(\mathbf{x})\,\partial_{\mathbf{x}}^\alpha \phi(\mathbf{x})\,d\mathbf{x},$$

for all $\phi \in C_c^\infty(D)$. u_α is called the $L^p(D)$ weak derivative $\partial_{\mathbf{x}}^\alpha$ derivative of u.

The set of $L^p(D)$ functions with weak derivatives in $L^p(D)$ of order k is denoted as $W^{k,p}(D)$.

Theorem 2.12. *For any $k \in \mathbb{N}$, $W^{k,2}(\mathbb{R}^n) = H^k(\mathbb{R}^n)$.*

Proof. Our argument above implies that $H^k(\mathbb{R}^n) \subset W^{k,2}(\mathbb{R}^n)$.

For the converse, fix a non-negative $\rho \in C_c^\infty(B_1(0))$ with $\int_{B_1(0)} \rho(\mathbf{x})\,d\mathbf{x} = 1$ and $\rho(\mathbf{x}) = \rho(0) = \max_{B_1(0)} \rho$ for $\mathbf{x} \in B_{\frac{1}{2}}(0)$. Define $\rho_\epsilon(\mathbf{x}) = \epsilon^{-n}\rho(\epsilon^{-1}\mathbf{x})$ for $\epsilon > 0$. Then $\rho_\epsilon \in C_c^\infty(B_\epsilon(0))$. For any $u \in W^{k,2}(\mathbb{R}^n)$, our conclusion would follow if we can establish

(i). $\rho_\epsilon * u \to u$ and $\rho_\epsilon * u_\alpha \to u_\alpha$ in $L^2(\mathbb{R}^n)$ as $\epsilon \to 0$ for all α with $|\alpha| \le k$.

(ii). $\rho_\epsilon * u \in C^\infty(\mathbb{R}^n)$ and $\partial_{\mathbf{x}}^\alpha (\rho_\epsilon * u) = \rho_\epsilon * u_\alpha$ for all α with $|\alpha| \le k$.

(iii). $\rho(0)^{-1}\rho(\epsilon\mathbf{x})\rho_\epsilon * u \in C_c^\infty(\mathbb{R}^n)$ converges to u in $H^k(\mathbb{R}^n)$.

(i) is proved using an integral version of the Minkowski inequality as follows.

$$\begin{aligned}
&\|\rho_\epsilon * u(\mathbf{x}) - u(\mathbf{x})\|_{L^2_{\mathbf{x}}(\mathbb{R}^n)}\\
=&\|\int_{\mathbb{R}^n} \rho_\epsilon(\mathbf{y})\left(u(\mathbf{x}-\mathbf{y}) - u(\mathbf{x})\right) d\mathbf{y}\|_{L^2_{\mathbf{x}}(\mathbb{R}^n)}\\
\le&\int_{\mathbb{R}^n} \rho_\epsilon(\mathbf{y})\|u(\mathbf{x}-\mathbf{y}) - u(\mathbf{x})\|_{L^2_{\mathbf{x}}(\mathbb{R}^n)}\, d\mathbf{y}\\
=&\int_{\|\mathbf{y}\|\le\epsilon} \rho_\epsilon(\mathbf{y})\|u(\mathbf{x}-\mathbf{y}) - u(\mathbf{x})\|_{L^2_{\mathbf{x}}(\mathbb{R}^n)}\, d\mathbf{y}.
\end{aligned}$$

$u \in L^2(\mathbb{R}^n)$ has the property that for any $\sigma > 0$ there exists $\epsilon_0 > 0$ such that $\|u(\mathbf{x}-\mathbf{y}) - u(\mathbf{x})\|_{L^2_{\mathbf{x}}(\mathbb{R}^n)} < \sigma$ for all $\|\mathbf{y}\| \le \epsilon < \epsilon_0$. Using this and $\int_{\|\mathbf{y}\|\le\epsilon} \rho_\epsilon(\mathbf{y})\, d\mathbf{y} = 1$ we conclude (i).

The first statement of (ii) is a standard property of convolution. For the second part, in the defining property for u_α we replace $\phi(\mathbf{x})$ by $\phi(\mathbf{x}-\mathbf{y})$, $\mathbf{y}$ is any vector in $\mathbb{R}^n$. After a change of variables this leads to

$$\int_{\mathbb{R}^n} u_\alpha(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\, d\mathbf{x} = (-1)^{|\alpha|}\int_{\mathbb{R}^n} u(\mathbf{x}-\mathbf{y})\, \partial_{\mathbf{x}}^\alpha \phi(\mathbf{x})\, d\mathbf{x}.$$

Since $\rho_\epsilon * u \in C^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned}
&(-1)^{|\alpha|}\int_{\mathbb{R}^n} \partial_{\mathbf{x}}^\alpha\left(\rho_\epsilon * u(\mathbf{x})\right)\phi(\mathbf{x})\, d\mathbf{x}\\
=&\int_{\mathbb{R}^n} \left(\rho_\epsilon * u(\mathbf{x})\right)\partial_{\mathbf{x}}^\alpha\phi(\mathbf{x})\, d\mathbf{x}\\
=&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_\epsilon(\mathbf{y})u(\mathbf{x}-\mathbf{y})\partial_{\mathbf{x}}^\alpha\phi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{y}\\
=&(-1)^{|\alpha|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_\epsilon(\mathbf{y})u_\alpha(\mathbf{x}-\mathbf{y})\phi(\mathbf{x})\, d\mathbf{x}\, d\mathbf{y}\\
=&(-1)^{|\alpha|}\int_{\mathbb{R}^n} \rho_\epsilon * u_\alpha\, \phi(\mathbf{x})\, d\mathbf{x}
\end{aligned}$$

In the above the integrals in both $\mathbf{x}$ and $\mathbf{y}$ are done in a compact set over which the integrands are absolutely integrable so the interchange of integrals is justified. Since this holds for all $\phi \in C_c^\infty(\mathbb{R}^n)$, it follows that $\partial_{\mathbf{x}}^\alpha\left(\rho_\epsilon * u(\mathbf{x})\right) = \rho_\epsilon * u_\alpha$.

For (iii), we first note that

$$\begin{aligned}
&\|\rho(0)^{-1}\rho(\epsilon\mathbf{x})\rho_\epsilon * u - u\|_{L^2(\mathbb{R}^n)}\\
\le&\|\rho(0)^{-1}\rho(\epsilon\mathbf{x})\left(\rho_\epsilon * u - u\right)\|_{L^2(\mathbb{R}^n)} + \|\left(\rho(0)^{-1}\rho(\epsilon\mathbf{x}) - 1\right)u\|_{L^2(\mathbb{R}^n)}\\
\le&\|\rho_\epsilon * u - u\|_{L^2(\mathbb{R}^n)} + \|\left(\rho(0)^{-1}\rho(\epsilon\mathbf{x}) - 1\right)u\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Using $\|\rho_\epsilon * u - u\|_{L^2(\mathbb{R}^n)} \to 0$ as $\epsilon \to 0$ and

$$\|\left(\rho(0)^{-1}\rho(\epsilon\mathbf{x}) - 1\right)u\|_{L^2(\mathbb{R}^n)} \le \left(\int_{(2\epsilon)^{-1}\le\|\mathbf{x}\|\le\epsilon^{-1}} u(\mathbf{x})^2\, dx\right)^{1/2} \to 0$$

as $\epsilon \to 0$, it follows that $\|\rho(0)^{-1}\rho(\epsilon\mathbf{x})\rho_\epsilon * u - u\|_{L^2(\mathbb{R}^n)} \to 0$ as $\epsilon \to 0$. The convergence in $L^2(\mathbb{R}^n)$ of the derivatives follows in a similar fashion. □

Note that by the Plancherel Theorem for Fourier Transforms,

$$||\partial_{\mathbf{x}}^\alpha u||^2_{L^2(\mathbb{R}^n)} = (2\pi)^n ||\widehat{\partial_{\mathbf{x}}^\alpha u}||^2_{L^2(\mathbb{R}^n)} = (2\pi)^n ||\boldsymbol{\xi}^\alpha \widehat{u}(\boldsymbol{\xi})||^2_{L^2(\mathbb{R}^n)},$$

so

$$||u||_{H^k(\mathbb{R}^n)} = (2\pi)^{n/2} \left(\int_{\mathbb{R}^n} \left[\sum_{|\alpha|=0}^{k} |\boldsymbol{\xi}^\alpha|^2 \right] |\widehat{u}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right)^{1/2}$$

which is equivalent to $||(1+|\boldsymbol{\xi}|)^k \widehat{u}(\boldsymbol{\xi})||_{L^2(\mathbb{R}^n)}$. If $k > n/2$, using $||(1+|\boldsymbol{\xi}|)^{-k}||_{L^2(\mathbb{R}^n)} < \infty$, we see that for any $u \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} |u(\mathbf{x})| \le (2\pi)^n \int_{\mathbb{R}^n} |\widehat{u}(\boldsymbol{\xi})| d\boldsymbol{\xi} \le (2\pi)^n ||(1+|\boldsymbol{\xi}|)^k \widehat{u}(\boldsymbol{\xi})||_{L^2(\mathbb{R}^n)} ||(1+|\boldsymbol{\xi}|)^{-k}||_{L^2(\mathbb{R}^n)} \\ \le C(n,k)||u||_{H^k(\mathbb{R}^n)}. \end{aligned} \tag{2.39}$$

(2.39) continues to hold for any $u \in H^k(\mathbb{R}^n)$. This is part of **Sobolev imbedding** Theorem.

In fact, even for non-integer $s \in \mathbb{R}$, $||(1+|\boldsymbol{\xi}|)^s \widehat{u}(\boldsymbol{\xi})||_{L^2(\mathbb{R}^n)}$ defines a norm on $C_c^\infty(\mathbb{R}^n)$. The completion of $C_c^\infty(\mathbb{R}^n)$ in this norm is denoted as $H^s(\mathbb{R}^n)$. (2.39) continues to hold if k there is replaced by a non-integer s as long as $s > n/2$.

Theorem 2.13. *For any $s > n/2$, and any $u \in H^s(\mathbb{R}^n)$, u must be bounded and continuous in $\mathbb{R}^n$, and (2.39) holds. This is expressed as $H^s(\mathbb{R}^n) \subset C_b(\mathbb{R}^n)$. Furthermore, if $s - n/2 > m \in \mathbb{N}$, then any $u \in H^s(\mathbb{R}^n)$ has bounded and continuous derivatives for orders up to m in $\mathbb{R}^n$, and for each α with $|\alpha| \le m$,*

$$\sup_{\mathbb{R}^n} |\partial_{\mathbf{x}}^\alpha u(\mathbf{x})| \le C(n,s)||u||_{H^s(\mathbb{R}^n)}. \tag{2.40}$$

A version of (2.40) holds for functions $W^{k,p}(D)$ when $kp > n$ under suitable regularity assumption on D.

We remark that the norm on $H^s(\mathbb{R}^n)$ is induced by an inner product defined as

$$(u,v) = \int_{\mathbb{R}^n} |(1+|\boldsymbol{\xi}|)^{2s} \widehat{u}(\boldsymbol{\xi})\overline{\widehat{v}(\boldsymbol{\xi})}\, d\boldsymbol{\xi} \quad \text{for } u, v \in H^s(\mathbb{R}^n).$$

Using the inequality

$$\left| \int_{\mathbb{R}^n} \widehat{u}(\boldsymbol{\xi})\widehat{v}(-\boldsymbol{\xi})\, d\boldsymbol{\xi} \right| \le \left(\int_{\mathbb{R}^n} |(1+|\boldsymbol{\xi}|)^{2s} |\widehat{u}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right)^{1/2} \left(\int_{\mathbb{R}^n} |(1+|-\boldsymbol{\xi}|)^{-2s} |\widehat{v}(-\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \right)^{1/2}$$

and the relation $\int_{\mathbb{R}^n} u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} = (2\pi)^n \int_{\mathbb{R}^n} \widehat{u}(\boldsymbol{\xi})\widehat{v}(-\boldsymbol{\xi})\,d\boldsymbol{\xi}$ for $u, v \in C_c^\infty(\mathbb{R}^n)$ (based on (iv) of Theorem 2.9), we can use the right hand side to extend the definition of $\int_{\mathbb{R}^n} u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x}$ to $u \in H^s(\mathbb{R}^n), v \in H^{-s}(\mathbb{R}^n)$, which gives a well-defined bilinear pairing between $H^s(\mathbb{R}^n)$ and $H^{-s}(\mathbb{R}^n)$ defined by

$$\langle u, v\rangle := (2\pi)^n \int_{\mathbb{R}^n} |(1+|\boldsymbol{\xi}|)^s \widehat{u}(\boldsymbol{\xi})(1+|\boldsymbol{\xi}|)^{-s}\widehat{v}(-\boldsymbol{\xi})\,d\boldsymbol{\xi} \quad \text{for } u \in H^s(\mathbb{R}^n), v \in H^{-s}(\mathbb{R}^n),$$

such that, for each $v \in H^{-s}(\mathbb{R}^n)$, $u \mapsto \langle u, v\rangle$ defines a continuous linear functional on $u \in H^s(\mathbb{R}^n)$. This notion of pairing will become clearer after we introduce tempered distribution later.

Noting that $H^t(\mathbb{R}^n) \subset H^{t'}(\mathbb{R}^n)$ whenever $t \geq t'$, we see that if $s + t \geq 0$, then $\int_{\mathbb{R}^n} u(\mathbf{x})v(\mathbf{x})\,d\mathbf{x}$ extends to a well-defined pairing between $u \in H^s(\mathbb{R}^n)$ and $v \in H^t(\mathbb{R}^n)$. Finally $u \in H^s(\mathbb{R}^n) \mapsto \partial_{\mathbf{x}} u \in H^{s-1}(\mathbb{R}^n)$ is naturally defined, which allows us to discuss linear partial differential operators (with appropriately smooth coefficients) in the framework of $H^s(\mathbb{R}^n)$.

Example 2.2. We now construct solutions to the Schrödinger equation (1.27), normalized in the form $(i\partial_t + \partial_x^2)u(x,t) = 0$ for $(x,t) \in \mathbb{R}\times\mathbb{R}$. If we use separation of variables to look for a solution of the form $T(t)e^{i\xi x}$ for some $\xi \in \mathbb{R}$, we find $iT'(t) - \xi^2 T(t) = 0$, from which we get $T(t) = T(0)e^{-i\xi^2 t}$. Thus for every $\xi \in \mathbb{R}$, $u(x,t) = e^{i(\xi x - \xi^2 t)}$ is a solution. This is equivalent to taking the Fourier transform in the x-variable on both sides, which would give us $i\partial_t \hat{u}(\xi,t) - \xi^2\hat{u}(\xi,t) = 0$.

The above process gives us $\hat{u}(\xi,t) = \hat{u}(\xi,0)e^{-i\xi^2 t}$. Thus we construct a solution in the form of

$$u(x,t) = \int_{\mathbb{R}} \hat{u}(\xi,0)e^{i(\xi x - \xi^2 t)}\,d\xi.$$

Note that $|\hat{u}(\xi,t)| = |\hat{u}(\xi,0)|$, so if $u(\cdot,0) \in H^s(\mathbb{R})$, then so is $u(\cdot,t)$ with $||u(\cdot,t)||_{H^s(\mathbb{R})} = ||u(\cdot,0)||_{H^s(\mathbb{R})}$.

To make $u(x,t)$ a twice continuously differentiable function in x, it suffices to work with a $u(x,0)$ such that $\hat{u}(\xi,0)$ decays sufficiently fast as $|\xi| \to \infty$. $u(x,0) \in C_c^\infty(\mathbb{R}^n)$ will certainly do. But we use the preservation of the $H^s(\mathbb{R})$ to produce classical solutions for a bigger class of initial data: the Sobolev's Theorem above says that it suffices to work with $H^s(\mathbb{R})$ with $s > 2 + 1/2$. It turns out that under this condition, the $u(x,t)$ is also once continuously differentiable in t and that it satisfies the Schrödinger equation in the classical sense.

The sense in which this solution takes on its initial data is given by examining

$$||u(\cdot,t) - u(\cdot,0)||_{H^s(\mathbb{R})} = \left(\int_{\mathbb{R}} (1+|\xi|)^{2s}|\hat{u}(\xi,0)|^2|e^{-i\xi^2 t} - 1|^2\,d\xi\right)^{1/2}.$$

Since $\int_{\mathbb{R}}(1+|\xi|)^{2s}|\hat{u}(\xi,0)|^2\,d\xi<\infty$ and $|e^{-i\xi^2 t}-1|\le 2$, with $|e^{-i\xi^2 t}-1|\to 0$ as $t\to 0$, we see that $||u(\cdot,t)-u(\cdot,0)||_{H^s(\mathbb{R})}\to 0$ as $t\to 0$ (using an argument similar to that at the end of Proof of Theorem 2.2 or Lebesgue's Dominated Convergence Theorem).

For $u(\cdot,0)\in H^s(\mathbb{R})$ but $s\le 2+1/2$, we can use smooth solutions with $u_j(\cdot,0)\in C_c^\infty(\mathbb{R})$ and $u_j(\cdot,0)\to u(\cdot,0)\in H^s(\mathbb{R})$ to generate solutions that are converging in the $H^s(\mathbb{R})$ norm and define the limit as a generalized solution in $H^s(\mathbb{R})$.

Formally the operator $i\partial_t+\partial_x^2$ arises from the operator for the heat equation, $-\partial_t+\partial_x^2$, after substituting t by it. Using this formal relation, we set

$$u(x,t)=\int_{\mathbb{R}}K(x-y,it)g(y)\,dy=\int_{\mathbb{R}}\frac{e^{\frac{i|x-y|^2}{4t}}}{\sqrt{4\pi it}}g(y)\,dy$$

and can verify that it defines a solution of $(i\partial_t+\partial_x^2)u(x,t)=0$ at least for $g\in C_c^\infty(\mathbb{R})$. It is possible to prove that $u(x,0)=g(x)$ in $L^2(\mathbb{R})$ sense, namely, $\|u(\cdot,t)-g(\cdot)\|_{L^2(\mathbb{R})}\to 0$ as $t\searrow 0$, but a direct proof using this representation is not easy; it is also much harder to prove $u(x,t)\to g(x)$ as $t\searrow 0$ in the point wise sense.

Exercises

Exercise 2.10.1. Provide a detailed proof for (2.40).

Exercise 2.10.2. Assume that $\alpha=s-n/2$ satisfies $0<\alpha<1$. Show that there exists a constant $C=C(n,s)>0$ such that for all $u\in C_c^\infty(\mathbb{R}^n)$ and $\mathbf{h}\in\mathbb{R}^n$, there holds

$$|u(\mathbf{x}+\mathbf{h})-u(\mathbf{x})|\le C||u||_{H^s(\mathbb{R}^n)}\|\mathbf{h}\|^\alpha,$$

therefore proving that the above inequality continues to hold for $u\in H^s(\mathbb{R}^n)$. Hint: Use the representation

$$u(\mathbf{x}+\mathbf{h})-u(\mathbf{x})=(2\pi)^n\int_{\mathbb{R}^n}\hat{u}(\boldsymbol{\xi})e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\left(e^{i\mathbf{h}\cdot\boldsymbol{\xi}}-1\right)\,d\boldsymbol{\xi},$$

break up the integral into the sum of $\int_{\|\boldsymbol{\xi}\|\le\|\mathbf{h}\|^{-1}}$ and $\int_{\|\boldsymbol{\xi}\|>\|\mathbf{h}\|^{-1}}$ and estimate them separately.

Exercise 2.10.3. Assume $g\in L^1(\mathbb{R})$. Prove that $(x,t)\mapsto\int_{\mathbb{R}}K(x-y,it)g(y)\,dy$ is continuous in $(x,t)\in\mathbb{R}\times(\mathbb{R}\setminus\{0\})$, and that

$$\|\int_{\mathbb{R}}K(x-y,it)g(y)\,dy\|_{L^\infty(\mathbb{R})}\le\frac{\|g\|_{L^1(\mathbb{R})}}{\sqrt{4\pi|t|}}.$$

2.11 More Sobolev Imbedding Theorems

Theorem 2.13 is about function spaces with L^2 derivatives. We summarize here a few basic imbedding theorems of $W^{k,p}(D)$ allowing $p \neq 2$. The proofs for these imbedding theorems will not depend on Fourier transforms. Sections 4.2.3 and 4.3 also contain some simple cases of Sobolev spaces in the context of applications related to the energy method and variational method. Only case (b) of Theorem 2.15 will be used at the end of chapter six.

We first state a basic approximation property for functions in $W^{k,p}(D)$ without proof—it is an analogue of Theorem 2.12.

Theorem 2.14. *Suppose that $D \subset \mathbb{R}^n$ is an open domain such that any point $P \in \partial D$ has an open neighborhood U in $\mathbb{R}^n$, a hypercube $K_a = \{\mathbf{x} \in \mathbb{R}^n : |x_i| < a\}$, a C^k diffeormorphism Φ from K_a onto U, and some $m \leq n$, such that (i). $\Phi(\mathbf{0}) = P$; (ii). $\Phi^{-1}(U \cap D) = K_a^m := \{\mathbf{x} \in K_a : x_i > 0, i = 1, \cdots, m\}$. Then for any function $u \in W^{k,p}(D)$, there exists a sequence $u_j \in W^{k,p}(D) \cap C^\infty(D)$ such that $u_j \to u$ in $W^{k,p}(D)$.*

A most common domain satisfying the assumption of Theorem 2.14 is a domain whose boundary is a C^k-hypersurface, which corresponds to the case of $m = 1$. A domain satisfying the assumption of Theorem 2.14 is called a domain with piecewise C^k boundary. We denote by $W_0^{k,p}(D)$ the completion of $C_c^\infty(D)$ in the $W^{k,p}(D)$ norm.

With Theorem 2.14, we can prove properties of $W^{k,p}(D)$ (respectively $W_0^{k,p}(D)$) by proving the same properties in $W^{k,p}(D) \cap C^\infty(D)$ (respectively $C_c^\infty(D)$).

Another relevant geometric property of a domain is the following.

Definition. For any open subset $\Omega \subset \mathbb{S}^{n-1}$, let $C_\Omega = \{\mathbf{x} = r\omega : \omega \in \Omega, r > 0\}$ denote the cone with opening Ω, and $C_{\Omega,R} = \{\mathbf{x} = r\omega : \omega \in \Omega, R > r > 0\}$ denote the truncated cone of radius R.

A domain D is said to have the cone property if there exists $c > 0$ such that any point $P \in D$ has a truncated cone $C_{\Omega_P,R}$ such that the translated cone with vertex at P, $P + C_{\Omega_P,R} \subset D$ and $|\Omega_P| \geq c, R \geq c$.

A domain D is said to have the strong cone property if there exist $c > 0$ and $\rho > 0$ such that for any $P, Q \in D$ with $|P - Q| < \rho$, there exist truncated cones $P + C_{\Omega_P,R} \subset D$ and $Q + C_{\Omega_Q,R} \subset D$ such that $R \leq c^{-1}|P - Q|$ and

$$|(P + C_{\Omega_P,R}) \cap (Q + C_{\Omega_Q,R})| \geq c|P - Q|^n.$$

A bounded domain with piecewise C^1 boundary satisfies the cone and the strong cone property.

Recall that for $0 < \alpha < 1$, the $C^\alpha(D)$-seminorm is defined by

$$[u]_{\alpha;D} := \sup_{P,Q \in D} \frac{|u(P) - u(Q)|}{|P - Q|^\alpha}.$$

Theorem 2.15 (Morrey and Sobolev Imbedding). *(a). If $p > n$, then there exists $C = C(p, n) > 0$ such that for any $u \in W_0^{1,p}(D)$*

$$\max_D |u| \le CR^{1-n/p} \|\nabla u\|_{L^p(D)}, \text{ where } R \text{ is the diameter of } D, \tag{2.41}$$

and

$$[u]_{\alpha;D} \le C \|\nabla u\|_{L^p(D)}, \text{ where } \alpha = 1 - \frac{n}{p}. \tag{2.42}$$

(b). If $1 \le p < n$, then there exists $C = C(p, n) > 0$ such that for any $u \in W_0^{1,p}(D)$

$$\|u\|_{L^{p^*}(D)} \le C \|\nabla u\|_{L^p(D)}, \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}. \tag{2.43}$$

(c). If D satisfies the cone property and $p > n$, then there exists $C > 0$ such that any $u \in W^{1,p}(D)$ satisfies

$$\sup_{\mathbf{x} \in D} |u(\mathbf{x})| \le C \|u\|_{W^{1,p}(D)}. \tag{2.44}$$

(d). If D satisfies both the cone and the strong cone property and $p > n$, then there exists $C > 0$ such that any $u \in W^{1,p}(D)$ satisfies

$$[u]_{\alpha,D} \le C \|u\|_{W^{1,p}(D)}, \alpha = 1 - \frac{n}{p}. \tag{2.45}$$

Proofs for (a), (c) and (d) rely on the following Lemma.

Lemma 2.16. *There exists $c = c(n) > 0$ such that*

(i). if $u \in C_c^1(B_R)$, then for any $\mathbf{x} \in B_R$,

$$|u(\mathbf{x})| \le c \int_{B_R} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{x} - \mathbf{z}|^{n-1}} \, d\mathbf{z}; \tag{2.46}$$

(ii). if $u \in C^1(B_R)$, then for any $\mathbf{x} \in B_R$,

$$|u(\mathbf{x})| \le c \int_{B_R} \frac{|\nabla u(\mathbf{z})| + R^{-1}|u(\mathbf{z})|}{|\mathbf{x} - \mathbf{z}|^{n-1}} \, d\mathbf{z}; \tag{2.47}$$

(iii). if $u \in C^1(B_R(\mathbf{x}))$, then

$$|B_R(\mathbf{x})|^{-1}\int_{B_R(\mathbf{x})}|u(\mathbf{x})-u(\mathbf{z})|\,d\mathbf{z} \le c\int_{B_R(\mathbf{x})}\frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z}. \tag{2.48}$$

(2.47) *and* (2.48) *continue to hold if B_R is replaced by a truncated cone $C_{\Omega,R}$ of radius R, in which case the constant c is bounded above by $|\Omega|^{-1}$.*

Proof. For (i), we take any $u \in C_c^1(B_R)$ and any $\boldsymbol{\omega} \in \mathbb{S}^{n-1}$, then

$$u(\mathbf{x}) = -\int_0^\infty \frac{d}{dr}u(\mathbf{x}+r\boldsymbol{\omega})\,dr.$$

Integrating this over $\boldsymbol{\omega} \in \mathbb{S}^{n-1}$, we obtain

$$|\mathbb{S}^{n-1}||u(\mathbf{x})| \le \int_{\boldsymbol{\omega}\in\mathbb{S}^{n-1}}\int_0^\infty |\nabla u(\mathbf{x}+r\boldsymbol{\omega})\,drd\boldsymbol{\omega}.$$

Setting $\mathbf{z} = \mathbf{x} + r\boldsymbol{\omega}$ and converting the integral in terms of $\mathbf{z}$, we find

$$|\mathbb{S}^{n-1}||u(\mathbf{x})| \le \int_{B_R}\frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z}.$$

For (ii), we take a standard radial cut-off function η supported in $B_1(0)$ and $\eta = 1$ on $B_{1/2}(0)$, then apply the above estimate to $u(\mathbf{z})\eta(|\mathbf{z}-\mathbf{x}|/R)$. Using $|\nabla_{\mathbf{z}}(u(\mathbf{z})\eta(|\mathbf{z}-\mathbf{x}|/R))| \le |\nabla u(\mathbf{z})| + 4R^{-1}|u(\mathbf{z})|$, we establish (2.47).

For (iii), we write $\mathbf{z} = \mathbf{x} + r\boldsymbol{\omega}$ and use

$$u(\mathbf{z}) - u(\mathbf{x}) = \int_0^r \frac{d}{ds}u(\mathbf{x}+s\boldsymbol{\omega})\,ds = \int_0^r \nabla u(\mathbf{x}+s\boldsymbol{\omega})\cdot\boldsymbol{\omega}\,ds,$$

and integrate over $\mathbf{z} \in B_R(\mathbf{x})$ after taking absolute values on both sides to obtain

$$\begin{aligned}
\int_{\mathbf{z}\in B_R(\mathbf{x})}|u(\mathbf{z})-u(\mathbf{x})|\,d\mathbf{z} &\le \int_{\mathbf{z}\in B_R(\mathbf{x})}\int_0^r |\nabla u(\mathbf{x}+s\boldsymbol{\omega})|\,dsd\mathbf{z}\\
&\le \int_{\boldsymbol{\omega}\in\mathbb{S}^{n-1}}\int_0^R\int_0^r |\nabla u(\mathbf{x}+s\boldsymbol{\omega})|r^{n-1}\,ds\ dr\ d\boldsymbol{\omega}\\
&\le \int_{\boldsymbol{\omega}\in\mathbb{S}^{n-1}}\int_0^R \frac{R^n}{n}|\nabla u(\mathbf{x}+s\boldsymbol{\omega})|\,ds\ d\boldsymbol{\omega}\\
&= \frac{R^n}{n}\int_{B_R(\mathbf{x})}\frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z}.
\end{aligned}$$

This proves (2.48) with $c = |\mathbb{S}^{n-1}|^{-1}$.

The modification of proof needed for the case that B_R is replaced by a truncated cone $C_{\Omega,R}$ of radius R is straightforward. $\square$

Proof of (a), (c), (d) of Theorem 2.15. For (a), if R is the diameter of D, then for any $\mathbf{x} \in D$, we can apply (2.47) at $\mathbf{x}$ but the domain of integration can still be D, which is contained in $B_R(\mathbf{x})$. Note that

$$\int_D \frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z} \le \left(\int_D |\nabla u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p}\left(\int_{B_R(\mathbf{x})} |\mathbf{x}-\mathbf{z}|^{-(n-1)p/(p-1)}\,d\mathbf{z}\right)^{1-1/p}$$

with

$$\left(\int_{B_R(\mathbf{x})} |\mathbf{x}-\mathbf{z}|^{-(n-1)p/(p-1)}\,d\mathbf{z}\right)^{1-1/p} = \left(\frac{p-1}{p-n}|\mathbb{S}^{n-1}|\right)^{1-1/p} R^{1-n/p}.$$

This shows (2.41).

Next, we take any $\mathbf{x} \neq \mathbf{y}$ and set $\rho = |\mathbf{x}-\mathbf{y}|$, then apply (2.48) on $B_\rho(\mathbf{x})$ and $B_\rho(\mathbf{y})$. Using $|B_\rho(\mathbf{x}) \cap B_\rho(\mathbf{y})| \ge c(n)|B_\rho(\mathbf{x})| = c(n)|B_\rho(\mathbf{y})|$, we have

$$\begin{aligned}
&|u(\mathbf{x})-u(\mathbf{y})| \\
\le& |B_\rho(\mathbf{x}) \cap B_\rho(\mathbf{y})|^{-1} \int_{B_\rho(\mathbf{x}) \cap B_\rho(\mathbf{y})} (|u(\mathbf{x})-u(\mathbf{z})| + |u(\mathbf{z})-u(\mathbf{y})|)\,d\mathbf{z} \\
\le& c(n)^{-1}\left(|B_\rho(\mathbf{x})|^{-1}\int_{B_\rho(\mathbf{x})} |u(\mathbf{x})-u(\mathbf{z})|\,d\mathbf{z} + |B_\rho(\mathbf{y})|^{-1}\int_{B_\rho(\mathbf{y})} |u(\mathbf{y})-u(\mathbf{z})|\,d\mathbf{z}\right) \\
\le& c\left(\int_{B_\rho(\mathbf{x})} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z} + \int_{B_\rho(\mathbf{y})} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{y}-\mathbf{z}|^{n-1}}\,d\mathbf{z}\right).
\end{aligned}$$

Using the estimate established above

$$\int_{B_\rho(\mathbf{x})} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z} \le \left(\int_{B_\rho(\mathbf{x})} |\nabla u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p} c(p,n)R^{1-n/p},$$

and an identical estimate for $\int_{B_R(\mathbf{y})} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{y}-\mathbf{z}|^{n-1}}\,d\mathbf{z}$, (2.42) follows with a C comparable to $\left(\frac{p-1}{p-n}|\mathbb{S}^{n-1}|\right)^{1-1/p}$.

For (c), we apply (2.47) on $C_{\Omega_\mathbf{x},R}$ and use

$$\left(\int_{C_{\Omega_\mathbf{x},R}} |\mathbf{x}-\mathbf{z}|^{-(n-1)p/(p-1)}\,d\mathbf{z}\right)^{1-1/p} \le \left(\frac{p-1}{p-n}|\Omega_\mathbf{x}|\right)^{1-1/p} R^{1-n/p},$$

to obtain

$$|u(\mathbf{x})| \le \left(\frac{p-1}{p-n}\right)^{1-1/p}\left\{R\left(|C_{\Omega_\mathbf{x},R}|^{-1}\int_{C_{\Omega_\mathbf{x},R}} |\nabla u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p} + \left(|C_{\Omega_\mathbf{x},R}|^{-1}\int_{C_{\Omega_\mathbf{x},R}} |u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p}\right\}.$$

For (d), for any $\mathbf{x}, \mathbf{y} \in D$, if $|\mathbf{x}-\mathbf{y}| < \rho$, where $\rho > 0$ is as in the strong cone property, we set $V = (\mathbf{x} + C_{\Omega_{\mathbf{x}},R}) \cap (\mathbf{y} + C_{\Omega_{\mathbf{y}},R})$, where $C_{\Omega_{\mathbf{x}},R}$ and $C_{\Omega_{\mathbf{y}},R}$ are given as in the strong cone property. Then

$$\begin{aligned}
|V||u(\mathbf{x})-u(\mathbf{y})| \leq& \int_V |u(\mathbf{x})-u(\mathbf{z})|\,d\mathbf{z} + \int_V |u(\mathbf{y})-u(\mathbf{z})|\,d\mathbf{z}\\
\leq& \int_{C_{\Omega_{\mathbf{x}},R}} |u(\mathbf{x})-u(\mathbf{z})|\,d\mathbf{z} + \int_{C_{\Omega_{\mathbf{y}},R}} |u(\mathbf{y})-u(\mathbf{z})|\,d\mathbf{z}\\
\leq& c|C_{\Omega_{\mathbf{x}},R}| \int_{C_{\Omega_{\mathbf{x}},R}} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z} + c|C_{\Omega_{\mathbf{y}},R}| \int_{C_{\Omega_{\mathbf{y}},R}} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{y}-\mathbf{z}|^{n-1}}\,d\mathbf{z}.
\end{aligned}$$

Our earlier argument gives

$$\int_{C_{\Omega_{\mathbf{x}},R}} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{x}-\mathbf{z}|^{n-1}}\,d\mathbf{z} \leq \left(\frac{p-1}{p-n}\right)^{1-1/p} R^{1-n/p}|\Omega_{\mathbf{x}}|^{1-1/p} \left(\int_{C_{\Omega_{\mathbf{x}},R}} |\nabla u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p}$$

and a similar one for $\int_{C_{\Omega_{\mathbf{y}},R}} \frac{|\nabla u(\mathbf{z})|}{|\mathbf{y}-\mathbf{z}|^{n-1}}\,d\mathbf{z}$. Using $|V| \geq c|\mathbf{x}-\mathbf{y}|^n$ and $c||C_{\Omega_{\mathbf{y}},R}| \leq R^n \leq c^{-n}|\mathbf{x}-\mathbf{y}|^n$, we obtain

$$|u(\mathbf{x})-u(\mathbf{y})| \leq C(n,p,c)|\mathbf{x}-\mathbf{y}|^{1-n/p} \left\{ \left(\int_{C_{\Omega_{\mathbf{x}},R}} |\nabla u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p} + \left(\int_{C_{\Omega_{\mathbf{y}},R}} |\nabla u(\mathbf{z})|^p\,d\mathbf{z}\right)^{1/p} \right\}.$$

If $|\mathbf{x}-\mathbf{y}| \geq \rho$, then

$$\frac{|u(\mathbf{x})-u(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^{1-n/p}} \leq 2\rho^{n/p-1}\max\{|u(\mathbf{x})|, |u(\mathbf{y})|\},$$

and we can estimate $\max\{|u(\mathbf{x})|, |u(\mathbf{y})|\}$ in terms of $||u||_{W^{1,p}(D)}$ as done in proving (c). This concludes our proof of (2.45). □

Proof of (b) of Theorem 2.15. (b) was first proved for $p > 1$ using (2.46) and a Hardy-Littlewood inequality generalizing Young's inequality on convolution dealing with the issue that $|\mathbf{x}-\mathbf{y}|^{1-n}$ in (2.46) just fails to be in $L^{\frac{n}{n-1}}$. The $p = 1$ case was first given by Gagliardo and Nirenberg independently, which can also be used to derive the $1 < p < n$ cases.

We may take $u \in C_c^1(D)$ and treat it as in $C_c^1(\mathbb{R}^n)$. For any $\mathbf{x} = (x_1, \cdots, x_n)$ and $1 \leq i \leq n$, we have

$$|u(\mathbf{x})| \leq \int_{-\infty}^{\infty} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\,dy_i.$$

So

$$|u(\mathbf{x})|^{\frac{n}{n-1}} \leq \prod_{i=1}^n \left(\int_{-\infty}^{\infty} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\,dy_i\right)^{\frac{1}{n-1}}.$$

Integrating over $x_1 \in \mathbb{R}$ and applying Hölder's inequality on the last $(n-1)$-factors, we get

$$\int_{-\infty}^{\infty} |u(\mathbf{x})|^{\frac{n}{n-1}}\, dx_1$$

$$\leq \left(\int_{-\infty}^{\infty} |\partial_{x_1} u(y_1, x_2, \cdots, x_n)|\, dy_1\right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dy_i\, dx_1\right)^{\frac{1}{n-1}}.$$

Repeating this for $x_i, i = 2, \cdots, n$, we obtain

$$\int_{\mathbb{R}^n} |u(\mathbf{x})|^{\frac{n}{n-1}}\, dx_1 \cdots dx_n \leq \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dx_1 \cdots dx_{i-1} dy_i dx_{i+1} \cdots dx_n\right)^{\frac{1}{n-1}}$$

Since the integrations are effectively done in D, it then follows that

$$\|u\|_{L^{\frac{n}{n-1}}(D)} \leq \prod_{i=1}^{n} \left(\int_{D} |\partial_{x_i} u(x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n)|\, dx_1 \cdots dx_{i-1} dy_i dx_{i+1} \cdots dx_n\right)^{\frac{1}{n}}.$$

Applying the geometric-arithmetic mean inequality to the right hand side, we obtain

$$\|u\|_{L^{\frac{n}{n-1}}(D)} \leq \frac{1}{n}\sum_{i=1}^{n} \int_D |\partial_{x_i} u(\mathbf{x})|\, d\mathbf{x} \leq \frac{1}{\sqrt{n}} \int_D \left(\sum_{i=1}^{n} |\partial_{x_i} u(\mathbf{x})|^2\right)^{1/2} d\mathbf{x},$$

which is the $p = 1$ case of (5.23).

For the case of $1 < p < n$, for any $u \in C_c^1(D)$, we consider $v := |u|^{r-1}u$ for some $r > 1$ to be determined. Then $\nabla v = r|u|^{r-2}u\nabla u$, and the $p = 1$ case of (5.23) applied to v gives

$$\left(\int_D |u|^{r\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq c(n)r\int_D |u|^{r-1}|\nabla u| \leq c(n)r\left(\int_D |u|^{(r-1)p'}\right)^{1/p'}\left(\int_D |\nabla u|^p\right)^{1/p},$$

where $1/p' + 1/p = 1$. Choose r such that $(r-1)p' = r\frac{n}{n-1} =: q$, which gives q from $1/q = 1/p - 1/n$, and leads to

$$\left(\int_D |u|^q\right)^{1/q} \leq c(p,n)\left(\int_D |\nabla u|^p\right)^{1/p}.$$

□

2.12 Additional Problems

Problem 2.12.1. This problem deals with the construction of solutions to the Dirichlet problem on a round disk for the **Helmholtz** equation, namely,

$$\begin{cases} \Delta u(x) + cu(x) = 0 & x \in B_{r_0}(0), \\ u(x) = g(x) & x \in \partial B_{r_0}(0), \end{cases} \tag{2.49}$$

for some given constant c and (continuous) function g on $\partial B_{r_0}(0)$.

(i). If we look for a solution of $\Delta u(x) + cu(x) = 0$ of the form $u = R(r)\Theta(\theta)$. Verify that $R(r)$ and $\Theta(\theta)$ must satisfy

$$\left[R''(r) + \frac{R'(r)}{r}\right]\Theta(\theta) + \frac{R(r)}{r^2}\Theta''(\theta) + cR(r)\Theta(\theta) = 0,$$

from which deduce that $\Theta(\theta)$ satisfies the same (2.13) for some λ and $R(r)$ satisfies

$$R''(r) + \frac{R'(r)}{r} + \left[c - \frac{\lambda}{r^2}\right]R(r) = 0.$$

Although the problem here only needs $R(r)$ for $0 < r < r_0$, this ODE for $R(r)$ is a linear one, so we will consider its behavior for $r \in [0, \infty)$.

Recall that (2.13) has a non-trivial solution only iff $\lambda = k^2$ for some $k \in \mathbb{Z}$. If $c > 0$, then verify that $J(r) = R(r/\sqrt{c})$ satisfies

$$J''(r) + \frac{J'(r)}{r} + \left[1 - \frac{\alpha^2}{r^2}\right]J(r) = 0, \tag{2.50}$$

with $\alpha^2 = \lambda$. (2.50) is called **Bessel**'s equation of order α. Many problems in mathematical physics are related to these functions so they are extensively studied—separable solutions to the higher dimensional Helmholtz equation would lead to an equation similar to (2.50) in structure, and can in fact be reduced to (2.50); see Exercises in sections 5.4 and 6.8.

(2.50) has two linearly independent solutions. Their leading behavior near $r = 0$ are determined by r^β, where β is determined so that r^β is a solution of an Euler type ODE derived from (2.50) by keeping only the leading order terms near $r = 0$: $R''(r) + \frac{1}{r}R'(r) - \frac{\alpha^2}{r^2}R(r) = 0$. So $\beta = \pm\alpha$. For $\beta = \alpha > 0$, (2.50) has a solution of the form

$$J_\alpha(r) = \left(\frac{r}{2}\right)^\alpha \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha + m + 1)}\left(\frac{r}{2}\right)^{2m} \quad \text{for } r > 0,$$

where $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$ is the Gamma function, with $\Gamma(m) = (m-1)!$ when $m \in \mathbb{N}$. When $\beta = -\alpha \notin \mathbb{Z}$, a second linearly independent solution of (2.50) is given by $J_{-\alpha}$, which is obtained by simply replacing α by $-\alpha$ in the above. When $\beta = -\alpha = -d \in \mathbb{Z}_{<0}$, J_{-d} as given above is not well defined due to the poles of Γ at $\mathbb{Z}_{<0}$; a second linearly independent solution of (2.50) can be given in the form of

$$Y_\alpha(r) = \frac{2}{\pi}J_\alpha(r)\ln\left(\frac{r}{2}\right) - \frac{1}{\pi}\left(\frac{r}{2}\right)^{-\alpha}\sum_{m=0}^{\infty} a_m \left(\frac{r}{2}\right)^{2m},$$

where the a_n's are given explicitly. The same conclusion holds also for $\alpha = 0$. Note that the leading order term of $Y_0(r)$ near $r = 0$ is $\frac{2}{\pi}\ln\left(\frac{r}{2}\right)$.

(ii). Verify that in the case $c < 0$, a similar transformation leads to

$$J''(r) + \frac{J'(r)}{r} + \left[-1 - \frac{\alpha^2}{r^2}\right] J(r) = 0. \tag{2.51}$$

This is called modified Bessel equation of order α. Its solutions are spanned by $J_\alpha(ir)$ and $Y_\alpha(ir)$ — the same J_α and Y_α as in the previous part; other than the possibly multi-valued nature of $\left(\frac{r}{2}\right)^{\pm\alpha}$ and $\ln\left(\frac{r}{2}\right)$, the remaining construction of $J_\alpha(r)$ and $Y_\alpha(r)$ are given by power series of r with infinite radius of convergence, so extends to the entire complex plane $z = x + iy$ (with $z \in \mathbb{C}$ extending $x = r \geq 0$); (2.50) thus holds for $z = x + iy \in \mathbb{C} \setminus \{\mathbb{R}_{\leq 0}\}$, with the derivatives taken in the sense for complex analytic functions. When $z = ir$, $r \in \mathbb{R}^+$, using $\frac{dJ}{dz} = \frac{dJ}{idr}$, $\frac{d^2J}{dz^2} = -\frac{d^2J}{dr^2}$, we see that, when $J(r)$ and $Y(r)$ are solutions to (2.50), $J(ir)$ and $Y(ir)$ are solutions to the modified Bessel's equation. Their leading behavior near $r = 0$ are determined by $(ir)^{\pm\alpha}$ (except for $Y_0(ir)$, which has leading order term $\frac{2}{\pi}\ln\left(\frac{ir}{2}\right)$); but their behavior near $r = \infty$ is distinct from that of $J(r)$ and $Y(r)$, respectively—the leading behavior near $r = \infty$ of solutions to the modified Bessel's equation is determined by $R''(r) - R(r) = 0$, while the leading behavior near $r = \infty$ of solutions to the Bessel's equation is determined by $R''(r) + R(r) = 0$; the precise asymptotic behavior of Bessel's functions are more subtle than what these two equations may suggest:

$$J_\alpha(r) \sim \sqrt{\frac{2}{\pi r}}\left(\cos(r - \frac{\alpha\pi}{2} - \frac{\pi}{4}) + O(r^{-1})\right) \quad \text{as } r \to \infty,$$
$$Y_\alpha(r) \sim \sqrt{\frac{2}{\pi r}}\left(\sin(r - \frac{\alpha\pi}{2} - \frac{\pi}{4}) + O(r^{-1})\right) \quad \text{as } r \to \infty.$$

(iii). In constructing a solution of (2.49) in the case $c < 0$, separation of variables leads to consideration of a proposed solution of the form

$$\sum_{k=0}^{\infty} \left[a_k e^{ik\theta} + b_k e^{-ik\theta}\right] J_k(i\sqrt{|c|}r).$$

Here we do not use Y_k, as it would be singular near $r = 0$. Note that $J_k(i\sqrt{|c|}r)$ has no zero for $r \in (0, \infty)$ for any k (this can be proved by the maximum principle or the energy method to be introduced later). Prove that (2.49) in this case has a solution for any give $g \in C(\partial B_{r_0}(0))$ (other methods for proving the existence of a solution of this problem, such as the variational method, will be introduced later).

(iv). In constructing a solution of (2.49) in the case $c > 0$, one would work with a proposed solution of the form $\sum_{k=0}^{\infty} \left[a_k e^{ik\theta} + b_k e^{-ik\theta}\right] J_k(\sqrt{c}r)$. Its boundary value on $\partial B_{r_0}(0)$ would be given by $\sum_{k=0}^{\infty} \left[a_k e^{ik\theta} + b_k e^{-ik\theta}\right] J_k(\sqrt{c}r_0)$.

Note that, this time, each J_k has a discrete but infinite number of zeros $0 < r_{k,1} < r_{k,2} < \cdots$, with $r_{k,l} \to \infty$ as $l \to \infty$. If c and r_0 are such that $\sqrt{c}r_0 = r_{k,l}$ for some k, l, verify that $v = \left[a_k e^{ik\theta} + b_k e^{-ik\theta}\right] J_k(\sqrt{c}r)$ is a solution of (2.49), which is not trivial if a_k and b_k are not all zero. A non-trivial solution of (2.49) is called an eigenfunction of Δ on $B_{r_0}(0)$ with zero Dirichlet boundary condition. You will be asked below to prove that in such a case there will be necessary conditions on g in order for (2.49) to have a solution—note that in such a case, the component $\left[a_k e^{ik\theta} + b_k e^{-ik\theta}\right] J_k(\sqrt{c}r)$ in the above construction will be 0 when $r = r_0$, so, on a heuristic level, such a construction would not be able take on a boundary value g if g has a non-zero component in $\left[a_k e^{ik\theta} + b_k e^{-ik\theta}\right]$.

Follow the instruction below to prove that a necessary condition for (2.49) to have a solution is that

$$\int_0^{2\pi} g(r_0 e^{i\theta}) e^{\pm ik\theta} d\theta = 0 \quad \text{for each } k \text{ such that } \sqrt{c}r_0 = r_{k,l} \text{ for some } l.$$

(a). Verify that for any pair of C^2 functions u and v,

$$[\Delta u(x) + cu(x)]\, v(x) - u(x)\, [\Delta v(x) + cv(x)] = \operatorname{div}\left[v(x)\nabla u(x) - u(x)\nabla v(x)\right].$$

(b). Suppose that $\sqrt{c}r_0 = r_{k,l}$ for some k, l. Set $v(x) = J_k(\sqrt{c}r)e^{\pm ik\theta}$. Note that $[\Delta v(x) + cv(x)] = 0$, and $v = 0$ on $\partial B_{r_0}(0)$. Use the divergence structure above and Green's theorem to prove that if u is a solution of (2.49), then

$$\int_{\partial B_{r_0}(0)} g(r_0 e^{i\theta}) \frac{\partial v(x)}{\partial r} r_0 d\theta = 0.$$

Note that $\frac{\partial v(x)}{\partial r}|_{\partial B_{r_0}(0)} = \sqrt{c} J_k'(\sqrt{c}r_0) e^{\pm ik\theta}$. Argue that $J_k'(\sqrt{c}r_0) \neq 0$, and use this to conclude the proof for the necessary condition.

Problem 2.12.2. This problem provides a different approach to the condition on a hyper surface across which a piecewise C^2 solution of a second order PDE of the form $\sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_ix_j}u(x) = f(x)$ may experience a jump discontinuity of the second order derivatives. More precisely, let Σ be a hyper surface, $u^\pm$ are C^2 functions on the two sides of Σ, respectively, and satisfy the above PDE in their respective region, and $u^\pm$, together with their first derivatives agree across Σ, but experience a jump discontinuity of some of their second derivatives.

Suppose that Σ is described by $\phi(x) = 0$ near $x_0 \in \Sigma$, where ϕ is a C^1 function and we may assume that $\partial_{x_n}\phi(x_0) \neq 0$. Thus the map $x \mapsto (y, \tau) := (x', \phi(x))$, $x' = (x_1, \cdots, x_{n-1})$, defines a local diffeomorphism. Define $v^\pm(y, \tau) = v^\pm(x', \phi(x)) = u^\pm(x)$. Then what is given is that $v^\pm(y, \tau)$ agree across $\tau = 0$, and equal a C^2 function of y: $v^\pm(y, 0) = g(y)$. This transformation from u to v transforms the hyper surface Σ into a piece of the flat hyper plane $\{(y, 0)\}$, and is a commonly used technique in theoretical investigations.

(i). Verify that

$$\begin{aligned}
u^\pm_{x_i}(x) = v^\pm_{y_i}(y,\tau) + v^\pm_\tau(y,\tau)\phi_{x_i}(x), 1 \le i \le n-1,\\
u^\pm_{x_n}(x) = v^\pm_\tau(y,\tau)\phi_{x_n}(x),\\
u^\pm_{x_ix_j}(x) = v^\pm_{y_iy_j}(y,\tau) + v^\pm_{y_i\tau}(y,\tau)\phi_{x_j}(x) + v^\pm_{y_j\tau}(y,\tau)\phi_{x_i}(x) + v^\pm_\tau(y,\tau)\phi_{x_ix_j}(x),\\
u^\pm_{x_ix_n}(x) = v^\pm_{y_i\tau}(y,\tau)\phi_{x_n}(x) + v^\pm_\tau(y,\tau)\phi_{x_ix_n}(x) + v^\pm_{\tau\tau}(y,\tau)\phi_{x_i}(x)\phi_{x_n}(x),\\
u^\pm_{x_nx_n}(x) = v^\pm_{\tau\tau}(y,\tau)(\phi_{x_n}(x))^2 + v^\pm_\tau(y,\tau)\phi_{x_nx_n}(x).
\end{aligned}$$

The assumption that $u^\pm$, together with their first derivatives agree across Σ, is equivalent to $v^\pm(y, \tau)$ and $v^\pm_\tau(y, \tau)$ agree across $\tau = 0$. Let $v^\pm_\tau(y, 0) = h(y)$. Then $v^\pm_{y_i\tau}(y, 0) = h_{y_i}(y)$ agree.

The above relations imply that if $v^\pm_{\tau\tau}(y, \tau)$ agree across $\tau = 0$, then all the second derivatives of u would agree across Σ. Thus a jump discontinuity of some of the second derivatives of u across Σ is equivalent to a jump discontinuity of $v^\pm_{\tau\tau}(y, \tau)$ across $\tau = 0$.

(ii). Verify that the PDE $\sum_{i,j=1}^n a_{ij}(x)\partial^2_{x_ix_j}u(x) = f(x)$, expressed in terms of $v(y, \tau)$ has the form

$$\left(\sum_{i,j=1}^n a_{ij}(x)\phi_{x_i}\phi_{x_j}(x)\right) v^\pm_{\tau\tau}(y,\tau) + R = f(x),$$

where R stands for sum of linear expressions in $v^\pm_{y_i\tau}(y, \tau)$, $v^\pm_{y_i}$ for $1 \le i \le n-1$, and in $v^\pm_\tau$. Since all these terms agree across $\tau = 0$, conclude that $v^\pm_{\tau\tau}(y, \tau)$ may experience a jump discontinuity across $\tau = 0$ only if $\sum_{i,j=1}^{n} a_{ij}(x)\phi_{x_i}\phi_{x_j}(x) = 0$ for $x \in \Sigma$. Note that $\nabla\phi(x)$ stands for a normal vector to Σ at $x \in \Sigma$, let $\nu(x)$ be a unit normal vector to Σ at $x \in \Sigma$, so this condition is expressed as $\sum_{i,j=1}^{n} a_{ij}(x)\nu_i(x)\nu_j(x) = 0$.

(iii). Conclude that for the wave equation $u_{tt}(x,t) - c^2\Delta u(x,t) = 0$ in $\mathbb{R}^n \times \mathbb{R}$, a piecewise C^2 solution may have a jump discontinuity of some of its second derivatives across a hyper surface of the form $t = \psi(x)$ only if $1 = c^2|\nabla\psi(x)|^2$.

(iv). Conclude that for the Laplace equation $\Delta u(x) = 0$ in $\mathbb{R}^n$, a piecewise C^2 solution can't have a jump discontinuity of any of its second derivatives across any hyper surface in its domain.

Chapter 3

Additional Elementary Solution Methods

3.1 Method of Characteristic Curves Applied to the One Dimensional Wave Equation

We next explore another approach related to the method of ODEs to find a solution formula for the one dimensional wave equation $u_{tt} - c^2 u_{xx} = 0$ for $(x,t) \in \mathbb{R}^2$. One ingredient is to recognize that $u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u$. So if we set $(\partial_t + c\partial_x)u = v(x,t)$, then we are to solve a system of first order equations

$$(\partial_t + c\partial_x)u = v(x,t), \tag{3.1}$$

$$(\partial_t - c\partial_x)v = 0. \tag{3.2}$$

A second ingredient is to think of the actions of the first order differential operators in (3.1)-(3.2) at each point (x,t) as a directional derivative. Thus, for (3.2), there is a vector-field which takes the value $(-c, 1)$ at every (x,t), along the integral curve of which v remains a constant. These integral curves are called the **characteristic curves** for the wave equation. Here, the integral curves satisfy $\frac{dx}{dt} = -c$, which implies that $x = -ct + x_0$. Thus $v(-ct + x_0, t)$ is a constant of t and is equal to $v(x_0, 0)$, which follows from

$$\frac{d}{dt}v(-ct+x_0,t) = -cv_x(-ct+x_0,t) + v_t(-ct+x_0,t) = (\partial_t - c\partial_x)v((-ct+x_0,t) = 0.$$

In other words, $v(x,t) = v(x+ct,0)$. Plugging this into (3.1), and applying the same method, we find

$$\frac{d}{dt}u(ct+x_1,t) = (\partial_t + c\partial_x)u(ct+x_1,t) = v(ct+x_1,t) = v(2ct+x_1,0).$$

Thus

$$u(ct+x_1,t) = u(x_1,0) + \int_0^t v(2c\tau + x_1, 0)d\tau = u(x_1,0) + \int_{x_1}^{x_1+2ct} v(y,0)\,dy/2c.$$

On setting $x = ct + x_1$, we see $x_1 = x - ct$, and

$$\begin{aligned}
u(x,t) &= u(x-ct,0) + \int_{x-ct}^{x+ct} v(y,0)\,dy/2c \\
&= u(x-ct,0) + \int_{x-ct}^{x+ct} \left[(\partial_t + c\partial_y)u(y,0)\right]\,dy/2c \\
&= u(x-ct,0) + \int_{x-ct}^{x+ct} u_t(y,0)\,dy/2c + \left[u(x+ct,0) - u(x-ct,0)\right]/2 \\
&= \left[u(x+ct,0) + u(x-ct,0)\right]/2 + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(y,0)\,dy.
\end{aligned}$$

This recovers the d'Alembert's formula.

Remark 3.1. The derivation process here also proves that a C^2 solution of (2.11) is unique, while that is not the case for the solution formula for the heat equation or for the Poisson formula, or even the method for finding the d'Alembert's formula in the last section.

This method can be easily extended to solve the non-homogeneous problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), \quad \text{in} \quad \mathbb{R}\times[0,\infty), \\ u(x,0) = g(x), \\ u_t(x,0) = h(x). \end{cases} \tag{3.3}$$

(3.2) now needs to be modified into

$$(\partial_t - c\partial_x)v = f(x,t).$$

But the same method gives now

$$v(x_0 - ct, t) = v(x_0, 0) + \int_0^t f(x_0 - c\tau, \tau)d\tau,$$

which can be rewritten as $v(x,t) = v(x+ct,0) + \int_0^t f(x+ct-c\tau,\tau)d\tau$, Then the same relation $\frac{d}{dt}u(ct+x_0,t) = v(ct+x_0,t)$ gives

$$\begin{aligned}
u(ct+x_0,t) &= u(x_0,0) + \int_0^t v(x_0 + c\tau, \tau)d\tau \\
&= u(x_0,0) + \int_0^t \left[v(2c\tau + x_0, 0) + \int_0^\tau f(x_0 + 2c\tau - cs, s)ds\right]d\tau \\
&= u(x_0,0) + \int_{x_0}^{x_0+2ct} v(y,0)\,dy/2c + \int_0^t\int_0^\tau f(x_0 + 2c\tau - cs, s)dsd\tau.
\end{aligned}$$

On setting $x = ct + x_0$, we see $x_0 = x - ct$, and

$$
\begin{aligned}
u(x,t) &= u(x-ct,0) + \int_{x-ct}^{x+ct} v(y,0)\,dy/2c + \int_0^t \int_0^\tau f(x-ct+2c\tau-cs,s)dsd\tau \\
&= u(x-ct,0) + \int_{x-ct}^{x+ct} \frac{(\partial_t + c\partial_y)u(y,0)}{2c}\,dy + \int_0^t \int_s^t f(x-ct+2c\tau-cs,s)d\tau ds \\
&= \frac{u(x+ct,0)+u(x-ct,0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(y,0)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds.
\end{aligned} \tag{3.4}
$$

Theorem 3.1. *Assume that $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, and $f \in C^1(\mathbb{R}\times[0,\infty))$. Then*

$$
u(x,t) = \frac{g(x+ct)+g(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds
$$

provides a $C^2(\mathbb{R}\times[0,\infty))$ solution of (3.3), and this solution is unique.

The process that leads to (3.4) does not seem to need differentiability of f, but the verification that the said formula indeed provides a $C^2(\mathbb{R}\times[0,\infty))$ solution of (3.3) would require some differentiability of f, as

$$
\partial_x \left(\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dyds \right) = \int_0^t [f(x+c(t-s),s) + f(x-c(t-s),s)]\,ds,
$$

and further differentiation in x would require some differentiability of f in x. In fact, when one checks through the proof here, only the C^1 differentiability of f in x is needed.

Corollary 3.2. *Suppose that $f \equiv 0$ in the region $|x| > R$ $0 \le t \le T$, and $g = h \equiv 0$ in the region $|x| > R$. Then the solution u to (3.3) is 0 in the region $|x| > R + ct$.*

Remark 3.2. In order to obtain a solution that is $C^2(\mathbb{R}\times[0,\infty))$, f is assumed to be $C^1(\mathbb{R}\times[0,\infty))$, which has one additional order of differentiability than the second order derivatives of the solution u. This “loss of differentiability” of the solution with respect to data f is a feature of the solution of the wave equation. One can check that, with $f(x,t) = |ct - |x| - 1|$, the solution as provided by the formulae above is not in C^2 near $(0, 1/c)$.

Remark 3.3. The method of characteristic curves is most effect on solving first order PDEs; we will discuss its general theory later on. It is applicable here because the one-dimensional wave operator $\partial_t^2 - c^2\partial_x^2$ happens to be the product of two first order linear differential operators. If we consider a perturbation to the wave operator of the form $\partial_t^2 - c^2\partial_x^2 + a\partial_t + b\partial_x + d$, then the method, as presented, may not apply readily.

Remark 3.4. We now supply a proof for the uniqueness of generalized solution to (2.10) using Holmgren's approach. It suffices to prove that if u is a generalized solution with $u(x,0) = u_t(x,0) = 0$, then u is identically 0. From (2.22) we have

$$\iint_{\mathbb{R}\times[0,\infty)} u(\eta_{tt} - c^2\eta_{xx})\,dxdt = 0,$$

for all $\eta \in C^2(\mathbb{R}\times[0,\infty))$ with compact support in $\mathbb{R}\times[0,\infty)$. If we can find η to solve $\eta_{tt} - c^2\eta_{xx} = u$, and are allowed to insert such an η in the above integral relation, it would lead to $u \equiv 0$. However, our solvability for such an equation requires some regularity on u; and we would need η to have compact support in $\mathbb{R}\times[0,\infty)$. This difficulty is resolved by the following approximation scheme.

We will assume that $u \in L^2(\mathbb{R}\times(0,T))$ for any $T > 0$. Fix any $T > 0$, then there exists a sequence $u_k \in C_c^1(\mathbb{R}\times(0,T))$ such that $\lim_{k\to\infty} ||u - u_k||_{L^2(\mathbb{R}\times(0,T))} = 0$. For each u_k, we can use d'Alembert's formula to construct a classical solution η_k to

$$\begin{cases} \eta_{tt} - c^2\eta_{xx} = u_k, & \text{for } (x,t) \in \mathbb{R}\times(0,T), \\ \eta(x,T) = 0, & \\ \eta_t(x,T) = 0. & \end{cases}$$

Note that the initial data for η is placed at $t = T$, and that η_k will have compact support in $\mathbb{R}\times[0,\infty)$ — $\eta(x,t) \equiv 0$ for $t \geq T$ would make $\eta \in C_c^2(\mathbb{R}\times[0,\infty))$, as u_k has compact support in $\mathbb{R}\times(0,T)$. Thus we can use these η_n as test functions to obtain

$$\iint_{\mathbb{R}\times(0,T)} uu_k = 0, \quad \text{for each } k.$$

Sending $k \to \infty$, we obtain $\iint_{\mathbb{R}\times(0,T)} u^2\,dxdt = 0$. Thus $u = 0$.

Exercise 3.1.1. Apply the method of characteristic curves to study the solvability of

$$\begin{cases} (\partial_t + c\partial_x)u = f(x,t) & (x,t) \in \mathbb{R}^+\times\mathbb{R}^+, \\ u(x,0) = g(x) & x \in \mathbb{R}^+, \\ u(0,t) = h(t) & t \in \mathbb{R}^+. \end{cases}$$

(a). Assume $f \equiv 0$ and $c > 0$. Derive a solutions formula for $u(x,t)$, and discuss conditions on g and h to make this u a continuous, and then a C^1, solution in $\mathbb{R}^+\times\mathbb{R}^+$.

(b). Repeat the discussion in (a) assuming $c < 0$.

(c). Repeat the discussion in (a) assuming $c > 0$ and a general f.

Exercise 3.1.2. Apply the method of characteristic curves to study the solvability of

$$\begin{cases} (\partial_t + cx\partial_x)u = f(x,t) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

Hint: The integral curves of the vector field $(cx, 1)$ satisfy $\frac{dx}{dt} = cx$.

Exercise 3.1.3. Apply the method of characteristic curves to study the solvability of

$$\begin{cases} (\partial_t + ct\partial_x)u = f(x,t) & (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases}$$

3.2 The Duhamel's Principle

We next explain the Duhamel's principle, which gives a procedure for constructing a solution of non-homogeneous linear PDEs based on the construction of solutions to the corresponding homogeneous equations, assuming homogeneous boundary condition(s) if we are solving an IBVP. We first illustrate this method in the context of solving an IVP for the wave equation.

Theorem 3.3. *Assume that $f(x,t) \in C(\mathbb{R} \times [0,\infty))$ and that for each t, $f(\cdot,t)$ is C^1 in $x \in \mathbb{R}$. For each parameter s, let $S(x,t;s)[f]$ be the unique solution of*

$$\begin{cases} \left[\partial_t^2 - c^2\partial_x^2\right] S(x,t;s)[f] = 0, \quad \text{in} \quad \mathbb{R} \times [s,\infty), \\ S(x,s;s)[f] = 0, \\ \partial_t S(x,t;s)[f]\big|_{t=s} = f(x,s). \end{cases} \tag{3.5}$$

Then

$$u(x,t) = \int_0^t S(x,t;s)[f]ds$$

solves

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t), \quad \text{in} \quad \mathbb{R} \times [0,\infty), \\ u(x,0) = 0, \\ u_t(x,0) = 0. \end{cases} \tag{3.6}$$

Remark 3.5. The conclusion above was written in a schematic form. In the particular case here, since $S(x,t;s)[f] = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$, we have

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds.$$

This reproduces the same formula as given in Theorem 3.1.

We now provide a justification for the solution representation derived from the Duhamel principle. Formally,

$$u_t(x,t) = S(x,t;t)[f] + \int_0^t S_t(x,t;s)[f]ds = \int_0^t S_t(x,t;s)[f]ds,$$
$$u_{tt}(x,t) = S_t(x,t;t)[f] + \int_0^t S_{tt}(x,t;s)[f]ds = f(x,t) + \int_0^t S_{tt}(x,t;s)[f]ds,$$
$$u_{xx}(x,t) = \int_0^t S_{xx}(x,t;s)[f]ds.$$

So

$$u_{tt}(x,t) - c^2 u_{xx}(x,t) = f(x,t).$$

Of course, *appropriate smoothness assumptions on $f(x,t)$ is needed to justify differentiation under the integral signs.* In the case of the wave equation continuous differentiability of $f(x,t)$ in x would guarantee the existence of $S_{tt}(x,t;s)[f]$ and $S_{xx}(x,t;s)[f]$. For some IVP, even when they exist for $t > s$, terms similar to $S_{tt}(x,t;s)[f]$ and $S_{xx}(x,t;s)[f]$ may not be well defined when $t = s$, or may contain a negative power of $t-s$ which is not integrable in s, without further regularity assumption on f—we will see this in the context of the heat equation.

This Duhamel construction also works for solutions of general IVP of PDEs.

Remark 3.6. It turns out that solutions to the special homogeneous wave equation (3.5) can be used to construct general solution to the Cauchy problem (3.3). A solution of (3.3) can be constructed as

$$S(x,t;0)[h] + \frac{d}{dt}S(x,t;0)[g] + \int_0^t S(x,t;s)[f]ds.$$

The only thing to be verified is that $v(x,t) = \frac{d}{dt}S(x,t;0)[g]$ solves (1.10) with $v(x,0) = g(x)$, and $v_t(x,0) = 0$. By construction $S(x,t;0)[g]$ solves (1.10). $\frac{d}{dt}$ and the wave operator commute, so as long as g has appropriate regularity (C^2 suffices), it's clear that $v(x,t) = \frac{d}{dt}S(x,t;0)[g]$ solves (1.10). Also by construction, $v(x,0) = \frac{d}{dt}S(x,0;0)[g] = g(x)$, and $v_t(x,0) = \frac{d^2}{dt^2}S(x,t;0)[g]|_{t=0} = c^2\partial_x^2 S(x,t;0)[g]|_{t=0} = 0$, as $S(x,0;0)[g] = 0$.

In the case here as well as in cases where the differential equation is invariant under translation in t, using the change of variable $t-s=\tau$ and the translation invariance in t, one sees that $S(x,t;s)[f]=S(x,t-s;0)[f(\cdot,s)]$. Thus (3.5) reduces to solving a special case of (2.10).

The Duhamel principle here has its origin in the solution to a non-homogeneous linear system of ODEs

$$\mathbf{w}'(t)-A(t)\mathbf{w}(t)=\mathbf{f}(t), \tag{3.7}$$

which is represented using solutions to the corresponding homogeneous linear system of ODEs $\mathbf{w}'(t)-A(t)\mathbf{w}(t)=0$. Here we assume that $\mathbf{w}(t)$ is an unknown vector valued function with n components, and $A(t)$ is a continuous $n\times n$ matrix valued function. Suppose that $W(t)$ is a matrix valued fundamental solution of $\mathbf{w}'(t)-A(t)\mathbf{w}(t)=0$, namely each column of $W(t)$ is a solution of $\mathbf{w}'(t)-A(t)\mathbf{w}(t)=0$, and $W(0)$ is non-degenerate (equals I, say). Then $\mathbf{w}(t)=\int_0^t W(t)W^{-1}(s)\mathbf{f}(s)ds$ is a solution of (3.7) with $\mathbf{w}(0)=\mathbf{0}$—it's an elementary fact that $W(t)$ remains invertible for all t. Denote $U(t;s)=W(t)W^{-1}(s)$, and $S(t;s)[\mathbf{f}]=W(t)W^{-1}(s)\mathbf{f}(s)=U(t;s)\mathbf{f}(s)$, then $\mathbf{w}(t)=\int_0^t S(t;s)[\mathbf{f}]ds$, and $U(t;s)$ as a (matrix valued) function of t is a solution to $\mathbf{U}'(\mathbf{t})-\mathbf{A}(\mathbf{t})\mathbf{U}(\mathbf{t})=\mathbf{0}$ with $U(s)=I$, $S(t;s)[\mathbf{f}]$ as a function of t is a solution to $\mathbf{u}'(t)-A(t)\mathbf{u}(t)=0$ with $\mathbf{u}(s)=\mathbf{f}(s)$.

We briefly review the derivation of the ODE case using the *variation of parameters* method, then give a different, heuristic, derivation, which gives a better idea on why $f(x,s)$ is placed in that initial condition of (3.5) and can be adapted to the PDE setting.

Recall that any solution $\mathbf{w}(t)$ to $\mathbf{w}'(t)-A(t)\mathbf{w}(t)=0$ can be expressed as $W(t)C$ for some column vector C. When $A(t)=A$ is a constant matrix, we can take $W(t)=e^{At}$.

The variation of parameters method suggests that we can look for a solution of (3.7) in the form of $\mathbf{w}(t)=W(t)C(t)$, where $C(t)$ is a column vector of coefficient functions to be determined; it follows routinely that $W(t)C'(t)=\mathbf{f}(\mathbf{t})$, so $C'(t)=W^{-1}(t)\mathbf{f}(t)$, from which we know that $C(t)=\int_0^t W^{-1}(s)\mathbf{f}(s)ds$ is a solution, and

$$\mathbf{w}(t)=W(t)\left(\int_0^t W^{-1}(s)\mathbf{f}(s)ds\right)=\int_0^t W(t)W^{-1}(s)\mathbf{f}(s)ds.$$

We now give a heuristic derivation of the Duhamel formula, explaining how the non-homogeneous term $f(t)$ in (3.7) gets converted into the initial condition and how this idea is adapted to the PDE setting*.

The key idea is that if $\mathbf{f}(t)$ is 0 except for a short burst of time interval, say, during $[s, s+\Delta s]$, then any solution $\mathbf{w}(t)$ to the non-homogeneous equation will solve the homogeneous equation $\mathbf{w}'(t) - A(t)\mathbf{w}(t) = 0$ for $t \le s$ and for $t \ge s + \Delta s$. The condition $\mathbf{w}(0) = 0$ and the homogeneous equation would give us $\mathbf{w}(t) = \mathbf{0}$ for $0 \le t \le s$. We will use the data of $\mathbf{f}(t)$ for $t \in [s, s+\Delta s]$ to approximate $\mathbf{w}(s+\Delta s)$ by $\mathbf{f}(s)\Delta s$, based on the equation $\mathbf{w}'(t) - A(t)\mathbf{w}(t) = \mathbf{f}(t)$ providing a rate of change for $\mathbf{w}(t)$ at $t = s$. Then we solve $\mathbf{w}(t)$ for $t \ge s+\Delta s$ using the homogeneous equation and $\mathbf{w}(s+\Delta s) = \mathbf{f}(s)\Delta s$ as initial data, which would give us $S(t; s+\Delta s)\mathbf{f}(s)\Delta s$.

To deal with general data $\mathbf{f}(t)$, we use partitions to decompose $\mathbf{f}$ as the superposition of "pulse funtions" as described above. Technically, we do a partition of $[0, t]$ into $0 = s_0 < s_1 < \cdots < s_N = t$, and construct the "pulse functions" $\mathbf{f}_j(\tau) = \mathbf{f}(s_j)$ when $s_{j-1} \le \tau < s_j$, and $\mathbf{f}_j(\tau) = 0$ elsewhere; namely, $\mathbf{f}_j(\tau) = \mathbf{f}(s_j)\chi\{s_{j-1} \le \tau < s_j\}$, where $\chi\{s_{j-1} \le \tau < s_j\} = 1$ if $s_{j-1} \le \tau < s_j$, and $= 0$ otherwise. If $\mathbf{f}$ is continuous, then, when the partition size tends to 0, $\sum_{j=1}^N \mathbf{f}_j(\tau)$ provides a good approximation to $\mathbf{f}(\tau)$ over $0 \le \tau \le t$, and if we construct a solution $\mathbf{v}_j(t)$ which solves the same type of IVP with $\mathbf{f}_j$ replacing $\mathbf{f}$, $\mathbf{v}_j(0) = 0$, $\sum_{j=1}^N \mathbf{v}_j(\tau)$ is then expected to be a good approximation for $\mathbf{w}(\tau)$ for $0 \le \tau \le t$.

Here, $\mathbf{f}_j(\tau)$ may have jump discontinues at $\tau = s_{j-1}, s_j$, so $\mathbf{v}_j(\tau)$ may not be C^1, but there is a well defined solution $\mathbf{v}_j(\tau)$ which is continuous, has a well defined left derivative and right derivative at each point, and is piecewise C^1.

Notice that, $\mathbf{f}_j(\tau) = 0$ except when $s_{j-1} \le \tau < s_j$, and outside $[s_{j-1}, s_j)$, $\mathbf{v}_j(\tau)$ solves the homogeneous DE $\mathbf{v}_j'(\tau) - A(\tau)\mathbf{v}_j(\tau) = 0$; furthermore, using the homogeneous DE satisfied by $\mathbf{v}_j$ over $[0, s_{j-1})$ and the initial condition $\mathbf{v}_j(0) = 0$, we conclude that $\mathbf{v}_j(\tau) = 0$ when $0 \le \tau \le s_{j-1}$, and $\mathbf{v}_j'(s_{j-1}+) = \mathbf{f}(s_j)$; in addition, when the partition is fine, namely, when $s_j - s_{j-1}$ is small, using $\mathbf{v}_j'(s_{j-1}+) = \mathbf{f}(s_j)$, we have

$$\mathbf{v}_j(s_j) \approx \mathbf{v}_j(s_{j-1}) + \mathbf{v}_j'(s_{j-1}+)(s_j - s_{j-1}) = \mathbf{f}(s_j)(s_j - s_{j-1}),$$

so $\mathbf{v}_j(\tau)$ can be approximated by $V_j(\tau)$ in the range $s_j \le \tau \le t$, where

$$V_j'(\tau) - A(\tau)V_j(\tau) = 0 \quad s_j \le \tau \le t; \quad V_j(s_j) = \mathbf{f}(s_j)(s_j - s_{j-1}).$$

But this implies that $V_j(\tau) = U(\tau; s_j)\mathbf{f}(s_j)(s_j - s_{j-1})$ for $s_j \le \tau \le t$. Thus $\sum_{j=1}^N V_j(t) = \sum_{j=1}^N U(t; s_j)\mathbf{f}(s_j)(s_j - s_{j-1})$, which is a Riemann sum for the integral $\int_0^t U(t; s)\mathbf{f}(s)ds$.

*The discussion here is only for building some intuitive understanding, and is not strictly needed for proving the validity of the Duhamel Principle.

Since we expect $\sum_{j=1}^N V_j(t)$ to approach $\sum_{j=1}^N v_j(t)$, which tends to $\mathbf{w}(t)$ when the partition size goes to 0, we thus expect $w(t) = \int_0^t U(t;s)\mathbf{f}(s)ds$.

Below we adapt the Duhamel principle to give a heuristic derivation for a solution of the wave equation. Intuitively, we can approximate f on the interval $\mathbb{R} \times [0,t]$ by the following piecewise defined function

$$\tilde{f}(x,\tau) = f(x,s_j), \quad \text{when } s_{j-1} \le \tau < s_j,$$

where $0 = s_0 < s_1 < \cdots < s_N = t$ is a partition of $[0,t]$. We can rewrite $\tilde{f}(x,\tau)$ as

$$\tilde{f}(x,\tau) = \sum_{j=1}^{N} f(x,s_j)\chi\{s_{j-1} \le \tau < s_j\},$$

where $\chi\{s_{j-1} \le \tau < s_j\} = 1$ if $s_{j-1} \le \tau < s_j$, and $= 0$ otherwise. When $s_j - s_{j-1} > 0$ is small, $f(x,s_j)\chi\{s_{j-1} \le \tau < s_j\}$ represents "turning on" $f(x,s_j)$ only for a short burst of time during $s_{j-1} \le \tau < s_j$. Let $v_j(x,\tau)$ be a solution of

$$\begin{cases} \partial_\tau^2 v_j(x,\tau) - c^2\partial_x^2 v_j(x,\tau) = f(x,s_j)\chi\{s_{j-1} \le \tau < s_j\}, \\ v_j(x,0) = 0, \\ \partial_\tau v_j(x,0) = 0. \end{cases}$$

Then, $v(x,\tau) = \sum_{j=1}^N v_j(x,\tau)$ is a solution of

$$\begin{cases} v_{\tau\tau}(x,\tau) - c^2 v_{xx}(x,\tau) = \tilde{f}(x,\tau), \quad \text{in} \quad \mathbb{R} \times [0,t], \\ v(x,0) = 0, \\ v_\tau(x,0) = 0. \end{cases}$$

We will construct an approximation to v_j using a solution of the homogeneous wave equation—note that v_j solves the homogeneous wave equation outside of $\tau \in [s_{j-1}, s_j)$. First, applying the d'Alembert's formula to v_j in the range $0 \le \tau \le s_{j-1}$, we obtain $v_j(x,s_{j-1}) = 0$, $\partial_\tau v_j(x,s_{j-1}) = 0$, thus $\partial_x^2 v_j(x,s_{j-1}) = 0$; but the PDE gives $\partial_\tau^2 v_j(x,s_{j-1}) = f(x,s_j)$. When $s_j - s_{j-1}$ is small, $v_j(x,s_j) \approx v_j(x,s_{j-1}) + \partial_\tau v_j(x,s_{j-1})(s_j - s_{j-1}) = 0$ (up to order $s_j - s_{j-1}$), and $\partial_\tau v_j(x,s_j) \approx \partial_\tau v_j(x,s_{j-1}) + \partial_\tau^2 v_j(x,s_{j-1})(s_j - s_{j-1}) = f(x,s_j)(s_j - s_{j-1})$. So for $\tau > s_j$, $v_j(x,\tau)$ is approximated by $V_j(x,t)$, which solves

$$\begin{cases} v_{\tau\tau}(x,\tau) - c^2 v_{xx}(x,\tau) = 0, \quad \text{for } s_j < \tau \le t, \\ v(x,s_j) = 0, \\ v_\tau(x,s_j) = f(x,s_j)(s_j - s_{j-1}). \end{cases}$$

$V_j(x,\tau) = v(x,\tau)$ depends on $f(x, s_j)$ linearly, so we may write

$$V_j(x,\tau) = S(x,\tau;s_j)[f](s_j - s_{j-1}),$$

where $S(x,\tau;s_j)[f]$ solves the same problem as $V_j(x,\tau) = v(x,\tau)$, with the only difference being the second initial condition $\partial_\tau S(x,\tau;s_j)[f]\big|_{\tau=s_j} = f(x,s_j)$. Since

$$\sum_{j=1}^{N} V_j(x,t) = \sum_{j=1}^{N} S(x,t;s_j)[f](s_j - s_{j-1})$$

is a Riemann sum for the integral $\int_0^t S(x,t;s)[f]ds$, we expect $u(x,t) = \int_0^t S(x,t;s)[f]ds$ to be a solution of (3.6), with $u(x,0) = u_t(x,0) = 0$.

Remark 3.7. Since $S(x,t;s)[f]$ depends on f linearly and f plays a role by its value at s, we can adapt the idea used above to further treat $f(\cdot,s)$ as the superposition of pulse-like function in the spatial variable x so that $S(x,t;s)[f]$ is the superposition of solutions with pulse-like function as initial data, which motivates the notion of a fundamental solution to the IVP.

This is done by using the approximation $f(x,s) \approx \sum f(y_j,s)\chi_\epsilon(x-y_j)$, where y_j is a point in the j-th box (or ball) of a partition of the domain of $f(x,s)$ of size ϵ, and $\chi_\epsilon(x-y_j)$ is the characteristic function of this j-th box, equal to 1 on this box, and 0 elsewhere—namely, using spatial "pulse functions" to approximate $f(y,s)$, then $S(x,t;s)[f]$ will be approximated by $\sum f(y_j,s)S(x,t;s)[\chi_\epsilon(x-y_j)]$. Namely, we can reduce the construction of a solution to a non-homogeneous IVP to one of solutions to the corresponding homogenous PDE with a special kind of initial data: temporaly and spatially localized near a point.

Since $\alpha_j(\epsilon) \overset{\text{def}}{=} \int \chi_\epsilon(x-y_j)\ dx \to 0$ and the support of $\chi_\epsilon(x-y_j)$ shrinks to y_j as $\epsilon \to 0$, $\alpha_j(\epsilon)^{-1}\chi_\epsilon(x-y_j) \to \delta(x-y_j)$ in the distribution sense as $\epsilon \to 0$, and in favorable cases, $S(x,t;s)[\alpha_j(\epsilon)^{-1}\chi_\epsilon(x-y_j)]$ has a limit as $\epsilon \to 0$, labeled as $S(x,t;s)[\delta(\cdot - y_j)]$. Furthermore, it is reasonable to expect

$$S(x,t;s)[f] = \lim_{\epsilon\to 0}\sum f(y_j,s)S(x,t;s)[\alpha_j(\epsilon)^{-1}\chi_\epsilon(x-y_j)]\alpha_j(\epsilon) = \int U(x,t;y,s)f(y,s)dy,$$

where $U(x,t;y,s) = S(x,t;s)[\delta(\cdot - y)]$ as a function of (x,t) solves the same problem as $V_j(x,t) = v(x,t)$ in the heuristic discussion, with the only difference being the second initial condition

$$\partial_t U(x,t;y,s)\big|_{t=s} = \delta(x-y).$$

So the solution $u(x,t) = \int_0^t S(x,t;s)[f]ds$ can also be expressed as

$$\int_0^t \int U(x,t;y,s)f(y,s)dyds.$$

Here $U(x,t;y,s)$ is a so called fundamental solution.

In the case of the wave equation, since it is defined as the limit of $S(x,t;s)[\alpha(\epsilon)^{-1}\chi_\epsilon(x-y)]$ as $\epsilon \to 0$, and

$$S(x,t;s)[\alpha(\epsilon)^{-1}\chi_\epsilon(x-y)] = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} \alpha(\epsilon)^{-1}\chi_\epsilon(x'-y)\,dx',$$

it can be worked out easily as

$$\begin{aligned} U(x,t;y,s) &= \begin{cases} \frac{1}{2c} & \text{if } x-c(t-s) < y < x+c(t-s) \text{ and } t > s, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \frac{1}{2c} & \text{if } -c(t-s) < x-y < c(t-s) \text{ and } t > s, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The advantage of introducing $U(x,t;y,s)$ is that it depends only on the wave equation, not on f; the price to be paid is that it has jump discontinuities along $x-y = \pm c(t-s), t \geq s$. But $\int U(x,t;y,s)f(y,s)dy = \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$ regains differentiability in x and t if f is continuous.

Exercises

Exercise 3.2.1. This problem deals with the Duhamel principle applied to the non-homogeneous version of IBVP (2.1), replacing 0 on the right hand side of the equation by $f(x,t)$. Note that the Fourier series solution of (2.1) can be expressed as an integral involving the initial data g as follows. Using $c_n = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})\,dx$ in $u(x,t) = \sum_{n=1}^\infty c_n \sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t}$, prove that for $t>0$, $u(x,t) = \int_0^l g(y)G(x,y,t)\,dy$, where

$$G(x,y,t) = \frac{2}{l}\sum_{n=1}^\infty \sin(\frac{n\pi y}{l})\sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t}.$$

Denote $\int_0^l g(y)G(x,y,t)\,dy$ by $H[g](x,t)$. Derive that

$$H[g](x,t)+\int_0^t H[f(\cdot,s)](x,t-s)ds = \int_0^l g(y)G(x,y,t)\,dy+\int_0^t\int_0^l G(x,y,t-s)f(y,s)\,dyds$$

is a (formal) solution of

$$\begin{cases} u_t - u_{xx} = f(x,t), & \text{for } (x,t) \in (0,l)\times\mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in (0,l). \end{cases}$$

Exercise 3.2.2. This problem deals with the Duhamel principle applied to the Cauchy problem for the heat equation:

$$\begin{cases} u_t - u_{xx} = f(x,t), \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = 0, \quad \text{for } x \in \mathbb{R}. \end{cases} \tag{3.8}$$

Note that $U(x, \tau; s) = \int_{\mathbb{R}} K(x-y, \tau - s) f(y,s)\, dy$ is a solution of

$$\begin{cases} U_\tau - U_{xx} = 0, \quad \text{for } (x,\tau) \in \mathbb{R} \times (s, \infty), \\ U(x,s;s) = f(x,s), \quad \text{for } x \in \mathbb{R}. \end{cases}$$

Prove that, if f is C^1 in its variables, bounded, and with bounded first derivative in x, then

$$u(x,t) = \int_0^t \int_{\mathbb{R}} K(x-y, t-s) f(y,s)\, dyds \in C(\mathbb{R} \times \overline{\mathbb{R}^+}) \cap C^{2,1}_{x,t}(\mathbb{R} \times \mathbb{R}^+)$$

is a solution of (3.8), where

$$C^{2,1}_{x,t}(\mathbb{R} \times \mathbb{R}^+) = \{u(x,t) : \partial_x^a \partial_t^b u \in C(\mathbb{R} \times \mathbb{R}^+) \text{ for all } a + 2b \le 2\},$$

and $K(x,t)$ is the heat kernel to the heat equation as introduced in (2.18). (*Remark: Differentiation under the integral sign can not be justified if one merely assumes f to be continuous, as the best one can get under this assumption is $|U_t(x,t;s)| \le \frac{C \max |f|}{t-s}$, which is not sufficient to justify $\frac{d}{dt} \int_0^t U(x,t;s)ds = \int_0^t U_t(x,t;s)ds$. (*HINT*: Justify*

$$U_t(x,t;s) = \int_{\mathbb{R}} K_t(x-y, t-s) f(y,s)\, dy = -\int_{\mathbb{R}} \nabla_y K(x-y, t-s) \cdot \nabla_y f(y,s)\, dy$$

and use it to get an improved bound on $|U_t(x,t;s)|$. As we will discuss later, some Hölder continuity on f is enough to justify differentiation under the integral sign.)

3.3 Reflection Method and Compatibility Conditions: Applied to One Dimensional Wave Equation on the Half Line

We use the one dimensional wave equation on the half line to illustrate the reflection method and discuss the issue of compatibility conditions. Consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & \text{for} \quad (x,t) \in \mathbb{R}^+ \times [0,\infty), \\ u(x,0) = g(x), & \text{for} \quad x > 0, \\ u_t(x,0) = h(x), & \text{for} \quad x > 0, \\ u(0,t) = 0, & \text{for} \quad t > 0. \end{cases} \tag{3.9}$$

We want to use the d'Alembert's formula for the Cauchy problem on the entire axis to construct a solution of the above problem on the half axis. In order to satisfy the boundary condition $u(0,t) = 0$, for $t > 0$, it is natural to do an odd extension of the initial data and the right hand side of the equation:

$$\tilde{g}(x) = \begin{cases} g(x), & \text{if } x \geq 0; \\ -g(-x), & \text{if } x < 0. \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x), & \text{if } x \geq 0; \\ -h(-x), & \text{if } x < 0. \end{cases}$$

$$\tilde{f}(x,t) = \begin{cases} f(x,t), & \text{if } x \geq 0; \\ -f(-x,t), & \text{if } x < 0. \end{cases}$$

Then

$$\tilde{u}(x,t) = \frac{1}{2}\left[\tilde{g}(x+ct) + \tilde{g}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{h}(y)\,dy + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \tilde{f}(y,\tau)\,dy \tag{3.10}$$

is the solution formula provided by the d'Alembert's formula—it is a smooth solution of the wave equation provided $\tilde{g}(x)$, $\tilde{h}(x)$ and $\tilde{f}(x,t)$ have sufficient smoothness. Due to the odd symmetry, $\tilde{u}(0,t) = 0$ for all $t > 0$. In the region $x > ct$, $\tilde{u}(x,t)$ is expressed in terms of the given initial data and $f(x,t)$ on $x > 0$:

$$\tilde{u}(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(y,\tau)\,dy.$$

In the region $x < ct$, we can convert the integral $\int_{x-ct}^{x+ct} \tilde{h}(y)\,dy$ to $\int_{ct-x}^{x+ct} h(y)\,dy$ to obtain

$$\tilde{u}(x,t) = \frac{1}{2}\left[g(x+ct) - g(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} h(y)\,dy + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} \tilde{f}(y,\tau)\,dy$$

where the last integral can also be expressed in terms of integrals of $f(y,t)$ in intervals of $y > 0$. Note that when $0 < x < ct$, the characteristic curve through (x,t) is a straight line which intersects the t-axis at $(0, t - \frac{x}{c})$, and it gets reflected into a straight line which intersects the x-axis at $ct - x$.

From Theorem 3.1, in order for $\tilde{u}(x,t)$ to be a C^2 solution for all (x,t), a sufficient condition is to assume $\tilde{g}$ to be C^2, and $\tilde{h}$, $\tilde{f}$ to be C^1. This amounts to

$$\begin{gathered} g \in C^2[0,\infty), g(0) = 0, \qquad g''(0) = 0; \\ h \in C^1[0,\infty), h(0) = 0; \\ f \in C^1([0,\infty) \times [0,\infty)), f(0,t) = 0, \qquad \text{for all } t > 0. \end{gathered}$$

But these conditions are too restrictive. Assuming only that $g \in C^2[0,\infty)$, $h \in C^1[0,\infty)$, and $f \in C^1([0,\infty) \times [0,\infty))$, the formula in (3.10) provides a function which is C^2 in $(x,t) \in [0,\infty) \times [0,\infty)$ with possibly the exception along $x = ct$.

In order for $\tilde{u}$ to be continuous at the corner point $(0,0)$, a necessary condition is that $\lim_{x\to 0+} \tilde{u}(x,0) = \lim_{t\to 0+} \tilde{u}(0,t)$, *i.e.*, $g(0) = 0$. In order for $\tilde{u}$ to be C^1 at the corner point $(0,0)$, a necessary condition is that $\lim_{x\to 0+} \tilde{u}_t(x,0) = \lim_{t\to 0+} \tilde{u}_t(0,t)$, *i.e.*, $h(0) = 0$. In order for $\tilde{u}$ to be C^2 at the corner point $(0,0)$, a necessary condition is that $\lim_{x\to 0+} \tilde{u}_{xx}(x,0) = \lim_{t\to 0+} \tilde{u}_{xx}(0,t)$. Since $\lim_{x\to 0+} \tilde{u}_{xx}(x,0) = g''(0)$, but $\lim_{t\to 0+} \tilde{u}_{xx}(0,t)$ can be calculated through the equation as

$$\lim_{t\to 0+} \left[u_{tt}(0,t) - f(0,t)\right]/c^2 = -f(0,0)/c^2,$$

we have $g''(0) = -f(0,0)/c^2$.

$g(0) = 0, h(0) = 0$, and $g''(0) = -f(0,0)/c^2$ are the compatibility conditions of up to second order derivatives for (3.9) at $(0,0)$. Note that these compatibility conditions are obtained by examining the boundary/initial conditions, as well as the PDE at the corner point $(0,0)$—all these conditions are supposed to hold on the closure of the domain, in particular, at the corner point $(0,0)$ for a solution which is C^2 in $[0,\infty) \times [0,\infty)$. It turns out that these compatibility conditions are also sufficient conditions for $\tilde{u}$ to be a C^2 solution in $(x,t) \in [0,\infty) \times [0,\infty)$.

Remark 3.8. From both physical and mathematical points of view, it is too restrictive to demand to deal only with C^2 solutions. Examination of the above discussion shows that if we assume $g(0) = 0$, then $\tilde{u}(x,t)$ will be continuous in $[0,\infty) \times [0,\infty)$; and if we we assume $h(0) = 0$, in addition to $g(0) = 0$, then $\tilde{u}(x,t)$ will be C^1 in $[0,\infty) \times [0,\infty)$. In both cases the formula (3.10) should be regarded as providing a generalized solution—we will discuss later (in section 3.5) on the appropriate rules in defining generalized solutions; but first see Exercise 3.3.3 below.

Remark 3.9. The compatibility issue also arises in the initial-boundary value problem (2.1) for the heat equation. Theorem 2.2 provides solutions that are smooth in $[0,l] \times (0,\infty)$ without necessarily requiring $g(x) = u(x,0)$ to be compatible at $x = 0$ or l with the boundary conditions $u(0,t) = u(l,t) = 0$ for $t > 0$; but the resulting solution may not be in $C([0,l] \times [0,\infty))$. In order to obtain a solution that is in $C([0,l] \times [0,\infty))$, we need to assume $g(0) = g(l) = 0$; and in order to obtain a solution that is in $C^2([0,l] \times [0,\infty))$, we need to assume further that $g''(0) = g''(l) = 0$, as the equation in this case $u_t(x,t) - u_{xx}(x,t) = 0$ would need to be satisfied at $(0,0)$ and $(l,0)$.

Exercise 3.3.1. Apply the reflection method to study the IBVP

$$\begin{cases} u_t - u_{xx} = f(x,t), & \text{for} \quad (x,t) \in \mathbb{R}^+ \times [0,\infty), \\ u(x,0) = g(x), & \text{for} \quad x > 0, \\ u(0,t) = h(t), & \text{for} \quad t > 0. \end{cases}$$

(a). Assume $h \equiv 0$. Use the method of this section to establish a solution formula for $u(x,t)$ in terms of $g(x)$ and $f(x,t)$. Then discuss compatibility of the boundary and initial conditions at $(x,t) = (0,0)$, and the sense in which the solution $u(x,t)$ takes on the initial value $g(x)$.

(b). Repeat the same discussion for a general $h(t)$. (You may assume $h(t)$ to be continuously differentiable and carry out the substitution $v(x,t) = u(x,t) - h(t)$.)

Exercise 3.3.2. Adapt the reflection method to study the IBVP

$$\begin{cases} u_t - u_{xx} = f(x,t), & \text{for} \quad (x,t) \in \mathbb{R}^+ \times [0,\infty), \\ u(x,0) = g(x), & \text{for} \quad x > 0, \\ u_x(0,t) = 0, & \text{for} \quad t > 0. \end{cases}$$

Exercise 3.3.3. This Exercise provides some details for the justification that under the assumption that $g \in C^2[0,\infty)$ with $g(0) = 0$, $h \in C^1[0,\infty)$ with $h(0) = 0$, and $f \in C^1([0,\infty)\times[0,\infty))$ with $g''(0) = -f(0,0)/c^2$, then (3.10) provides a C^2 solution in $(x,t) \in [0,\infty)\times[0,\infty)$ to (3.9). Note that the odd extension $\tilde{h}(x)$ of $h(x)$ is $C^1(\mathbb{R})$, so $V(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\tilde{h}(y)\,dy$ provides a $C^2([0,\infty)\times[0,\infty))$ solution of the homogeneous version of (3.9) with $V(x,0)=0$, $V_t(x,t) = h(x)$ for $x \in \mathbb{R}^+$. Let $U(x,t) = \frac{1}{2}\left[\tilde{g}(x+ct)+\tilde{g}(x-ct)\right]$, and $W(x,t) = \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)}\tilde{f}(y,\tau)\,dy$. We will see that both $U(x,t)$ and $W(x,t)$ are $C^1([0,\infty)\times[0,\infty))$, but their second derivatives may experience a jump discontinuity across $x = ct$; however, $U(x,t) + W(x,t)$ is $C^2([0,\infty)\times[0,\infty))$!

(a). Prove that

$$W_x(x,t) = \begin{cases} (2c)^{-1}\int_0^t \left[f(x+c(t-s),s) - f(x-c(t-s),s)\right]ds & \text{if } x > ct,\\ (2c)^{-1}\int_0^{t-\frac{x}{c}} \left[f(x+c(t-s),s) + f(c(t-s)-x,s)\right]ds & \\ +(2c)^{-1}\int_{t-\frac{x}{c}}^t \left[f(x+c(t-s),s) - f(x-c(t-s),s)\right]ds & \text{if } 0 \le x < ct; \end{cases}$$

$$W_t(x,t) = \begin{cases} 2^{-1}\int_0^t \left[f(x+c(t-s),s) + f(x-c(t-s),s)\right]ds & \text{if } x > ct,\\ 2^{-1}\int_0^{t-\frac{x}{c}} \left[f(x+c(t-s),s) - f(c(t-s)-x,s)\right]ds & \\ +2^{-1}\int_{t-\frac{x}{c}}^t \left[f(x+c(t-s),s) + f(x-c(t-s),s)\right]ds & \text{if } 0 \le x < ct; \end{cases}$$

and that they are continuous over $[0,\infty)\times[0,\infty)$.

(b). Prove that

$$W_{xx}(x,t) = \begin{cases} (2c)^{-1}\int_0^t \left[f_x(x+c(t-s),s) - f_x(x-c(t-s),s)\right]ds & \text{if } x > ct,\\ (2c)^{-1}\int_0^t \left[f_x(x+c(t-s),s) - f_x(|c(t-s)-x|,s)\right]ds & \\ -c^{-2}f(0,t-\frac{x}{c}) & \text{if } 0 \le x < ct, \end{cases}$$

$$W_{tt}(x,t) = \begin{cases} \frac{c}{2}\int_0^t \left[f_x(x+c(t-s),s) - f_x(x-c(t-s),s)\right]ds + f(x,t) & \text{if } x > ct,\\ \frac{c}{2}\int_0^t \left[f_x(x+c(t-s),s) - f_x(|c(t-s)-x|,s)\right]ds + f(x,t) & \\ -f(0,t-\frac{x}{c}) & \text{if } 0 \le x < ct. \end{cases}$$

Thus both $W_{xx}(x,t)$ and $W_{tt}(x,t)$ experience a jump discontinuity across $x = ct$, if $f(0,0) \neq 0$.

(c). Prove that

$$U_{xx}(x,t) = \begin{cases} \frac{1}{2}\left[g''(x-ct)+g''(x+ct)\right] & \text{if } x > ct,\\ \frac{1}{2}\left[-g''(ct-x)+g''(x+ct)\right] & \text{if } 0 \le x < ct, \end{cases}$$

$$U_{tt}(x,t) = \begin{cases} \frac{c^2}{2}\left[g''(x-ct)+g''(x+ct)\right] & \text{if } x > ct, \\ \frac{c^2}{2}\left[-g''(ct-x)+g''(x+ct)\right] & \text{if } 0 \le x < ct. \end{cases}$$

Thus both $U_{xx}(x,t)$ and $U_{tt}(x,t)$ experience a jump discontinuity across $x = ct$, if $g''(0) \neq 0$.

(d). Using the condition that $g''(0) = -f(0,0)/c^2$ to prove that $U(x,t) + W(x,t)$ is $C^2([0,\infty) \times [0,\infty))$.

3.4 The Method of Eigenfunction Expansion

For non-homogeneous linear PDEs, in addition to the Duhamel principle, we can also use the outcome of the separation of variables method as applied to the corresponding homogeneous PDE and adapt the idea of variation of parameters method in the theory of ODEs to form the method of eigenfunction expansions.

More specifically, when the $n \times n$ matrix A is diagonalized by the set of vectors $\{\vec{v}_1, \cdots, \vec{v}_n\}$, then not only can we use them to generate all solutions to the homogeneous system $\vec{w}'(t) = A\vec{w}(t)$, we can also construct solutions to the non-homogeneous system $\vec{w}'(t) = A\vec{w}(t) + \vec{f}(t)$ in the form of $\sum_{i=1}^n c_i(t)\vec{v}_i$. Since $\{\vec{v}_1, \cdots, \vec{v}_n\}$ forms a basis for $\mathbb{R}^n$, we can also write $\vec{f}(t) = \sum_{i=1}^n f_i(t)\vec{v}_i$. Then each $c_i(t)$ would satisfy $c_i'(t) = \lambda_i c_i(t) + f_i(t)$, where λ_i is the eigenvalue of A associated with $\vec{v}_i$. For the wave or heat equation, we can regard $e^{ix\cdot\xi}$ (or $\sin(\frac{n\pi x}{l})$ in the case of homogeneous Dirichlet boundary condition on $[0,l]$) as generalized eigenfunctions of Δ that "diagonalize" Δ and "form a basis for appropriate function space".

We illustrate this method via (2.11) with a non-homogeneous right hand side $f(x,t)$. Note that the boundary conditions $u(0,t) = u(l,t) = 0$ are homogeneous, and the associated eigenfunctions $\{\sin(\frac{n\pi x}{l})\}$ have incorporated these homogeneous boundary conditions in solving the homogeneous equation.

Since any $L^2[0,l]$ function can be expanded as a series in $\{\sin(\frac{n\pi x}{l})\}$ which is convergent in $L^2[0,l]$, we may look for a solution $u(x,t)$ such that at any t, $u(\cdot,t)$, $u_{tt}(\cdot,t)$, and $u_{xx}(\cdot,t) \in L^2[0,l]$:

$$u(\cdot,t) = \sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi x}{l}),$$
$$u_{tt}(\cdot,t) = \sum_{n=1}^{\infty} \alpha_n(t)\sin(\frac{n\pi x}{l}),$$
$$u_{xx}(\cdot,t) = \sum_{n=1}^{\infty} \beta_n(t)\sin(\frac{n\pi x}{l}).$$

We also assume that

$$f(\cdot,t) = \sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi x}{l}).$$

Thus

$$\alpha_n(t) - c^2\beta_n(t) = f_n(t), \quad \text{for all } n.$$

But

$$\alpha_n(t) = \frac{2}{l}\int_0^l u_{tt}(x,t)\sin(\frac{n\pi x}{l})\,dx = u_n''(t),$$

and

$$\begin{aligned}
\beta_n(t) &= \frac{2}{l}\int_0^l u_{xx}(x,t)\sin(\frac{n\pi x}{l})\,dx \\
&= \frac{2}{l}u_x(x,t)\sin(\frac{n\pi x}{l})|_{x=0}^{x=l} - \frac{2}{l}\int_0^l \frac{n\pi}{l}u_x(x,t)\cos(\frac{n\pi x}{l})\,dx \\
&= -\frac{2}{l}\frac{n\pi}{l}u(x,t)\cos(\frac{n\pi x}{l})|_{x=0}^{x=l} - \frac{2}{l}\int_0^l \left(\frac{n\pi}{l}\right)^2 u(x,t)\sin(\frac{n\pi x}{l})\,dx \\
&= -\left(\frac{n\pi}{l}\right)^2 u_n(t) \quad \text{(using the condition } u(0,t) = u(l,t) = 0.)
\end{aligned}$$

Thus $u_n(t)$ satisfies

$$u_n''(t) + c^2\left(\frac{n\pi}{l}\right)^2 u_n(t) = f_n(t).$$

This can also be obtained formally from

$$\begin{aligned}
\left[\partial_t^2 - c^2\partial_x^2\right]\left\{\sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi x}{l})\right\} &= \sum_{n=1}^{\infty}\left[\partial_t^2 - c^2\partial_x^2\right]\left\{u_n(t)\sin(\frac{n\pi x}{l})\right\} \\
&= \sum_{n=1}^{\infty}\left[u_n''(t) + c^2\left(\frac{n\pi}{l}\right)^2 u_n(t)\right]\sin(\frac{n\pi x}{l}) \\
&= \sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi x}{l});
\end{aligned}$$

the above steps provide a mechanism for justifying the interchange of differentiation and summation.

We can use the initial conditions to obtain

$$u_n(0) = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})\,dx, \quad u_n'(0) = \frac{2}{l}\int_0^l h(x)\sin(\frac{n\pi x}{l})\,dx.$$

Thus our problem has been reduced to solving an infinite system of (decoupled) ODEs.

To complete the construction of a solution, the central issue is again the convergence of the series $\sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi x}{l})$ and its differentiability. Instead of trying to prove directly the convergence of this infinite series, we will examine the truncated finite sums as an approximate solution: define

$$\begin{aligned} u_{(N)}(x,t) &= \sum_{j=1}^{N} u_n(t)\sin(\frac{n\pi x}{l}),\\ f_{(N)}(x,t) &= \sum_{j=1}^{N} f_n(t)\sin(\frac{n\pi x}{l}),\\ g_{(N)}(x,t) &= \sum_{j=1}^{N} g_n\sin(\frac{n\pi x}{l}),\\ h_{(N)}(x,t) &= \sum_{j=1}^{N} h_n\sin(\frac{n\pi x}{l}). \end{aligned}$$

Then

$$\begin{cases} \partial_t^2 u_{(N)} - c^2\partial_x^2 u_{(N)} = f_{(N)}(x,t), & \text{on } (x,t) \in [0,l]\times\mathbb{R}^+,\\ u_{(N)}(0,t) = u_{(N)}(l,t) = 0, & \text{for } t > 0,\\ u_{(N)}(x,0) = g_{(N)}(x), & \text{for } x \in [0,l],\\ \partial_t u_{(N)}(x,0) = h_{(N)}(x), & \text{for } x \in [0,l]. \end{cases}$$

We then need to study the convergence of $\{u_{(N)}(x,t)\}$ in an appropriate sense. The most elementary notion of convergence is that of uniform convergence of $\{u_{(N)}(x,t)\}$ and the sequences consisting of its derivatives of order up to two. This can be done if we impose sufficient differentiability of f, g, and h.

In a later section we will apply the energy estimates to be developed there to $u_{(N)} - u_{(N')}$ to prove that $\{u_{(N)}(x,t)\}$, $\{\partial_t u_{(N)}(x,t)\}$ and $\{\partial_x u_{(N)}(x,t)\}$ are Cauchy in $C([0,T],L^2[0,l])$ for any $T > 0$, so each has a limit in $C([0,T],L^2[0,l])$—the energy estimates are fairly easy to obtain and are quite robust so as to be extendible to variable coefficient wave equations or even certain nonlinear wave equations.

Let $u(\cdot,t)$ denote the limit of $\{u_{(N)}(x,t)\}$ in $C([0,T],L^2[0,l])$, then the limits of $\{\partial_t u_{(N)}(x,t)\}$ and $\{\partial_x u_{(N)}(x,t)\}$ in $C([0,T],L^2[0,l])$ are respectively the ∂_t and ∂_x derivatives of $u(\cdot,t)$ in an appropriate sense, and one can define $u(\cdot,t)$ to satisfy the wave equation in an appropriate integral sense.

Remark 3.10. The eigenfunction expansion method can also be applied to the initial-boundary value problem with non-homogeneous boundary conditions. For instance, consider the IBVP of the heat equation

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = f(x,t) & \text{for } (x,t) \in (0,l)\times\mathbb{R}^+, \\ u(0,t) = g_0(t) \quad u(l,t) = g_l(t) & \text{for } t\in[0,\infty), \\ u(x,0) = g(x) & \text{for } x \in [0,l]. \end{cases} \tag{3.11}$$

The usual strategy is to reduce (3.11) into several sub-problems of similar type, where parts of the boundary conditions (or source terms) are homogeneous. For example, we can first find an extension $h(x,t)$ on $[0,l]\times[0,\infty)$ of $g_0(t)$ and $g_l(t)$ with sufficient regularity — let's assume that we can find $h \in C([0,l]\times[0,\infty)) \cap C^{2,1}_{x,t}((0,l)\times\mathbb{R}^+)$ such that $h(0,t) = g_0(t)$ and $h(l,t) = g_l(t)$ for $t \in [0,\infty)$ (this may require some regularity on $g_0(t)$ and $g_l(t)$, especially if we require derivatives of $h(x,t)$ up to certain order to be continuous in the closed region $[0,l]\times[0,\infty)$). Then we look for $u(x,t)$ in the form of $u(x,t) = h(x,t) + v(x,t)$, where $v(x,t)$ would solve

$$\begin{cases} v_t(x,t) - v_{xx}(x,t) = f(x,t) - h_t(x,t) + h_{xx}(x,t) := \tilde{f}(x,t) & \text{for } (x,t) \in (0,l)\times\mathbb{R}^+, \\ v(0,t) = v(l,t) = 0 & \text{for } t \in \mathbb{R}^+, \\ v(x,0) = g(x) - h(x,0) := \tilde{g}(x) & \text{for } x\in[0,l]. \end{cases} \tag{3.12}$$

We can now apply the eigenfunction expansion method to (3.12).

The key to the eigenfunction expansion method is that the PDE fits into the pattern $u_t - L[u] = f$, where $L[u]$ is a differential operator in the space variable x with its coefficients independent of t, and the domain of definition of L may incorporate the boundary conditions for u; then when one first looks for separable solutions $T(t)X(x)$ to the homogeneous version of the PDE $u_t - L[u] = 0$, one is faced with the eigenvalue problem $L[X] = \lambda X$, and $T'(t) = \lambda T(t)$; and finally, if L is "diagonalizable" in the sense that it has a complete set of eigenfunctions $X_n(x) : L[X_n] = \lambda_n X_n$, namely, any function in an appropriate function space can be expanded as a "Fourier series" in terms of these eigenfunctions, then we can expand $f(x,t) = \sum f_n(t)X_n(x)$, and construct a solution of $u_t - L[u] = f$ in the form of $\sum T_n(t)X_n(x)$, which would lead to the system of ODEs $T_n'(t) - \lambda_n T_n(t) = f_n(t)$.

This scheme extends to PDEs that are higher order in t, such as the wave equation. For many IBVP's which involve a compact spatial region, the corresponding operator L can often be related to a compact self-adjoint operator, for which there is a standard theory of "diagonalizability".

Specific examples of such operators include the Schrödinger type operators $-\Delta + V(x)$ on a bounded domain acting on functions with zero boundary value, and their variable coefficient versions such as the spherical Laplace operator $\Delta_{\mathbb{S}^2} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$ on the round sphere, or a domain on the round sphere—acting on functions with zero boundary value.

For instance, if we consider an "annulus region" on the round sphere defined by $\{(\theta,\phi) : \theta_0 < \theta < \theta_1, 0 \le \phi \le 2\pi\}$, and consider either the heat operator or wave operator associated with $\Delta_{\mathbb{S}^2}$, with homogeneous boundary condition, we would need to understand whether $\Delta_{\mathbb{S}^2}$ has a "complete" set of eigenfunctions under this boundary condition.

If $X(\theta,\phi)$ is an eigenfunction on the annulus region: $\Delta_{\mathbb{S}^2} X(\theta,\phi) = -\lambda X(\theta,\phi)$, then we can first do an eigenfunction expansion in the ϕ variable at each θ, as we know that any function of ϕ (square integrable in ϕ over $[0,2\pi]$) can be expanded in terms of $\{e^{in\phi}\}_{n\in\mathbb{Z}}$: $X(\theta,\phi) = \sum_{n=-\infty}^{\infty}\Theta_n(\theta)e^{in\phi}$, then $\Theta_n(\theta)$ would satisfy

$$\frac{1}{\sin\theta}\left(\sin\theta\,\Theta_n'(\theta)\right)' - \frac{n^2}{\sin^2\theta}\Theta_n(\theta) + \lambda\Theta_n(\theta) = 0,$$

with appropriate boundary condition at $\theta = \theta_0$, and θ_1. If $0 < \theta_0 < \theta_1 < \pi$, then the eigenvalue problem we need to solve (with homogeneous Dirichlet boundary condition) is

$$\begin{cases} \frac{1}{\sin\theta}\left(\sin\theta\,\Theta_n'(\theta)\right)' - \frac{n^2}{\sin^2\theta}\Theta_n(\theta) + \lambda\Theta_n(\theta) = 0,\ \theta_0 < \theta < \theta_1, \\ \Theta_n(\theta_0) = \Theta_n(\theta_1) = 0. \end{cases} \tag{3.13}$$

This is an example of a **regular Sturm-Liouville eigenvalue problem**

$$\begin{cases} (p(x)u'(x))' - q(x)u(x) + \lambda w(x)u(x) = 0, a < x < b, \\ \alpha_1 u(a) + \alpha_2 u'(a) = 0, \\ \beta_1 u(b) + \beta_2 u'(b) = 0, \end{cases} \tag{3.14}$$

where $p(x), q(x)$, and $w(x)$ are real valued, continuous on $[a,b]$, with $p(x), w(x) > 0$ on $[a,b]$, and $(\alpha_1,\alpha_2) \neq (0,0)$, $(\beta_1,\beta_2) \neq (0,0)$. The continuity assumptions in $p(x), q(x)$, and $w(x)$ are not as essential as the positive lower bound conditions implied by the positivity assumptions on $p(x)$ and $w(x)$: there exists $m > 0$ such that $p(x), w(x) \geq m$ on $[a, b]$. We will assume that $p(x)$, $q(x)$ and $w(x)$ are bounded: there exists some $M > 0$ such that $p(x), |q(x)|, w(x) \leq M$ for all $x \in [a, b]$. The eigenvalue problem associated with **Exercise** 1.5.2 is an example where the $p(x)$ is discontinuous.

In the eigenvalue problem for the spherical Laplace operator, if $\theta_0 = 0$ or $\theta_1 = \pi$, then the coefficient in front of $\Theta_n'(\theta)$ as well $\Theta_n(\theta)$ become singular, one no longer prescribes a homogeneous boundary condition at the corresponding end; instead, one requires $\Theta_n(\theta)$ remain bounded in a neighborhood of that end. The corresponding eigenvalue problem is an example of a **singular Sturm-Liouville eigenvalue problem**.

Sturm and Liouville first studied these problems in the 1830's when they extended Fourier's method to variable coefficient PDE problems. It turns out that for a regular Sturm-Liouville eigenvalue problem,

(a). All its eigenvalues are real, and they are distributed discretely on the real axis;

(b). Its eigenfunctions associated with distinct eigenvalues are orthogonal in $L^2(a, b)$ with $w(x)$ as a **weight**, namely, $\int_a^b X(x)Y(x)w(x)\,dx = 0$ if $X(x)$ and $Y(x)$ are eigenfunctions of (3.14) associated with distinct eigenvalues—for this reason it's more meaningful to work with $L^2_w(a, b)$, the set of functions square integrable with w as a weight; and

(c). It has a complete set of eigenfunctions which span $L^2_w(a, b)$, namely, any function in $L^2_w(a, b)$ can be approximated in the mean square sense by a linear combination of the eigenfunctions. As a consequence, any $L^2_w(a, b)$ function can be expanded in a "Fourier series" in terms of the eigenfunctions, and there is a Parseval type identity. In other words, (i) and (ii) of Theorem 2.1 holds in this general context—the integrals there and in the Parseval identity need to use the w-weighted integrals.

(a) and (b) are elementary, and will be left as an exercise; we will provide a proof for (c) later.

The regular Sturm-Liouville problem has the additional property that the eigenspace corresponding to any eigenvalue is one dimensional. This follows from the uniqueness of IVP for the ODE: the solution space to the ODE without any boundary or initial condition is two dimensional; if the eigenspace were two dimensional, it would mean that all solutions to the ODE would satisfy the two homogeneous boundary conditions, which is not true.

When solving an IVP on $\mathbb{R}^n \times (0, T]$, the corresponding operator is no longer directly related to a compact self-adjoint operator, but we have used informally that $e^{ix\cdot\xi}$ for $\xi \in \mathbb{R}^n$ can be regarded as "generalized eigenfunction" of Δ on $\mathbb{R}^n$, and that any $L^2(\mathbb{R}^n)$ function can be expanded in terms of $e^{ix\cdot\xi}$ via the Fourier transform. We can adapt the eigenfunction expansion method to this setting as well. Here is an example to illustrate this adaptation.

Example 3.1. We will find an integral representation for a solution of

$$\begin{cases} u_t - u_{xx} = f(x,t) & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = g(x) & x \in \mathbb{R}. \end{cases} \tag{3.15}$$

At each $t > 0$, we expand $f(x,t) = \int_{\mathbb{R}} \hat{f}(\xi,t)e^{ix\xi}d\xi$ and $u(x,t) = \int_{\mathbb{R}} \hat{u}(\xi,t)e^{ix\xi}d\xi$; then treating $\hat{u}(\xi,t)e^{ix\xi} = T(t)e^{ix\xi}$ as the component corresponding to the Fourier mode $e^{ix\xi}$, we obtain, for each ξ as a parameter, an ODE in $\hat{u}(\xi,t)$:

$$\begin{cases} \dfrac{d\hat{u}(\xi,t)}{dt} + \xi^2\hat{u}(\xi,t) = \hat{f}(\xi,t), \\ \hat{u}(\xi,0) = \hat{g}(\xi), \end{cases}$$

where $\hat{g}(\xi)$ is given in terms of $g(x) = \int_{\mathbb{R}} \hat{g}(\xi)e^{ix\xi}d\xi$. Note that the procedure is essentially the method of variation of parameters, with $\hat{u}(\xi,t)$ serving as the coefficient function in front of $e^{ix\xi}$.

Many treatments derive the above ODE by taking Fourier transforms; but setting up appropriate framework in which Fourier transforms (in partial variables) can be taken and interchanged with differential operators would take considerable technical preparation, we have found this informal eigenfunction expansion approach very efficient—either approach would usually require some additional justifications, and since our main aim is to discover possible integral representation for a solution, we are happy to take this efficient approach.

The solution of the above ODE IVP is given by

$$\hat{u}(\xi,t) = \hat{g}(\xi)e^{-|\xi|^2 t} + \int_0^t e^{-|\xi|^2(t-s)}\hat{f}(\xi,s)ds.$$

We can now obtain $u(x,t)$ as

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}} \left[\hat{g}(\xi)e^{-|\xi|^2 t} + \int_0^t e^{-|\xi|^2(t-s)}\hat{f}(\xi,s)ds\right] e^{ix\xi}d\xi \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \left[\int_{\mathbb{R}} g(y)e^{i(x-y)\xi-|\xi|^2 t}\,dy + \int_0^t\int_{\mathbb{R}} e^{-|\xi|^2(t-s)+i(x-y)\xi}f(y,s)\,dy\,ds\right]d\xi \\ &= \frac{1}{2\pi}\left[\int_{\mathbb{R}} g(y)\left(\int_{\mathbb{R}} e^{i(x-y)\xi-|\xi|^2 t}d\xi\right)dy + \int_0^t\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-|\xi|^2(t-s)+i(x-y)\xi}d\xi\right)f(y,s)\,dy\,ds\right] \end{aligned}$$

Since $\frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x-y)\xi-|\xi|^2 t}d\xi = K(x-y,t)$ is the heat kernel, we have now obtained

$$u(x,t) = \int_{\mathbb{R}} g(y)K(x-y,t)\,dy + \int_0^t \int_{\mathbb{R}} K(x-y,t-s)f(y,s)\,dy\,ds.$$

Exercises

Exercise 3.4.1. Obtain formal series solutions to (3.12); then use the corresponding energy estimates to prove that the series converges in $C([0,T];L^2[0,l])$ under the assumption that $\tilde{g} \in L^2[0,l]$ and $\tilde{f} \in L^2([0,l]\times[0,T])$.

Exercise 3.4.2. Use the eigenfunction expansion method to find formal series solutions $\sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi x}{l})$ to

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) = f(x,t) & \text{for } (x,t) \in (0,l)\times\mathbb{R}^+, \\ v(0,t) = v(l,t) = 0 & \text{for } t \in \mathbb{R}^+, \\ v(x,0) = g(x) & \text{for } x \in (0,l), \\ v_t(x,0) = h(x) & \text{for } x \in (0,l)\ . \end{cases}$$

Then use the corresponding energy estimates to prove that, if $u^{(N)} = \sum_{n=1}^{N} u_n(t)\sin(\frac{n\pi x}{l})$, then $\{u^{(N)}\}$ converges in $C([0,l]\times[0,T])$, and $\{u_t^{(N)}\}$ and $\{u_x^{(N)}\}$ converge in $C([0,T],L^2[0,l])$ under the assumption that g is absolutely continuous over $[0,l]$ with $g(0) = g(l) = 0$, $g', h \in L^2[0,l]$ and $f \in L^2([0,l]\times[0,T])$. (Review Remark 2.3 to understand the need for the conditions on g; you may also find (1.30) helpful.)

Exercise 3.4.3. Set up an eigenfunction expansion scheme for constructing solutions to

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) - \gamma u_x(x,t) = f(x,t), \quad 0 < x < \pi, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = g(x), \quad 0 < x < \pi, \end{cases}$$

where γ is some non-zero constant. What are the eigenfunctions and their associated Sturm-Liouville BVPs written in a standard form as (3.14)?

Exercise 3.4.4. Prove that any eigenvalue of (3.14) must be real valued. (Hint: Multiple the equation of $X(x)$ by $\overline{X(x)}$, and integrate over (a,b), using the boundary conditions.)

Exercise 3.4.5. Suppose that $X(x)$ and $Y(x)$ are eigenfunctions of (3.14) associated with distinct eigenvalues λ and μ, respectively. Based on the previous exercise, one may assume that $X(x)$ and $Y(x)$ are real valued. Prove that $\int_a^b X(x)Y(x)w(x)\,dx = 0$. (HINT: Multiple the equation of $X(x)$ by $Y(x)$ and the equation of $Y(x)$ by $X(x)$, subtract the resulting equations, and integrate by parts over (a,b), using the boundary conditions.)

3.5 Some Remarks on Generalized Solutions

We now add a few more words on the concept of generalized solutions. Here are the guiding principles for the concept of generalized solutions:

(A). Classical solutions must be generalized solutions;

(B). Reasonable limits of classical solutions (in appropriate norms) should be generalized solutions;

(C). Uniqueness should hold for the generalized solutions, and there should be some kind of continuous dependence (in appropriate norms) of the generalized solutions on data.

Generalized solutions may allow the solutions to have some singularities (discontinuity of the solutions or their derivatives, or the size of the solutions or their derivatives become infinite on some part of the region). Depending on the problem, there may be different kinds of singularities that are relevant. For instance, there may be situations where discontinuity in the solutions need to be considered; while in other situations, one may need to consider solutions that are continuous, but have discontinuous derivatives. For problems coming from physical background, one often goes back to the physical principles to find the appropriate notion of generalized solutions.

Earlier, we have already introduced a notion of a generalized solution for the one dimensional homogeneous wave equation. It's straightforward to extend that notion to the corresponding non-homogeneous wave equation.

Definition. Given two integrable (or $L^2(\mathbb{R})$) functions $g(x)$ and $h(x)$. We say $u \in L^2_{\text{local}}(\mathbb{R}\times[0,\infty))$ is a generalized solution to (3.3) if

$$\begin{aligned}&\iint_{\mathbb{R}\times[0,\infty)} f(x,t)\zeta(x,t)\,dxdt\\ &= \iint_{\mathbb{R}\times[0,\infty)} [u(\zeta_{tt} - c^2\zeta_{xx})]\,dxdt - \int_{\mathbb{R}} [h(x)\zeta(x,0) - g(x)\zeta_t(x,0)]\,dx\end{aligned}$$

for all $\zeta \in C^2(\mathbb{R}\times[0,\infty))$ with compact support.

In the first section of this chapter, we have already given an argument for the uniqueness of generalized solution of (3.3). Below is an L^2 type estimate on generalized solutions.

Lemma 3.4. *Prove that, if u is a (generalized) solution of (3.3) or (2.11) with a non-homogeneous right hand side $f(x,t)$, then for any $T > 0$, there is a constant $M = M(T) > 0$ such that*

$$||u||^2_{L^2(\mathbb{R}\times(0,T))} \leq M\left[||f||^2_{L^2(\mathbb{R}\times(0,T))} + ||u(\cdot,0)||^2_{L^2(\mathbb{R})} + ||u_t(\cdot,0)||^2_{L^2(\mathbb{R})} + ||u_x(\cdot,0)||^2_{L^2(\mathbb{R})}\right]$$

The proof for this Lemma will be left as an exercise. With the help of **Lemma** 3.4, we can give an argument for the existence of a generalized solution to (3.3) when $f \in L^2_{\text{local}}(\mathbb{R}\times[0,\infty))$, g, g', and $h \in L^2(\mathbb{R})$.

For any $T > 0$, we can take a sequence $f_n \in C_c^\infty(\mathbb{R}\times[0,T])$ approximating f in $L^2(\mathbb{R}\times[0,T])$, and a sequence g_n, $h_n \in C_c^\infty(\mathbb{R})$ such that $g_n - g \to 0, g_n' - g' \to 0$, and $h_n - h \to 0$ in $L^2(\mathbb{R})$. Then we can use d'Alembert's formula to construct a classical solution $u_n(x,t)$ on $\mathbb{R}\times[0,T]$ to (3.3) with f_n, g_n, h_n substituing for f, g, h, respectively. Then **Lemma** 3.4 and the energy estimates imply that $\{u_n\}$, $\{\partial_t u_n\}$ and $\{\partial_x u_n\}$ are Cauchy in $L^2(\mathbb{R}\times[0,T])$. So there is a limit $u \in L^2(\mathbb{R}\times[0,T])$, $v \in L^2(\mathbb{R}\times[0,T])$, and $w \in L^2(\mathbb{R}\times[0,T])$ such that

$$u_n \to u, \quad \partial_t u_n \to v, \quad \text{and} \quad \partial_x u_n \to w \quad \text{in } L^2(\mathbb{R}\times[0,T]).$$

For u_n, we have

$$\begin{aligned}&\iint_{\mathbb{R}\times[0,\infty)} f_n(x,t)\zeta(x,t)\,dxdt\\ =&\iint_{\mathbb{R}\times[0,\infty)} [u_n(\zeta_{tt} - c^2\zeta_{xx})]\,dxdt - \int_{\mathbb{R}} [\partial_t u_n(x,0)\zeta(x,0) - u_n(x,0)\zeta_t(x,0)]\,dx,\end{aligned}$$

and

$$\iint_{\mathbb{R}\times[0,\infty)} u_n(x,t)\zeta_t(x,t)\,dxdt = -\iint_{\mathbb{R}\times[0,\infty)} \partial_t u_n(x,t)\zeta(x,t)\,dxdt - \int_{\mathbb{R}} u_n(x,0)\zeta(x,0)\,dx,$$

$$\iint_{\mathbb{R}\times[0,\infty)} u_n(x,t)\zeta_x(x,t)\,dxdt = -\iint_{\mathbb{R}\times[0,\infty)} \partial_x u_n(x,t)\zeta(x,t)\,dxdt,$$

for all $\zeta \in C^2(\mathbb{R}\times[0,\infty))$ with compact support. Taking limit as $n \to \infty$, we obtain that

$$\begin{aligned}&\iint_{\mathbb{R}\times[0,\infty)} f(x,t)\zeta(x,t)\,dxdt\\ =&\iint_{\mathbb{R}\times[0,\infty)} [u(\zeta_{tt} - c^2\zeta_{xx})]\,dxdt - \int_{\mathbb{R}} [h(x)\zeta(x,0) - g(x)\zeta_t(x,0)]\,dx,\end{aligned}$$

and

$$\iint_{\mathbb{R}\times[0,\infty)} u(x,t)\zeta_t(x,t)\,dxdt = -\iint_{\mathbb{R}\times[0,\infty)} v(x,t)\zeta(x,t)\,dxdt - \int_{\mathbb{R}} g(x)\zeta(x,0)\,dx,$$

$$\iint_{\mathbb{R}\times[0,\infty)} u(x,t)\zeta_x(x,t)\,dxdt = -\iint_{\mathbb{R}\times[0,\infty)} w(x,t)\zeta(x,t)\,dxdt.$$

The first equality shows that u is a generalized solution of (3.3); the remaining two equalities show that we can identify v and w to be the generalized $L^2(\mathbb{R}^2)$ derivatives $\partial_t u$, and $\partial_x u$, respectively. Thus this construction shows that we have obtained a generalized solution which has a notion of generalized first derivatives, not merely an $L^2(\mathbb{R}^2)$ generalized solution.

Note that it's difficult to get a good sense in which this notion of generalized solution takes on the initial data—we expect some sense of continuity in t of the generalized solution, and it's not easy to make sense of such a continuity in this notion of generalized solution.

As will be seen later on, having generalized $L^2(\mathbb{R}^2)$ first derivatives in the integral sense defined above will provide some sense of continuity for the function. It thus makes sense to incorporate the notion of generalized derivatives in the notion of a generalized solution. For instance, we may require a generalized solution of have generalized $L^2(\mathbb{R}^2)$ first derivatives in the integral sense defined above (in particular, $u(x,t)$ takes on initial value $g(x)$ through the integral relation above) and satisfy

$$\iint_{\mathbb{R}\times[0,\infty)} f(x,t)\zeta(x,t)\,dxdt = -\iint_{\mathbb{R}\times[0,\infty)} [u_t\zeta_t - c^2 u_x\zeta_x]\,dxdt - \int_{\mathbb{R}} h(x)\zeta(x,0)\,dx$$

for all $\zeta \in C_c^1(\mathbb{R}\times[0,\infty))$.

Generalized solutions are also called weak solutions. Further study of the notion of generalized derivatives and generalized solutions will be undertaken later on.

Concluding Remarks

Here are a few key features that have emerged.

(i). It is natural and fruitful to approach a PDE by first finding formal, exploratory prototype solutions, and then try to build more general solutions. When the equation has a nice structure (for instance, with constant coefficients, and is homogeneous), and the domain has appropriate geometry, separation of variables is often effective. This method often reduces the problem to an eigenvalue boundary value problem for an ODE. The key for building a general solution for a linear PDE through superposition is to find the right notion of convergence.

(ii). The convergence of the (prototype or later approximate) solutions can be established if we have appropriate a priori estimates for smooth solutions such as the energy estimates for solutions of the wave equation. Note that we don't have to have an explicit formula for the solution of derive a useful estimate. We also learned that it is not fruitful to always insist on point-wise or uniform convergence, that convergence in other (often integral) norms are often forced upon us by the structure of the problem.

As we move forward, here are a few questions that we should keep in mind.

- We have been able to solve the three prototype equations when the domain has special geometry. How to deal with the situation when the domain has no special geometry?
- We have been been able to obtain solvability for these prototype equations with non-homogeneous right hand side when it is sufficiently nice. Can we reduce the smoothness assumptions on it?
- How do we solve problems which are variations of the prototype equations (with some additional terms added, or with variable coefficients)?
- Given a general PDE which does not bear much resemblance to any of our prototype equations, how do we go about investigating whether it is reasonable (well-posed?) to study the boundary value or initial value problem for it? How do we go about investigating whether its solution is behaving like those of the heat equation or wave equation, or something completely different?

3.6 Additional Problems

Problem 3.6.1. It is known that, for each $n \in \mathbb{Z}$, (3.13) has a sequence of eigenvalues $\lambda_{l,n} \to \infty$ as $l \to \infty$, such that for each $\lambda_{l,n}$ the eigenspace is one dimensional and spanned by $\Theta_{l,n}(\theta)$, say.

(i). Make the change of variables $z = \cos\theta$ and $Z(\cos\theta) = \Theta(\theta)$. Verify that $Z(z)$ satisfies the following general **Legendre** ODE

$$(1-z^2)Z''(z) - 2zZ'(z) + \left(\lambda_{l,n} - \frac{n^2}{1-z^2}\right) Z(z) = 0, \cos\theta_1 < z < \cos\theta_0.$$

(ii). Suppose that $\Theta_1(\theta)$ and $\Theta_2(\theta)$ are two eigenfunctions of (3.13) corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$. Verify that $\int_{\theta_0}^{\theta_1} \Theta_1(\theta)\Theta_2(\theta) \sin\theta d\theta = 0$. Set $Z_1(\cos\theta) = \Theta_1(\theta)$ and $Z_2(\cos\theta) = \Theta_2(\theta)$. Verify that $\int_{\cos\theta_0}^{\cos\theta_1} Z_1(z)Z_2(z)\, dz = 0$.

(iii). When $n \in \mathbb{Z}$, $\theta_0 = 0$ and $\theta_1 = \pi$, look up ODE texts to confirm that the above ODE has a non-trivial, bounded solution iff $\lambda_{l,n} = l(l+1)$ for some $l \in \mathbb{N}$ with $l \geq |n|$, and that we can identify $Z(x)$ to be an associated Legendre (generalized) polynomial

$$P_l^n(z) = \frac{(-1)^n}{2^l l!}(1-z^2)^{n/2}\frac{d^{n+l}}{dz^{n+l}}(z^2-1)^l.$$

(iv). Write down a formal eigenfunction expansion in terms of $\{\Theta_{l,n}(\theta)e^{in\phi}\}$ for a solution $u(\theta,\phi,t)$ to $(\partial_t - \Delta_{\mathbb{S}^2})u(\theta,\phi,t) = f(\theta,\phi,t)$ for $\theta_0 < \theta < \theta_1$, $0 \leq \phi \leq 2\pi$, $t > 0$, and with $u(\theta_0,\phi,t) = u(\theta_1,\phi,t) = 0$ for all $0 \leq \phi \leq 2\pi$, $t > 0$, $u(\theta,\phi,0) = g(\theta,\phi)$ given.

Problem 3.6.2. Verify that a general Legendre type differential equation

$$(1-x^2)Q''(x) - bxQ'(x) + \left(d - \frac{c}{1-x^2}\right)Q(x) = 0 \tag{3.16}$$

can be transformed, with the change of variables $S(x) = (1-x^2)^{\frac{b-2}{4}}Q(x)$, to

$$(1-x^2)S''(x) - 2xS'(x) + \left(d + \frac{(b-2)b}{4} - \frac{c + \frac{(b-2)^2}{4}}{1-x^2}\right)S(x) = 0. \tag{3.17}$$

This relation can be used to construct $Q(x)$ in terms of $S(x)$.

Review ODE texts on the behavior of linear ODEs near a regular singular point to confirm that the leading order behavior of $Q(x)$ near $x = \pm 1$ are determined by

$$(1-x^2)^{-\frac{b-2\pm\sqrt{(b-2)^2+4c}}{4}},$$

and that (3.16) has a solution of the form

$$Q(x) = (1-x^2)^{-\frac{b-2\pm\sqrt{(b-2)^2+4c}}{4}} \cdot (\text{a polynomial in } x)$$

iff there exists some $k \in \mathbb{Z}_{\geq 0}$ such that $d = (\sigma+k)(\sigma+b+k-1)$, where $\sigma = -\frac{b-2\pm\sqrt{(b-2)^2+4c}}{2}$.

Problem 3.6.3. Assume that the Laplace operator on the unit n-dimensional round sphere $\mathbb{S}^n$ is given by

$$\Delta_{\mathbb{S}^n} u = \frac{\left(\sin^{n-1}\theta\, u_\theta\right)_\theta}{\sin^{n-1}\theta} + \frac{\Delta_{\mathbb{S}^{n-1}} u}{\sin^2\theta},$$

where θ is the angle between the point and the North Pole (namely, the geodesic distance between them), so the metric on $\mathbb{S}^n$ can be expressed as $g_{\mathbb{S}^n} = d\theta^2 + \sin^2\theta \, d\boldsymbol{\omega}^2_{\mathbb{S}^{n-1}}$.

(i). Verify that in looking for separable solution $u = \Theta(\theta)\Omega(\boldsymbol{\omega})$ to $\Delta_{\mathbb{S}^n} u = -\lambda u$, if we assume $\Delta_{\mathbb{S}^{n-1}}\Omega(\boldsymbol{\omega}) = -\mu\Omega(\boldsymbol{\omega})$, then we will be led to

$$\Theta''(\theta) + (n-1)\cot\theta \, \Theta'(\theta) + \left(\lambda - \frac{\mu}{\sin^2\theta}\right)\Theta(\theta) = 0. \tag{3.18}$$

(ii). Setting $x = \cos\theta$ and $X(\cos\theta) = \Theta(\theta)$, verify that

$$\Theta'(\theta) = -\sin\theta \, X'(x), \quad \Theta''(\theta) = \sin^2\theta \, X''(x) - \cos\theta \, X'(x),$$

so we arrive at

$$(1-x^2)X''(x) - nxX'(x) + \left(\lambda - \frac{\mu}{1-x^2}\right)X(x) = 0. \tag{3.19}$$

This is a general Legendre type differential equation.

(iii). Use the information from the previous exercise to confirm that if $\mu = m(m+n-2)$ and $\lambda = l(l+n-1)$ for some $l = m+k$, $k \in \mathbb{Z}_{\geq 0}$, then (3.19) has a bounded solution on $(-1,1)$ of the form $(1-x^2)^{m/2}\cdot$ (a polynomial in x). Using induction, this gives $\lambda = l(l+n-1)$, $l \in \mathbb{Z}_{\geq 0}$ as eigenvalues of $\Delta_{\mathbb{S}^n}$.

Chapter 4

First Study of the Maximum Principle, Energy Method, and Variational Method

4.1 The Maximum Principle and Applications to Uniqueness and Estimation of Solutions

The maximum principle is a powerful tool in the study of heat and Laplace equations and their generalizations: it is used in proving uniqueness, in bounding the solution in terms of data, and in proving geometric properties (such as monotonicity and symmetry) of solutions. What makes it so useful is that it is often proved using elementary means without using any representation formula for a solution. Here is the maximum principle in its simplest form.

Theorem 4.1. *Suppose that $U \subset \mathbb{R}^n$ is bounded and $u \in C^2(U) \cap C(\overline{U})$ satisfies $\Delta u(x) \geq 0$ in U. Then $\max_{\overline{U}} u = \max_{\partial U} u$.*

Corollary 4.2. *Suppose that $U \subset \mathbb{R}^n$ is bounded. Then there is at most one solution in the class $C^2(U) \cap C(\overline{U})$ to the Dirichlet problem*

$$\begin{cases} -\Delta u(x) = f(x) & \text{for } x \in U, \\ u(x) = g(x) & \text{for } x \in \partial U. \end{cases} \tag{4.1}$$

Proof of Corollary. Let $u_1(x), u_2(x) \in C^2(U) \cap C(\overline{U})$ be solutions to (4.1). Then

$v(x) = u_1(x) - u_2(x) \in C^2(U) \cap C(\overline{U})$ satisfies

$$\begin{cases} -\Delta v(x) = 0 & \text{for } x \in U, \\ v(x) = 0 & \text{for } x \in \partial U. \end{cases}$$

Applying the maximum principle to $v(x)$, we conclude that $\max_{\overline{U}} v = 0$; then applying the maximum principle to $-v(x)$, we conclude that $\max_{\overline{U}}(-v) = 0$. Thus $v \equiv 0$ in U and $u_1(x) \equiv u_2(x)$ in U. □

The proof for Theorem 4.1 is also elementary.

Proof for Theorem 4.1. First suppose that $\Delta u(x) > 0$ in U. Then $\max_{\overline{U}} u$ must be attained at some $x_0 \in \overline{U}$ and $x \notin U$; for, if $x_0 \in U$, then $\Delta u(x_0) = \sum_{i=1}^n u_{x_i x_i}(x_0) \le 0$, contradicting our hypothesis. Since $x_0 \in \partial U$, we now have $\max_{\overline{U}} u = u(x_0) \le \max_{\partial U} u$. Since $\max_{\partial U} u \le \max_{\overline{U}} u$ trivially, we conclude that $\max_{\overline{U}} u = \max_{\partial U} u$ in this case.

Now for any $\epsilon > 0$, the function $u_\epsilon = u(x) + \epsilon x_1^2$ satisfies $\Delta u_\epsilon(x) > 0$ in U, therefore $\max_{\overline{U}} u_\epsilon = \max_{\partial U} u_\epsilon$. Sending $\epsilon \to 0$, we see that $\max_{\overline{U}} u = \max_{\partial U} u$. □

Maximum principle can also be used to estimate the size of the solution.

Theorem 4.3. *Suppose $U \subset \mathbb{R}^n$ is bounded and $u \in C^2(U) \cap C(\overline{U})$ satisfies*

$$\begin{cases} \Delta u(x) = f(x) & \textit{in } U, \\ u(x) = g(x) & \textit{on } \partial U. \end{cases}$$

Then there exists a constant $C > 0$ depending on U only such that

$$\max_{\overline{U}} |u| \le \max_{\partial U} |g| + C \sup_U |f(x)|.$$

Proof. The key is to construct a function $v \ge 0$ in U satisfying $\Delta v(x) \le -1$ in U—$v(x) = (d^2 - x_1^2)/2$ will do here, if we assume $U \subset \{x \in \mathbb{R}^n : 0 \le x_1 \le d\}$. Then $w = (\sup_U |f(x)|)\, v(x) + \sup_{\partial U} |g| \pm u(x)$ satisfies

$$\Delta w \le -\sup_U |f(x)| \pm f(x) \le 0 \quad \text{in } U,$$

and on ∂U, $w \ge \sup_{\partial U} |g| \pm g(x) \ge 0$, so we can apply Theorem 4.1 to $-w$ to conclude that $w(x) \ge 0$ in U, from which follows our claimed bound on u. □

Remark 4.1. Maximum principle is the first instance for us to work out the properties of solutions directly from the equations, instead of relying on a representation formula. Most of the more robust methods we will develop later on also have this feature. This section presents the maximum principle in its simplest form; we will extend it to more general settings, obtain stronger results, and discuss more applications later on.

Here is a simple version of the maximum principle for the heat equation.

Theorem 4.4. *Suppose that $U \subset \mathbb{R}^n$ is bounded. Denote $U_T = U \times (0, T]$ and $\partial' U_T = \{(x, 0) : x \in \overline{U}\} \cup \{(x, t) : x \in \partial U, 0 < t \le T\}$. Suppose that $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) \le 0 & \text{in } U_T, \\ u(x,t) \le 0 & \text{on } \partial' U_T. \end{cases}$$

Then $u \le 0$ in U_T.

Theorem 4.5. *Suppose U and U_T are the same as in Theorem 4.4 and $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = 0 & \text{in } U_T, \\ u(x,t) = 0 & \text{on } \partial' U_T. \end{cases}$$

Then $u \equiv 0$ in U_T.

Proof for Theorem 4.5. We can apply Theorem 4.4 to u and $-u$ to conclude that $u \equiv 0$ in U_T. □

Proof for Theorem 4.4. Again, if $\partial_t u(x,t) - \Delta u(x,t) < 0$ in U_T, then $\max_{\overline{U}_T} u$ must be attained at some point $(x_0, t_0) \in \partial' U_T$; for if $(x_0, t_0) \in \overline{U}_T \setminus \partial' U_T$, then we would have $x_0 \in U$ and $t_0 > 0$, which would imply $u_t(x_0, t_0) \ge 0$ and $u_{x_i x_i}(x_0, t_0) \le 0$ and then $\partial_t u(x_0, t_0) - \Delta u(x_0, t_0) \ge 0$. Now that $(x_0, t_0) \in \partial' U_T$, we conclude that $\max_{\overline{U}_T} u = u(x_0, t_0) \le 0$.

Now for any $\epsilon > 0$, $u_\epsilon(x,t) = u(x,t) - \epsilon t$ satisfies $(\partial_t - \Delta)\, u_\epsilon(x,t) < 0$ in U_T, and $u_\epsilon(x,t) \le 0$ on $\partial' U_T$, so we can conclude that $u_\epsilon(x,t) = u(x,t) - \epsilon t \le 0$. Sending $\epsilon \to 0$, we conclude that $u \le 0$ in U_T.

□

Estimation on the solution of the heat equation also follows routinely.

Theorem 4.6. *Suppose $U \subset \mathbb{R}^n$ is bounded and $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = f(x,t) & \text{in } U_T, \\ u(x,t) = g(x,t) & \text{on } \partial' U_T. \end{cases}$$

Then

$$\max_{U_T} |u| \leq \left[T \sup_{U_T} |f| + \max_{\partial' U_T} |g| \right]. \tag{4.2}$$

Proof. Note that $(\partial_t - \Delta)[u - t\sup_{U_T}|f| - \max_{\partial' U_T}|g|] \leq 0$ in U_T, and $u - t\sup_{U_T}|f| - \max_{\partial' U_T}|g| \leq 0$ on $\partial' U_T$. Thus by the maximum principle,

$$u(x,t) \leq t \sup_{U_T} |f| + \max_{\partial' U_T} |g|, \quad \text{in } U_T.$$

Similarly

$$-u(x,t) \leq t \sup_{U_T} |f| + \max_{\partial' U_T} |g|, \quad \text{in } U_T.$$

Thus (4.2) holds. □

Remark 4.2. It was not easy to use the Fourier series solution directly to produce a solution of (2.1) that is continuous over $[0,l] \times [0,\infty)$ if we only assume $g \in C[0,l]$ with $g(0) = g(l) = 0$. The maximum principle provides a tool which would allow us to produce such a solution assuming only $g \in C[0,l]$ with $g(0) = g(l) = 0$. Even though the Fourier (sine) series $\sum_{n=1}^{\infty} g_n \sin\left(\frac{n\pi x}{l}\right)$ of $g(x)$ may not converge to $g(x)$ point-wise under this assumption, there is a Fejér theorem in the theory of Fourier series which implies that there is a finite Fourier sine series of the form $g^{(N)}(x) = \sum_{n=1}^{N} g_n^{(N)} \sin\left(\frac{n\pi x}{l}\right)$ which converges to $g(x)$ uniformly over $[0,l]$ as $N \to \infty$. Let $u^{(N)}(x,t) = \sum_{n=1}^{N} g_n^{(N)} e^{-(\frac{n\pi}{l})^2 t} \sin\left(\frac{n\pi x}{l}\right)$ be the finite series solution of the heat equation (2.1) with initial data $g^{(N)}(x)$. Then the maximum principle applied to $u^{(N)}(x,t) - u^{(N')}(x,t)$ implies that $\{u^{(N)}(x,t)\}_N$ is Cauchy in $C([0,l] \times [0,T])$, so there is a limit function $u(x,t)$ in $C([0,l] \times [0,T])$ such that $u^{(N)}(x,t) \to u(x,t)$ uniformly in $C([0,l] \times [0,T])$. Therefore, $u(x,0) = g(x)$ for $x \in [0,l]$. To prove that $u \in C^2([0,l] \times (0,T])$ and satisfies the equation in (2.1), we use the following estimate: for any $0 < \tau < T$, there exists a constant $C > 0$ depending on τ such that

$$\max_{[0,l]\times[\tau,T]} |\nabla^{\alpha,\beta}_{x,t} u^{(N)}(x,t)| \leq C \max_{[0,l]} |g^{(N)}(x)| \quad \text{for } |\alpha| + 2|\beta| \leq 2.$$

The same estimate also applies to $u^{(N)}(x,t) - u^{(N')}(x,t)$. Thus $\{\nabla^{\alpha,\beta}_{x,t} u^{(N)}(x,t)\}$ is Cauchy in $C([0,l] \times [\tau,T])$ for each $|\alpha| + 2|\beta| \leq 2$, which implies that the limit $u(x,t) \in C^{2,1}_{x,t}([0,l] \times [\tau,T])$ and satisfies the heat equation there. Since $\tau > 0$ is arbitrary, this proves that $u(x,t)$ is in $C^{2,1}_{x,t}([0,l]\times(0,T])$ and satisfies the heat equation there.

The estimate above is proved by using the exponential decay of $e^{-(\frac{n\pi}{l})^2 t}$ in n when $t \geq \tau > 0$:

$$\begin{aligned}
|\nabla^{\alpha,\beta}_{x,t} u^{(N)}(x,t)| &\leq \sum_{n=1}^{N} |g_n^{(N)}| \left(\frac{n\pi}{l}\right)^{|\alpha|+2|\beta|} e^{-(\frac{n\pi}{l})^2 t} \\
&\leq 2 \max_{[0,l]} |g^{(N)}(x)| \sum_{n=1}^{N} \left(\frac{n\pi}{l}\right)^{|\alpha|+2|\beta|} e^{-(\frac{n\pi}{l})^2 \tau} \\
&\leq C \max_{[0,l]} |g^{(N)}(x)|,
\end{aligned}$$

where $C = 2\sum_{n=1}^{\infty} \left(\frac{n\pi}{l}\right)^{|\alpha|+2|\beta|} e^{-(\frac{n\pi}{l})^2\tau} < \infty$ depends only on τ (and l).

Remark 4.3. The above derivative estimate on $u^{(N)}(x,t)$ was done using its explicit form, but we will later introduce Bernstein's method, which proves a similar estimate for any sufficiently smooth solution of (2.1) using the maximum principle only.

Generalization of the maximum principle (and uniqueness) for both the Laplace equation and heat equation on unbounded (spatial) domains are available, but they require appropriate growth restrictions on the solutions. For example, $u(x) = x_n$ is harmonic on $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$, and $u(x) = 0$ when $x = (x', x_n) \in \partial\mathbb{R}^n_+$, yet $u(x)$ is not identically zero on $\mathbb{R}^n_+$.

Here is an example of how to adapt the maximum principle to solutions on unbounded domains.

Example 4.1. Let u be a bounded harmonic function on $U = \{x = (x', x_n) : 0 < x_n < h\}$ such that $u(x', x_n) = 0$ when $x_n = 0$ or h. We will prove that $u(x', x_n) \equiv 0$ in U. The boundedness assumption on u here is needed, as the conclusion does not hold for the family of unbounded harmonic functions $\exp(\frac{\pi x_1}{h})\sin(\frac{\pi x_n}{h})$. For simplicity of notation, we will assume $n = 2$.

Let $M > 0$ be such that $|u(x_1, x_2)| \leq M$ for all $(x_1, x_2) \in U$. We will use the positive harmonic function $\cosh(x_1/h)\cos(x_2/h)$. For any $\epsilon > 0$, we will apply the usual maximum principle to $v = \pm u - \epsilon\cosh(x_1/h)\cos(x_2/h)$ on $U_R = \{x = (x_1, x_2) : |x_1| < R, 0 < x_2 < h\}$ for $R > 0$ large enough so that $\epsilon\cosh(x_1/h)\cos(x_2/h) \geq M$ when $|x_1| = R$. Then the harmonic function $v \leq 0$ on ∂U_R, and therefore $v \leq 0$ in U_R. This implies that $|u| \leq \epsilon\cosh(x_1/h)\cos(x_2/h)$ in U_R.

The choice for $R > 0$ depends on ϵ, but for any $x \in U$ and for any $\epsilon > 0$, x will be in U_R as long as $R > |x|$, so $|u(x)| \leq \epsilon\cosh(x_1/h)\cos(x_n/h)$ for all $\epsilon > 0$, which leads to $u(x) = 0$. From the proof one can see that one can modify it so that the same conclusion continues to hold as long as there exists some $h' > h$ such that $\limsup_{|x_1|\to\infty} \exp(-\frac{\pi|x_1|}{h'})|u(x_1, x_2)| = 0$ uniformly over $0 < x_2 < h$. This is similar to the Phragmén-Lindelöf type theorems in complex analysis.

For general dimension, we can modify $\cosh(x_1/h)\cos(x_n/h)$ into

$$\cosh(ax_1/h)\cdots\cosh(ax_{n-1}/h)\cos(x_n/h)$$

with $(n-1)a^2 = 1$.

Remark 4.4. The Fourier series solution of (2.1) can also be expressed as an integral involving the initial data g as follows. Using $c_n = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})\,dx$ in $u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t}$, we obtain for $t > 0$

$$\begin{aligned} u(x,t) &= \frac{2}{l}\sum_{n=1}^{\infty}\int_0^l g(y)\sin(\frac{n\pi y}{l})\sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t}\,dy \\ &= \frac{2}{l}\int_0^l g(y)\left[\sum_{n=1}^{\infty}\sin(\frac{n\pi y}{l})\sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t}\right]dy, \end{aligned}$$

where we have used the uniform convergence of

$$G(x,y,t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin(\frac{n\pi y}{l})\sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t}$$

over $t \ge \tau > 0$ for any given $\tau > 0$. Thus the solution $u(x,t)$ to (2.1) can be represented as

$$u(x,t) = \int_0^l g(y)G(x,y,t)\,dy.$$

Using the exponentially fast decay of $e^{-(\frac{n\pi}{l})^2 t}$ in n over $t \ge \tau > 0$, one can see easily that $G(x,y,t)$ is a smooth function of (x,y,t) over $t > 0$, and

$$\begin{aligned} G_t(x,y,t) - G_{xx}(x,y,t) &= 0, \quad \text{over } t > 0; \\ G_t(x,y,t) - G_{yy}(x,y,t) &= 0, \quad \text{over } t > 0; \\ G(x,y,t) &= 0, \quad \text{if } t > 0 \text{ and either } x \text{ or } y = 0 \text{ or } l. \end{aligned}$$

This $G(x,y,t)$ is called the **Green's function** for (2.1). It turns out that $G(x,y,t) > 0$ for $x, y \in (0,l)$ and $t > 0$, which is not entirely clear from its series representation.

We would like to apply the Maximum Principle to $u(x,t)$ to obtain information on $G(x,y,t)$, but our Maximum Principle is established for solutions in the class $C([0,l]\times[0,T])$, while the solution constructed here is known to be in $C([0,T], L^2[0,l])$

when $g \in C[0,l]$. However, when g further satisfies $g(0) = g(l) = 0$, the argument in the Remark 4.2 proves the existence of solution in the class $C([0,l] \times [0,T]) \cap C^2([0,l]\times(0,T])$, which is obviously in the class $C([0,T], L^2[0,l])$; since we have proved the uniqueness of solution to (2.1) in this class via the energy method, we conclude that these two solutions must be identical, and that the Fourier series solution is in $C([0,l]\times[0,T])$. Thus we can apply the the Maximum Principle to $u(x,t)$ to conclude that

$$\begin{aligned} &u(x,t) = \int_0^l g(y)G(x,y,t)\,dy \geq 0 \quad \text{at every } (x,t) \in [0,l] \times (0,T], \\ &\text{if } g(y) \in C[0,l], g(0) = g(l) = 0 \text{ and } g(y) \geq 0 \text{ for } y \in [0,l]. \end{aligned} \tag{4.3}$$

It follows from this property that

$$G(x,y,t) = \frac{2}{l}\sum_{n=1}^{\infty} \sin(\frac{n\pi y}{l})\sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t} \geq 0 \quad \text{for all } x,y \in [0,l] \text{ and } t > 0; \tag{4.4}$$

but it is not clear at all from the summation for $G(x,y,t)$ that this property holds.

If one carries out the computation for (a) of **Exercise** 2.7.3, one would find that $u(x,t) = \int_{\mathbb{R}} K(x-y,t)\tilde{g}(y)\,dy$, where $\tilde{g}$ is obtained from g by first extending it to an odd function on $[-l,l)$, then extending this odd function as a $2l$-periodic function on $\mathbb{R}$. Thus

$$\begin{aligned} u(x,t) &= \sum_{k=-\infty}^{\infty}\left[\int_{-l+2kl}^{2kl} K(x-y,t)\tilde{g}(y)\,dy + \int_{2kl}^{l+2kl} K(x-y,t)\tilde{g}(y)\,dy\right] \\ &= \sum_{k=-\infty}^{\infty}\left[-\int_{-l}^{0} K(x-y-2kl,t)g(-y)\,dy + \int_{0}^{l} K(x-y-2kl,t)g(y)\,dy\right] \\ &= \int_0^l \sum_{k=-\infty}^{\infty} \left[K(x-y-2kl,t) - K(x+y-2kl,t)\right] g(y)\,dy \\ &= \int_0^l \sum_{k=-\infty}^{\infty} \frac{e^{-\frac{|x-y-2kl|^2}{4t}} - e^{-\frac{|x+y-2kl|^2}{4t}}}{\sqrt{4\pi t}} g(y)\,dy. \end{aligned} \tag{4.5}$$

It follows now that for $t > 0$

$$\frac{2}{l}\sum_{n=1}^{\infty} \sin(\frac{n\pi y}{l})\sin(\frac{n\pi x}{l})e^{-(\frac{n\pi}{l})^2 t} = \sum_{k=-\infty}^{\infty} \frac{e^{-\frac{|x-y-2kl|^2}{4t}} - e^{-\frac{|x+y-2kl|^2}{4t}}}{\sqrt{4\pi t}}. \tag{4.6}$$

As a consequence, for any fixed $x \in (0, l)$, $G(x, y, t)$ is continuous in $(y, t) \in ([0, l] \times [0, \infty)) \setminus \{(x, 0)\}$, and $G(x, y, 0) = 0$ for $y \neq x$. In appropriate sense $G(x, y, t) \to \delta(x - y)$ as $t \searrow 0$.

(4.6) is related to the Jacobi identify for ϑ-functions. A more familiar form of the Jacobi identify can be derived from solving the heat equation with periodic boundary conditions by using the Fourier series method and by using the heat kernel, and by equating the solution obtained by these two methods, as done above. Taking the $l = 1/2$ case, one would arrive at the more familiar Jacobi identify

$$\frac{\sum_{k=-\infty}^{\infty} e^{-\frac{|z-k|^2}{4t}}}{\sqrt{4\pi t}} = 1 + \sum_{n=1}^{\infty} \left[e^{2n\pi z i} + e^{-2n\pi z i}\right] e^{-4n^2\pi^2 t} = 1 + 2\sum_{n=1}^{\infty} \cos(2n\pi z) e^{-4n^2\pi^2 t}. \tag{4.7}$$

Exercise 4.1.1. Suppose that U is a bounded domain in $\mathbb{R}^n$, $c(x) \geq 0$ in U.

(a). Prove the following extension of Theorem 4.1: suppose that $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} \Delta u(x) - c(x)u(x) \geq 0 & \text{in } U, \\ u(x) \leq 0 & \text{on } \partial U, \end{cases}$$

then $u(x) \leq 0$ in U.

(b). Suppose that $u \in C^2(U) \cap C(\bar{U})$ satisfies

$$\begin{cases} \Delta u(x) - c(x)u(x) \geq 0 & \text{in } U, \\ u(x) = g(x) & \text{on } \partial U, \end{cases}$$

where $g \in C(\partial U)$. Prove that $\max_{\bar{U}} u \leq \max\{0, \max_{\partial U} g\}$; and that it's possible for $\max_{\bar{U}} u > \max_{\partial U} g$ (Think of the 1-D case).

(c). Prove that the Dirichlet problem

$$\begin{cases} \Delta u(x) - c(x)u(x) = f(x) & \text{in } U, \\ u(x) = g(x) & \text{on } \partial U, \end{cases}$$

has at most one solution in $C^2(U) \cap C(\bar{U})$.

Exercise 4.1.2. Prove that, for $\alpha \geq 0$, the solution $J_\alpha(ir)$ to the modified Bessel equation, (2.51), has no zero in $\{r : r > 0\}$. Prove, in addition, that $J_\alpha(ir) \to \infty$ as $r \to \infty$. Is the conclusion valid for any solution of (2.51) on $\mathbb{R}^+$? (Note that, up to multiplication by a constant, $J_\alpha(ir)$ can be made into real valued. When $\alpha = k \in \mathbb{N}$, $u = J_\alpha(ir)\cos(k\theta)$ is a solution of (2.49) with $c = -1$, and one can use the maximum principle to complete a proof. When α is not a positive integer, try to adapt the argument for the maximum principle to (2.51), or adapt the energy method of the next section.)

Exercise 4.1.3. Suppose that U is a bounded domain in $\mathbb{R}^n$, $c(x) \geq 0$ in U. Prove that there exists $C > 0$ depending on U such that for any $u \in C^2(U) \cap C(\bar{U})$ satisfying

$$\begin{cases} \Delta u(x) - c(x)u(x) = f(x) & \text{in } U, \\ u(x) = 0 & \text{on } \partial U, \end{cases}$$

where $f(x)$ is bounded over U, then there holds $\max_{\bar{U}} |u(x)| \leq C \sup_U |f(x)|$.

Exercise 4.1.4. Suppose U is a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, and $x_0 \in \partial U$. Let $u \in C(\overline{U} \setminus \{x_0\})$ be a bounded harmonic function in U such that $u \equiv 0$ on $\partial U \setminus \{x_0\}$. Prove that $u \equiv 0$ in U.

Exercise 4.1.5. Prove that if $u(x', x_n)$ is a bounded harmonic function on $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$ in the class $C(\overline{\mathbb{R}^n_+})$, and $u(x', 0) = 0$ for all $x' \in \mathbb{R}^{n-1}$, then $u(x', x_n) \equiv 0$ in $\mathbb{R}^n_+$.

Exercise 4.1.6. Suppose $U \subset \mathbb{R}^n$ is bounded and $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) = f(x) & \text{in } U_T, \\ u(x,t) = g(x,t) & \text{on } \partial' U_T. \end{cases}$$

Suppose, further, that $f(x) \geq 0$ in U. Prove that $u_t(x,t) \geq 0$ in U_T.

Exercise 4.1.7. Using (4.3) to prove (4.4). Also prove that, if $g \in C[0,l]$, then for any $0 < x_0 < l$, $\int_0^l g(y) G(x,y,t)\, dy \to g(x_0)$ as $(x,t) \to (x_0, 0)$, $t > 0$. What happens if $x_0 = 0$ or l?

Exercise 4.1.8. Suppose that one can establish the existence of a constant $C > 0$ such that for any solution $u(x,t)$ on $|x| \leq 1, 0 \leq t \leq 1$ of $u_t(x,t) - \Delta_x u(x,t) = f(x,t)$, the following holds

$$|\nabla_x u(0,1)| \leq C \left(\max_{|x| \leq 1, 0 \leq t \leq 1} |u(x,t)| + \max_{|x| \leq 1, 0 \leq t \leq 1} |f(x,t)| \right).$$

Deduce that for any solution $u(x,t)$ to $u_t(x,t) - \Delta_x u(x,t) = f(x,t)$ on $x \in \mathbb{R}^n, 0 < t$, the following holds

$$\sqrt{t} |\nabla_x u(x,t)| \leq C \left(\max_{|y-x| \leq \sqrt{t}, 0 \leq s \leq t} |u(y,s)| + t \max_{|y-x| \leq \sqrt{t}, 0 \leq s \leq t} |f(y,s)| \right).$$

4.2 The Energy Method and Applications to Uniqueness and Existence of Solutions

The energy method is related to the variational characterization of solutions. It can often be used to prove uniqueness of solutions — so far, with the exception of the maximum principle and the characteristic curves method in finding solutions to the one-dimensional wave equation, other solution methods (separation of variables and Fourier expansion) have not provided uniqueness.

4.2.1 The Simplest Cases of the Energy Method

Earlier we proved the uniqueness for (2.1) by using the energy method. We now recall the derivation and point out further consequences of the estimate.

Multiplying both sides of the first equation in (2.1) by $u(x,t)$ and integrating over $x \in [0,l]$, we find that for $t > 0$

$$0 = \int_0^l u(x,t)\,(u_t(x,t) - u_{xx}(x,t))\,dx = \int_0^l \left[\left(\frac{u^2(x,t)}{2}\right)_t - u(x,t)u_{xx}(x,t)\right] dx$$
$$= \frac{1}{2}\frac{d}{dt}\left(\int_0^l u^2(x,t)\,dx\right) + \int_0^l u_x^2(x,t)\,dx,$$

where we have used integration by parts and the homogeneous boundary conditions $u(0,t) = u(l,t) = 0$; the computation also assumes that $u(x,t), u_x(x,t), u_{xx}(x,t) \in C[0,l]$ for $t > 0$ and $u_t(x,t)$ is jointly continuous in $(x,t) \in [0,l] \times (0,\infty)$. It now follows by integrating in t the above relation

$$\int_0^l u^2(x,t)\,dx + 2\int_0^t \int_0^l u_x^2(x,\tau)\,dx d\tau = \int_0^l u^2(x,0)\,dx. \tag{4.8}$$

The derivation here has assumed that $t \mapsto \int_0^l u^2(x,t)\,dx$ is in $C[0,T]$ and $t \mapsto \int_0^l u_x^2(x,t)\,dx$ is in $C(0,T)$ for some $T > t > 0$, which is the case for the solution constructed by the Fourier series.

Remark 4.5. The energy method is another instance, where we work out properties of the solutions without using any solution representation formula.

This method trivially generalizes to the initial-boundary value problems for the heat equation in higher spatial dimensions. Let $u(x,t)$ in $C^{2,1}_{x,t}(\overline{\Omega}\times(0,\infty))\cap C([0,\infty), L^2(\Omega))$

or in $C^{2,1}_{x,t}(\overline{\Omega}\times(0,\infty))\cap C(\overline{\Omega}\times[0,\infty))$ be a solution of

$$\begin{cases} u_t - \Delta_x u = 0, & \text{for} \quad (x,t)\in\Omega\times(0,\infty) \\ u(x,t)=0, & \text{for} \quad (x,t)\in\partial\Omega\times(0,\infty) \\ u(x,0)=g(x) & \text{for} \quad x\in\Omega. \end{cases} \tag{4.9}$$

Then

$$\frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega |\nabla_x u(x,\tau)|^2\,dxd\tau = \int_\Omega u^2(x,0)\,dx. \tag{4.10}$$

(4.8) is useful not only for proving uniqueness, but it will also be useful for constructing solutions by using the bounds $\int_0^l u^2(x,t)\,dx$ and $\int_0^t\int_0^l u_x^2(x,\tau)\,dxd\tau$ for the solution u in terms of $\int_0^l u^2(x,0)\,dx$, and it is also a form of stability estimate for the solution.

In fact, even if we allow the PDE in (2.1) to include a non-homogenous term $f(x,t)$: $u_t(x,t)-u_{xx}(x,t)=f(x,t)$, we can adapt the method above to obtain a corresponding energy estimate for u as follows.

Proposition 4.7. *Suppose that $u\in C([0,T],L^2[0,T])\cap C^{2,1}_{x,t}([0,l]\times(0,T])$ is a solution of*

$$\begin{cases} u_t - u_{xx} = f(x,t), & \textit{for} \quad (x,t)\in(0,l)\times(0,\infty), \\ u(0,t)=u(l,t)=0, & \textit{for} \quad t>0, \\ u(x,0)=g(x) & \textit{for} \quad x\in[0,l]. \end{cases}$$

Then

$$\begin{aligned} &\int_0^l u^2(x,t)\,dx + 2\int_0^t\int_0^l u_x^2(x,\tau)\,dxd\tau \\ \le{}& e^t\left[\int_0^l u^2(x,0)\,dx + \int_0^t\int_0^l f^2(x,\tau)\,dxd\tau\right] \end{aligned} \tag{4.11}$$

Proof. First, we multiply both sides of the first equation above by $u(x,t)$, integrate in $x\in[0,l]$, and integrate by parts as above to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\int_0^l u^2(x,t)\,dx\right) + \int_0^l u_x^2(x,t)\,dx &= \int_0^l u(x,t)f(x,t)\,dx \\ &\le \frac{1}{2}\int_0^l u^2(x,t)\,dx + \frac{1}{2}\int_0^l f^2(x,t)\,dx. \end{aligned} \tag{4.12}$$

Next, we set $G(t)=\int_0^l u^2(x,t)\,dx$, $H(t)=\int_0^l u_x^2(x,t)\,dx$, and $F(t)=\int_0^l f^2(x,t)\,dx$. Then $G\in C^1(0,T)\cap C[0,T]$, and

$$G'(t)+2H(t)\le G(t)+F(t). \tag{4.13}$$

(4.13) is often referred to as a **Gronwall** type inequality: it relates the growth rate of $G(t)$ with $G(t)$ in this fashion. It follows now that $G(t) + 2\int_0^t e^{t-\tau}H(\tau)d\tau \leq e^t \left[G(0) + \int_0^t e^{-\tau}F(\tau)d\tau\right]$, namely

$$\int_0^l u^2(x,t)\,dx + 2\int_0^t\int_0^l e^{t-\tau}u_x^2(x,\tau)\,dxd\tau \leq e^t\left[\int_0^l u^2(x,0)\,dx + \int_0^t\int_0^l e^{-\tau}f^2(x,\tau)\,dxd\tau\right],$$

from which (4.11) follows. □

Remark 4.6. A previous variational characterization of solutions to the (homogeneous) heat equation (with homogeneous Dirichlet boundary condition) examined

$$\begin{aligned}\frac{d}{dt}\int_0^l |u_x(x,t)|^2\,dx &= 2\int_0^l u_x(x,t)\cdot u_{xt}(x,t)\;dx\\ &= -2\int_0^l u_{xx}(x,t)u_t(x,t)\,dx\\ &= -2\int_0^l |u_t(x,t)|^2\,dx \leq 0,\end{aligned}$$

which also provides uniqueness to (2.1), albeit with some higher regularity assumptions on the solution u to justify the differentiation under the integral sign and integration by parts above; in particular, this argument would require $t \to \int_0^l |u_x(x,t)|^2\,dx$ be continuous in $t \in [0,\infty)$, which may not be the case even if $u(x,0)$ is smooth in $(0,l)$ (need to watch out for the matching, or its failure, of $u(0,0)$ and $u(l,0)$ with the boundary condition $u(0 \,\text{or}\, l,t) = 0$; examine the case $u(x,0) \equiv 1$).

There are versions of the energy estimates for solutions defined on $\Omega\times(0,\infty)$ where Ω is unbounded; but some decay assumptions on the solution is needed, as there are non-trivial smooth solutions to the heat equation $u_t - \Delta_x u = 0$ on $\mathbb{R}^n \times [0,\infty)$ with $u(x,0) \equiv 0$, which were first discovered by Tychonoff.

The energy method can be easily applied to the Poisson equation $-\Delta u(x) = f(x)$ to prove the uniqueness for the Dirichlet problem

$$\begin{cases} -\Delta u(x) = f(x) & \text{for } x \in \Omega \\ u(x) = g(x) & \text{for } x \in \partial\Omega. \end{cases} \tag{4.14}$$

Theorem 4.8. *Suppose that Ω is a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\Omega$ is piecewise C^1. Then there is at most one solution of (4.14) in the class $C^2(\Omega)\cap C^1(\overline{\Omega})$.*

The proof boils down to proving that

$$\begin{cases} -\Delta u(x) = 0 & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

has $u = 0$ as the only solution in the class $C^2(\Omega) \cap C^1(\overline{\Omega})$. But that follows from multiplying both sides of the equation and integrating by parts:

$$0 = -\int_\Omega u(x)\Delta u(x)\,dx = \int_\Omega |\nabla u(x)|^2\,dx - \int_{\partial\Omega} u(x)\frac{\partial u(x)}{\partial n(x)}\,d\sigma(x) = \int_\Omega |\nabla u(x)|^2\,dx,$$

which implies that $\int_\Omega |\nabla u(x)|^2\,dx = 0$. This, together with the boundary condition $u(x) = 0$ on $\partial\Omega$, implies that $u(x) = 0$ in Ω. Note that this proof requires slightly more regularity on the solution: in $C^2(\Omega) \cap C^1(\overline{\Omega})$ instead of $C^2(\Omega) \cap C(\overline{\Omega})$.

Next we introduce the energy method for the wave equation. Based on physical considerations for solutions to (2.11), we define

$$E[u(\cdot,t)] = \frac{1}{2}\int_0^l \left(u_t^2(x,t) + c^2u_x^2(x,t)\right)\,dx$$

to be the energy of the solution u at time t. Then for $t > 0$

$$\begin{aligned}\frac{dE[u(\cdot,t)]}{dt} &= \int_0^l \left(u_tu_{tt} + c^2u_xu_{xt}\right)\,dx\\ &= \int_0^l \left(u_tu_{tt} - c^2u_{xx}u_t\right)\,dx\\ &= 0, \quad \text{if } u_{tt} - c^2u_{xx} = 0.\end{aligned}$$

The integration by parts can be justified if we consider $C^2([0,l]\times(0,\infty)) \cap C^1([0,l]\times[0,\infty))$ solutions to (2.11).

Theorem 4.9. *Let $u \in C^2([0,l]\times(0,\infty)) \cap C^1([0,l]\times[0,\infty))$ be a solution of (2.11). Then*

$$E[u(\cdot,t)] = \frac{1}{2}\int_0^l \left(u_t^2(x,t) + c^2u_x^2(x,t)\right)\,dx \text{ is a constant in } t. \tag{4.15}$$

In particular, if $u(x,0) = 0$ and $u_t(x,0) = 0$ for $x \in [0,l]$, then $u(x,t) = 0$ for $(x,t) \in [0,l]\times[0,\infty)$.

Another consequence of the energy estimates is the continuous dependence in L^2 norm of the (derivatives of) solution on initial data.

Such energy estimates on the solutions can also be used to prove existence of solutions. For instance, when the initial data $g(x)$ and $h(x)$ are trigonometric sine polynomials, the Fourier series method readily provides a genuine solution of (2.11). For more general initial data, suppose that $g(x)$ and $h(x)$ are such that we can use trigonometric sine polynomials $g_n(x)$ and $h_n(x)$ to approximate $g(x)$ and $h(x)$, respectively, in the following L^2 norm:

$$||g_n - g||_{L^2} = \left[\int_0^l |g_n(x) - g(x)|^2 \, dx\right]^{1/2} \to 0 \quad \text{and } \{g_n'(x)\} \text{ is Cauchy in } L^2[0,l];$$

and

$$||h_n - h||_{L^2} = \left[\int_0^l |h_n(x) - h(x)|^2 \, dx\right]^{1/2} \to 0,$$

as $n \to \infty$. It follows that there is some $\dot{g} \in L^2[0,l]$ such that $g_n' \to \dot{g}$ in $L^2[0,l]$. This $\dot{g}$ is the $L^2[0,l]$ derivative of g; for, each g_n satisfies $g_n(x) = \int_0^x g_n'(y)\, dy$, and since $g_n' \to \dot{g}$ in $L^2[0,l]$, $\{\int_0^x g_n'(y)\, dy\}_n$ is converging to $\int_0^x \dot{g}(y)\, dy$ uniformly in $x \in [0,l]$, thus $g_n(x)$ converges uniformly to $g(x)$ in $[0,l]$ and $g(x) = \int_0^x \dot{g}(y)\, dy$. This implies that $g(x)$ is absolutely continuous in $[0,l]$, with $g'(x) = \dot{g}(x)$ almost everywhere in $[0,l]$.

Let $u_n(x,t)$ denote the corresponding unique solution of (2.11) with $u_n(x,0) = g_n(x)$ and $\partial_t u_n(x,0) = h_n(x)$, then the energy estimate above says $\{\partial_t u_n(x,t)\}$ and $\{\partial_x u_n(x,t)\}$ are Cauchy sequences in the above L^2 norm, more precisely, in $C([0,T], L^2[0,l])$, so we expect to find an appropriate limit, which should satisfy the equation in some form.

To obtain the convergence of $\{u_n(x,t)\}$ itself, we make use of the inequalities (1.30). (1.30) applied to $u_n(\cdot,t) - u_m(\cdot,t)$, together with the energy estimates, implies that $\{u_n(x,t)\}$ is also Cauchy in the $C([0,l] \times [0,T])$ norm. So there exists $u \in C([0,l] \times [0,T])$ such that $u_n \to u$ uniformly in $[0,l] \times [0,T]$.

If g and h are such that we can take the approximating sine polynomials g_n and h_n with the further property that $\{\partial_x h_n\}$ and $\{\partial_x^2 g_n\}$ are also Cauchy in $L^2[0,l]$, then we can apply the energy estimates to $\partial_t u_n(x,t)$, which solves the homogeneous wave equation, to obtain that $\{\partial_x^2 u_n\}$, $\{\partial_{xt}^2 u_n\}$, and that $\{\partial_t^2 u_n\}$ are also Cauchy in $C([0,T], L^2[0,l])$. In such cases, we expect the limit to have some notion of second derivatives. This approach will lead to the L^2 weak derivatives and weak solutions. The sacrifice is that we may not get a C^2 limit as a solution. But many natural properties for the wave equation suggest that L^2 space is a more natural space to work with for the wave equation. We will explore the ideas here in more detail later on.

The energy estimates above can be modified to deal with solutions to non-homogeneous equations by modifying our earlier derivation.

Theorem 4.10. *Let $u(x,t) \in C^2([0,l]\times(0,\infty)) \cap C^1([0,l]\times[0,\infty))$ be a solution of*

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & \text{on } (x,t) \in [0,l]\times\mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ u_t(x,0) = h(x), & \text{for } x \in [0,l], \end{cases} \tag{4.16}$$

then

$$E[u(\cdot,t)] \le e^t \left(E[u(\cdot,0)] + \int_0^t \int_0^l \frac{|f(x,\tau)|^2}{2}\, dx d\tau \right). \tag{4.17}$$

Proof. We have

$$\begin{aligned} \frac{dE[u(\cdot,t)]}{dt} &= \int_0^l \left[u_t(x,t)u_{tt}(x,t) + c^2 u_x(x,t)u_{xt}(x,t) \right] dx \\ &= \int_0^l u_t(x,t) \left[u_{tt}(x,t) - c^2 u_{xx}(x,t) \right] dx \\ &= \int_0^l u_t(x,t) f(x,t)\, dx \\ &\le E[u(\cdot,t)] + \frac{1}{2}\int_0^l |f(x,t)|^2\, dx, \end{aligned}$$

from which it follows that

$$\begin{aligned} E[u(\cdot,t)] &\le e^t \left(E[u(\cdot,0)] + \int_0^t \int_0^l \frac{e^{-\tau}}{2} |f(x,\tau)|^2\, dx d\tau \right) \\ &\le e^t \left(E[u(\cdot,0)] + \int_0^t \int_0^l \frac{|f(x,\tau)|^2}{2}\, dx d\tau \right). \end{aligned}$$

□

4.2.2 Some Improvements and Modifications of the Energy Method

In both (4.11) and (4.17), the exponential factor e^t is an undesirable feature. It came about from simple algebraic manipulation such as in (4.12). Both (4.11) and (4.17) are good enough for proving uniqueness and finite time stability of solutions; if we desire better estimates than those provided for by (4.11) or (4.17), there are ways to exploit the algebra more carefully.

First modification to (4.11). Our earlier derivation didn't exploit the term $\int_0^l u_x^2(x,t)\,dx$ on the left hand side of (4.13). Making use of (1.30), we can estimate

$$\begin{aligned}\int_0^l u(x,t)f(x,t)\,dx &\le \frac{1}{2l^2}\int_0^l |u(x,t)|^2\,dx + \frac{l^2}{2}\int_0^l f^2(x,t)\,dx\\ &\le \frac{1}{2}\int_0^l |u_x(x,t)|^2\,dx + \frac{l^2}{2}\int_0^l f^2(x,t)\,dx.\end{aligned}$$

Then (4.12) can be modified into

$$G'(t) + H(t) \le l^2 F(t),$$

so it follows that

$$\int_0^l |u(x,t)|^2\,dx + \int_0^t\int_0^l u_x^2(x,\tau)\,dxd\tau \le \int_0^l |u(x,0)|^2\,dx + l^2\int_0^t\int_0^l f^2(x,\tau)\,dxd\tau. \tag{4.18}$$

Below is a further modification if we use $G(t) \le l^2 H(t)$, which is a consequence of (1.30). (4.12) can be modified into

$$\frac{1}{2}G'(t) + \frac{1}{l^2}G(t) \le \sqrt{G(t)}\sqrt{F(t)},$$

from which it follows that

$$\sqrt{G(t)} \le e^{-\frac{t}{l^2}}\left(\sqrt{G(0)} + \int_0^t e^{\frac{\tau}{l^2}}\sqrt{F(\tau)}d\tau\right),$$

namely,

$$\left(\int_0^l |u(x,t)|^2\,dx\right)^{\frac{1}{2}} \le e^{-\frac{t}{l^2}}\left[\left(\int_0^l |u(x,0)|^2\,dx\right)^{\frac{1}{2}} + \int_0^t e^{\frac{\tau}{l^2}}\left(\int_0^l |f(x,\tau)|^2\,dx\right)^{\frac{1}{2}}d\tau\right].$$

This approach would not work for a problem on an unbounded domain, as we would not have an estimate as given in (1.30); but we can still estimate

$$\int_0^l u(x,t)f(x,t)\,dx \le \frac{\lambda}{2}\int_0^l |u(x,t)|^2\,dx + \frac{1}{2\lambda}\int_0^l f^2(x,t)\,dx$$

for appropriately chosen $\lambda > 0$ (it can even depend on t; allowing $l = \infty$). If we choose to work with a constant $\lambda > 0$, we would get

$$\begin{aligned}&\int_0^l |u(x,t)|^2\,dx + 2\int_0^t\int_0^l e^{\lambda(t-\tau)}u_x^2(x,\tau)\,dxd\tau\\ \le& e^{\lambda t}\left[\int_0^l |u(x,0)|^2\,dx + \int_0^t\int_0^l \frac{e^{-\lambda\tau}}{\lambda}f^2(x,\tau)\,dxd\tau\right].\end{aligned}$$

We try to choose a $\lambda > 0$ which would give a more favorable estimate: for a given $t > 0$, if we choose $\lambda = t^{-1}$, then we would get

$$\begin{aligned}&\int_0^l |u(x,t)|^2\,dx + 2\int_0^t\int_0^l e^{1-\tau/t}u_x^2(x,\tau)\,dxd\tau\\ \le& e\left[\int_0^l |u(x,0)|^2\,dx + \int_0^t\int_0^l te^{-\tau/t}f^2(x,\tau)\,dxd\tau\right],\end{aligned}$$

which would imply the cleaner estimate, which is also valid for $l = \infty$ case.

$$\begin{aligned}&\int_0^l |u(x,t)|^2\,dx + 2\int_0^t\int_0^l u_x^2(x,\tau)\,dxd\tau\\ \le& e\left[\int_0^l |u(x,0)|^2\,dx + t\int_0^t\int_0^l f^2(x,\tau)\,dxd\tau\right].\end{aligned} \tag{4.19}$$

Below is another variant of the energy method.

If we exploit $|\int_0^l u(x,t)f(x,t)\,dx| \le \sqrt{G(t)}\sqrt{F(t)}$, in the absence of $G(t) \le l^2H(t)$ (when $l = \infty$ for example), we still have

$$\frac{1}{2}G'(t) \le \sqrt{G(t)}\sqrt{F(t)},$$

from which it follows that $\sqrt{G(t)} \le \sqrt{G(0)} + \int_0^t \sqrt{F(\tau)}d\tau$, and

$$\begin{aligned}&\int_0^l u^2(x,t)\,dx + 2\int_0^t\int_0^l u_x^2(x,\tau)\,dxd\tau\\ =&\int_0^l u^2(x,0)\,dx + 2\int_0^t\int_0^l u(x,\tau)f(x,\tau)\,dxd\tau\\ \le&\int_0^l u^2(x,0)\,dx + 2\int_0^t \sqrt{G(\tau)}\sqrt{F(\tau)}d\tau\\ \le&\int_0^l u^2(x,0)\,dx + 2\int_0^t\left(\sqrt{G(0)} + \int_0^\tau \sqrt{F(s)}ds\right)\sqrt{F(\tau)}d\tau\\ \le&\int_0^l u^2(x,0)\,dx + 2\sqrt{G(0)}\int_0^t \sqrt{F(\tau)}d\tau + \left(\int_0^t \sqrt{F(\tau)}d\tau\right)^2\\ \le&2\left[\int_0^l u^2(x,0)\,dx + \left(\int_0^t\left\{\int_0^l f^2(x,\tau)\,dx\right\}^{\frac{1}{2}}d\tau\right)^2\right].\end{aligned}$$

The technique for getting these estimates seems somewhat ad hoc; below we introduce a more systematic approach: scaling.

Second modification to (4.11). To obtain an estimate for $\int_0^l |u(x,T)|^2\,dx$, we introduce the normalizing scaling $t = Ts$ for $0 \le s \le 1$ and consider $v(x,s) = u(x,Ts)$.

Then

$$v_s(x,s) = Tu_t(x,Ts) = T\left[u_{xx}(x,Ts) + f(x,Ts)\right] = Tv_{xx}(x,s) + Tf(x,Ts).$$

(4.11) applied to $v(x,s)$ at $s = 1$ gives

$$\begin{aligned} &\int_0^l |v(x,1)|^2\,dx + 2T\int_0^1\int_0^l v_x^2(x,\tau)\,dxd\tau \\ \leq & e\left[\int_0^l |v(x,0)|^2\,dx + \int_0^1\int_0^l T^2 f^2(x,T\tau)\,dxd\tau\right]. \end{aligned}$$

But

$$\int_0^l |v(x,1)|^2\,dx = \int_0^l |u(x,T)|^2\,dx,$$

$$T\int_0^1\int_0^l v_x^2(x,\tau)\,dxd\tau = \int_0^T\int_0^l u_x^2(x,\tau)\,dxd\tau,$$

and

$$\int_0^1\int_0^l T^2 f^2(x,T\tau)\,dxd\tau = T\int_0^T\int_0^l f^2(x,\tau)\,dxd\tau,$$

so we arrive at (4.19) for $t = T$.

4.2.3 Using the Energy Estimates to Construct Solutions of the Wave Equation

We now apply the above energy estimates (4.17) to complete the construction of a solution of

$$\begin{cases} \partial_t^2 u - c^2\partial_x^2 u = f(x,t), & \text{on } (x,t) \in [0,l]\times\mathbb{R}^+, \\ u(0,t) = u(l,t) = 0, & \text{for } t > 0, \\ u(x,0) = g(x), & \text{for } x \in [0,l], \\ \partial_t u(x,0) = h(x), & \text{for } x \in [0,l]. \end{cases}$$

using eigenfunction expansion. Recall that

$$u_{(N)}(x,t) = \sum_{j=1}^{N} u_n(t)\sin(\frac{n\pi x}{l}),$$

$$f_{(N)}(x,t) = \sum_{j=1}^{N} f_n(t)\sin(\frac{n\pi x}{l}),$$

$$g_{(N)}(x,t) = \sum_{j=1}^{N} g_n\sin(\frac{n\pi x}{l}),$$

$$h_{(N)}(x,t) = \sum_{j=1}^{N} h_n\sin(\frac{n\pi x}{l}),$$

solve

$$\begin{cases} \partial_t^2 u_{(N)} - c^2\partial_x^2 u_{(N)} = f_{(N)}(x,t), & \text{on } (x,t) \in [0,l]\times\mathbb{R}^+, \\ u_{(N)}(0,t) = u_{(N)}(l,t) = 0, & \text{for } t > 0, \\ u_{(N)}(x,0) = g_{(N)}(x), & \text{for } x \in [0,l], \\ \partial_t u_{(N)}(x,0) = h_{(N)}(x), & \text{for } x \in [0,l]. \end{cases}$$

Then (4.17) applied to $u_{(N)} - u_{(N')}$ implies that $\{\partial_t u_{(N)}(x,t)\}$ and $\{\partial_x u_{(N)}(x,t)\}$ are Cauchy in $C([0,T],L^2[0,l])$ for any $T > 0$, as

$$\int_0^T\int_0^l |f(x,s)|^2\,dxds = \int_0^T\sum_{j=1}^{\infty}\frac{l}{2}f_n^2(s)ds = \frac{l}{2}\sum_{j=1}^{\infty}\int_0^T f_n^2(s)ds < \infty,$$

so for $N > N'$,

$$\int_0^T\int_0^l |f_{(N)}(x,s) - f_{(N')}(x,s)|^2\,dxds = \int_0^T\sum_{j=N'+1}^{N}\frac{l}{2}f_n^2(s)ds \to 0 \quad \text{as } N' \to \infty.$$

Therefore $\{\partial_t u_{(N)}(x,t)\}$ and $\{\partial_x u_{(N)}(x,t)\}$ have limits in $C([0,T],L^2[0,l])$.

$\{u_{(N)}(x,t)\}$ is also Cauchy in the space $C([0,l]\times[0,T])$ based on the elementary inequality (1.30) and the energy estimate (4.17) applied to $u_{(N)} - u_{(N')}$.

Let $u(x,t) = \lim_{N\to\infty} u_{(N)}(x,t)$, $v(x,t) = \lim_{N\to\infty}\partial_t u_{(N)}(x,t)$ (in $C([0,T],L^2[0,l])$), and $w(x,t) = \lim_{N\to\infty}\partial_x u_{(N)}(x,t)$ (in $C([0,T],L^2[0,l])$. Then $v(x,t)$ will be the generalized ∂_t derivative of $u(x,t)$, $w(x,t)$ will be the generalized ∂_x derivative of $u(x,t)$, and $u(x,t)$ will be a generalized solution of (4.16) — we will elaborate on this later.

We used the notion of L^2 derivatives above and now provide more details of that notion in the one dimensional setting.

Definition. A function $g(x)$ defined on (a,b) is said to have L^p **derivative** for some $p \geq 1$ over (a,b), if $g(x)$ is **absolutely continuous** over any compact subinterval $[a',b']$ in (a,b) (so $g'(x)$ is defined almost everywhere over (a,b) and is Lebesgue integrable over any compact subinterval) and $g'(x) \in L^p(a,b)$. We will use the notation $g' \in L^p(a,b)$ to mean that g has an L^p derivative over (a,b).

Remark 4.7. A function that is absolutely continuous over any compact subinterval $[a',b']$ in (a,b) may fail to be continuous over $[a,b]$, or fail to be extended as a continuous function over $[a,b]$; but a function having an L^p derivative for some $p \geq 1$ over (a,b) can be extended to $[a,b]$ as an absolutely continuous function on $[a,b]$, as the integral representation $g(x) = g(c) + \int_c^x g'(y)\,dy$ can be extended to include $x = a$ or b, when $g' \in L^p(a,b)$.

Note that the condition that $g(x)$ having an $L^p(a,b)$ derivative is not quite the same as the existence of $g'(x)$ in (a,b), with the exception of a finite number of points, and that $|g'(x)|^p$ is integrable on (a,b). For example, for any step function $s_c(x)$ defined to be $=1$ for $a \leq x < c$, and $=0$ for $c \leq x \leq b$, $s_c'(x)$ is not considered to be in $L^p(a,b)$, although $s_c'(x)$ fails to exist only at $x = c$.

An equivalent way to define $g'(x) \in L^p(a,b)$ is that there exists a sequence of $\{g_k(x)\} \subset C^1[a,b]$ such that $||g_k(x) - g(x)||_{L^p(a,b)} + ||g_k'(x) - g'(x)||_{L^p(a,b)} \to 0$ as $k \to \infty$. This follows from passing to the limit in the relation $g_k(x) = g_k(x_0) + \int_{x_0}^x g_k'(y)\,dy$ for any $x_0, x \in [a,b]$ to obtain $g(x) = g(x_0) + \int_{x_0}^x g'(y)\,dy$, from which it follows that $g(x)$ is absolutely continuous over $[a,b]$, and that $g' \in L^p(a,b)$. The latter way of defining a function having L^p derivative will be used in higher dimensional settings. When $g'(x)$ exists with the exception of a finite number of points and is piecewise continuous in $[a,b]$, and in addition, $g(x)$ is continuous everywhere in $[a,b]$, then $g'(x) \in L^p[a,b]$ for any $1 \leq p < \infty$ in the sense just defined.

4.2.4 Energy Estimates for Solutions to the Cauchy Problem of the Wave Equation

We next extend the above energy estimates to solutions to the Cauchy problem (3.3). The standard approach so far relies on carrying out integration by parts on finite intervals, which can be justified if the solution satisfies appropriate boundary conditions, or has compact support in x for each t. We will prove below that the solution will remain compactly in x if we assume $u(x,0)$ and $u_t(x,0)$ to have compact support.

We modify the energy estimate in the following way to prove this. Notice that

$$u_t(u_{tt} - c^2 u_{xx}) = (\frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2)_t - (c^2 u_x u_t)_x.$$

So for a solution u to the homogeneous wave equation, the vector-field

$$(P, Q) = (-c^2 u_x u_t, \frac{1}{2}u_t^2 + \frac{c^2}{2}u_x^2)$$

is divergence free. For any given (X, T), we form the triangle with vertices (X, T), $(X - cT, 0)$, and $(X + cT, 0)$, and also consider the trapezoid

$$D_0^\tau(X, T) = \{(x, t) : 0 \le t \le \tau, |x - X| \le c(T - t)\}$$

for any $0 < \tau < T$. Integrating the above divergence free vector-field (P, Q) on $D_0^\tau(X, T)$, we obtain

$$0 = \int_{\partial D_0^\tau(X,T)} (P, Q) \cdot (n_x, n_t) ds.$$

On the $t = \tau$ portion of $\partial D_0^\tau(X, T)$, $X - c(T - \tau) \le x \le X + c(T - \tau)$, and

$$(P, Q) \cdot (n_x, n_t) = \frac{1}{2}u_t^2(x, \tau) + \frac{c^2}{2}u_x^2(x, \tau),$$

while on the $t = 0$ portion of $\partial D_0^\tau(X, T)$, $X - cT \le x \le X + cT$,

$$(P, Q) \cdot (n_x, n_t) = -\left[\frac{1}{2}u_t^2(x, 0) + \frac{c^2}{2}u_x^2(x, 0)\right].$$

On the lateral portion of $\partial D_0^\tau(X, T)$, $(x, t) = (X \pm c(T - t), t)$, so $ds = \sqrt{1 + c^2}dt$, $(n_x, n_t) = \frac{(\pm 1, c)}{\sqrt{1+c^2}}$, and

$$(P, Q) \cdot (n_x, n_t) = \frac{c}{2\sqrt{c^2 + 1}} \left[u_t(X \pm c(T - t), t) \mp c u_x(X \pm c(T - t), t)\right]^2.$$

The key feature is that $(P, Q) \cdot (n_x, n_t) \ge 0$ along $\partial D_0^\tau(X, T)$, as long as we take $n_t \ge 0$, so

$$\begin{aligned}
&\int_{X-c(T-\tau)}^{X+c(T-\tau)} \left[\frac{1}{2}u_t^2(x, \tau) + \frac{c^2}{2}u_x^2(x, \tau)\right] dx \\
&+ \int_0^\tau \frac{c}{2}\left[u_t(X + c(T - t), t) - c u_x(X + c(T - t), t)\right]^2 dt \\
&+ \int_0^\tau \frac{c}{2}\left[u_t(X - c(T - t), t) + c u_x(X - c(T - t), t)\right]^2 dt \\
&= \int_{X-cT}^{X+cT} \left[\frac{1}{2}u_t^2(x, 0) + \frac{c^2}{2}u_x^2(x, 0)\right] dx.
\end{aligned}$$

Thus we have the local version of the energy estimate

$$\int_{X-c(T-\tau)}^{X+c(T-\tau)} \left[\frac{1}{2}u_t^2(x,\tau)+\frac{c^2}{2}u_x^2(x,\tau)\right] dx \le \int_{X-cT}^{X+cT} \left[\frac{1}{2}u_t^2(x,0)+\frac{c^2}{2}u_x^2(x,0)\right] dx.$$

As a consequence, if

$$\int_{X-cT}^{X+cT} \left[\frac{1}{2}u_t^2(x,0)+\frac{c^2}{2}u_x^2(x,0)\right] dx = 0,$$

then

$$\int_{X-c(T-\tau)}^{X+c(T-\tau)} \left[\frac{1}{2}u_t^2(x,\tau)+\frac{c^2}{2}u_x^2(x,\tau)\right] dx = 0,$$

for all $0<\tau<T$, which then implies that $u(x,t)=0$ for $(x,t)\in D_0^T(X,T)$. A direct consequence of the energy estimate is the uniqueness property: if $u(x,0)\equiv u_t(x,0)\equiv 0$, then $u(x,t)\equiv 0$—one simply applies the above energy estimate on any finite trapezoid as in the proof above.

Exercises

Exercise 4.2.1. Let $u(x,t)$ be a solution of (2.10) such that $u(x,0)=0$ and $u_t(x,0)=0$ for $x\le a$ or $x\ge b$. Prove that $u(x,t)=0$ for $x\ge b+ct$ or $x\le a-ct$. This is a statement that the speed of propagation is not faster than c. (Hint: modify the above local energy estimate in appropriately chosen triangles/trapezoids. While the conclusion can be read off from the solution formula given in Theorem 3.1, the energy estimate approach can be extended to higher dimensions.)

Exercise 4.2.2. Suppose that $g(x)$ is continuous and piecewise differentiable over $[0,\pi]$. Let $\sum_{n=1}^{\infty} g_n\sin(nx)$ be the Fourier sine series of $g(x)$ over $(0,\pi)$ and $u(x,t)=\sum_{n=1}^{\infty} g_n e^{-n^2t}\sin(nx)$ be the Fourier series solution of (2.1) for $(x,t)\in(0,\pi)\times(0,\infty)$. Prove that

(a). $t\mapsto\int_0^\pi|\partial_x u(x,t)|^2\,dx$ is non-decreasing in $t\in(0,\infty)$; and

(b). $\lim_{t\to 0}\int_0^\pi|\partial_x u(x,t)|^2\,dx<\infty$ iff $g(0)=g(\pi)=0$.

Exercise 4.2.3. Here is a version of the energy estimates for solutions to the non-homogeneous heat equation with homogeneous Dirichlet boundary condition: Let $U\subset\mathbb{R}^n$ be a bounded domain with piecewise C^1 boundary and $v(x,t)\in C^1(\overline{U}\times[0,T])\cap C^{2,1}_{x,t}(U\times(0,T])$ be a solution of

$$\begin{cases} v_t(x,t)-\Delta v(x,t)=f(x,t) & \text{for } (x,t)\in U\times\mathbb{R}^+,\\ v(x,t)=0 & \text{for } (x,t)\in\partial U\times\mathbb{R}^+,\\ v(x,0)=g(x) & \text{for } x\in U.\end{cases}$$

Then there exists a constant $C > 0$ depending on U, T such that

$$\max_{0\le t\le T}\left(\int_U v^2(x,t)\,dx\right)+\int_0^T\int_U |\nabla v(x,t)|^2\,dxdt \le C\left(\int_0^T\int_U |f(x,t)|^2\,dxdt+\int_U g^2(x)\,dx\right).$$

Exercise 4.2.4. Here is a version of the energy estimates for solutions to the wave equation (3.3) on $\mathbb{R}\times\mathbb{R}_+$ with non-homogeneous right hand side. Let $u \in C^2(\mathbb{R}\times(0,T)\cap C^1(\mathbb{R}\times[0,T])$ be a solution of (3.3) such that $g(x) = u(x,0)$, $h(x) = u_t(x,0)$, and $f(x,t)$ satisfy $\int_{\mathbb{R}}(|g'(x)|^2+|h(x)|^2)\,dx < \infty$, and $\int\int_{\mathbb{R}\times[0,T]} f^2(x,\tau)\,dxd\tau < \infty$. Prove that there exists $M = M(T) > 0$ such that for any $0 < t \le T$,

$$\int_{\mathbb{R}}\left[u_t^2(x,t)+c^2u_x^2(x,t)\right]\,dx \le M\left\{\int_{\mathbb{R}}\left[c^2|g'(x)|^2+h^2(x)\right]\,dx+\iint_{\mathbb{R}\times[0,t]} f^2(x,\tau)\,dxd\tau\right\},$$

and

$$\begin{aligned}&\iint_{\mathbb{R}\times[0,T]}\left[u_t^2(x,t)+c^2u_x^2(x,t)\right]\,dx\,dt\\ \le& MT\left\{\int_{\mathbb{R}}\left[c^2|g'(x)|^2+h^2(x)\right]\,dx+\iint_{\mathbb{R}\times[0,T]} f^2(x,\tau)\,dxd\tau\right\}.\end{aligned}$$

(Hint: carry out the energy estimate on the trapezoid $D_0^T(0,\tau)$ for $\tau\to\infty$.)

Exercise 4.2.5. Let $\{g_k(x)\}\subset C^1[a,b]$ be a sequence such that both $\{g_k(x)\}$ and $\{g_k'(x)\}$ are Cauchy in $L^p(a,b)$. Prove that $\{g_k(x)\}$ is Cauchy in $C[a,b]$, and that there exists an absolutely continuous function g over $[a,b]$ such that $||g_k(x)-g(x)||_{L^p(a,b)} + ||g_k'(x)-g'(x)||_{L^p(a,b)} \to 0$ as $k\to\infty$; furthermore, $g(x) = g(a)+\int_a^x g'(y)\,dy$ for all $x\in[a,b]$. Hint: First prove that $\{\int_a^b g_k(x)\,dx\}$ is Cauchy, then use $g_k(x) = (b-a)^{-1}\left[\int_a^b g_k(z)\,dz+\int_a^b\int_z^x g_k'(y)\,dydz\right]$.

4.3 Variational Method

Recall that the Dirichlet principle says that a solution $u(x)$ to

$$\begin{cases}-\Delta u = 0 & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega.\end{cases} \tag{4.20}$$

can be found as a minimizer to the variational problem

$$E[u] = \min_{w\in M_g} E[w], \tag{4.21}$$

where $E[w] = \int_\Omega \frac{1}{2}|\nabla w(x)|^2\,dx$ and $M_g = \{w \in C^2(\Omega) \cap C^1(\overline{\Omega}) : w = g \text{ on } \partial\Omega\}$.

To find a minimizer of $E[w]$ over M_g, note that $E[w]$ is non-negative, so $\inf_{w\in M_g} E[w]$ is well defined. A minimizing sequence for $E[w]$ on M_g, namely, $\{u_j(x)\} \subset M_g$ such that $E[u_j] \to \inf_{w\in M_g} E[w]$, is bounded in $\int_\Omega |\nabla u_j(x)|^2\,dx$. However, M_g is not finite dimensional; and there is no clear mechanism that would prove that such a minimizing sequence $\{u_j\}$ is convergent (in C^2 norm), or at least has a subsequence which is convergent.

We note that $E[w]$ is (strictly) convex over M_g. More specifically, note that $\frac{u_j+u_k}{2} \in M_g$ and

$$E[u_j] + E[u_k] = 2E[\frac{u_j+u_k}{2}] + \frac{1}{4}||\nabla(u_j-u_k)||^2_{L^2(\Omega)},$$

so we have

$$\begin{aligned}\frac{1}{8}||\nabla(u_j-u_k)||^2_{L^2(\Omega)} =& \frac{E[u_j]+E[u_k]}{2} - E[\frac{u_j+u_k}{2}] \\ \le& \frac{E[u_j]+E[u_k]}{2} - \inf_{u\in M_0} E[u] \to 0,\end{aligned}$$

when $j,k\to\infty$, as $E[u_j], E[u_k] \to \inf_{u\in M_0} E[u]$ when $j,k\to\infty$. This proves that a minimizing sequence is Cauchy in the semi-norm $||\nabla w||_{L^2(\Omega)}$ ($||\nabla w - \nabla v||_{L^2(\Omega)}$ is in fact a metric on M_g)! What remains is to identify this limit and its properties.

The early difficulties of the variational approach lie mostly with trying to work directly with convergence in M_g in the traditional C^2 norm or its minor variations. The computations above suggest strongly that it's much more natural and advantageous to work with convergence in the semi-norm $||\nabla w||_{L^2(\Omega)}$.

Definition. For a bounded domain Ω in $\mathbb{R}^n$, define $H^1_0(\Omega)$ to be the completion* in the norm $||\nabla u||_{L^2(\Omega)}$ of the space $C^1_c(\Omega)$. For a domain Ω in $\mathbb{R}^n$ with piecewise C^1 boundary, define $H^1(\Omega)$ to be the completion in the norm $||\nabla u||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}$ of the space $C^1(\overline{\Omega})$.

Remark 4.8. It is possible to define $H^1(\Omega)$ for domains without requiring its boundary being piecewise C^1; however, in order to get approximations by functions in $C^1(\overline{\Omega})$, certain regularity assumptions on $\partial\Omega$ are needed. There are complications

*We are using a basic fact that $||\nabla u||_{L^2(\Omega)}$ is a norm on $C^1_c(\Omega)$, and that $L^p(\Omega)$ is complete in the sense that any Cauchy sequence $\{u_k\}$ of $L^p(\Omega)$ has a limit function $u \in L^p(\Omega)$ such that $\|u_k - u\|_{L^p(\Omega)} \to 0$ as $k\to\infty$. In this context, a sequence $\{u_k\}$ in $C^1_c(\Omega)$ such that each $\{\partial_\alpha u_k\}$, $\alpha = 1,\cdots,n$, is Cauchy in $L^2(\Omega)$ has a limit function $u_{[\alpha]}$ in $L^2(\Omega)$ such that $\|\partial_\alpha u_k - u_{[\alpha]}\|_{L^2(\Omega)} \to 0$ as $k\to\infty$. These $u_{[\alpha]}$'s will be the partial derivatives of u in an integral sense.

when $\partial\Omega$ has pieces which have codimension higher than 1, or have codimension 1 pieces with both sides lying in Ω; such boundary components create difficulties for approximations by functions in $C^1(\overline{\Omega})$.

One could use $||\nabla w - \nabla v||_{L^2(\Omega)}$ as a metric on M_g and discuss its completion. The main technical issue is to discuss the sense in which the functions in the completion take on the boundary value g. Discussion later in this section gives a sense of the boundary value of functions in the completion on codimension one boundary components, but see Exercise 4.3.3 for an example where the boundary value on a codimension two component of the boundary is not preserved in the completion.

The definition for $H_0^1(\Omega)$ only includes the norm $||\nabla u||_{L^2(\Omega)}$. The fact that a sequence which is complete in this norm is also complete in the norm $||u||_{L^2(\Omega)}$ follows from the following Poincaré inequality.

Poincaré inequality: For any bounded domain Ω in $\mathbb{R}^n$, there is a constant $C > 0$ depending on Ω, such that for any function u in $C_c^1(\Omega)$, we have

$$\int_\Omega |u(x)|^2\, dx \le C \int_\Omega |\nabla u(x)|^2\, dx. \tag{4.22}$$

Remark 4.9. $H_0^1(\Omega)$ and $H^1(\Omega)$ are defined in terms of completion of the space of appropriate C^1 functions in the L^2 norms of their derivatives. When inequalities such as (1.30) or (4.22) are valid for the $C^1(\overline{\Omega})$ functions, for which the completion is to be taken, these inequalities continue to hold for functions in the completion. Let's take $\Omega = (0, l)$ to carry out the analysis. A sequence of functions $\{u_j\} \subset C_c^1(0, l)$ which is Cauchy in the norm $||u_j'(x)||_{L^2(0,l)}$ is also Cauchy in $||u_j(x)||_{C[0,l]}$ due to (1.30), so it has a limit $u \in C[0, l]$ and $\{u_j'\}$ has a limit v in $L^2[0, l]$, and this v is the L^2 weak derivative of u. Furthermore, based on the following stronger inequality for functions in $C^1[0, l]$,

$$|u(x_1) - u(x_2)| \le \left(\int_{x_1}^{x_2} |u'(x)|^2\, dx\right)^{1/2} |x_1 - x_2|^{1/2}, \tag{4.23}$$

(4.23) continues to hold for functions in $H^1(0, l)$, where $u'(x)$ represents the weak L^2 derivative of u. This kind of argument identifies elements in $H^1(\Omega)$ with functions having some traditional regularity. When Ω is a higher dimensional domain, we may no longer have (1.30) or (4.23), but there will be variations of this kind of argument. By definition for $H_0^1(\Omega)$, (4.22) is valid for functions in $H_0^1(\Omega)$ as well.

Proof of Poincaré inequality. We may suppose that $\Omega \subset \{x \in \mathbb{R}^n : 0 < x_1 < L\}$. Then for any function u in $C_c^1(\Omega)$, we may treat it as in $C_c^1(\{x \in \mathbb{R}^n : 0 < x_1 < L\})$, and it follows from $u(x) = \int_0^{x_1} u_{x_1}(y, x_2, \cdots, x_n)\,dy$ that $|u(x)| \le \int_0^L |u_{x_1}(y, x_2, \cdots, x_n)|\,dy$. Squaring both sides and applying the Cauchy-Schwarz inequality on the right hand side, we obtain

$$|u(x)|^2 \le \left(\int_0^L |u_{x_1}(y, x_2, \cdots, x_n)|^2\,dy\right)\left(\int_0^L 1\,dy\right).$$

Integrating both sides over $x \in \Omega$, we obtain

$$\int_\Omega |u(x)|^2\,dx \le L^2\left(\int_\Omega |u_{x_1}(x)|^2\,dx\right) \le L^2\left(\int_\Omega |\nabla u(x)|^2\,dx\right).$$

□

Remark 4.10. Note that if $C(h) = \{x \in \mathbb{R}^n : a < x_1 < a + h, (x_2, \cdots, x_n) \in D\}$ is a cylinder with D as base, then for any $a \le a_1 < a_2 \le a + h$, similar to (4.23), for any $u \in C^1(\overline{C(h)})$, we have

$$\begin{aligned}&\int_D |u(a_1, x_2, \cdots, x_n) - u(a_2, x_2, \cdots, x_n)|^2\,dx_2 \cdots dx_n\\ \le&(a_2 - a_1)\left(\int_D \int_{a_1}^{a_2} |u_{x_1}(x)|^2\,dx\right) \le (a_2 - a_1)\left(\int_D \int_{a_1}^{a_2} |\nabla u(x)|^2\,dx\right).\end{aligned}$$

This inequality also holds for $H^1(C(h))$ functions and shows that $x_1 \mapsto u(x_1, \cdot) \in L^2(D)$ is Hölder continuous for an $H^1(C(h))$ function, and implies that H^1 functions have well defined restrictions as L^2 functions to hypersurfaces in its domain; in particular, they have well defined boundary values as functions in $L^2(\partial\Omega)$ when the domain has piecewise C^1 boundary.

An element u in $H_0^1(\Omega)$ or $H^1(\Omega)$ has weak L^2 derivatives in the following sense: there exist $u_{[\alpha]} \in L^2(\Omega)$ for each $\alpha = 1, \cdots, n$, such that for any $\eta \in C_c^1(\Omega)$

$$\int_\Omega u(x)\partial_\alpha \eta(x)\,dx = -\int_\Omega u_{[\alpha]}(x)\eta(x)\,dx.$$

This is true because there exists $\{v_j\} \subset C_c^1(\Omega)$ (or $C^1(\Omega)$ in the case $u \in H^1(\Omega)$) such that $v_j \to u$ in $L^2(\Omega)$ and $\{\nabla v_j\}$ is Cauchy in $L^2(\Omega)$, so there exists $u_{[\alpha]} \in L^2(\Omega)$ for each $\alpha = 1, \cdots, n$, such that $\partial_\alpha v_j \to u_{[\alpha]}$ in $L^2(\Omega)$ as $j \to \infty$. Since, for each j,

$$\int_\Omega v_j(x)\partial_\alpha \eta(x)\,dx = -\int_\Omega \partial_\alpha v_j(x)\eta(x)\,dx,$$

using

$$\left|\int_\Omega [v_j(x) - u(x)]\,\partial_\alpha \eta(x)\,dx\right| \le \left(\int_\Omega |v_j(x) - u(x)|^2\,dx\right)^{1/2}\left(\int_\Omega |\partial_\alpha \eta(x)|^2\,dx\right)^{1/2},$$

and a similar estimate for $\left|\int_\Omega \partial_\alpha \left[v_j(x) - u_{[\alpha]}(x)\right] \eta(x)\, dx\right|$, we see that

$$
\begin{aligned}
\int_\Omega u(x)\partial_\alpha \eta(x)\, dx &= \lim_{j\to\infty} \int_\Omega v_j(x)\partial_\alpha \eta(x)\, dx \\
&= -\lim_{j\to\infty} \int_\Omega \partial_\alpha v_j(x)\eta(x)\, dx \\
&= -\int_\Omega u_{[\alpha]}(x)\eta(x)\, dx.
\end{aligned}
$$

We return to our discussion of finding a minimizer for $E[u]$ in M_g. Since $M_g \subset C^1(\overline{\Omega})$, we should consider the closure $\overline{M_g}$ of M_g in $H^1(\Omega)$. Because $H^1(\Omega)$ is the completion in the norm $||\nabla u||_{L^2(\Omega)} + ||u||_{L^2(\Omega)}$, $E[u]$ extends to $\overline{M_g}$ and is a continuous functional on $\overline{M_g}$. Our earlier argument on finding a minimizer for $E[u]$ now works on $\overline{M_g}$; the only difference is that the minimizer $u \in \overline{M_g}$ is only known to have weak L^2 derivatives in Ω, and takes on the prescribed boundary value g only on codimension-one boundary component and in the generalized L^2 sense as discussed in Remark 4.10.

Below is an alternative approach that also works for the Poisson equation (4.24) below. Assume that g is the restriction to $\partial\Omega$ of a $C^2(\overline{\Omega})$ function $\tilde{g}$. (Although we already discussed that an H^1 function on a domain with piecewise C^1 boundary has well defined boundary value as an L^2 function, it is not easy at this stage to characterize those functions on the boundary which are boundary values of an H^1 function in the domain (not every $L^2(\partial\Omega)$ or $C(\partial\Omega)$ function can be the boundary value of an H^1 function; some fractional order differentiability is needed.) We look for a solution of (4.20) in the form of $u = \tilde{g} + v$ for some $v \in H^1_0(\Omega)$, then

$$
\begin{cases}
-\Delta v = f(x) \stackrel{\text{def}}{=} \Delta \tilde{g} & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{4.24}
$$

(4.24) has a variational structure too. A solution v to (4.24) is a minimizer to the variational problem

$$
I[v] = \min_{w \in H^1_0(\Omega)} I[w],
\tag{4.25}
$$

where $I[w] = \int_\Omega \left\{\frac{1}{2}|\nabla w(x)|^2 - f(x)w(x)\right\} dx$; and we can look for a minimizer to $I[w]$ as a mechanism to find a solution of (4.24).

First we need to show that $\inf_{w\in H^1_0(\Omega)} I[w]$ is well defined, namely, $I[w]$ is bounded from below on $H^1_0(\Omega)$. This can be verified with the help of Poincaré's inequality.

It follows from (4.22) that

$$\begin{aligned} I[w] \geq & \frac{1}{2}||\nabla w||^2 - \sqrt{C}||f||_{L^2(\Omega)}||\nabla w||_{L^2(\Omega)} \\ \geq & \frac{1}{2}||\nabla w||^2_{L^2(\Omega)} - \frac{1}{4}||\nabla w||^2_{L^2(\Omega)} - C||f||^2_{L^2(\Omega)} \\ = & \frac{1}{4}||\nabla w||^2_{L^2(\Omega)} - C||f||^2_{L^2(\Omega)}. \end{aligned} \tag{4.26}$$

$I[w]$ differs from $E[w]$ only by a linear term, so $I[w]$ carries the same convexity property as $E[w]$. In particular, a minimizing sequence $\{w_j\} \subset H_0^1(\Omega)$ for $I[w]$ is Cauchy in the $H_0^1(\Omega)$ norm, therefore there exists $v \in H_0^1(\Omega)$ such that $w_j \to v$ in $H_0^1(\Omega)$. Furthermore, $I[v] \leq \lim_{j\to\infty} I[w_j] = \inf_{w\in H_0^1(\Omega)} I[w]$; and is the unique minimizer due to the strict convexity of $I[w]$.

Since v attains $\inf_{w\in H_0^1(\Omega)} I[w]$, we have, for any $w \in H_0^1(\Omega)$, that

$$\left.\frac{d}{dt}\right|_{t=0} I[v+tw] = \int_\Omega [\nabla v(x) \cdot \nabla w(x) - f(x)w(x)]\, dx = 0,$$

which is the weak form of the equation to $-\Delta v(x) = f(x)$. It remains to prove that v, satisfying the weak form of the equation above, is in fact sufficiently regular to satisfy $-\Delta v(x) = f(x)$ in the classical sense.

The same approach works if we change $I[w]$ into

$$I[w] = \int_\Omega \left[\frac{1}{2}\left(\sum_{i,j=1}^n a_{ij}(x) w_{x_i}(x) w_{x_j}(x) + c(x)w^2(x)\right) - f(x)w(x)\right] dx,$$

where the $a_{ij}(x)$'s satisfy

$$m|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq M|\xi|^2 \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega, \tag{4.27}$$

for some $0 < m < M$, and we may assume for now $0 \leq c(x) \leq M$ for $x \in \Omega$. Such kind of $a_{ij}(x)$'s arise naturally in dealing with a medium which is anisotropic (the energy density may vary depending on the orientation of ∇w) or non-homogeneous (the energy density may vary depending on the location x), or both. Simplest examples include

$$(a_{ij}(x)) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda(x) & 0 \\ 0 & \lambda(x) \end{bmatrix} \text{ or } \begin{bmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(x) \end{bmatrix},$$

where $m \leq \lambda_j$, $\lambda(x)$, and $\lambda_j(x) \leq M$ (they need not be continuous in x).

Under these assumptions, $I[w]$ is still well defined on $H^1(\Omega)$, is continuous with respect to the $H^1(\Omega)$ metric and strictly convex, and $I[w]$ has a unique minimizer v in $H^1_0(\Omega)$. This v satisfies

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} I[v+tw] &= \int_\Omega \left[\sum_{i,j=1}^n \frac{1}{2}\left(a_{ij}(x)+a_{ji}(x)\right)v_{x_i}(x)w_{x_j}(x) + \left(c(x)v(x)-f(x)\right)w(x)\right] dx \\ &= 0,\end{aligned}$$

for any $w \in H^1_0(\Omega)$. We may as well assume $a_{ij}(x) = a_{ji}(x)$ for all $x \in \Omega$—this will not affect $I[w]$. Thus v satisfies

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x)v_{x_i}(x)w_{x_j}(x) + \left(c(x)v(x)-f(x)\right)w(x)\right] dx = 0, \tag{4.28}$$

for any $w \in H^1_0(\Omega)$. If the terms $a_{ij}(x)v_{x_i}(x)$ are in $C^1(\Omega)$, or have $L^2(\Omega)$ derivatives, we would have

$$\int_\Omega \left\{-\sum_{i,j=1}^n \left(a_{ij}(x)v_{x_i}(x)\right)_{x_j} + c(x)v(x) - f(x)\right\} w(x)\, dx = 0,$$

for any $w \in H^1_0(\Omega)$. We say that v is an $H^1_0(\Omega)$ weak solution of

$$-\sum_{i,j=1}^n \left(a_{ij}(x)v_{x_i}(x)\right)_{x_j} + c(x)v(x) - f(x) = 0, \tag{4.29}$$

when it satisfies the integral form (4.28).

(4.27) is called **ellipticity** condition for (4.29), and in such a case (4.29) is called a second order elliptic equation. Thus the variational approach has provided a mechanism to produce a weak solution of (4.29); what remains is to prove that, when the $a_{ij}(x)$, $c(x)$, and $f(x)$ have additional regularity, the weak solution $v(x)$ has improved regularity than being in $H^1_0(\Omega)$; in particular, when the $a_{ij}(x)$, $c(x)$, and $f(x)$ are sufficiently regular, the $v(x)$ becomes a classical $C^2(\Omega) \cap C(\overline{\Omega})$ solution.

Remark 4.11. Although the variational method is quite robust, there are some subtle issues which are not revealed in a conspicuous way by this approach. For instance, for certain domains Ω and certain boundary value $g \in C(\partial\Omega)$, (4.20) does not have a solution. $\Omega = B_1(0) \setminus \{0\}$ with $g(x) = 0$ for $|x| = 1$ and $g(0) = 1$ is one such example. How is this reflected in the variational approach? In seeking a minimizer in M_g, or its appropriate completion, of $E[w] = \frac{1}{2}\int_\Omega |\nabla w(x)|^2 dx$, a minimizing sequence $\{u_j\}$

in M_g is still a Cauchy sequence under the metric $||\nabla u_j - \nabla u_k||_{L^2(\Omega)}$. Why isn't the limit a solution to (4.20) here? It turns out that completion of M_g under this metric can maintain the property of $w(x) = 0$ for $|x| = 1$ in appropriate sense*, but can't maintain the property that $w(0) = 1$ in the case here. In other words, boundary properties of functions in M_g may not be all preserved in the completion when the boundary of the domain has a component that is not a hypersurface. In order for the completion to maintain the boundary value at $x = 0$, there must be a constant $C > 0$ such that for a minimizing sequence $w_j(x)$, $\int_{\mathbb{S}^{n-1}} |w_j(r\boldsymbol{\omega}) - w_j(0)|\, d\sigma(\boldsymbol{\omega}) \leq C||\nabla w_j||_{L^2(\Omega)}$—this would be an analogue of (1.30) in the one dimensional case. But such an inequality is impossible, even for radially symmetric functions.

Exercises

Exercise 4.3.1. Define $I[w]$ as on the previous page and assume (4.27). In addition, assume that $c(x) \geq m$ for all $x \in \Omega$. Prove that $\inf\{I[w] : w \in H^1(\Omega)\}$ is attained by a unique $v \in H^1(\Omega)$, and that v satisfies (4.29). Assume further that $\partial\Omega \in C^1$ and that $v \in C^1(\overline{\Omega})$, prove that $\sum_{i,j=1}^n a_{ij}(x) v_{x_i}(x)\nu_j(x) = 0$ at each $x \in \partial\Omega$, where $\nu(x)$ stands for the unit exterior normal to $\partial\Omega$ at $x \in \partial\Omega$.

Exercise 4.3.2. Verify that the infimum of $I[u] \stackrel{\text{def}}{=} \int_{-1}^1 x^2|u'(x)|^2 dx$ over $X = \{u \in H^1(-1,1) : u(-1) = -1, u(1) = 1\}$ is equal to 0 and is not attained by any $u \in X$. This example is based on Weierstrass' example pointing out a defect in Riemann's application of the Dirichlet principle. The same conclusion holds if the condition $u \in H^1(-1,1)$ in X above is replaced by either $u \in C^1[-1,1]$ or $u \in AC[-1,1]$.

Let Y denote the completion of $AC[-1,1]$ under the norm $|u(-1)| + |u(1)| + \left(\int_{-1}^1 x^2|u'(x)|^2 dx\right)^{1/2}$ and $Y_1 = \{u \in Y : u(-1) = -1, u(1) = 1\}$ (Note that if $u \in Y$, then for any $0 < c < 1$, $u \in AC[-1,-c] \cap AC[c,1]$, so $u(x)$ is point-wise well defined over $[-1,0) \cup (0,1]$). Verify that the function

$$u_1(x) = \begin{cases} -1 & \text{if } x \in [-1,0), \\ 1 & \text{if } x \in (0,1], \end{cases}$$

is in Y_1 and that the infimum of the same $I[u]$ over Y_1 is attained by u_1. Note that, unlike $H^1(-1,1)$, which is also obtained by a completion procedure, and whose members are in $AC[-1,1]$, Y and Y_1 admit functions which are not absolutely continuous (not even continuous) over $[-1,1]$.

*E.g., one can easily establish $\int_{|x|=r} |w(x)|^2 d\sigma(x) \leq C(1-r)\int_{|x|\geq r} |\nabla w(x)|^2\, dx$ for $0 < r < 1$ and $w \in M_g$, and use this in the completion process to describe the sense in which $w(x) = 0$ on $|x| = 1$ for any limit function $w(x)$ in the completion.

Exercise 4.3.3. Define X to be the completion of the space M_g of $C^1[0,1]$ functions with $u(0) = g(0) = 1$ and $u(1) = g(1) = 0$ with respect to the metric induced by the semi-norm $||u|| := \left(\int_0^1 |u'(r)|^2 r\, dr\right)^{1/2}$ (This is actually an H^1 norm of radial functions in the two dimensional unit disc with boundary value 0). Verify that the infimum of $I[u] := \int_0^1 |u'(r)|^2 r\, dr$ over X is equal to 0 by constructing a sequence u_k in $C^1[0,1]$ functions with $u_k(0) = 1$ and $u_k(1) = 0$ such that $I[u_k] \to 0$ and $||u||_{L^p[0,1]} \to 0$ as $k \to \infty$, where $1 \leq p < \infty$ is arbitrary. This example shows that the completion may not keep the boundary value of functions in M_g. Note, however, that the boundary value $u(1) = 0$ is preserved in X in the sense that, for $u \in X$, $|u(s)| \leq \sqrt{\ln s^{-1}} \left(\int_s^1 |u'(r)|^2 r\, dr\right)^{1/2}$ holds for $0 < s < 1$. Hint: Since two-dimensional harmonic functions are related to making $I[u]$ the least possible, one should make use of two-dimensional radial harmonic functions (perhaps on appropriate annulus region) in the construction of u_k.

Exercise 4.3.4. Prove that if $u \in H_0^1(0,1)$, then $\dfrac{u(x)}{\sqrt{x}} \to 0$ as $x \to 0^+$.

Exercise 4.3.5. Note that (4.23) shows that any function in $H^1(0,1)$ is Hölder continuous with exponent $\frac{1}{2}$, and in particular, is in $C[0,1]$. Prove that if $u \in H^1(0,1)$ and $u(0) = u(1) = 0$, then $u \in H_0^1(0,1)$.

Exercise 4.3.6. Suppose that $u \in H^1(0,1)$ satisfies

$$\int_0^1 a(x)u'(x)\eta'(x)\, dx = 0$$

for all $\eta \in H_0^1(0,1)$, where $M \geq a(x) \geq m > 0$ for $x \in (0,1)$.

(a). Show that $u'(x)$ needs not be continuous. (Study the case where $a(x)$ is piecewise constant.)

(b). Show that, if, in addition, $a(x) \in C^1(0,1)$, then $u(x) \in C^2(0,1)$.

Exercise 4.3.7. For any $c \in (0,1)$, prove that $v \in H_0^1(0,1) \mapsto v(c)$ is a continuous linear functional on $H_0^1(0,1)$: this is clearly linear in v; to prove that it is a continuous linear functional, it remains to prove the existence of some $C > 0$ such that $|v(c)| \leq C||v'||_{L^2(0,l)}$ for any $v \in H_0^1(0,1)$. In addition, prove that

$$I[v] := \frac{1}{2}\int_0^1 |v'(x)|^2 dx - v(c)$$

has a unique minimizer $u \in H_0^1(0,1)$, that $u(x)$ is piecewise C^1 on $(0,1)$, and $u'(c+) - u'(c-) = -1$, where $u'(c\pm)$ are the right and left limits of $u'(x)$ at c. Finally determine $u(x)$ explicitly.

Exercise 4.3.8. This exercise provides a mechanism to prove the existence of a weak solution to a modified (4.29) which includes first derivative terms of $v_{x_j}(x)$:

$$-\sum_{i,j=1}^{n}\left(a_{ij}(x)v_{x_i}(x)\right)_{x_j}+\sum_{j=1}^{n}b_j(x)v_{x_j}(x)+c(x)v(x)-f(x)=0, \tag{4.30}$$

where we may assume $b_j(x)$ to be bounded measurable, say, $|b_j(x)|\le M$ for all $x\in\Omega$, and $f\in L^2(\Omega)$. First, we set up a bilinear form $B[v,w]$ on $H_0^1(\Omega)$ as

$$B[v,w]=\int_\Omega\left(\sum_{i,j=1}^{n}a_{ij}(x)v_{x_i}(x)w_{x_j}(x)+\left[\sum_{j=1}^{n}b_j(x)v_{x_j}(x)+c(x)v(x)\right]w(x)\right)dx.$$

If $v\in H_0^1(\Omega)$ is such that $B[v,w]=\int_\Omega f(x)w(x)dx$ for all $w\in H_0^1(\Omega)$, then we say v is a weak solution of (4.30).

(a) Prove that there exists $C>0$ such that $|B[v,w]|\le C||v||_{H_0^1(\Omega)}||w||_{H_0^1(\Omega)}$ for any $v,w\in H_0^1(\Omega)$.

(b) $B[v,w]$ is said to be coercive if there exists $\alpha>0$ such that $B[v,v]\ge\alpha||v||^2_{H_0^1(\Omega)}$ for all $v\in H_0^1(\Omega)$. Prove that if $c(x)\ge nM^2/(2m)$ for all $x\in\Omega$, then $B[v,w]$ defined above is coercive. Note that $B[v,w]$ is in general not symmetric in v and w; otherwise, one could have used the variational method of this section to prove the existence of a weak solution of (4.30), namely, some $u\in H_0^1(\Omega)$ such that $B[u,w]=\int_\Omega f(x)w(x)dx$ for all $w\in H_0^1(\Omega)$.

(c) The following abstract formulation, due to Lax and Milgram, will be used to prove the existence of a weak solution of (4.30), under appropriate conditions.

> Suppose that H is a Hilbert space and $B[v,w]$ is a bilinear form on H satisfying (i). there exists some $C>0$ such that $|B[v,w]|\le C||v||||w||$ for all $v,w\in H$, and (ii). there exists some $\alpha>0$ such that $B[v,v]\ge\alpha||v||^2$ for all $v\in H$. Then for any bounded linear functional l of H, there is a unique $u\in H$ such that $B[u,w]=\langle l,w\rangle$ for all $w\in H$.

Follow the instruction to prove this Lax-Milgram theorem. First, prove that for any $v\in H$, there is a unique $T(v)\in H$ such that $B[v,w]=(T(v),w)$ for all $w\in H$. Here $(T(v),w)$ is the inner product of H. Second, prove that T is a linear operator and $\alpha||v||\le||T(v)||\le C||v||$ for all $v\in H$. Third, prove that $T:H\mapsto H$ is onto by showing that $T(H)$ is closed in H, and that if w is such that $(T(v),w)=0$ for all $v\in H$, then $w=0$. Lastly, prove that for any bounded linear functional l of H, there is a unique $u \in H$ such that $B[u, w] = (T(u), w) = \langle l, w \rangle$ for all $w \in H$.

(d) Prove that under our assumptions, including those made in (b), there exists a unique weak solution to (4.30).

Chapter 5

Laplace and Poisson Equations

Overview. In this chapter, we aim to develop methods to solve boundary value problem of the kind

$$\begin{cases} \Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{5.1}$$

In the first chapter, we discussed the separation of variables method to solve (5.1) in the case U is a round disk in $\mathbb{R}^2$ and $f \equiv 0$ in U. The same method can also be made to work on higher dimensional round balls. Eigenfunction expansion method can be used to solve (5.1) on such domains when f is not identically 0. However these methods leave us no clue as for how to approach (5.1) on general domains.

We will first do some reductions and try to understand the solvability of the reduced problems. The first is to try to construct solutions to

$$\Delta u = f \quad \text{in } U, \tag{5.2}$$

for reasonably behaved f without worrying whether the boundary values of u equals g, or to construct solutions to

$$\begin{cases} \Delta u = 0 & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{5.3}$$

for reasonably behaved g; we will first carry this out for a round ball or a half space in general dimensions. Next we will try to construct solutions to

$$\begin{cases} \Delta u = f(x) & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{5.4}$$

for well behaved f.

Once we understand how to solve (5.3) and (5.4), a solution of (5.1) can be constructed as $u_1(x)+u_2(x)$, where $u_1(x)$ is a solution of (5.3) and $u_2(x)$ is a solution of (5.4). If we also know how to construct solutions to (5.2), then (5.3) and (5.4) are essentially equivalent. For example, let $v(x)$ be a solution of (5.2) such that $v \in C(\overline{U})$, then a solution of (5.4) can be constructed as $v(x) + w(x)$, where w is a solution of (5.3) with boundary value $g(x) - v(x)$.

After we have some understanding for solving (5.2), (5.3) and (5.4) for a class of data (source term or boundary value) and domain, we move on to tackle their solvability in general cases. One main focus will be to develop tools to understand how solutions converge when the data or domains converge. We should also keep an eye on properties and methods which may hold for solutions to equations that are modifications of (5.2), such as the Helmholtz equation $\Delta u(x) + c(x)u(x) = 0$, or its variable coefficients variants.

We already saw several applications of the maximum principle for solutions to (5.1). An immediate consequence of the maximum principle is the uniqueness of solution of (5.1) in the class $C^2(U) \cap C^0(\bar{U})$: suppose that U is a bounded domain and that $u, v \in C^2(U) \cap C^0(\bar{U})$ are solutions to (5.1), then $u - v \in C^2(U) \cap C^0(\bar{U})$ is harmonic in U with $(u-v)|_{\partial U} = 0$, therefore $u - v \equiv 0$ in U. A consequence of the uniqueness of solution of (5.1) is that one *can't* prescribe both the boundary value $u(x)$ on ∂U and its normal derivative $\frac{\partial u}{\partial \nu}$ on ∂U.

Another consequence of the maximum principle is the estimate on the size of $|u|_{C^0(\bar{U})}$ in terms of $|u|_{C^0(\partial U)}$ and $|\Delta u|_{C^0(\bar{U})}$: for a bounded domain U, there exists a constant $C > 0$ depending only on the domain U such that

$$|u|_{C^0(\bar{U})} \leq |u|_{C^0(\partial U)} + C|\Delta u|_{C^0(\bar{U})}. \tag{5.5}$$

A further consequence of the maximum principle is the following convergence property: suppose that $u_k \in C^2(U) \cap C^0(\bar{U})$ is the unique solution to (5.1) with f_k, g_k replacing f, g, respectively, and suppose that there exist $f \in C^0(\bar{U})$ and $g \in C^0(\partial U)$ such that $f_k \to f$ in $C^0(\bar{U})$ and $g_k \to g$ in $C^0(\partial U)$, then we see through (5.5) applied to $u_k - u_l$ that $\{u_k\}$ is Cauchy in $C^0(\bar{U})$, therefore, there exists a limit $u \in C^0(\bar{U})$ with $u = g$ on ∂U.

In order for u to solve (5.1) in the classical sense, we need to find conditions which guarantee that $u_k \to u$ not only in $C^0(\bar{U})$, but also in $C^2_{\text{local}}(U)$, at least after extracting a subsequence. This can be achieved if we could obtain versions of (5.5) where the left hand side is replaced by appropriate norms of derivatives of order two or higher of the solution; then we could prove that derivatives up to order two of the solutions u_k is Cauchy in appropriate space, or argue via Arzela-Ascoli theorem that derivatives of u_k up to second order are equicontinuous on any compact subset of U. Finding and proving such appropriate derivative estimates will be a main focus.

Such estimates can be proved relatively easily for solutions to (5.1) when $f \equiv 0$ in U, namely, for harmonic functions. See Theorem 5.8 below. The precise statements on the convergence of solutions to (5.1) are given in Theorems 5.10 and 5.20 below.

The equicontinuity of the second derivatives of solutions of (5.1) for general f requires some control on the modulus of continuity of f and can be developed using potential representation. There is a theory, called the Schauder theory, that generalizes such estimates to solutions of elliptic equations with Hölder continuous variable coefficients. We may only have time to touch on some rudimentary aspect of this theory and leave the full development of this theory to a subsequent course.

Our main immediate focus in this course will be to find conditions on U and g such that we have a reasonably complete result on the solvability of (5.1) for the case $f \equiv 0$. Although the convergence result in Theorem 5.20 looks like a plausible approach, a complete result would require the solvability of (5.1) for a dense set of data. This can be provided by Poincaré's method of balayage. An alternative approach is to use Perron's method which involves the concept of subharmonic functions.

In order to develop the derivatives estimates for harmonic functions in Theorem 5.8, we first discuss Green's representation and some of its consequences, including the regularity properties of harmonic functions, the concept of Green's functions, and the construction of Green's function on round balls and half spaces in any dimension.

5.1 Elementary Examples of Harmonic Functions

First let's work out some sample solutions to $\Delta u = 0$.

(i) The real and imaginary parts of complex analytic functions on domains in $\mathbb{C}$ are harmonic functions.

(ii) Polynomial solutions of degree 2 or less: $\sum_{i,j=1}^{n} a_{ij}x_ix_j + \sum_{i=1}^{n} b_ix_i + c$ satisfies $\Delta u = 0$ iff $\sum_{i=1}^{n} a_{ii} = 0$. This is a vector space of dimension $\frac{n(n+1)}{2} + n$. In fact, using an essentially linear algebra argument, one can see that the linear map $\Delta : \mathscr{P}_k \mapsto \mathscr{P}_{k-2}$ is onto and has non-trivial kernel whose dimension is $\dim \mathscr{P}_k - \dim \mathscr{P}_{k-2} = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}$. Here $\mathscr{P}_k$ is the space of homogeneous polynomials of degree k in $\mathbb{R}^n$. Polynomials $p(x)$ in $\mathscr{P}_k$ satisfying $\Delta p(x) = 0$ are called homogeneous harmonic polynomials of degree k.

(iii) Separable solutions of the form $\Phi(x)\Psi(y)$ on $\mathbb{R}^2$, where $x, y \in \mathbb{R}$ and for some constant λ, $\Phi''(x) + \lambda\Phi(x) = 0$ and $\Psi''(y) - \lambda\Psi(y) = 0$. When $\lambda = \xi^2$, we find solutions of the form $e^{ix\xi \pm |\xi| y}$. This also works in higher dimensions, with $x \in \mathbb{R}^k$, $\xi = (\xi_1, \cdots, \xi_k)$ replacing x and ξ above, and $x \cdot \xi$ replacing $x\xi$.

(iv) Superpositions of known solutions. In the case of (iii) above, we can form

$$\sum \left(A(\xi)e^{ix\cdot\xi+|\xi|y} + B(\xi)e^{ix\cdot\xi-|\xi|y} \right).$$

When we would like to form superposition with infinitely many terms, in particular when allowing the parameter $\xi \to \infty$, we should be concerned about the exponential growth in $|\xi|$ of $e^{|\xi|y}$ for $y > 0$ (and that of $e^{-|\xi|y}$ for $y < 0$). If we consider constructing solutions to (5.3) on $\mathbb{R}^n_+ = \{(x, y) \in \mathbb{R}^n : y > 0\}$, then it's reasonable to work with $\int_{\mathbb{R}^{n-1}} B(\xi)e^{ix\cdot\xi-|\xi|y}d\xi$, and hope to use $B(\xi)$ to achieve the desired boundary value. Formally $B(\xi)$ should be chosen so that $\int_{\mathbb{R}^{n-1}} B(\xi)e^{ix\cdot\xi}d\xi = g(x)$ for $x \in \mathbb{R}^{n-1}$. Based on knowledge of Fourier transforms, $B(\xi) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} g(z)e^{-iz\cdot\xi}\, dz$. Thus we can formally represent the solution in terms of g as

$$\begin{aligned} &(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}^{n-1}} g(z)e^{-iz\cdot\xi}\, dz \right) e^{ix\cdot\xi-|\xi|y}d\xi \\ =&(2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} g(z) \left(\int_{\mathbb{R}^{n-1}} e^{i(x-z)\cdot\xi-|\xi|y}d\xi \right) dz \\ =&\int_{\mathbb{R}^{n-1}} g(z)P(x-z, y)\, dz \end{aligned}$$

where $P(x - z, y) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} e^{i(x-z)\cdot\xi-|\xi|y}d\xi$.

When $n = 2$, we find easily $P(x - z, y) = \frac{y}{\pi(y^2+|x-z|^2)}$; while for $n > 2$, it would take some effort by Fourier transform to find an explicit expression for $P(x-z, y)$; but we will encounter and find $P(x-z, y)$ by another method below.

(v) Separable solutions of the form $\Phi(r)\Psi(\theta)$ in the sector $\Sigma_{\theta_0} = \{z \in \mathbb{C} : 0 < \arg(z) < \theta_0\}$ of $\mathbb{C}$, where, for some constant λ,

$$\begin{cases} \Psi''(\theta) + \lambda\Psi(\theta) = 0, \quad 0 < \theta < \theta_0 \\ \Psi(0) = \Psi(\theta_0) = 0, \end{cases} \tag{5.6}$$

and

$$r^2\Phi''(r) + r\Phi'(r) - \lambda\Phi(r) = 0 \quad \text{for } r > 0. \tag{5.7}$$

Here we have used the fact that the Laplace operator $\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$ in polar coordinates in dimension 2 and have also imposed the homogeneous Dirichlet boundary condition on $\partial\Sigma_{\theta_0}$. We know that (5.6) has non-trivial solutions only when $\lambda = \left(\frac{k\pi}{\theta_0}\right)^2$ for $k \in \mathbb{N}$, with solutions being scalar multiples of $\Psi_k(\theta) = \sin\left(\frac{k\pi\theta}{\theta_0}\right)$. Setting $\lambda_1 = \left(\frac{\pi}{\theta_0}\right)^2$, we see that the corresponding solutions to (5.7) are $\Phi_k(r) = r^{\pm k\sqrt{\lambda_1}}$. Thus both $r^{k\sqrt{\lambda_1}}\sin\left(\frac{k\pi\theta}{\theta_0}\right)$ and $r^{-k\sqrt{\lambda_1}}\sin\left(\frac{k\pi\theta}{\theta_0}\right)$ are harmonic functions in Σ_{θ_0} with vanishing boundary value on $\partial\Sigma_{\theta_0}$, which are non-trivial and exhibit some growth either at ∞ or near 0

This approach generalizes to higher dimensions as well. When θ runs over the entire $\mathbb{S}^{n-1}$, there will be no boundary condition in the generalization to (5.6), the corresponding solutions $\Psi(\theta)$ to $\Delta_\theta\Psi(\theta) + \lambda\Psi(\theta) = 0$ on $\mathbb{S}^{n-1}$ will be called (surface) spherical harmonics. Harmonic polynomials in (ii) appear in this approach, for, if $p(x) \in \mathscr{P}_k$ is a degree k harmonic polynomial, then $p(x) = |x|^k\Psi(\theta)$, with $\theta = \frac{x}{|x|}$, for some $\Psi(\theta)$ on $\mathbb{S}^{n-1}$, and $\Delta p(x) = |x|^{k-2}\left[k(k+n-2)\Psi(\theta) + \Delta_\theta\Psi(\theta)\right]$, which implies that $k(k+n-2)\Psi(\theta) + \Delta_\theta\Psi(\theta) = 0$; the converse direction also holds.

Next we state the solvability of (5.2) and (5.4) when f is a polynomial, and U is a round ball in the case of (5.4). This is based on (ii) above. Consider the map $\mathscr{P}_{\le k} \mapsto \mathscr{P}_{\le k}$: $p(x) \mapsto \Delta\left[(|x|^2-1)p(x)\right]$, where $\mathscr{P}_{\le k}$ is the space of polynomials of degree $\le k$. We claim that this map is an isomorphism. First, it has a trivial kernel, for, if $\Delta\left[(|x|^2-1)p(x)\right] = 0$, then $u(x) := (|x|^2-1)p(x)$ is a harmonic function on $\mathbb{R}^n$ with $u(x) = 0$ on $|x| = 1$, and by the maximum principle, $u(x) \equiv 0$ for all $x \in B_1(0)$, so $p(x) \equiv 0$ for all $x \in B_1(0)$; but $p(x)$ is homogeneous of degree k, so $p(x) \equiv 0$ on $\mathbb{R}^n$. Since $\mathscr{P}_{\le k}$ is finite dimensional, it follows that the above map is an isomorphism, and (5.4) has a (polynomial) solution when f is a polynomial and U is a round ball in the case of (5.4)

It further follows from the above that $|x|^2\mathscr{P}_{k-2}$ is a subspace of $\mathscr{P}_k$ and $\Delta : |x|^2\mathscr{P}_{k-2} \mapsto \mathscr{P}_{k-2}$ is an isomorphism. Take any $q \in \mathscr{P}_{k-2}$, it follows from the solution of (5.4) described above that there is some $p(x) \in \mathscr{P}_{\le k-2}$ such that $\Delta\left[(|x|^2-1)p(x)\right] = q(x)$. Writing out $p(x) = \sum_{j=0}^{k-2} p_j(x)$, where $p_j(x) \in \mathscr{P}_j$, we see that

$$\Delta\left[(|x|^2-1)p(x)\right] = \sum_{j=0}^{k-2}\left\{\Delta\left[|x|^2 p_j(x)\right] - \Delta p_j(x)\right\} = q(x).$$

$\Delta\left[|x|^2 p_j(x)\right] \in \mathscr{P}_j$ and $\Delta p_j(x) \in \mathscr{P}_{j-2}$ for $j = 0, \cdots, k-2$, so $\Delta\left[|x|^2 p_{k-2}(x)\right] = q(x)$ and

$$\begin{aligned} \Delta\left[|x|^2 p_{k-2-l}(x)\right] &= 0 \quad \text{for all odd } 1 \le l \le k-2, \text{ and} \\ \Delta\left[|x|^2 p_{k-2-l}(x)\right] - \Delta\left[p_{k-l}(x)\right] &= 0 \quad \text{for all even } 2 \le l \le k-2. \end{aligned}$$

The solvability of $\Delta\left[|x|^2 p_{k-2}(x)\right] = q(x)$ shows that $\Delta : |x|^2\mathscr{P}_{k-2} \mapsto \mathscr{P}_{k-2}$ is onto. Since $\mathscr{P}_{k-2}$ is finite dimensional, this map must be injective as well, thus it is an isomorphism. Furthermore, we also have from the above that $p_{k-2-l}(x) = 0$ for all odd $1 \le l \le k-2$, and that $h_{k-l} := p_{k-l}(x) - |x|^2 p_{k-2-l}(x) \in \mathscr{P}_{k-l}$ is harmonic on $\mathbb{R}^n$, for all even $2 \le l \le k-2$. This establishes the direct sum decomposition $\mathscr{P}_k = |x|^2\mathscr{P}_{k-2} \oplus \mathscr{H}_k$, where $\mathscr{H}_k = \mathrm{Ker}(\Delta)$ is the space of homogeneous harmonic polynomials of degree k. An iteration of this decomposition further gives $\mathscr{P}_k = \mathscr{H}_k \oplus |x|^2\mathscr{H}_{k-2} \oplus |x|^4\mathscr{H}_{k-4} + \cdots$.

Exercises

Exercise 5.1.1. Suppose that $u \in C^k(B_R(0))$ is harmonic in $B_R(0)$. Let $T_l(u)(x)$ be the degree l Taylor expansion of u at $x = 0$, for $l = 0, 1, \cdots, k$. Show that $T_l(u)(x) \in \mathscr{H}_l$.

Exercise 5.1.2. Verify that if $h \in \mathscr{H}_k$ and $l \in \mathbb{N}$, then

$$\Delta\left[|x|^{2l} h(x)\right] = 2l(2l + 2k + n - 2)|x|^{2l-2} h(x).$$

Use this and the decomposition $\mathscr{P}_k = \mathscr{H}_k \oplus |x|^2\mathscr{H}_{k-2} \oplus |x|^4\mathscr{H}_{k-4} + \cdots$ to produce an algorithm to compute $p(x) \in \mathscr{P}_{k+2}$ solving $\Delta p(x) = q(x) \in \mathscr{P}_k$.

Exercise 5.1.3. Note that $\Delta : \mathscr{P}_k \mapsto \mathscr{P}_{k-2}$ is nilpotent; in fact, $\Delta^m = 0$ for $m > k/2$. Use the information in the precious exercise to produce a basis in $\mathscr{H}_k$ with respect to which the representation of Δ is a Jordan canonical form. Also prove that $|x|^2\Delta : \mathscr{P}_k \mapsto \mathscr{P}_k$ is diagonalized.

5.2 Maximum Principle for subharmonic functions and Applications

To further develop properties of solutions to (5.2), (5.3) and (5.4), we will use the Green's identities for the Laplace operator. We first state the Green's identity.

Proposition 5.1. *Suppose that U is a bounded domain with piecewise C^1 boundary, and $u, v \in C^2(\bar{U})$. Then*

$$\int_U [u(x)\Delta v(x)]\, dx = -\int_U \nabla u(x) \cdot \nabla v(x)\, dx + \int_{\partial U} \left[u(x)\frac{\partial v(x)}{\partial n(x)}\right] d\sigma(x), \tag{5.8}$$

and

$$\int_U [u(x)\Delta v(x) - v(x)\Delta u(x)]\, dx = \int_{\partial U} \left[u(x)\frac{\partial v(x)}{\partial n(x)} - v(x)\frac{\partial u(x)}{\partial n(x)}\right] d\sigma(x), \tag{5.9}$$

here $n(x)$ denotes the unit exterior normal to ∂U at $x \in \partial U$.

(5.8) follows from the divergence theorem by noting that

$$u(x)\Delta v(x) + \nabla u(x) \cdot \nabla v(x) = \nabla [u(x)\nabla v(x)];$$

while (5.9) from taking the difference between (5.8) and a corresponding version by interchanging u and v.

Remark 5.1. (5.9) continues to hold when u and v satisfy the weaker condition that $u, v \in C^1(\bar{U}) \cap C^2(U)$ as long as Δu and Δv are in $L^1(U)$. Also, we often only need to apply (5.9) on subdomains $V \subset\subset U$ with piecewise C^1 boundary, such as balls $B_r(x_0) \subset\subset U$.

We now proceed to prove the maximum principle and strong maximum principle for the so called subharmonic (superhamonic) functions, using (5.9).

Definition. A $C^2(U)$ function is called subharmonic (superhamonic) in U, if $\Delta u \geq (\leq) 0$ in U.

Proposition 5.2. *(i). If U is a bounded domain and $u \in C^2(U) \cap C(\overline{U})$ is subharmonic (superhamonic) in U, then*

$$\max_{\overline{U}} u = \max_{\partial U} u \left(\min_{\overline{U}} u = \min_{\partial U} u \text{ for the superharmoinc case} \right).$$

Furthermore, if U is connected, u can not attain $\max_{\overline{U}} u$ ($\min_{\overline{U}} u$ for the superharmonic case) in U unless u is a constant in U.

(ii). Given $u \in C^2(U) \cap C(\overline{U})$*, then*

$$\max_{\overline{U}} |u| \le \max_{\partial U} |u| + C \sup_{U} |\Delta u|,$$

where C *is a constant depending only on the diameter of* U*.*

We proved the maximum principle for harmonic functions earlier by more elementary means. This is a stronger form of the maximum principle, called the strong maximum principle, and has important applications. A first application is

Corollary 5.3. *Suppose that* U *is connected and* $u \in C^2(U) \cap C(\overline{U})$ *satisfies* $\Delta u \le 0$ *in* U *and* $u \ge 0$ *but not* $\equiv 0$ *on* ∂U*, then* $u(x) > 0$ *in* U*.*

We first prove the mean value property of (sub/super) harmonic functions using (5.9).

Proposition 5.4. *Suppose that* $u \in C^2(U)$ *is subharmonic in* $B_R(x_0) \subset U$*. Then for any* $0 < r < R$*,*

$$u(x_0) \le \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(x_0 + r\boldsymbol{\omega}) d\boldsymbol{\omega} = \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} u(x)\ d\sigma(x),$$

and

$$u(x_0) \le \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(x)\, dx.$$

If $u \in C^2(U)$ *is harmonic in* $B_R(x_0) \subset U$*, then the above two inequalities are equalities.*

Proof. Suppose that $u \in C^2(U)$ is subharmonic in $B_R(x_0) \subset U$, then applying (5.9) to u and $v \equiv 1$ on $B_r(x_0)$ for any $0 < r < R$, we obtain

$$\begin{aligned} 0 &\le \int_{\partial B_r(x_0)} \frac{\partial u(x)}{\partial n(x)}\ d\sigma(x) \\ &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u(x_0 + r\boldsymbol{\omega})}{\partial r} d\boldsymbol{\omega} \qquad \text{using } d\sigma(x) = r^{n-1} d\boldsymbol{\omega} \text{ in } x = x_0 + r\boldsymbol{\omega} \\ &= r^{n-1} \frac{\partial}{\partial r} \left(\int_{\partial B_1(0)} u(x_0 + r\boldsymbol{\omega}) d\boldsymbol{\omega} \right), \end{aligned}$$

from which it follows that $\int_{\partial B_1(0)} u(x_0 + r\boldsymbol{\omega}) d\boldsymbol{\omega}$ is a non-decreasing function of r for $0 < r < R$. Since $\int_{\partial B_1(0)} u(x_0 + r\boldsymbol{\omega}) d\boldsymbol{\omega} \to |\partial B_1(0)| u(x_0)$ as $r \to 0$, we conclude that $\int_{\partial B_1(0)} u(x_0 + r\boldsymbol{\omega}) d\boldsymbol{\omega} \ge |\partial B_1(0)| u(x_0)$ for all $0 < r < R$, where $|\partial B_1(0)|$ stands for the volume of $\partial B_1(0)$. This proves the first equality.

Next, for any $0 < r < R$, we apply the first inequality for $0 < s \leq r$, multiply both sides by s^{n-1}, and integrate in s from 0 to r, to get

$$\begin{aligned}
&\int_{B_r(x_0)} u(x)\,dx \\
=&\int_0^r \int_{\partial B_1(0)} u(x_0 + s\boldsymbol{\omega})s^{n-1}\,d\boldsymbol{\omega}ds \\
\geq&\int_0^r |\partial B_1(0)|u(x_0)s^{n-1}\,ds \\
=&\frac{|\partial B_1(0)|}{n}r^n u(x_0) \\
=&|B_r(x_0)|u(x_0) \quad \text{using } |B_r(x_0)| = \frac{|\partial B_1(0)|}{n}r^n.
\end{aligned}$$

This concludes the proof. □

Proof of Proposition 5.2. For (i), it suffices to prove that if there exists $x_0 \in U$ such that $u(x_0) = \max_{\bar{U}} u$, then the set $\{x \in U : u(x) = \max_{\bar{U}} u\}$ is open in U. For any $x \in U$ with $u(x) = \max_{\bar{U}} u = M$, let $B_r(x) \subset\subset U$, then the mean value property of u in $B_r(x)$ implies that

$$M = u(x) \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} u(y)\,dy \leq \frac{1}{|B_r(x)|}\int_{B_r(x)} M\,dy = M,$$

from which it follows that $u(y) = M$ for all $y \in B_r(x)$.

For (ii) set $v = \max_{\partial U}|u| + \frac{1}{2}(d^2 - x_1^2)\sup_U|\Delta u|$, where we may assume that $U \subset \{x : 0 \leq x_1 \leq d\}$, then $\Delta(v \pm u) \leq 0$ in U and $(v \pm u) \geq 0$ on ∂U, then apply (i). □

An immediate corollary of the maximum principle is the uniqueness of the Dirichlet problem.

Corollary 5.5. *Suppose that U is a bounded domain and that $u_1, u_2 \in C^2(U) \cap C(\bar{U})$ are both solutions to*

$$\begin{cases} \Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

Then $u_1 \equiv u_2$ in U.

Remark 5.2. A corollary of the uniqueness of the Dirichlet problem is that the following Cauchy problem in U

$$\begin{cases} \Delta u = f & \text{in } U, \\ u = g & \text{on } \partial U, \\ \dfrac{\partial u}{\partial n} = h & \text{on } \partial U. \end{cases} \tag{5.10}$$

is not well-posed, as a solution would be determined by f and g alone and one can not prescribe h arbitrarily.

Exercises

Exercise 5.2.1. Suppose that U is a bounded domain in $\mathbb{R}^n$ such that $-x \in U$ whenever $x \in U$, and that $g \in C(\partial U)$ satisfies $g(-x) = g(x)$. Prove that any solution $u \in C^2(U) \cap C(\overline{U})$ to (5.3) satisfies $u(-x) = u(x)$.

Exercise 5.2.2. Let $\Sigma_{\theta_0} = \{z \in \mathbb{C} : 0 < \arg(z) < \theta_0\}$ of $\mathbb{C}$. Suppose that $u \in C(\overline{\Sigma_{\theta_0}}) \cap C^2(\Sigma_{\theta_0})$ is harmonic in Σ_{θ_0} and satisfies, for $0 < \gamma < \frac{\pi}{\theta_0}$ and some $M, C > 0$,

$$u(x) \leq M \text{ on } \partial\Sigma_{\theta_0}, \text{ and } u(x) \leq |x|^\gamma \text{ for } x \in \Sigma_{\theta_0}.$$

Then $u(x) \leq M$ for $x \in \Sigma_{\theta_0}$. (Hint: For sufficiently small $\epsilon > 0$ and $\gamma < \gamma' < \frac{\pi-\epsilon}{\theta_0}$, apply the maximum principle to the harmonic function $u(x) - \epsilon|x|^{\gamma'}\sin(\gamma'\theta + \epsilon)$ in Σ_{θ_0} on the intersection of Σ_{θ_0} with a sufficiently large disc, where θ is the polar angle of x.)

Exercise 5.2.3. Suppose that $u \in C(\overline{\Sigma_{\theta_0}}) \cap C^2(\Sigma_{\theta_0})$ is harmonic in Σ_{θ_0} and satisfies for some M

$$u(x) \leq M \text{ on } \partial\Sigma_{\theta_0}, \text{ and } \limsup_{|x|\to\infty} \frac{u(x)}{|x|^{\frac{\pi}{\theta_0}}} \leq 0 \text{ uniformly for } 0 \leq \theta \leq \theta_0.$$

Then $u(x) \leq M$ for $x \in \Sigma_{\theta_0}$. (Hint: For sufficiently small $\epsilon > 0$, show that

$$M_\epsilon := \sup_{\theta=\theta_0/2, r\geq 0} \left\{ u(r\cos\theta, r\sin\theta) - \epsilon r^{\frac{\pi}{\theta_0}} \sin\left(\frac{\pi\theta}{\theta_0}\right) \right\}$$

is attained at some $r_\epsilon \geq 0$, and use the method of the previous problem to show that $u(x) \leq \max\{M, M_\epsilon\}$ in Σ_{θ_0}; then show that $M_\epsilon \leq M$ using the strong maximum principle.)

Exercise 5.2.4. Let $S_h = \{x = (x_1, \ldots, x_n) : 0 < x_n < h, (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}$. Suppose that $u \in C(\overline{S_h}) \cap C^2(S_h)$ is harmonic in S_h and satisfies for some M

$$u(x) \leq M \text{ on } \partial S_h, \text{ and } \limsup_{|x|\to\infty} \frac{u(x)}{\exp\{\frac{\pi}{h}|x|\}} \leq 0 \text{ uniformly for } 0 \leq x_n \leq h.$$

Then $u(x) \leq M$ for $x \in S_h$.

5.3 Fundamental Solution of the Laplace Operator

A function $\Phi(x)$ is called a fundamental solution of the Laplace operator if it satisfies

$$-\Delta\Phi(x) = \delta(x), \tag{5.11}$$

where $\delta(x)$ refers to the Dirac delta distribution at 0.

A more formal way to define (5.11) is to make sense of it in the distributional sense: $\Phi(x)$ is integrable on any compact subset of $\mathbb{R}^n$, and

$$\eta(0) = -\int_{\mathbb{R}^n} \Phi(x)\Delta\eta(x)\,dx, \quad \text{for any test function } \eta \in C_c^2(\mathbb{R}^n). \tag{5.12}$$

Here is a heuristic discussion on why we are interested in a fundamental solution. When discussing Duhamel's principle, we already saw that in order to solve a non-homogeneous linear PDE, it is helpful to regard the source term as a superposition of locally constant functions which equal 0 except on (vanishingly) small boxes, and construct a solution to the PDE with this localized source term. Namely, for any x_a and a family of functions $f_\epsilon(x)$ whose support is shrinking to x_a as $\epsilon \to 0+$, say, supported in the ball $B(x_a, \epsilon)$, and with $\int f_\epsilon(x)\,dx = 1$, we look for a solution to

$$-\Delta_x E_\epsilon(x; x_a) = f_\epsilon(x)$$

and examine its limit $E(x; x_a) = \lim_{\epsilon\to 0+} E_\epsilon(x; x_a)$. Since $\Delta E_\epsilon(x; x_a) = 0$ outside $B(x_a, \epsilon)$, so we expect $\Delta_x E(x; x_a) = 0$ for $x \neq x_a$. The family $f_\epsilon(x)$ in the limit of $\epsilon \to 0+$ is the unit source at x_a, namely, $\delta(x - x_a)$, in the following sense: integrate against a test function $\eta \in C_c^\infty(\mathbb{R}^n)$:

$$\eta(x_a) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} \eta(x) f_\epsilon(x)\,dx = \int_{\mathbb{R}^n} \eta(x)\delta(x - x_a)\,dx\,(\text{definition}).$$

Suppose that we can apply (5.9) to $\eta(x)$ and $E_\epsilon(x; x_a)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta(x) f_\epsilon(x)\,dx \\
&= -\int_{\mathbb{R}^n} \eta(x)\Delta_x E_\epsilon(x; x_a)\,dx \\
&= -\int_{\mathbb{R}^n} [\Delta_x \eta(x)]\, E_\epsilon(x; x_a)\,dx,
\end{aligned}$$

and that $E_\epsilon(x; x_a)$ has a limit $E(x; x_a)$ as $\epsilon \to 0+$ in the sense that

$$\lim_{\epsilon\to 0+} \int_{\mathbb{R}^n} [\Delta_x \eta(x)]\, E_\epsilon(x; x_a)\,dx = \int_{\mathbb{R}^n} [\Delta_x \eta(x)]\, E(x; x_a)\,dx,$$

then

$$\eta(x_a) = -\int_{\mathbb{R}^n} [\Delta_x \eta(x)]\, E(x; x_a)\, dx.$$

We are choosing $\Phi(x)$ to be a fundamental solution of $-\Delta$, instead of Δ, because it turns out that such a $\Phi(x)$ would have a limiting behavior of $\Phi(x) \to +\infty$ as $x \to 0$. Because Δ is a constant coefficient differential operator, $\Phi(x-y)$ would satisfy

$$-\Delta_x \Phi(x-y) = \delta(x-y),$$

so we expect $E(x;\xi) = \Phi(x-\xi)$ in such a case, and for a function $f(x)$ under appropriate conditions,

$$-\Delta_x \left(\int f(y)\Phi(x-y)\, dy \right) = \int f(y)\delta(x-y)\, dy = f(x), \tag{5.13}$$

at least formally at this point. In other words, integration against $\Phi(x-y)$ provides a right inverse operator for $-\Delta$. Of course the validity of this needs some regularity assumptions on f and needs to be carefully justified.

One way to make sense of (5.13) is to take $f \in C_c^2(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} f(y)\Phi(x-y)\, dy = \int_{\mathbb{R}^n} f(x-y)\Phi(y)\, dy$, so

$$\begin{aligned} &-\Delta_x \left(\int_{\mathbb{R}^n} f(y)\Phi(x-y)\, dy \right) \\ =& -\Delta_x \left(\int_{\mathbb{R}^n} f(x-y)\Phi(y)\, dy \right) \\ =& -\int_{\mathbb{R}^n} \Delta_x f(x-y)\Phi(y)\, dy \\ =& -\int_{\mathbb{R}^n} \Delta_y f(x-y)\Phi(y)\, dy = f(x), \end{aligned}$$

as derived above.

Note that if $\Phi(x)$ is a fundamental solution and $h(x)$ is a smooth solution on $\mathbb{R}^n$ to $\Delta h(x) = 0$, then $\Phi(x) + h(x)$ is also a fundamental solution. So some kind of normalizing condition would be picked to narrow down the choice for a fundamental solution. Also, Φ can't be a smooth harmonic function across 0 itself, as (5.11) implies that

$$\Delta\Phi(x) = 0 \quad \text{for } x \neq 0, \tag{5.14}$$

and

$$1 = -\int_{\partial B_r(0)} \frac{\partial \Phi(x)}{\partial n(x)}\, d\sigma(x) \tag{5.15}$$

for all $r > 0$, and a smooth harmonic function across 0 can't satisfy (5.15). (5.15) follows from (5.9), with $u(x) = \Phi(x)$ and $v \in C_c^2(\mathbb{R}^n)$ such that $v(x) \equiv 1$ over $B_r(0)$,

$$\begin{aligned} 1 &= -\int_{\mathbb{R}^n} \Delta_x v(x)\Phi(x)\, dx \\ &= -\int_{|x|\geq r} \Delta_x v(x)\Phi(x)\, dx \\ &= -\int_{\partial B_r(0)} \frac{\partial \Phi(x)}{\partial n(x)}\, d\sigma(x). \end{aligned}$$

Here we assumed that $\Phi(x) \in C^2(B_r(0)^c)$ so that the Green's identity (5.9) can be applied—this can be justified once we have developed some properties, but we are doing only a heuristic derivation to under properties which characterize a fundamental solution. Changing variable of integration $x = r\boldsymbol{\omega}$, $\boldsymbol{\omega} \in \partial B_1(0)$, the integral can be simplified as

$$\begin{aligned} 1 &= -\int_{\partial B_r(0)} \frac{\partial \Phi(x)}{\partial n(x)}\, d\sigma(x) \\ &= -r^{n-1}\int_{\partial B_1(0)} \frac{\partial \Phi(r\boldsymbol{\omega})}{\partial r} d\boldsymbol{\omega} = -r^{n-1}\partial_r\left(\int_{\partial B_1(0)} \Phi(r\boldsymbol{\omega})d\boldsymbol{\omega}\right), \end{aligned} \tag{5.16}$$

for all $r > 0$. In particular, this implies that $\partial_r\left(\int_{\partial B_1(0)} \Phi(r\boldsymbol{\omega})d\boldsymbol{\omega}\right) = -r^{1-n}$ for all $r > 0$, therefore

$$\int_{\partial B_1(0)} \Phi(r\boldsymbol{\omega})d\boldsymbol{\omega} = \begin{cases} \frac{r^{2-n}}{n-2} + c & \text{when } n \geq 3, \\ \ln\frac{1}{r} + c & \text{when } n = 2 \end{cases} \tag{5.17}$$

for some constant c.

Since both Δ and $\delta(x)$ are rotationally invariant, it is reasonable to look for a $\Phi(x)$ which is rotationally invariant, namely $\Phi(x) = \Phi(|x|)$. Such a Φ would then satisfy

$$\Phi(x) = \begin{cases} \frac{r^{2-n}}{(n-2)|S^{n-1}|} + c & \text{when } n \geq 3, \\ \frac{1}{2\pi}\ln\frac{1}{r} + c & \text{when } n = 2 \end{cases} \tag{5.18}$$

for some constant c. Of course, one needs to verify directly that the Φ as given indeed satisfies both (5.14) and (5.15).

Remark 5.3. The characterization (5.11) of a fundamental solution in fact includes some growth control of the form $|\Phi(x)| \leq C|x|^{2-n}$ near $x = 0$, in addition to (5.14) and (5.15), as there are functions $\Phi(x)$, such as $|x|^{2-n}/[(n-2)|\mathbb{S}^{n-1}|]+x_i/|x|^n$, which satisfy (5.14) and (5.15), but not (5.12) or (5.13).

A function Φ satisfying (5.14), (5.15), and the growth bounds $|\Phi(x)| \le C|x|^{2-n}$ and $|\nabla\Phi(x)| \le C|x|^{1-n}$ would satisfy (5.12)—We will soon show that the latter bound follows from (5.14) and the first bound. This follows from

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Phi(x)\Delta\eta(x)\,dx \\
=&\lim_{\epsilon\to 0}\int_{\mathbb{R}^n\setminus B_\epsilon(0)} \Phi(x)\Delta\eta(x)\,dx \\
=&-\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(0)} \left(\Phi(x)\frac{\partial\eta(x)}{\partial r} - \eta(x)\frac{\partial\Phi(x)}{\partial r}\right) d\sigma(x) \\
=&-\lim_{\epsilon\to 0}\int_{\mathbb{S}^{n-1}} \epsilon^{n-1}\left(\Phi(\epsilon\boldsymbol{\omega})\frac{\partial\eta(\epsilon\boldsymbol{\omega})}{\partial r} - \eta(\epsilon\boldsymbol{\omega})\frac{\partial\Phi(\epsilon\boldsymbol{\omega})}{\partial r}\right) d\boldsymbol{\omega}.
\end{aligned}$$

Using $|\Phi(x)| \le C|x|^{2-n}$ and the boundedness of $|\frac{\partial\eta(\epsilon\boldsymbol{\omega})}{\partial r}|$ for $(\epsilon,\boldsymbol{\omega}) \in (0,1]\times\mathbb{S}^{n-1}$, we see that

$$\lim_{\epsilon\to 0}\int_{\mathbb{S}^{n-1}} \epsilon^{n-1}\Phi(\epsilon\boldsymbol{\omega})\frac{\partial\eta(\epsilon\boldsymbol{\omega})}{\partial r}d\boldsymbol{\omega} = 0.$$

Using (5.15), the bound $|\nabla\Phi(x)| \le C|x|^{1-n}$, and the continuity of $\eta(x)$ at $x=0$, we see that

$$\lim_{\epsilon\to 0}\int_{\mathbb{S}^{n-1}} \epsilon^{n-1}\eta(\epsilon\boldsymbol{\omega})\frac{\partial\Phi(\epsilon\boldsymbol{\omega})}{\partial r}d\boldsymbol{\omega} = -\eta(0).$$

In the case here we can also use a scaling argument to get some scaling property of a fundamental solution. Replacing $\eta(x)$ in (5.12) by $\eta(\lambda x)$, where $\lambda > 0$ is a parameter, we find that

$$\eta(0) = -\int_{\mathbb{R}^n} \lambda^2\Phi(x)\Delta\eta(\lambda x)\,dx = -\int_{\mathbb{R}^n} \lambda^{2-n}\Phi(\frac{y}{\lambda})\Delta\eta(y)\,dy.$$

This argument is essentially a reflection of the property that "$\lambda^n\delta(\lambda x) = \delta(x)$". So $\lambda^{2-n}\Phi(\frac{x}{\lambda})$ would also be a fundamental solution for any $\lambda > 0$ and we could try to find a fundamental solution which is invariant under this transformation, namely $\lambda^{2-n}\Phi(\frac{x}{\lambda}) = \Phi(x)$ for any $\lambda > 0$. Such a Φ would satisfy $\Phi(x) = |x|^{2-n}\Phi(\boldsymbol{\omega})$, where $\boldsymbol{\omega} = x/|x| \in \mathbb{S}^{n-1}$. This is possible when $n \ge 3$ as seen above. In general what we can say is that

$$\int_{\mathbb{R}^n} \left[\lambda^{2-n}\Phi(\frac{y}{\lambda}) - \Phi(y)\right]\Delta\eta(y)\,dy = 0 \quad \text{for any test function } \eta \in C_c^2(\mathbb{R}^n),$$

from which we see that $\lambda^{2-n}\Phi(\frac{y}{\lambda}) - \Phi(y)$ is a harmonic function in $\mathbb{R}^n$ in the distributional sense.

We will see later that a harmonic function in the distributional sense is in fact smooth. So $\lambda^{2-n}\Phi(\frac{y}{\lambda}) - \Phi(y)$ would be an entire harmonic function in $\mathbb{R}^n$, and if we impose some normalizing condition as $x \to \infty$, we would see that $\lambda^{2-n}\Phi(\frac{y}{\lambda}) - \Phi(y)$ would be a constant, which may depend on λ, by the Liouville theorem for entire harmonic function in $\mathbb{R}^n$, soon to be discussed. We can try to see whether we can find a solution for which this constant is 0 independent of λ. It works out when $n \geq 3$, as seen above; when $n = 2$, the fundamental solution we have found $\Phi(x) = (2\pi)^{-1} \ln \frac{1}{|x|}$ satisfies $\Phi(\frac{x}{\lambda}) - \Phi(x) = (2\pi)^{-1} \ln \lambda$.

Choosing test functions η in (5.12) such that support of $\eta \cap \{0\} = \emptyset$, we see that Φ is a harmonic function in the distributional sense in $\mathbb{R}^n \setminus \{0\}$, so should be a smooth harmonic function there. This relates (5.12) to (5.14).

Remark 5.4. Another way to find a fundamental solution is to find general solutions to (5.14) and then choose the one(s) that also satisfies (5.15) and the growth control $|\Phi(x)| \leq C|x|^{2-n}$ near $x = 0$.

We can find separable solutions to $\Delta\Phi(x) = 0$ for $x \neq 0$ by setting $\Phi(r\boldsymbol{\omega}) = \phi(r)\psi(\boldsymbol{\omega})$, then

$$r^2\phi''(r) + (n-1)r\phi'(r) - \lambda\phi(r) = 0 \quad \text{for some constant } \lambda \text{ and all } r > 0, \tag{5.19}$$

and

$$\Delta_{\boldsymbol{\omega}}\psi(\boldsymbol{\omega}) + \lambda\psi(\boldsymbol{\omega}) = 0 \quad \text{for all } \boldsymbol{\omega} \in \mathbb{S}^{n-1}. \tag{5.20}$$

Solutions to (5.19) are spanned by solutions of the form $\phi(r) = r^\beta$, where $\beta(\beta + n - 2) - \lambda = 0$—when this indicial equation has only simple zeros; otherwise, a $\ln r$ factor needs to be put in.

Although we know very little at this stage about the existence of non-trivial solutions to (5.20), we can see the following two facts easily:

$$\text{(5.20) has non-trivial solution only when } \lambda \geq 0; \tag{5.21}$$

$$\text{For } \lambda \neq 0, \text{ any solution of (5.20) satisfies } \int_{\mathbb{S}^{n-1}} \psi(\boldsymbol{\omega})d\boldsymbol{\omega} = 0. \tag{5.22}$$

(5.21) follows by multiplying both sides of (5.19) by $\psi(\boldsymbol{\omega})$ and integrating over $\mathbb{S}^{n-1}$, while (5.22) follows by simply integrating both sides of (5.19) over $\mathbb{S}^{n-1}$ and using that $\Delta_{\boldsymbol{\omega}}\psi(\boldsymbol{\omega})$ is the divergence of $\nabla_{\boldsymbol{\omega}}\psi(\boldsymbol{\omega})$, so its integral over the closed manifold $\mathbb{S}^{n-1}$ is zero. Note that if $p_k(x)$ is a harmonic homogeneous polynomial of degree k, then if we define $\phi(\boldsymbol{\omega}) = p_k(\boldsymbol{\omega})$, we will have $p_k(x) = |x|^k\phi(\boldsymbol{\omega})$ and $\Delta_x p_k(x) = |x|^{k-2}[k(k+n-2)\phi(\boldsymbol{\omega}) + \Delta_{\boldsymbol{\omega}}\phi(\boldsymbol{\omega})]$, so $p_k(x)$ gives rise to a non-trivial solution $\phi(\boldsymbol{\omega})$ to $k(k+n-2)\phi(\boldsymbol{\omega}) + \Delta_{\boldsymbol{\omega}}\phi(\boldsymbol{\omega}) = 0$.

Note that for $\lambda > 0$, $\Phi(r\boldsymbol{\omega}) = \phi(r)\psi(\boldsymbol{\omega})$ wouldn't be able to satisfy $\Delta\Phi(x) = 0$ for $x \neq 0$ and $\int_{\partial B_r(0)} \frac{\partial\Phi(x)}{\partial n(x)}\, d\sigma(x) = -1$ for $r > 0$, as $\int_{\partial B_r(0)} \frac{\partial\Phi(x)}{\partial n(x)}\, d\sigma(x) = r^{n-1}\phi'(r)\int_{\mathbb{S}^{n-1}} \psi(\boldsymbol{\omega})d\boldsymbol{\omega} = 0$ for such cases. Thus Φ must include a term corresponding to $\lambda = 0$, in which case $\psi(\boldsymbol{\omega})$ would be a constant over $\mathbb{S}^{n-1}$ and we can set it to be 1. Now β satisfies $\beta(\beta + n - 2) = 0$. When $n > 2$, we have to take $\beta = 2 - n$, and when $n = 2$, we need to re-examine the solutions to $r^2\phi''(r) + r\phi'(r) = 0$ and find that we should take $\phi(r) = \ln r$.

One may want to examine whether one can add to the fundamental solution in (5.18) additional terms of the form $\phi(r)\psi(\boldsymbol{\omega})$, where ψ solves (5.20) for some $\lambda > 0$. If one chooses the positive root β from $\beta(\beta + n - 2) - \lambda = 0$, then it turns out that $\phi(r)\psi(\boldsymbol{\omega})$ would be a harmonic polynomial, so adding such a term would be admissible. If, however, one chooses the negative root from the above equation, it turns out that $\beta < 2 - n$, so one wouldn't be able to satisfy the bound $|\Phi(x)| \le C|x|^{2-n}$ near $x = 0$.

Exercises

Exercise 5.3.1. Construct a fundamental solution $\Phi(x)$ to $-\Phi''(x) = \delta(x)$ in $\mathbb{R}$.

Exercise 5.3.2. Adapt the methods of this section to construct a fundamental solution $E(x)$ to $(-\Delta + c)E(x) = \delta(x)$, where c is a constant. (Hint: Try to construct $E(x)$ as a function depending only on $|x|$, so as to reduce the construction to an appropriate solution of a (singular) linear ODE related to Bessel's equation; use an analog of (5.15) to establish an asymptotic boundary condition for $E'(|x|)$ as $|x| \to 0$.)

5.4 Initial Applications of the Fundamental solution of $-\Delta$

We will use $\Phi(x)$ to denote the (radial) fundamental solution of $-\Delta$ as determined from the previous section. We can now use $\Phi(x)$ to write down a Green's representation for the Laplace operator and discuss some initial applications.

Proposition 5.6. *Suppose that U is a bounded domain in $\mathbb{R}^n$ with piecewise C^1 boundary and $u \in C^2(\overline{U})$, then for any $x \in U$,*

$$u(x) = \iint_U -\Delta u(y)\, \Phi(x-y)\, dy + \int_{\partial U} \left[\Phi(x-y)\frac{\partial u(y)}{\partial n(y)} - u(y)\frac{\partial \Phi(x-y)}{\partial n(y)}\right] d\sigma(y). \tag{5.23}$$

If u is harmonic in U, then (5.23) holds for $\in C^1(\overline{U}) \cap C^2(U)$.

This is obtained from applying (5.9) to $u(y)$ and $\Phi(x-y)$ on $U \setminus \{B_\epsilon(x)\}$ for small $\epsilon > 0$ and sending $\epsilon \to 0$, noting that $\Delta_y \Phi(x-y) = 0$ in $U \setminus \{B_\epsilon(x)\}$ and $|\Phi(r)|r^{n-1} \to 0$, $\int_{\partial B_r(0)} \frac{\partial \Phi(r)}{\partial r} = -1$ as $r \to 0$. As a consequence (5.23), we have

Corollary 5.7. *If $u \in C^2(U)$ is harmonic, then it is smooth in U.*

Proof. For any proper subdomain V of U with C^1 boundary, we can use the Green's representation (5.23) on V to express $u(x)$, $x \in V$ as

$$\int_{\partial V} \left(\Phi(x-y) \frac{\partial u(y)}{\partial n(y)} - u(y) \frac{\partial \Phi(x-y)}{\partial n(y)} \right) d\sigma(y).$$

Since the integrand is a smooth function of $x \in V$ for $y \in \partial V$, with any of its derivatives uniformly bounded for $y \in \partial V$ as long as $x \in V$ stays away from ∂V, this shows that u is smooth in V. □

Theorem 5.8. *If $u \in C^2(B_R(x_0)) \cap C(\overline{B_R(x_0)})$ is harmonic in $B_R(x_0)$, then, for some $C = C(n) > 0$,*

$$|\nabla u(x_0)| \le \frac{C}{R} \max_{B_R(x_0)} |u|, \quad \text{and for } k > 1, \quad |\nabla^\alpha u(x_0)| \le \frac{(Ck)^k}{R^k} \max_{B_R(x_0)} |u|,$$

for all $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n = k$.

Remark 5.5. The higher derivative estimates imply the analyticity of $u(x)$. There are also versions of the gradient estimates where the factor $\max_{B_R(x_0)} |u|$ on the right hand side is replaced by $R^{-n}||u||_{L^1(B_R(x_0))}$. Our proof below will give this version.

Proof. First it suffices to reduce the proof to that for a harmonic function on a unit ball. For, if we set $v(y) = u(x_0 + Ry)$, then $v(y)$ is harmonic on $B_1(0)$, $||v||_{L^1(B_1(0))} = R^{-n}||u||_{L^1(B_R(x_0))}$, and $\nabla_y^\alpha v(y)|_{y=0} = R^{|\alpha|} \nabla_x^\alpha u(x)|_{x=x_0}$.

Next we will derive a variant of the Green's representation without involving ∇v in the boundary integral. Choose a smooth cut-off function η such that $\eta(y) \equiv 1$ for $|y| \le 1/2$ and is supported in $B_1(0)$. For any z with $|z| < 1/4$ and $0 < \epsilon < 1/4$, we apply the Green's identity to $v(y)$ and $\Phi(y-z)\eta(y)$ over $B_1(0) \setminus B_\epsilon(z)$ to obtain

$$\begin{aligned} &\int_{B_1(0) \setminus B_\epsilon(z)} v(y) \Delta_y \left(\Phi(y-z)\eta(y) \right) dy \\ = &- \int_{\mathbb{S}^{n-1}} \epsilon^{n-1} \left(v(z+\epsilon\boldsymbol{\omega}) \frac{\partial \left[\Phi(r\boldsymbol{\omega}) \eta(z+r\boldsymbol{\omega}) \right]}{\partial r} \Big|_{r=\epsilon} - \Phi(\epsilon\boldsymbol{\omega})\eta(z+\epsilon\boldsymbol{\omega}) \frac{\partial v(z+\epsilon\boldsymbol{\omega})}{\partial r} \Big|_{r=\epsilon} \right) d\boldsymbol{\omega}, \end{aligned}$$

here we have used that $\eta(y) = 0$ in a neighborhood of $\partial B_1(0)$ and expressed the integrals on $\partial B_\epsilon(z)$ using spherical polar coordinates $y = z + \epsilon\boldsymbol{\omega}$ with $\boldsymbol{\omega} \in \mathbb{S}^{n-1}$.

Since

$$\Delta_y \left[\Phi(y-z)\eta(y)\right] = 2\nabla_y \Phi(y-z) \cdot \nabla_y \eta(y) + \Phi(y-z)\Delta_y \eta(y),$$

and $\nabla_y \eta(y) = 0$, $\Delta_y \eta(y) = 0$ for $|y| < 1/2$, as well as $\eta(z+\epsilon\boldsymbol{\omega}) = 1$ and $\nabla\eta(z+\epsilon\boldsymbol{\omega}) = 0$ for $|z|, \epsilon < 1/4$, we can send ϵ to 0 to obtain

$$\begin{aligned} v(z) &= \int_{B_1(0)} v(y) \left[2\nabla_y \Phi(y-z) \cdot \nabla_y \eta(y) + \Phi(y-z)\Delta_y \eta(y)\right] \, dy \\ &= \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} v(y) \left[2\nabla_y \Phi(y-z) \cdot \nabla_y \eta(y) + \Phi(y-z)\Delta_y \eta(y)\right] \, dy, \end{aligned}$$

using

$$-\int_{\mathbb{S}^{n-1}} \epsilon^{n-1} v(z+\epsilon\boldsymbol{\omega}) \frac{\partial \left[\Phi(r\boldsymbol{\omega})\eta(z+r\boldsymbol{\omega})\right]}{\partial r}\Big|_{r=\epsilon} d\boldsymbol{\omega} \to v(z),$$

and

$$\Phi(\epsilon\boldsymbol{\omega})\eta(z+\epsilon\boldsymbol{\omega}) \frac{\partial v(z+\epsilon\boldsymbol{\omega})}{\partial r}\Big|_{r=\epsilon} d\boldsymbol{\omega} \to 0$$

as $\epsilon \to 0$.

Let $\hat{\Phi}(y,z) = 2\nabla_y \Phi(y-z) \cdot \nabla_y \eta(y) + \Phi(y-z)\Delta_y \eta(y)$, then $\hat{\Phi}(y,z) \in C^\infty(B_1(0) \setminus B_{\frac{1}{2}}(0) \times B_{\frac{1}{4}}(0))$, so for $z \in B_{\frac{1}{4}}(0)$,

$$\nabla^\alpha v(z) = \int_{B_1(0) \setminus B_{\frac{1}{2}}(0)} v(y) \nabla_z^\alpha \hat{\Phi}(y,z) \, dy.$$

It's easy to see that, with $m = (n-2)/2$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, and $k = |\alpha|$, we have

$$\nabla_z^\alpha \Phi(z-y) = (-1)^k |\mathbb{S}^{n-1}|^{-1} (m+1)(m+2)\cdots(m+k-1)|z-y|^{2-n-2k} p_k(z-y),$$

where p_k is a homogeneous polynomial of degree k. Using induction, it is easy to see that each of the coefficients of p_k is bounded in absolute values by 3^{k-1}. Using this and the observation that p_k has at most $\binom{n-1+k}{k}$ terms , we see that for $z \in B_{\frac{1}{4}}(0)$ and $y \in B_1(0) \setminus B_{\frac{1}{2}}(0)$,

$$|z-y|^{2-n-2k} |p_k(z-y)| \le \binom{n-1+k}{k} 3^{k-1} 4^{n-2+k},$$

$$|\nabla_z^\alpha \Phi(z-y)| \le [c(n)k]^k,$$

and

$$|\nabla_z^\alpha \hat{\Phi}(y,z)| \le [c(n)k]^k$$

with $k = |\alpha|$ and some $c(n)$. It now follows that $|\nabla^\alpha v(0)| \le [c(n)k]^k ||v||_{L^1(B_1(0))}$.

In fact one can directly work with (5.23) for $v(y)$ and $\Phi(y-z)$ over $B_{\frac{1}{2}}(0)$ to obtain estimates of $|\nabla^\alpha v(0)|$ in terms of $\max_{\partial B_{\frac{1}{2}}(0)} |v|$ and $\max_{\partial B_{\frac{1}{2}}(0)} |\nabla v|$; then use the first derivative estimate on $|\nabla v|$ to estimate $\max_{\partial B_{\frac{1}{2}}(0)} |\nabla v|$ in terms of $||v||_{L^1(B_1(0))}$. The technique of working with the fundamental solution multiplied by a smooth cut-off function (to avoid boundary integral terms) will be useful in other contexts.

□

Corollary 5.9 (Liouville)**.** *A bounded harmonic function on $\mathbb{R}^n$ must be a constant.*

This is proved by the gradient estimate: $R|\nabla u(x_0)| \le C \max_{B_R(x_0)} |u|$ holds for all $R > 0$; sending $R \to \infty$ implies that $|\nabla u(x_0)| = 0$ at any x_0, which proves that u must be a constant.

Remark 5.6. Liouville type theorems do not necessarily hold for bounded entire solutions to certain modifications of the Laplace equation, such as the **Helmholtz** equation: $\Delta u(x) + \lambda u(x)$ on $\mathbb{R}^n$, as $u(x) = \sin(kx)$ and its higher dimensional analogues are bounded entire solutions to such equations. There are still gradient estimates for solutions to equations such as the Helmholtz equation, but those estimates have a different scaling from that for harmonic functions and do not lead to a version which would imply the conclusion in the Liouville Theorem here. We will explore in the exercises which properties of harmonic functions can be generalized to solutions of the Helmholtz equation.

The more powerful consequences of the gradient estimates are the convergence theorems.

Theorem 5.10. *(i). Uniform limit of a sequence of harmonic functions is harmonic. (ii). A bounded sequence of harmonic functions on U must have a subsequence that converges on any compact subset of U to a harmonic function.*

Proof. The proof follows from the derivative estimates in Theorem (5.8) and Arzela-Ascoli theorem. It suffices to prove the conclusions for any ball B_R such that $\overline{B}_{2R} \subset U$. Suppose that $\{u_k(x)\}$ is a sequence of harmonic functions in U converging uniformly to $u(x)$ over $\overline{B}_{2R}$ as $k \to \infty$. Then $u_k(x) - u_l(x)$ is a Cauchy sequence in $C(\overline{B}_{2R})$. The higher derivatives estimates, applied to $u_k(x) - u_l(x)$ on $\overline{B}_{2R}$, implies that up to two derivatives of $u_k(x) - u_l(x)$ is a Cauchy sequence in $C(\overline{B}_R)$, as a consequence, $\Delta u_k(x)$ converges to $\Delta u(x)$ uniformly over $\overline{B}_R$, which implies that $u(x)$ is harmonic in B_R.

Another approach is to use the information that $\max_{\overline{B}_{2R}} |u_k(x)|$ remains bounded independent of k. Then the higher derivatives estimates for $\{u_k(x)\}$ implies that up to three derivatives of $u_k(x)$ are uniformly bounded in $\overline{B}_R$. So by Arzela-Ascoli theorem a subsequence of $\{u_k(x)\}$ has the property that it, together with sequences consisting of the derivatives of up to second order, converges uniformly in B_R, which shows that the limit function $u(x)$ is harmonic in B_R. (ii) is proved in a similar fashion.

□

Corollary 5.11. *If $u \in C^2(U)$ is harmonic, then it is real analytic in U.*

Proof. This follows from the same representation formula, using the analytic expansion of $\Phi(x-y)$ at any $x_0 \in U$ for $y \in \partial U$: fix any $x_0 \in U$, then there exists $r_0 > 0$ such that

$$\Phi(x-y) = \sum_{\alpha} a_\alpha(x_0 - y)(x-x_0)^\alpha,$$

with uniform convergence for $|x - x_0| \le r_0$ and $y \in \partial U$ (we should have chosen a subdomain V with C^1 boundary such that $x_0 \in V \subset\subset U$, as done in the proof of the previous corollary, to apply the Green's representation on V so as to avoid the need for $u \in C^2(\bar{U})$, but will assume that this reduction has been done and will write U for V). In the above, the notation z^α stands for $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ if $z = (z_1, z_2, \cdots, z_n)$ and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, with $\alpha_j \in \mathbb{Z}_{\ge 0}$. We will also denote $\alpha_1 + \alpha_2 + \cdots + \alpha_n = |\alpha|$. The above expansion can be obtained by noting that

$$\begin{aligned}
|x-y|^{2-n} &= \left[|x - x_0 + x_0 - y|^2\right]^{(2-n)/2} \\
&= \left[|x-x_0|^2 + 2(x-x_0)\cdot(x_0 - y) + |x_0 - y|^2\right]^{(2-n)/2} \\
&= |x_0 - y|^{2-n}\left[1 + \frac{|x-x_0|^2 + 2(x-x_0)\cdot(x_0-y)}{|x_0-y|^2}\right]^{(2-n)/2},
\end{aligned}$$

and using that $d > |x_0 - y| > \delta > 0$ for some $d > \delta > 0$ and for all $y \in \partial U$, so for $|x - x_0|$ sufficiently small, we can do a binomial expansion of the above power to get a power series in terms of $x - x_0$, with coefficients in terms of $x_0 - y$.

Thus for any $\epsilon > 0$, there is N such that

$$|\Phi(x-y) - \sum_{|\alpha| \le N} a_\alpha(x_0 - y)(x - x_0)^\alpha| \le \epsilon,$$

uniformly for $|x - x_0| \le r_0$ and $y \in \partial U$. Multiplying by $\frac{\partial u(y)}{\partial n(y)}$ and integrating over $y \in \partial U$, we obtain

$$|\int_{\partial U} \Phi(x-y)\frac{\partial u(y)}{\partial n(y)}\, d\sigma(y) - \sum_{|\alpha|\le N} A_\alpha(x_0)(x-x_0)^\alpha| \le \epsilon \int_{\partial U} |\frac{\partial u(y)}{\partial n(y)}|\, d\sigma(y),$$

with $A_\alpha(x_0) = \int_{\partial U} a_\alpha(x_0 - y)\frac{\partial u(y)}{\partial n(y)}\, d\sigma(y)$. Similarly, the other integral is also analytic in $x \in U$, showing that u is analytic in U. □

Remark 5.7. Since for each fixed y, $|x-y|^{2-n}$ is a harmonic function for $x \in \mathbb{R}^n (n \geq 3), x \neq y$, we may try to use the superposition principle to construct harmonic functions by an integral

$$\int_E |x-y|^{2-n} f(y)\, dy$$

over some set E. The proofs of the above two corollaries already use this idea and indicate that it works when E is taken to be ∂U, which is assumed to be a piecewise C^1 hyper-surface*. When E is taken to be an open domain U, however, $\int_U |x-y|^{2-n} f(y)\, dy$ is in general not a harmonic function of $x \in U$. This is because $\partial_{x_i x_j}|x-y|^{2-n}$ is no longer an integrable function of $y \in U$ when $x \in U$, so we can't differentiate twice in x on the integral and pass the differentiation inside the integral to conclude that $\int_U |x-y|^{2-n} f(y)\, dy$ is harmonic for $x \in U$.

This is also a good place to insert a comment on (5.23). It was derived under the assumption that $u \in C^2(\overline{U})$. However, we will see that, for $f \in C(\overline{U})$, $\int_U \Phi(x-y)f(y)\, dy$ may not be a C^2 function of $x \in U$; and with given $g, h \in C(\partial U)$, the function

$$u(x) = \int_U -f(y)\,\Phi(x-y)\, dy + \int_{\partial U} \left[\Phi(x-y)h(y) - g(y)\frac{\partial \Phi(x-y)}{\partial n(y)}\right] d\sigma(y),$$

may not be a $C^2(U)$ solution of (5.10), as already commented earlier. We do have

Proposition 5.12. *Suppose that U is a bounded domain. If $f \in C^1(\overline{U})$, then $\int_U \Phi(x-y)f(y)\, dy$ is a C^2 function of $x \in U$ and*

$$-\Delta_x \int_U \Phi(x-y) f(y)\, dy = f(x), \qquad \text{for } x \in U.$$

Remark 5.8. A proof for Proposition 5.12 will be provided below. With mere continuity of f in $\overline{U}$ (in fact just with $f \in L^\infty(U)$), the Newton potential of f, $\int_U \Phi(x-y)f(y)\, dy$, is in $C^1(\overline{U})$, but may not be C^2 function of $x \in U$. However, in order for the Newton potential of f to be a C^2 function of x, the regularity requirement of f can be weakened to the following: for some $0 < \alpha \leq 1$ and $C > 0$,

$$|f(y_1) - f(y_2)| \leq C|y_1 - y_1|^\alpha \quad \text{for all } y_1, y_2 \in U. \tag{5.24}$$

*When E has a component Γ with codimension > 1, then $\int_E |x-y|^{2-n} f(y)\, dy$ may tend to ∞ as $x \to x_* \in \Gamma$, as demonstrated when Γ is an isolated point of E.

Functions satisfying (5.24) in U (therefore in $\overline{U}$) with $0 < \alpha < 1$ are said to be Hölder continuous in $\overline{U}$ with exponent α, and the set of such functions is denoted as $C^\alpha(\overline{U})$; those functions satisfying (5.24) in U (therefore in $\overline{U}$) with $\alpha = 1$ are said to be Lipschitz continuous in $\overline{U}$, and the set of such functions is denoted as $Lip(\overline{U})$. $C^\alpha(U)$ (respectively $Lip(U)$) usually denotes the set of functions satisfying (5.24) on any compact subsets of U (the constant C may depend on the compact subset).

Proof. * We will prove that, if $f \in L^\infty(U)$, then

$$\frac{\partial u(x)}{\partial x_i} = \int_U \frac{\partial[\Phi(x-y)]}{\partial x_i} f(y)\, dy, \text{ for any } x \in U, \tag{5.25}$$

and if $f \in C^1(\overline{U})$, then

$$\frac{\partial^\alpha u(x)}{\partial x^\alpha} = -\int_U \frac{\partial[\Phi(x-y)]}{\partial y_i} \frac{\partial f(y)}{\partial y_j}\, dy + \int_{\partial U} n_j(y) \partial_{y_i} \Phi(x-y) f(y)\, d\sigma(y) \tag{5.26}$$

for any $x \in U$ and $\alpha = (i, j)$ with $|\alpha| = 2$, where $n_j(y)$ are the coordinate components of the exterior unit normal vector $n(y)$ to ∂U at $y \in \partial U$. It then follows that

$$\begin{aligned}
\Delta_x u(x) &= -\int_U \sum_{i=1}^n \frac{\partial[\Phi(x-y)]}{\partial y_i} \frac{\partial f(y)}{\partial y_i}\, dy + \int_{\partial U} \sum_{j=1}^n n_j(y) \partial_{y_j} \Phi(x-y) f(y)\, d\sigma(y) \\
&= -\lim_{\epsilon \to 0} \int_{U \setminus B_\epsilon(x)} \sum_{i=1}^n \frac{\partial[\Phi(x-y)]}{\partial y_i} \frac{\partial f(y)}{\partial y_i}\, dy + \int_{\partial U} \sum_{j=1}^n n_j(y) \partial_{y_j} \Phi(x-y) f(y)\, d\sigma(y) \\
&= \lim_{\epsilon \to 0} \left\{ \int_{U \setminus B_\epsilon(x)} \sum_{i=1}^n \frac{\partial^2[\Phi(x-y)]}{\partial y_i^2} f(y)\, dy + \int_{\partial B_\epsilon(x)} \frac{\partial[\Phi(x-y)]}{\partial n(y)} f(y)\, d\sigma(y) \right\} \\
&= -f(x),
\end{aligned}$$

using $\int_{\partial B_\epsilon(x)} \frac{\partial[\Phi(x-y)]}{\partial n(y)}\, d\sigma(y) = -1$.

The key difficulty in proving (5.25) and (5.26) is that the $\Phi(x-y)$ in the integrand is singular at $y = x$; more precisely, $\partial_x \Phi(x-y) f(y)$ is still integrable in $y \in U$, but $\partial_x^2 \Phi(x-y) f(y)$ is not integrable for $y \in U$ due to the strength of the singularity in $\partial_x^2 \Phi(x-y) \sim |x-y|^{-n}$.

First assume $f \in C_c^1(U)$, then

$$\int_U \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \Phi(z) f(x-z)\, dz,$$

*May be omitted on a first reading.

and

$$
\begin{aligned}
&\partial_{x_i} \int_U \Phi(x-y) f(y)\, dy \\
=&\int_{\mathbb{R}^n} \Phi(z) \partial_{x_i} f(x-z)\, dz \\
=&-\lim_{\epsilon\to 0} \int_{|z|\geq\epsilon} \Phi(z) \partial_{z_i} f(x-z)\, dz \\
=&\lim_{\epsilon\to 0} \left\{ \int_{|z|\geq\epsilon} \partial_{z_i}\Phi(z) f(x-z)\, dz + \int_{|z|=\epsilon} \nu_i(z)\Phi(z) f(x-z)\ d\sigma(z) \right\} \\
=&\int_{\mathbb{R}^n} \partial_{z_i}\Phi(z) f(x-z)\, dz \\
=&\int_{\mathbb{R}^n} \partial_{x_i}\Phi(x-y) f(y)\ dy.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\partial_{x_i x_j} \int_U \Phi(x-y) f(y)\, dy \\
=&\partial_{x_j} \int_{\mathbb{R}^n} \partial_{z_i}\Phi(z) f(x-z)\, dz \\
=&\int_{\mathbb{R}^n} \partial_{z_i}\Phi(z) \partial_{x_j} f(x-z)\, dz \\
=&\int_U \partial_{x_i}\Phi(x-y) \partial_{y_j} f(y)\, dy \\
=&-\int_U \partial_{y_i}[\Phi(x-y)] \partial_{y_j} f(y)\, dy.
\end{aligned}
$$

Next, for any $f \in C^1(\overline{U})$ and $x_0 \in U$, we can split $f(x) = f_1(x) + f_2(x)$, where $f_i \in C^1(\overline{U})$, but $f_1(x) = f(x)$ in a neighborhood of x_0 and has compact support in U. Thus $f_2(x) = 0$ in a neighborhood of x_0, but $f_2(x) = f(x)$ for x near ∂U. Then

$$
\int_U \Phi(x-y) f(y)\, dy = \int_U \Phi(x-y) f_1(y)\, dy + \int_U \Phi(x-y) f_2(y)\, dy,
$$

where $\int_U \Phi(x-y) f_2(y)\, dy$ is smooth in x near x_0, therefore $\partial^2_{x_i x_j} \int_U \Phi(x-y) f_2(y)\, dy = \int_U \partial^2_{x_i x_j} \Phi(x-y) f_2(y)\, dy$ holds for any α. We can apply the above argument to $\int_U \Phi(x-y) f_1(y)\, dy$ to conclude that it is a C^2 function of x near x_0 and

$$
\begin{aligned}
&\partial_{x_i x_j} \int_U \Phi(x-y) f_1(y)\, dy \\
=&-\int_U \partial_{y_i}[\Phi(x-y)] \partial_{y_j} f_1(y)\, dy.
\end{aligned}
$$

Since $f_2(x) = 0$ in a neighborhood of x_0, we also have, for x near x_0,

$$\begin{aligned}
&\int_U \partial^2_{x_i x_j}\Phi(x-y) f_2(y)\,dy \\
&\qquad = -\int_U \partial_{y_i}[\Phi(x-y)]\partial_{y_j} f_2(y)\,dy + \int_{\partial U} \partial_{y_i}[\Phi(x-y)] n_j(y) f_2(y)\,dy \\
&\qquad = -\int_U \partial_{y_i}[\Phi(x-y)]\partial_{y_j} f_2(y)\,dy + \int_{\partial U} \partial_{y_i}[\Phi(x-y)] n_j(y) f(y)\,dy
\end{aligned}$$

Putting these together we have proved (5.26) □

Remark 5.9. If we note that in (5.26)

$$\begin{aligned}
&\int_U \partial_{y_i}[\Phi(x-y)]\partial_{y_j} f(y)\,dy \\
&= \int_U \partial_{y_i}[\Phi(x-y)]\partial_{y_j}[f(y)-f(x)]\,dy \\
&= -\int_U \partial^2_{y_i y_j}\Phi(x-y)[f(y)-f(x)]\,dy + \int_{\partial U} \partial_{y_i}[\Phi(x-y)] n_j(y)[f(y)-f(x)]\,d\sigma(y),
\end{aligned}$$

we can rewrite (5.26) as

$$\partial^2_{x_i x_j}\int_U \Phi(x-y) f(y)\,dy = \int_U \partial^2_{y_i y_j}\Phi(x-y)[f(y)-f(x)]\,dy + f(x)\int_{\partial U}\partial_{y_i}[\Phi(x-y)] n_j(y)\,d\sigma(y).$$

The right hand side is defined as long as $f \in C^\alpha(U)$ for some $0 < \alpha \le 1$, and as a function of x, it converges uniformly if $f_k(x) \to f(x)$ in $C^\alpha(U)$, so this equality also holds for $f \in C^\alpha(U)$.

Remark 5.10. To solve the Dirichlet problem (5.3) on U, one can still try to take some E disjoint from U and use $\int_E \Phi(x-y) g(y)\,dy$ to construct harmonic functions in U. The question becomes:

(i) whether such harmonic functions extends continuously to $\overline{U}$?

(ii) whether one can achieve all (continuous) boundary value functions on ∂U by choosing $g \in C(E)$?

Such harmonic functions indeed extend continuously to $\overline{U}$ when $E = \partial U$, ∂U is a piecewise C^1 hypersurface, and $g \in C(\partial U)$. However, even in this case, for $\bar{x} \in \partial U$, in general,

$$\lim_{x \in U,\, x \to \bar{x}} \int_{\partial U} \Phi(x-y) g(y)\,d\sigma(y) \neq g(\bar{x}).$$

So even though we can construct a harmonic function $x \mapsto \int_{\partial U}\Phi(x-y)g(y)\,dy$ for $x \in U$, this harmonic function may not solve (5.1) with a prescribed boundary value for the case $f \equiv 0$. In fact the map $g \in C(\partial U) \mapsto \int_{\partial U} \Phi(x-y)g(y)\,d\sigma(y) \in C(\partial U)$ is a compact linear map*, so $\int_{\partial U} \Phi(x-y)g(y)\,d\sigma(y)$ can not possibly take on all continuous boundary value functions when g runs through $C(\partial U)$. It turns out a modification of this idea, using the so called double layer potential, $\int_{\partial U} \frac{\partial \Phi(x-y)}{\partial n(y)} g(y)\,d\sigma(y)$, one can solve the Dirichlet problem (5.1) this way.

Exercises

Exercise 5.4.1. Provide detailed proofs for Theorem 5.10.

Exercise 5.4.2. Use (5.23) and (5.26) to prove that if $\Delta u(x) = \lambda u(x)$ in U, where λ is a constant. Then

(i). $u \in C^\infty(U)$. (Hint: This involves a bootstrap process: first prove $u \in C^3(U)$ by an extension of (5.26), using $u \in C^2(U)$; then prove $u \in C^4(U)$, using the knowledge that $u \in C^3(U)$—one technique is to use $\Delta u_{x_i}(x) = \lambda u_{x_i}(x)$; repeat this process.)

(ii). Suppose that $\overline{B_R(x_0)} \subset U$. Prove that there exists $c = c(n) > 0$ such that

$$
\begin{aligned}
R \max_{B_{R/2}(x_0)} |\nabla u| &\le c(1 + R^2|\lambda|) \max_{B_R(x_0)} |u| \\
R^2 \max_{B_{R/2}(x_0)} |\nabla^2 u| &\le c(1 + R^2|\lambda|)^2 \max_{B_R(x_0)} |u| \\
R^3 \max_{B_{R/2}(x_0)} |\nabla^3 u| &\le c(1 + R^2|\lambda|)^3 \max_{B_R(x_0)} |u|
\end{aligned}
$$

(iii). Suppose that $\{u_k\}$ is a sequence of solutions to $\Delta u(x) = \lambda u(x)$ in U and converges uniformly to $u(x)$ on compact subset of U. Prove that $u(x)$ satisfies the same equation in U.

5.5 Green's Function and Poisson's Formula

We continue to explore the consequences of (5.23). Let $\phi(y)$ be a $C^2(U) \cap C^1(\overline{U})$ harmonic function in U, then applying the Green's theorem to ϕ and u on U, we have

$$
0 = \int_U -\Delta u(y)\phi(y)\,dy + \int_{\partial U} \left(\phi(y)\frac{\partial u(y)}{\partial n(y)} - u(y)\frac{\partial \phi(y)}{\partial n(y)} \right) d\sigma(y).
$$

*This means that for any bounded sequence $g_j \in C(\partial U)$, $\int_{\partial U} \Phi(x-y)g_j(y)\,d\sigma(y)$ has a subsequence converging in $C(\partial U)$.

Combining this with (5.23), we have

$$\begin{aligned} u(x) = &\int_U -\Delta u(y)\,[\Phi(x-y)-\phi(y)]\,dy \\ &+ \int_{\partial U} \left\{ [\Phi(x-y)-\phi(y)]\frac{\partial u(y)}{\partial n(y)} - u(y)\frac{\partial\,[\Phi(x-y)-\phi(y)])}{\partial n(y)} \right\} d\sigma(y). \end{aligned}$$

If, for each $x \in U$, we can choose a harmonic function $\phi(y) = \phi^x(y) \in C^1(\overline{U})$, which depends on x, such that $\Phi(x-y) - \phi^x(y) \equiv 0$ for all $y \in \partial U$, then we have

$$u(x) = \int_U -\Delta u(y)\,[\Phi(x-y)-\phi^x(y)]\,dy - \int_{\partial U} u(y)\frac{\partial\,[\Phi(x-y)-\phi^x(y)]}{\partial n(y)}\,d\sigma(y).$$

That is, we can represent u in U by its Dirichlet data. When this is possible, we call $G(x,y) = \Phi(x-y) - \phi^x(y)$ the **Green's function** (for the Laplace operator with Dirichlet boundary condition) for the region U. We summarize the above discussion as

Proposition 5.13. *Suppose that, for each $x \in U$, there exists $\phi^x(y) \in C^2(U) \cap C^1(\overline{U})$ such that*

$$\begin{cases} \Delta_y \phi^x(y) = 0, & \text{for } y \in U, \\ \phi^x(y) = \Phi(x-y), & \text{for } y \in \partial U. \end{cases}$$

Let $G(x,y) = \Phi(x-y) - \phi^x(y)$. Then, for $n \geq 3$,

$$\begin{cases} \Delta_y G(x,y) = 0, & \text{for } y \in U \setminus \{x\}, \\ G(x,y) = 0, & \text{for } y \in \partial U, \\ \lim_{y\to x} |x-y|^{n-2} G(x,y) = \dfrac{1}{(n-2)|\mathbb{S}^{n-1}|}, \\ \lim_{y\to x} |x-y|^{n-2} (x-y)\cdot\nabla_y G(x,y) = \dfrac{1}{|\mathbb{S}^{n-1}|}, \end{cases}$$

and for any $u \in C^2(\overline{U})$, there holds

$$u(x) = \int_U -\Delta u(y)\,G(x,y)\,dy - \int_{\partial U} u(y)\frac{\partial G(x,y)}{\partial n(y)}\,d\sigma(y). \tag{5.27}$$

For the $n = 2$ case, the last two conditions on $G(x,y)$ as $y \to x$ need to be modified appropriately.

For some special domains, we can write out $G(x,y)$ explicitly. First, when $U = \mathbb{R}^n_+$, for each $x \in \mathbb{R}^n_+$, we define $\phi^x(y) = \Phi(x^* - y)$, where x^* is the mirror image of x in $\partial\mathbb{R}^n_+$. Then

$$\Phi(x-y) - \Phi(x^*-y) \equiv 0 \quad \text{for } y \in \partial\mathbb{R}^n_+,$$

and we identify $G(x,y) = \Phi(x-y) - \Phi(x^*-y)$; furthermore, we define

$$-\frac{\partial G(x,y)}{\partial n(y)} = \frac{\partial G(x,y)}{\partial y_n} = \frac{2x_n}{|\mathbb{S}^{n-1}||x-y|^n} := K(x,y),$$

for $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$. $K(x,y)$ is called the Poisson kernel for $\mathbb{R}^n_+$. Even though our earlier discussion was for a compact domain, and the argument may not directly carry over to a non-compact domain such as $\mathbb{R}^n_+$, we still have

Theorem 5.14. *Assume $g \in C(\partial\mathbb{R}^n_+) \cap L^\infty(\partial\mathbb{R}^n_+)$. Define $u(x)$ for $x \in \mathbb{R}^n_+$ by*

$$u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y)g(y)\,dy = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\partial\mathbb{R}^n_+} \frac{2x_n}{|x-y|^n} g(y)\,d\sigma(y). \tag{5.28}$$

Then

(i). $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$,

(ii). $\Delta_x u(x) = 0$, *in* $x \in \mathbb{R}^n_+$,

(iii). $\lim_{x \to \bar{x} \in \partial\mathbb{R}^n_+,\, x \in \mathbb{R}^n_+} u(x) = g(\bar{x})$.

And there is only one solution u satisfying (i)–(iii) above.

The last part of the above theorem is proved by maximum principle and is left as an exercise. Notice that, although the Green's representation was derived under stricter requirement on u: $u \in C^2(\overline{U})$, we can apply the representation (5.28) for any continuous boundary data, which produces a solution which may not be C^2 up to the boundary but is C^2 in the interior of the domain and is continuous up to the boundary. It's also true that any bounded harmonic function on $\mathbb{R}^n_+$ (and continuous on $\overline{\mathbb{R}^n_+}$) has a representation in the form of (5.28). The key properties used in proving (iii) above are that the family $\{K(x,y)\}$ when $x \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$ satisfies

(a). $\int_{\partial\mathbb{R}^n_+} K(x,y)\,dy = 1$ for all $x \in \mathbb{R}^n_+$;

(b). $K(x,y) \geq 0$;

(c). For any $\delta > 0$, $\int_{y \in \partial\mathbb{R}^n_+, |y| \geq \delta} K(x,y)\,dy \to 0$ as $x \to 0$ in $\mathbb{R}^n_+$.

Next, when $U = B_R(0)$ in $\mathbb{R}^n$. It turns out that for each $x \neq 0 \in B_R(0)$, if we define $x^* = \frac{R^2}{|x|^2}x$, then $\phi^x(y) = \Phi(\frac{|x|}{R}(x^*-y))$ would work, because $|y-x|/|y-x^*|$ is independent of $y \in \partial B_R(0)$, and equals $|x|/R$, so $\Phi(x-y) - \Phi(\frac{|x|}{R}(x^*-y)) = 0$ for all

$y \in \partial B_R(0)$. It works out that for any $y \neq 0$ in $\overline{B}_R(0)$, $\phi^x(y) = \Phi(\frac{|x|}{R}(x^* - y)) \to \Phi(R)$ as $x \to 0$, so the construction of $\phi^x(y)$ continues to work in the case $x = 0$ and

$$-\frac{\partial\left[\Phi(x-y) - \phi^x(y)\right])}{\partial n(y)} = \frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}||x-y|^n} := K(x,y), \qquad \text{for } x \in B_R(0) \text{ and } y \in \partial B_R(0).$$

$K(x, y)$ is called the Poisson kernel for $B_R(0)$. This generalizes our Poisson kernel on two dimensional discs, which was found based on separation of variables.

Theorem 5.15. *Assume $g \in C(\partial B_R(0))$. Define $u(x)$ for $x \in B_R(0)$ by*

$$u(x) = \int_{\partial B_R(0)} \frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}||x-y|^n} g(y)\, d\sigma(y). \tag{5.29}$$

Then

(i). $u \in C^\infty(B_R(0))$,

(ii). $\Delta_x u(x) = 0$, *in* $x \in B_R(0)$,

(iii). $\lim_{x \to \bar{x} \in \partial B_R(0), x \in B_R(0)} u(x) = g(\bar{x})$.

And there is only one solution u satisfying (i)–(iii) above.

Remark 5.11. Although the Poisson kernel

$$K(x,y) = -\frac{\partial G(x,y)}{\partial n(y)}$$

is defined asymmetrically: x in the interior of the domain, and y on the boundary, the Green's function $G(x, y)$ is defined for both x and y inside the domain, $x \neq y$ and is a harmonic function in y.

A useful property for $G(x, y)$ is

Proposition 5.16. *For all $x, y \in U$, $x \neq y$, we have*

$$G(x, y) = G(y, x).$$

Thus $G(x, y)$ is also a smooth function in $U \times U \setminus \{(z, z) : z \in U\}$, extends to a continuous function in $\overline{U} \times \overline{U} \setminus \{(z, z) : z \in \bar{U}\}$, and is harmonic in $x \in U$, for $x \neq y$.

Proof. For any $x \neq y$ in U, choose $\epsilon > 0$ such that $B_\epsilon(x), B_\epsilon(y)$ are non-overlapping proper subdomains of U, and apply the Green's identity to $u(z) = G(x,z)$ and $v(z) = G(y,z)$ on the domain $U \setminus \{B_\epsilon(x) \cup B_\epsilon(y)\}$. Since $\Delta_z G(x,z) = \Delta_z G(y,z) = 0$ in this domain and $G(x,z) = G(y,z) = 0$ for z on ∂U, we obtain

$$\begin{aligned}
&\int_{\partial B_\epsilon(x)} \left[G(x,z)\frac{\partial G(y,z)}{\partial n(z)} - G(y,z)\frac{\partial G(x,z)}{\partial n(z)} \right] d\sigma(z) \\
=&\int_{\partial B_\epsilon(y)} \left[G(y,z)\frac{\partial G(x,z)}{\partial n(z)} - G(x,z)\frac{\partial G(y,z)}{\partial n(z)} \right] d\sigma(z).
\end{aligned}$$

We note that

$$\begin{aligned}
&\int_{\partial B_\epsilon(x)} \frac{\partial G(x,z)}{\partial n(z)}\, d\sigma(z) \\
=&\int_{\partial B_\epsilon(x)} \frac{\partial \Phi(x-z)}{\partial n(z)}\, d\sigma(z) - \int_{\partial B_\epsilon(x)} \frac{\partial \phi^x(z)}{\partial n(z)}\, d\sigma(z) \\
=&-1 + \int_{B_\epsilon(x)} \Delta_z \phi^x(z)\, dz \\
=&-1
\end{aligned}$$

and similarly, $\int_{\partial B_\epsilon(y)} \frac{\partial G(y,z)}{\partial n(z)}\, d\sigma(z) = -1$. Using these and that fact that $|\frac{\partial G(y,z)}{\partial n(z)}|$ and $\epsilon^{n-2}G(x,z)$ are bounded on $\partial B_\epsilon(x)$, independent of $\epsilon > 0$ when it is small, and $|\frac{\partial G(y,z)}{\partial n(z)}|$ and $\epsilon^{n-2}G(y,z)$ are bounded on $\partial B_\epsilon(y)$, it follows by sending $\epsilon \to 0$ that $G(x,y) = G(y,x)$. □

Proposition 5.17. *A C^0 function u in U which satisfies the mean value property for any balls $B \subset U$ is a smooth harmonic function.*

Proof. Let u be such a C^0 function. For any ball $B \subset\subset U$, by Theorem 5.15, there is a harmonic function $v \in C(\overline{B}) \cap C^2(B)$ such that $v = u$ on ∂B. Since both u and v satisfy the maximum principle, we conclude now that $v = u$ in B, i.e., u is harmonic in B. Since this holds for any ball $B \subset\subset U$, we conclude that u is harmonic in U. □

Exercises

Exercise 5.5.1. (Schwarz reflection principle) Suppose that $u \in C(\overline{B_R^+}) \cap C^2(B_R^+)$ is harmonic in B_R^+, where $B_R^+ = \{x \in B_R(0) : x_n > 0\}$, and that $u(x', 0) = 0$ for $(x', 0) \in \overline{B_R^+}$. Extend u to $B_R^- = \{x \in B_R(0) : x_n < 0\}$ by $u(x', x_n) = -u(x', -x_n)$ when $(x', x_n) \in B_R^-$. Prove that this extension is a harmonic function on $B_R(0)$.

Exercise 5.5.2. (Kelvin transform) The transformation $y = I(x) = \frac{x}{|x|^2}$ is called the inversion with respect to the unit sphere $\mathbb{S}^{n-1}$.

(i). Prove that $I \circ I(x) = x$ for all x, and that I maps $\{(x', x_n) : x_n \geq h > 0\}$ onto the ball $B_{\frac{1}{2h}}(\frac{e_n}{2h})$.

(ii). Set $v(x) = |x|^{2-n} u(\frac{x}{|x|^2})$. Prove that $\Delta_x v(x) = |x|^{-2-n} \Delta u(\frac{x}{|x|^2})$. (HINT: Computing in polar coordinates may be easier.)

Note that this transformation allows us to transform a question of boundary value problem for the Laplace equation on a ball into a question of the boundary value problem for the same equation on a half space, and vice versa. Can one say the same thing on the Helmholtz equation?

Exercise 5.5.3. This exercise works out the justification that any bounded harmonic function u on $\mathbb{R}^n_+$ (and continuous on $\overline{\mathbb{R}^n_+}$) has a representation in the form of (5.28). Fix any $x \in \mathbb{R}^n_+$, one would like to apply Green's identity to $u(X)$ and $G(x, X)$ on the sequence of compact regions $B_R(0) \cap \mathbb{R}^n_+$ exhausting $\mathbb{R}^n_+$ and establish a Green's representation for $u(x)$ in terms of data on the boundary of this sequence of regions. However, the Green's identify would involve boundary integrals of $u(X)\partial_\nu G(x, X) - G(x, X)\partial_\nu u(X)$, and we don't have good enough estimates on $|\partial_\nu u(X)|$, especially when X is near $\partial\mathbb{R}^n_+$, which would demonstrate that the portion of the boundary integral approaching ∞ is tending to 0. We bypass this difficulty in one of two ways.

(i). Use $g = u|_{\partial\mathbb{R}^n_+}$ and the right hand side of (5.28) to construct a solution, and use the maximum principle to prove that u is identical to this solution.

(ii). Fix a standard cut-off function η such that it is supported in $|x| \leq 2$ and $\eta = 1$ in $B_1(0)$. Then for $R >> 1$ large, use the Green's identity to $u(X)$ and $G(x, X)\eta(X/R)$ over $B_{2R}(0) \cap \mathbb{R}^n_+$ to establish a Green's representation for $u(x)$ on such domains and sending $R \to \infty$ to justify (5.28). Here one needs to use the improved decay rate of $G(x, X)$ when $|X|$ is large: for any $x \in \mathbb{R}^n_+$, there exist $C > 0$ and $R > 0$ depending on x such that $|G(x, X)| \leq C|X|^{1-n}$ and $|\nabla_X G(x, X)| \leq C|X|^{-n}$ when $|X| > R$. Note that this second approach involves considerably more technical arguments.

Exercise 5.5.4. Consider the Helmholtz equation $u_{xx} + u_{yy} - m^2 u = 0$ on $\mathbb{R}^2_+$ with the Dirichlet condition on $\partial\mathbb{R}^2_+$, where $m > 0$ is a real parameter.

(i). Use the method of image as above for the Laplace operator on the upper half plane to construct the Green's function for the above Helmholtz equation, namely, $G(\xi, \eta; x, y)$ such that $G_{xx}(\xi, \eta; x, y) + G_{yy}(\xi, \eta; x, y) - m^2 G(\xi, \eta; x, y) = -\delta(x - \xi, y - \eta)$ for $(x, y), (\xi, \eta) \in \mathbb{R}^2_+$, and $G(\xi, \eta; x, 0) = 0$.

(ii). Then use the Green's function to establish a Green's representation for a solution in terms of its boundary value (assuming necessary decay condition for the solution; identify the decay conditions needed).

(iii). Compare the result with that obtained by Fourier's method. Is the Green's function positive for $(x, y) \in \mathbb{R}^2_+$? Is there uniqueness in the class of bounded solutions?

(iv). Does this approach work when the domain is a round disk?

(v). Also carry out the analysis for solutions of $u_{xx} + u_{yy} + m^2 u = 0$.

HINT: Review the relevant exercises in section 5.3 and some basic properties of Bessel's functions; try separable solutions in studying the uniqueness of $u_{xx} + u_{yy} + m^2 u = 0$.

5.6 Harnack Estimates for Harmonic Functions

Another consequence of the Poisson representation formula on $B_R(0)$ is the Harnack estimates.

Theorem 5.18 (Harnack inequality)**.** (Local Version) *If u is a nonnegative harmonic function in $B_R(0)$, then*

$$\frac{R^{n-2}(R - |x|)}{(R + |x|)^{n-1}} u(0) \le u(x) \le \frac{R^{n-2}(R + |x|)}{(R - |x|)^{n-1}} u(0), \quad \textit{for all } x \in B_R(x_0).$$

In particular, for any $x \in B_{\frac{R}{2}}(0)$,

$$\frac{2^{n-2}}{3^{n-1}} u(0) \le u(x) \le 2^{n-2} 3\, u(0).$$

(Global Version) *For each connected open set $V \subset\subset U$, there exists a positive constant C depending on V and U, such that*

$$\sup_V u \le C \inf_V u$$

for all nonnegative harmonic function u in U.

Proof. The local version follows from (5.15) as follows. The Poisson kernel $K(x, y)$ for the ball $B_R(0)$ has the property

$$\frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}|(R + |x|)^n} \le K(x, y) \le \frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}|(R - |x|)^n} \quad \text{for all } y \in \partial B_R(0).$$

It then follows that

$$\begin{aligned} u(x) &= \int_{\partial B_R(0)} K(x,y)u(y)\,d\sigma(y) \\ &\le \int_{\partial B_R(0)} \frac{R^2-|x|^2}{R|\mathbb{S}^{n-1}|(R-|x|)|^n} u(y)\,d\sigma(y) \quad (\text{using } u(y)\ge 0) \\ &= \frac{R^{n-2}(R^2-|x|^2)}{(R-|x|)^n} u(0), \end{aligned}$$

where we used the mean value property for u; and

$$\begin{aligned} u(x) &= \int_{\partial B_R(0)} K(x,y)u(y)\,d\sigma(y) \\ &\ge \int_{\partial B_R(0)} \frac{R^2-|x|^2}{R|\mathbb{S}^{n-1}|(R+|x|)|^n} u(y)\,d\sigma(y) \\ &= \frac{R^{n-2}(R^2-|x|^2)}{(R+|x|)^n} u(0). \end{aligned}$$

The global version follows from a covering argument. There exists a finite cover of V by balls $\{B_r(x_i)\}_{i=1}^N$ such that each $B_{3r}(x_i) \subset U$. Further, the connectedness of V implies that we can choose these balls such that for each $i, 1 \le i \le N$, there exist $y_1, \cdots, y_m \in \{x_1, \cdots, x_N\}$, such that $m \le N$, $y_1 = x_1$, $y_m = x_i$, and $B_r(y_j) \cap B_r(y_{j+1}) \ne \emptyset$, for $1 \le j \le m-1$. In other words, each $B_r(x_i)$ is on a string of balls from this finite number of covering balls such that the consecutive ones have non-empty overlap, and the first one is $B_r(x_1)$. This guarantees that one can apply the local version of the Harnack estimate on each $B_{3r}(y_j)$ to get a fixed upper and lower bound, depending only on the dimension n, for the ratio $u(y_j)/u(y_{j+1})$ for any non-negative harmonic function on $B_{3r}(y_j)$. Now for any non-negative harmonic function in U and any $P_1, P_2 \in V$, each $P_k \in B_r(x_{i_k})$ for some i_k, $k = 1, 2$, then we can compare $u(P_k)$ with $u(x_{i_k})$, and compare $u(x_{i_1})$ with $u(x_{i_2})$ via the connected path of segments between the centers of the balls $\{B_r(x_i)\}_{i=1}^N$ and apply the local Harnack estimates to u at any two consecutive vertices to conclude the global Harnack estimate. □

With Harnack estimate, we obtain a strengthened version of Liouville Theorem.

Corollary 5.19. *A harmonic function on $\mathbb{R}^n$ bounded from below (or above) must be a constant.*

Proof. The Harnack estimate implies that a harmonic function $u(x)$ on $\mathbb{R}^n$ bounded from below, say $u(x) \ge -c$, gives rise to a non-negative harmonic function $u(x) + c$. For any $x \in \mathbb{R}^n$, one can then apply the Harnack inequality on $B_R(0)$ with $R > |x|$, to imply that $u(x) + c \le C(u(0) + c)$, where $C = C(n) > 0$ is the constant in the local Harnack estimate. We can now apply the usual Liouville Theorem to conclude that $u(x) + c$ is a constant. □

Exercises

Exercise 5.6.1. Suppose that $\{u_k\}$ is a sequence of harmonic functions in a connected domain U such that $u_k(x) \le u_{k+1}(x)$ for any $x \in U$. Suppose also that for some $x_0 \in U$, $u_k(x_0)$ converges. Prove that there exists a harmonic function $u(x)$ on U such that $\{u_k\}$ converges to u uniformly on any compact subdomain of U.

Exercise 5.6.2. Suppose that $u \in C^2(\overline{B_R^+(0)})$ is harmonic in $B_R^+(0)$, where $B_R^+(0) = \{x \in B_R(0) : x_n > 0\}$, and that $u(x', 0) = 0$ for $(x', 0) \in \overline{B_R^+(0)}$. Establish the Poisson representation

$$u(x) = \frac{R^2 - |x|^2}{R|\mathbb{S}^{n-1}|} \int_{\partial' B_R(0)} \left\{ \frac{u(y)}{(|x' - y'|^2 + |x_n - y_n|^2)^{n/2}} - \frac{u(y)}{(|x' - y'|^2 + |x_n + y_n|^2)^{n/2}} \right\} d\sigma(y),$$

for $x = (x', x_n) \in B_R^+(0)$, where $\partial' B_R(0) = \{x = (x', x_n) \in \partial B_R(0) : x_n \ge 0\}$ and $y = (y', y_n)$.

Assume, in addition, that $u(x', x_n) \ge 0$ in $B_R^+(0)$. Use the above representation to prove that there exists $c(n) > 0$ such that

$$c(n)^{-1} u(0, R/2) x_n / R \le u(x', x_n) \le c(n) u(0, R/2) x_n / R \quad \text{for all } x \in B_{R/2}^+(0).$$

Exercise 5.6.3. Suppose that $u \in C(\overline{\mathbb{R}_+^n})$ is a non-negative harmonic function on $\mathbb{R}_+^n$, and that $u(x', 0) = 0$ for all $(x', 0) \in \partial\mathbb{R}_+^n$. Prove that there exists $c \ge 0$ such that $u(x', x_n) = c x_n$ for all $(x', x_n) \in \mathbb{R}_+^n$. Does the same conclusion necessarily hold if the non-negativity of u is dropped?

5.7 Limit Theorems for Harmonic Functions and Applications

We now discuss applications of the convergence theorems of harmonic functions.

Theorem 5.20. *If, for a sequence $g_k \in C(\partial U)$, there exists a (unique) solution u_k to*

$$\begin{cases} \Delta u_k = 0, & \text{in } U, \\ u_k = g_k, & \text{on } \partial U, \end{cases}$$

and $g_k \to g$ uniformly on ∂U, then the Dirichlet problem with g as boundary value has a unique solution.

Next we discuss how to use the convergence theorems above to solve the Dirichlet problem on a ball without using the Poisson kernel. This consists of three steps. Let $\mathcal{P}_{\le k}$ denote polynomials of degree $\le k$.

Step 1. For any polynomial $p \in \mathcal{P}_{\le k-2}$, there exists a unique solution $v \in \mathcal{P}_{\le k}$ satisfying

$$\begin{cases} \Delta v = p, & \text{in } B_R(0), \\ v = 0, & \text{on } \partial B_R(0), \end{cases}$$

This follows from considering the linear map $T : \mathcal{P}_{\le k-2} \mapsto \mathcal{P}_{\le k-2}$ by $T(u) = \Delta\left[(R^2 - |x|^2)u(x)\right]$. The maximum principle implies that T has trivial kernel, so must be an isomorphism from $\mathcal{P}_{\le k-2}$ to $\mathcal{P}_{\le k-2}$.

Step 2. For any $p \in \mathcal{P}_{\le k-2}$ and $g \in \mathcal{P}_{\le k}$, there exists $u \in \mathcal{P}_{\le k}$ such that

$$\begin{cases} \Delta u = p, & \text{in } B_R(0), \\ u = g, & \text{on } \partial B_R(0), \end{cases} \tag{5.30}$$

This follows from Step 1. Since $\Delta g \in \mathcal{P}_{\le k-2}$, by Step 1, there exists $v \in \mathcal{P}_{\le k}$ solving

$$\begin{cases} \Delta v = p - \Delta g, & \text{in } B_R(0), \\ v = 0, & \text{on } \partial B_R(0), \end{cases}$$

The $u = v + g$ is the solution.

Step 3. For any $g \in C(\partial B_R(0))$, take a sequence of polynomials g_k such that $g_k \to g$ uniformly over $\partial B_R(0)$ as $k \to \infty$, and let u_k be the unique harmonic function in $B_R(0)$ whose boundary value on $\partial B_R(0)$ equals g_k. Then the maximum principle and convergence theorems provide a limit in $C(\overline{B}_R(0)) \cap C^2(B_R(0))$ which is a harmonic function with g as boundary value.

We can attempt to solve (5.30) for more general right hand side, as we can already solve it for polynomials. Take f to be continuous on $\overline{B}_R(0)$ for instance. We can take polynomials p_k and g_k such that $p_k \to f$ uniformly in $\overline{B}_R(0)$, and $g_k \to g$ uniformly on $\partial B_R(0)$ as $k \to \infty$. Let u_k denote the corresponding (polynomial) solution. By Proposition 5.2, we have uniform bound on $||u_k||_{C(\overline{B}_R(0))}$. In fact, $\{u_k\}$ is Cauchy in $C(\overline{B}_R(0))$. But to prove the convergence of $\{u_k\}$ to a solution of (5.30), we need higher derivative estimates for $\{u_k\}$. This can be done using the Green's representations, under appropriate smoothness assumptions on f.

Remark 5.12. For any $p > n$, there exist a constant $C(p,n) > 0$ such that for any $u \in C^2(\overline{B_R(x_0)})$,

$$\begin{aligned} R|\nabla u(x_0)| &\leq C(p,n)\left(R^{-n}||u(x)||_{L^1(B_R(x_0)} + R^{2-\frac{n}{p}}||\Delta_x u||_{L^p(B_R(x_0))}\right), \\ R||\nabla u||_{C(B_{\frac{R}{2}}(x_0))} &\leq C(p,n)\left(R^{-n}||u(x)||_{L^1(B_R(x_0)} + R^{2-\frac{n}{p}}||\Delta_x u||_{L^p(B_R(x_0))}\right). \end{aligned} \tag{5.31}$$

For any $v \in C^2(\overline{B_1(0)})$, we already established the following variant of Green's representation on $v(z)$, $z \in B_{1/2}(0)$:

$$v(z) = \int_{B_1(0)} \left[v(y)\Delta_y\left(\Phi(y-z)\eta(y)\right) - \left(\Delta_y v(y)\right)\Phi(y-z)\eta(y)\right] dy.$$

Here $\eta(y)$ is a smooth cut-off function such that $\eta(y) \equiv 1$ in $B_{1/2}(0)$ and $z \in B_{1/2}(0)$. This can be further used to establish

$$\nabla_z v(z) = \int_{B_1(0)} \left[v(y)\nabla_z\Delta_y\left(\Phi(y-z)\eta(y)\right) - \left(\Delta_y v(y)\right)\nabla_z\Phi(y-z)\eta(y)\right] dy,$$

for $z \in B_{1/2}(0)$, which can be used to establish gradient estimate on $v(z)$ (either pointwise or integral), using $\Delta_y\left(\Phi(y-z)\eta(y)\right) = 2\nabla_y\Phi(y-z)\cdot\nabla_y\eta(y) + \Phi(y-z)\Delta_y\eta(y)$. For example, using $|\nabla_y| = \Delta_y\eta(y) = 0$ for $|y| \leq 1/2$, we see that $|\nabla_y^2\Phi(y)||\nabla_y\eta(y)| + +|\nabla_y\Phi(y)||\Delta_y\eta(y)| \leq C(n)$ for all $|y| \leq 1$, so it follows that

$$\begin{aligned} |\nabla_z v(0)| \leq &\int_{B_1(0)} \left[|v(y)|\left(|\nabla_y^2\Phi(y)||\nabla_y\eta(y)| + |\nabla_y\Phi(y)||\Delta_y\eta(y)|\right)\right. \\ &\left. +|\Delta_y v(y)||\nabla_y\Phi(y)||\eta(y)|\right] dy \\ \leq &C(n)\int_{B_1(0)} \left[|v(y)| + |\Delta_y v(y)||y|^{1-n}\right] dy. \end{aligned}$$

We can estimate $\int_{B_1(0)} |\Delta_y v(y)||y|^{1-n}\, dy$ under various conditions on $\Delta_y v(y)$. For example,

$$|\int_{B_1(0)} |\Delta_y v(y)||y|^{1-n}\, dy| \leq \max_{B_1(0)} |\Delta_y v(y)| \int_{B_1(0)} |y|^{1-n}\, dy \leq |\mathbb{S}^{n-1}| \max_{B_1(0)} |\Delta_y v(y)|;$$

while if we assume a bound on $||\Delta_y v(y)||_{L^p(B_1(0))}$ for $p > n$, then using Hölder's inequality, with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\begin{aligned} |\int_{B_1(0)} |\Delta_y v(y)||y|^{1-n}\, dy| &\leq ||\Delta_y v(y)||_{L^p(B_1(0))}\left(\int_{B_1(0)} |y|^{(1-n)p'}\, dy\right)^{\frac{1}{p'}} \\ &\leq C(p,n)||\Delta_y v(y)||_{L^p(B_1(0))}, \end{aligned}$$

since $\int_{B_1(0)} |y|^{(1-n)p'}\, dy = |\mathbb{S}^{n-1}| \int_0^1 r^{(1-n)p'+n-1} dr = |\mathbb{S}^{n-1}| \int_0^1 r^{\frac{1-n}{p-1}} dr < \infty$ when $p > n$.

Finally note that $\Delta_y v(y) = R^2 \Delta_x u(x_0 + Ry)$, the gradient estimate above turns into the dimensionless form (5.31).

We could obtain second derivative estimates of $v(z)$ in $B_{1/2}$ in terms of $||\Delta v||_{C^1(B_1)}$ using the Green's representation, but we take this opportunity to introduce Bernstein's Gradient Estimates.

Theorem 5.21. *Let $u \in C^4(\overline{B}_1)$ and denote Δu by f. Then for any $r \in (0,1)$, there is a constant $N = N(n,r)$ such that in B_r*

$$|u| + |\nabla u| + |\nabla^2 u| \leq N \left(\max_{B_1} |f| + \max_{B_1} |\nabla f| + \max_{B_1} |\nabla^2 f| + \max_{\partial B_1} |u| \right)$$

The idea of Bernstein is to verify that some auxiliary function involving u and its derivatives satisfies an appropriate differential inequality (subharmonic, for instance) thus satisfies the maximum principle. The advantage of this method is that it is very flexible and applies even to some nonlinear equations; the drawback is the requirement on the higher derivatives of f.

We will illustrate the method by verifying that for any smooth cut-off function ζ supported in B_1 and identically equal to 1 on B_r, the function $w = \zeta^2 |\nabla u|^2 + Cu^2$ satisfies in B_1, for C large depending on r,

$$\Delta w \geq -\zeta^2 |\nabla f|^2 - Cf^2 - Cu^2 \geq -\tilde{N} \left(\max_{B_1} |f|^2 + \max_{B_1} |\nabla f|^2 + \max_{B_1} |u|^2 \right) \tag{5.32}$$

for some $\tilde{N}$ depending on C. A generalization of Proposition 5.2 applied to (5.32) then implies that

$$\max_{B_1} w \leq N \left(\max_{B_1} |f|^2 + \max_{B_1} |\nabla f|^2 + \max_{B_1} |u|^2 \right) + \max_{\partial B_1} w \tag{5.33}$$

for some $N > 0$. Note that $\max_{\partial B_1} w = C \left(\max_{\partial B_1} |u|\right)^2$. Then using Proposition 5.2 again to estimate $\max_{B_1} |u|^2$ in terms of $\max_{B_1} |f|^2$ and $\max_{\partial B_1} |u|^2$, it follows from (5.33) by estimating the left hand side on B_r that

$$\max_{B_r} |\nabla u|^2 \leq \max_{B_1} w \leq N \left(\max_{B_1} |f|^2 + \max_{B_1} |\nabla f|^2 + \max_{\partial B_1} |u|^2 \right).$$

(5.32) follows by noting that

$$\Delta |\nabla u|^2 = 2 \sum_{i,j=1}^n u_{ij}^2 + 2 \sum_i u_i (\Delta u)_i,$$

and

$$\Delta u^2 = 2|\nabla u|^2 + 2u\Delta u.$$

In order to obtain estimates for $|\nabla u|^2$ not dependent on the boundary behavior of u, we work with

$$\begin{aligned}\Delta(\zeta^2|\nabla u|^2) =& 2\zeta^2(\sum_{i,j=1}^n u_{ij}^2 + \sum_i u_i(\Delta u)_i) \\ &+ (\Delta\zeta^2)|\nabla u|^2 - 8\zeta\sum_{i,j=1}^n \nabla_i\zeta\nabla_j u\nabla_{ij}u\end{aligned}$$

Using

$$-8\zeta\sum_{i,j=1}^n \nabla_i\zeta\nabla_j u\nabla_{ij}u \geq -\zeta^2(\sum_{i,j=1}^n u_{ij}^2) - 16|\nabla\zeta|^2|\nabla u|^2,$$

and $2\sum_i u_i(\Delta u)_i) \geq -|\nabla u|^2 - |\nabla f|^2$, we have

$$\Delta(\zeta^2|\nabla u|^2) \geq \zeta^2(\sum_{i,j=1}^n u_{ij}^2) - (2\zeta^2 - \Delta\zeta^2 + 16|\nabla\zeta|^2)|\nabla u|^2 - \zeta^2|\nabla f|^2.$$

Thus we can choose $C > 0$ depending on r and n to satisfy (5.32)—note that the bound on the right hand side now does not involve ∇u or $\nabla_{ij}u$.

A similar construction proves the second derivative estimate.

As a consequence of the derivative estimates obtained through Bernstein's method, we have

Corollary 5.22. *Suppose that $f \in C(\overline{B}_R) \cap C^2(B_R)$ and $g \in C(\partial B_R)$. Then there exists a unique $u \in C(\overline{B}_R) \cap C^2(B_R)$ solving*

$$\begin{cases} \Delta u(x) = f(x) & \text{in } B_R, \\ u(x) = g(x) & \text{on } \partial B_R. \end{cases} \tag{5.34}$$

Remark 5.13. The corollary above can be obtained under the weaker assumption of $f \in L^\infty(B_R) \cap C^1(B_R)$ with the help of Proposition 5.12. The advantage of Bernstein's method is its flexibility and robustness.

The idea of constructing solution through an approximation procedure can be applied in many other different contexts. The key is to obtain appropriate estimates for sufficiently smooth solutions, the so called *a priori* estimates. For instance, we already obtained the following estimate for any $u \in C^2(\overline{B}_R)$:

$$R\max_{\overline{B}_{R/2}}|\nabla u| \leq C(n)\left(\max_{\overline{B}_R}|u| + R^2\max_{\overline{B}_R}|\Delta u|\right). \tag{5.35}$$

(5.35) and (ii) of Proposition 5.2 imply that if $\{\Delta u_j\}$ is Cauchy in $C(\overline{U})$ and $\{u_j|_{\partial U}\}$ is Cauchy in $C(\partial U)$, then $\{u_j\}$ is Cauchy in $C(\overline{U})$ and $\{\nabla u_j\}$ is Cauchy in $C(V)$ in any subdomain $V \subset\subset U$. So if U is a bounded domain such that (5.34) has a (unique) solution for any sufficiently smooth f and g, then for any $f \in C(\overline{U})$ and $g \in C(\partial U)$, we can use smooth f_j and g_j to approximate f and g in $C(\overline{U})$ and $C(\partial U)$, respectively. Let u_j be the corresponding solution, then there is a limit $u_\infty \in C(\overline{U}) \cap C^1(U)$ such that $u_j \to u_\infty$ uniformly on $\overline{U}$. $u_\infty = g$ on ∂U obviously. u_∞ satisfies $\Delta u_\infty = f$ in U in the following sense:

$$\int_U u(x)\Delta\eta(x)\,dx = -\int_U \nabla u(x)\cdot\nabla\eta(x)\,dx = \int_U f(x)\eta(x)\,dx, \text{ for all } \eta \in C_c^2(U).$$

One question to be addressed is the issue of regularity: whether a function satisfying the above integral relation has appropriate improved regularity (higher differentiability) if f has some appropriate regularity assumptions. Another issue is the uniqueness. It is equivalent to the following: suppose $u \in C(\overline{U})$ satisfies $\int_U u(x)\Delta\eta(x)\,dx = 0$ for all $\eta \in C_c^2(U)$, and $u = 0$ on ∂U, then $u = 0$ in U.

This can be settled with the help of the following Weyl Lemma.

Lemma 5.23 (Weyl). *Suppose that $u \in C(U)$ satisfies $\int_U u(x)\Delta\eta(x)\,dx = 0$ for all $\eta \in C_c^2(U)$, then $u \in C^\infty(U)$ and $\Delta u(x) = 0$ in U.*

Proof. Let $\rho(x)$ be a smooth, non-negative, even, cut-off function on $\mathbb{R}^n$ such that $supp(\rho) \subset B_2$ and $\int_{B_2} \rho(x)\,dx = 1$. Define $\rho_\epsilon(x) = \epsilon^{-n}\rho(x/\epsilon)$. For any compact subdomain $V \subset\subset U$, there exists another compact subdomain W such that $V \subset\subset W \subset\subset U$. Define

$$\tilde{u}(x) = \begin{cases} u(x) & \text{for } x \in W \\ 0 & \text{otherwise} \end{cases}$$

and $\tilde{u}_\epsilon = \tilde{u} * \rho_\epsilon$, where $*$ is the convolution operation. Then $\tilde{u}_\epsilon$ is smooth in W, and $\tilde{u}_\epsilon \to u$ uniformly in V as $\epsilon \to 0$.

Next, we show that, in fact, $\tilde{u}_\epsilon$ is harmonic in V, thus it follows from the convergence theorems that u is harmonic in V. Since V is an arbitrary compact subdomain of U, this will prove that u is harmonic in U.

For any $\eta \in C_c^2(V)$ and $\epsilon > 0$ small, $\rho_\epsilon * \eta \in C_c^\infty(W)$, we have

$$\begin{aligned}
&\int_U \tilde{u}_\epsilon(x)\Delta\eta(x)\, dx \\
&= \int_U \left(\int_U \tilde{u}(y)\rho_\epsilon(x-y)\, dy\right) \Delta\eta(x)\, dx \\
&= \int_U \tilde{u}(y) \left(\int_U \rho_\epsilon(x-y)\Delta\eta(x)\, dx\right) dy \\
&= \int_U \tilde{u}(y) \left(\int_U \Delta_x\rho_\epsilon(x-y)\eta(x)\, dx\right) dy \\
&= \int_U \tilde{u}(y) \left(\int_U \Delta_y\rho_\epsilon(x-y)\eta(x)\, dx\right) dy \\
&= \int_U \tilde{u}(y)\Delta_y \left(\int_U \rho_\epsilon(x-y)\eta(x)\, dx\right) dy \\
&= \int_U \tilde{u}(y)\Delta_y \left(\rho_\epsilon * \eta(y)\right)\, dy.
\end{aligned}$$

Since $\rho_\epsilon * \eta \in C_c^\infty(W)$, the last integral above is equal to $\int_U u(y)\Delta_y\left(\rho_\epsilon * \eta(y)\right)\, dy$, which is 0 by the assumption on u. Thus

$$\int_U \Delta_x \tilde{u}_\epsilon(x)\eta(x)\, dx = \int_U \tilde{u}_\epsilon(x)\Delta_x\eta(x)\, dx = 0$$

for all $\eta \in C_c^2(V)$, which implies that $\tilde{u}_\epsilon(x)$ is a smooth harmonic function in V.

This argument will also work if one replaces the assumption that $u \in C(U)$ by $u \in L^1_{\text{local}}(U)$. Instead of getting $\tilde{u}_\epsilon \to u$ uniformly in V as $\epsilon \to 0$, we will first get $\tilde{u}_\epsilon \to u$ in $L^1(V)$, which, together with the mean value property for harmonic functions, as applied to $\tilde{u}_\epsilon$, will imply the $C^0(V)$ bound of $\tilde{u}_\epsilon$; in fact it implies that $\tilde{u}_\epsilon$ is Cauchy in $C^0(V)$, thus implying $C^0(V)$ convergence of $\tilde{u}_\epsilon \to u$, which then implies that $u \in C^\infty(V)$. □

Remark 5.14. Another possible way to define a generalized solution of (5.34) is to formulate an integral equation using the Green's representation (5.27) in terms of the Green's function $G(x, y)$ of the corresponding homogeneous problem: a $C^2(U) \cap C(\bar{U})$ solution $u(x)$ of (5.34) would satisfy

$$u(x) = -\int_U G(x, y)f(y)\, dy + \int_{\partial U} g(y)\frac{\partial G(x, y)}{\partial \nu(y)}\, d\sigma(y),$$

and we may simply use this representation as a generalized solution of (5.34). Recall that if $f \in C(\bar{U})$ or simply $L^\infty(U)$, we know that the right hand side is at least in

$C^1(\bar{U})$; we just don't know that it is in $C^2(U)$. We can even use such a formulation to set up a possible scheme to construction a solution which is a perturbation of (5.34):

$$\begin{cases} \Delta u(x) + \sum_{i=1}^{n} b_i(x) u_{x_i}(x) + c(x)u(x) = f(x) & x \in U, \\ u(x) = g(x) & x \in \partial U. \end{cases} \tag{5.36}$$

We can try to set up an iteration scheme $u_k(x) \mapsto u_{k+1}(x)$, where

$$u_{k+1}(x) = \int_U G(x,y) \left[\sum_{i=1}^{n} b_i(y)(u_k)_{x_i}(y) + c(y)u_k(y) - f(y) \right] dy + \int_{\partial U} g(y) \frac{\partial G(x,y)}{\partial \nu(y)} \, d\sigma(y),$$

and try to show that it has a fixed point. Based on our discussion, if $b_i(x), c(x) \in L^\infty(U)$, then any $u_k \in C^1(\bar{U})$ would certainly give $u_{k+1} \in C^1(\bar{U})$. When $||b_i||_{L^\infty(U)}$ and $||c||_{L^\infty(U)}$ are small, we can easily prove that the map defined by the right hand side is a contraction in $C^1(\bar{U})$, thus has a fixed point. We will discuss later situations where the smallness condition can be removed.

If one tries to extend this perturbative approach to allow perturbation to the highest order derivatives terms, say, replacing $\Delta u(x)$ by $\sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j}(x)$, where $(a_{ij}(x))$ is close to (δ_{ij}) in appropriate sense, then one encounters the difficulty that for $u \in C^2(\bar{U})$, it is not known that $\int_U G(x,y)\left(\delta_{ij} - a_{ij}(y)\right) u_{x_i x_j}(y)\, dy$ is in $C^2(\bar{U})$, so we can't set up an appropriate map in $C^2(\bar{U})$. As mentioned earlier, if $\left(\delta_{ij} - a_{ij}(y)\right) u_{x_i x_j}(y)$ has some Hölder continuity, then the integral operator would return a function whose second derivatives have the same Hölder continuity. This discussion indicates that to solve a similar BVP for a more general equation of the kind $\sum_{i,j=1}^n a_{ij}(x) u_{x_i x_j}(x) + \sum_{i=1}^n b_i(x) u_{x_i}(x) + c(x)u(x) = f(x)$, there is a need to work in the class where the solutions $u_{x_i x_j}(x)$ and the coefficients $a_{ij}(x)$, $b_i(x), c(x)$, have some Hölder continuity.

Exercises

Exercise 5.7.1. Provide details for a proof of the upper bound for $|\nabla^2 u|$ in Theorem 5.21.

Exercise 5.7.2. Prove that if λ is an eigenvalue of $\Delta_{\boldsymbol{\omega}}$ on the round unit sphere $\mathbb{S}^{n-1}$, namely, there exists $\phi(\boldsymbol{\omega})$, not identically 0 on $\mathbb{S}^{n-1}$, such that $\Delta_{\boldsymbol{\omega}} \phi(\boldsymbol{\omega}) + \lambda \phi(\boldsymbol{\omega}) = 0$ on $\mathbb{S}^{n-1}$, then $\lambda = k(k+n-2)$ for some $k \in \mathbb{Z}_{\geq 0}$. (Hint: Using $\lambda \geq 0$, one can find a $k \in \mathbb{R}_{\geq 0}$ such that $\lambda = k(k+n-2)$. Then prove $|x|^k \phi(x/|x|)$ is harmonic on $\mathbb{R}^n$ appealing to Weyl's Lemma, and use this to prove that $k \in \mathbb{Z}_{\geq 0}$.)

5.8 Poincaré's Balayage Method

Next we will use the maximum principle and the convergence theorems to find a harmonic function with prescribed (continuous) Dirichlet data on fairly general domains. I will describe Poincaré's balayage method, which can be considered as a precursor of the Perron method. The latter is presented in most modern treatment.

Poincaré's method, as well as Perron's, depends on the maximum principle, the convergence theorems, and the solvability of Dirichlet problem on balls. The last we obtained by Poisson's kernel or the approximation method. We need to extend the notion of subharmonic functions to C^0 class.

Definition. A $C^0(U)$ function u is called subharmonic (superharmonic) in U, if for every ball $B \subset\subset U$ and every harmonic function h in B satisfying $u \leq (\geq) h$ on ∂B, we also have $u \leq (\geq) h$ inside B.

The maximum principle extends to these functions in the following way:

(i). Suppose U is a bounded domain. If u is subharmonic in U and is continuous up to the boundary of U, then $\max_{\overline{U}} u \leq \max_{\partial U} u$. In fact, the strong maximum principle also holds. If v is superharmonic in U, and $u \leq v$ on ∂U, then in any connected component of U either $u < v$, or $u \equiv v$.

(ii). If u is subharmonic in U and B is a ball strictly contained in U, then the harmonic lifting $h_B(u)$ of u in B is subharmonic in U, where $h_B(u)$ is defined as

$$h_B(u) = \begin{cases} \overline{u}(x), & \text{for } x \in B \ , \\ u(x), & \text{for } x \in U \setminus B, \end{cases}$$

with $\overline{u}(x)$ being the harmonic function in B satisfying $\overline{u}(x) = u(x)$ on ∂B.

(iii). If u and v are subharmonic in U, then so is $u + v$; if u is superharmonic in U and v is subharmonic in U, then $v - u$ is subharmonic in U.

(iv) If u and v are subharmonic in U, then so is $\max\{u, v\}$.

Poincaré's method consists of several steps. Given a domain U and a continuous boundary function g.

Step 1. Assume first that there exists a subharmonic $u_0 \in C(\overline{U})$ such that $u_0|_{\partial U} = g$.

Step 2. Cover U by a countable number of balls $\{B_1, B_2, \cdots\}$ such that each $B_i \subset\subset U$. Starting from u_0, we will replace each with its harmonic lifting on successive balls to obtain a sequence of monotone increasing subharmonic functions that are bounded from above. Thus a limiting function u exists. We will prove that this u is harmonic in U and will study its boundary behavior.

First let $B^{(1)} = B_1$ and define $u_1 = h_{B^{(1)}}(u_0)$. Then

(a) $u_1(x) \geq u_0(x)$, for all $x \in \overline{U}$.

(b) u_1 is harmonic in $B^{(1)}$.

(c) $u_1(x) \leq \max_{\partial U} g$, and $u_1(x) = g(x)$ on ∂U.

(d) u_1 is still subharmonic in U.

Order the balls in the following way

$$\begin{array}{ccccccccc} & B_1 & \to & B_2 & \hookleftarrow & & & & \\ \hookrightarrow & B_1 & \to & B_2 & \to & B_3 & \hookleftarrow & & \\ \hookrightarrow & B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \hookleftarrow \\ \hookrightarrow & \cdots & & & & & & & \end{array}$$

Define inductively, for each $i \geq 2$, $u_i(x)$ on U by $u_i = h_{B^{(i)}}(u_{i-1})$. Then

(a) $u_i(x) \geq u_{i-1}(x)$, for all $x \in \overline{U}$.

(b) u_i is harmonic in $B^{(i)}$.

(c) $u_i(x) \leq \max_{\partial U} g$, and $u_i(x) = g(x)$ on ∂U.

(d) u_i is still subharmonic in U.

Therefore $\{u_i\}$ is a sequence of monotone increasing, subharmonic, continuous functions in U, and is bounded from above. Thus $u(x) = \lim_{i\to\infty} u_i(x)$ is well defined for all $x \in U$. To prove u is harmonic in U, notice that for each $x \in U$, there is a ball $B_{i(x)}$ such that $x \in B_{i(x)}$. Notice also that because of the ordering of the balls, a subsequence of $\{u_i\}$, called $\{h_j(x)\}$, is actually *harmonic* in $B_{i(x)}$, and monotone in j. $u(x)$ is defined as a point-wise limit. But at this stage, one can prove, using Harnack estimate, that on any smaller $B \subset\subset B_{i(x)}$, $h_j \Longrightarrow u$ uniformly on B; one could also use the convergence theorem to prove that a subsequence of $\{h_j(x)\}$ converges uniformly on B. Thus u is harmonic in $B_{i(x)}$.

Step 3. The continuity in $\overline{U}$ of the solution u in the above step is handled by the concept of barriers.

Definition. $w \in C(\overline{U})$ is called a barrier function at $\xi \in \partial U$ for the Dirichlet problem on U if

a). w is superharmonic in U.

b). $w(\xi) = 0$, and $w(x) > 0$ for $x \in \overline{U} \setminus \{\xi\}$.

Suppose a barrier function w at $\xi \in \partial U$ exists. For any given $\epsilon > 0$, by the continuity of g at ξ, we can find $r_0 > 0$ such that $g(x) \leq g(\xi) + \epsilon$ for all $x \in \partial U$ with $|x - \xi| \leq r_0$. There also exists $M > 0$ depending on r_0, w and g such that $g(x) - g(\xi) \leq Mw(x)$ for all $x \in \partial U$ with $|x - \xi| \geq r_0$. Thus

$$g(x) \leq g(\xi) + \epsilon + Mw(x), \quad \text{for all } x \in \partial U.$$

Note that $g(\xi) + \epsilon + Mw(x)$ is a superharmonic function on U. So in the steps above, we also have $u_0 \leq u_i \leq g(\xi) + \epsilon + Mw$ on $\overline{U}$ by (iii). By the continuity of w and u_0 at ξ, we can find $0 < r_1 < r_0$ such that when $x \in U$ and $|x - \xi| \leq r_1$, $Mw(x) \leq \epsilon$ and $u_0(x) \geq g(\xi) - \epsilon$. Thus

$$g(\xi) - \epsilon \leq u_0(x) \leq u(x) \leq g(\xi) + 2\epsilon, \quad \text{for all } x \in U \text{ with } |x - \xi| \leq r_1,$$

proving the continuity of u at ξ.

Step 4. To remove that assumption in Step 1, we first argue that for any polynomial g given, $g_1 = g + Ax_1^2$ and $g_2 = Ax_1^2$ are subharmonic in U for $A > 0$ sufficiently large. Thus the first three steps can be applied to show that the Dirichlet problem has solutions with g_1 and g_2 as boundary values, thus it also has one with $g_1 - g_2 = g$ as boundary value. Finally the convergence theorems can be used to prove the existence of solution of the Dirichlet problem with any given continuous boundary value, provided the barrier argument in Step 3 can be carried out. That turns out to depend only on the geometry of the domain U.

An easily verified criterion for the existence of a barrier at $\xi \in \partial U$ is the existence of some exterior ball, *i.e.*, there exists a ball B such that $\overline{B} \cap \overline{U} = \{\xi\}$. Let x_0 be the center of this ball and r be its radius, then $w(x) = r^{2-n} - |x - x_0|^{2-n}$ defines a barrier at ξ.

Definition. A boundary point ξ is called regular with respect to the Laplacian if there exists a barrier at that point.

We can now summarize our discussion as

Theorem 5.24. *The classical Dirichlet problem for the Laplacian in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.*

Points on components of the boundary with codimension 2 or higher are not regular. For instance, we can't solve the Dirichlet boundary value on the domain $B \setminus \{0\}$ with prescribed value everywhere on $\partial\{B \setminus \{0\}\} = \partial B \cup \{0\}$, as the following theorem shows

Theorem 5.25. *Suppose u is harmonic in $B \setminus \{0\}$, and satisfies $|u(x)| = o(|x|^{2-n})$ as $x \to 0$ (assume $n \geq 3$). Then u extends to a smooth harmonic function over B.*

Proof. We may assume that u is continuous up to ∂B. Let v be the unique solution in B to

$$\begin{cases} \Delta v = 0, & \text{in } B, \\ v = u, & \text{on } \partial B. \end{cases}$$

Then $w = u - v$ is still harmonic in $B \setminus \{0\}$, and satisfies $w(x)| = o(|x|^{2-n})$ as $x \to 0$. Furthermore $w(x) \equiv 0$ on ∂B. We prove $w \equiv 0$ in B in the following way: for any $\epsilon > 0$, we can find $r > 0$ such that $B_r \subset\subset B$, and on ∂B_r, $|w(x)| \leq \epsilon(|x|^{2-n} - r_0^{2-n})$ (r_0 is the radius of B). Then $\pm w(x) + \epsilon(|x|^{2-n} - r_0^{2-n})$ is a harmonic function on $B \setminus B_r$, and is nonnegative on $\partial(B \setminus B_r)$. Thus by the maximum principle $\pm w(x) + \epsilon(|x|^{2-n} - r_0^{2-n}) \geq 0$ in $B \setminus B_r$, i.e., $|w(x)| \leq \epsilon(|x|^{2-n} - r_0^{2-n})$ for all $x \in B \setminus B_r$. For any fixed $\bar{x} \in B \setminus \{0\}$, $\bar{x}$ is in $B \setminus B_r$ for all sufficiently small $\epsilon > 0$. Thus $|w(\bar{x})| \leq \epsilon(|\bar{x}|^{2-n} - r_0^{2-n})$, and by sending $\epsilon \to 0$, we conclude that $w(\bar{x}) = 0$. In conclusion, $w \equiv 0$ in B, and so $u \equiv v$, a smooth harmonic function in B. □

Remark 5.15. If one examines Poincarè's method, one finds that it would work in a setting where the following can be verified: maximum principle in the form of (i)–(iii) above; solvability of the Dirichlet problem on small domains; the convergence properties in Theorem 6 (part (ii) depends on Harnack estimates while the other parts depend only on gradient estimates); and regularity of the boundary. Perron made a modification of Poincarè' method so that one does not have to start with Step 1 above and needs not to have Harnack estimate; instead, the argument relies on the strong maximum principle to the difference of two solutions. The method gives a prospective solution for a boundary function that needs not be continuous and isolates the continuity of the constructed solution near a continuity point of the boundary function as a local problem. Gilbarg and Trudinger give a complete presentation of Perron's method in their monograph [7]. Below is a sketch of the main steps in Perron's method. Let

$$S_g = \{v \in C(\bar{U}) : v \text{ subharmonic in } U \text{ and } v \le g \text{ on } \partial U\},$$

and define $u(x) = \sup_{v\in S_g} v(x)$.

(i). Prove that $u(x)$ is well defined and is harmonic in U. This is done by picking any $y \in U$ and a ball $B_r(y)$ such that $\overline{B_r(y)} \subset U$, picking a sequence $v_j \in S_g$ such that $v_j(y) \to u(y)$ as $j \to \infty$, then replacing v_j by $h_{B_r(y)}(v_j)$ and showing that $h_{B_r(y)}(v_j) \in S_g$, $h_{B_r(y)}(v_j)(x) \to h(x)$ in $B_r(y)$, with $\Delta h(x) = 0$, $h(x) \le u(x)$, in $B_r(y)$ and $h(y) = u(y)$, and finally proving that $u(x) = h(x)$ in $B_r(y)$, therefore harmonic in $B_r(y)$. The last step is proved by showing that if $h(z) < u(z)$ for some $z \in B_r(y)$, then a similar argument centered at z would produce a harmonic function $h'(x)$ in $B_r(y)$ such that $h'(x) \ge h(x)$ in $B_r(y)$, $h'(y) = h(y)$, but $h'(z) = u(z) > h(z)$, which would be a contradiction to the strong maximum principle.

(ii). If g is continuous at $y \in \partial U$ and there exists a barrier function $w(x)$ at y, then for any $\epsilon > 0$ one can choose k such that

$$g(y) - \epsilon - kw(x) \le g(x) \le g(y) + \epsilon + kw(y) \text{ for } x \in \partial U,$$

and the $u(x)$ constructed in the previous step would satisfy

$$g(y) - \epsilon - kw(x) \le u(x) \le g(y) + \epsilon + kw(y) \text{ for } x \in U.$$

Now sending $x \to y$ and using the continuity of w at y shows the continuity of $u(x)$ at y.

The notion of local barrier is sometimes useful.

Definition. w is called a local barrier function at $\xi \in \partial U$ for the Dirichlet problem on U if there is a ball B centered at ξ such that $w \in C(\overline{U} \cap \overline{B})$ and

a). w is superharmonic in $U \cap B$.

b). $w(\xi) = 0$, and $w(x) > 0$ for $x \in \left(\overline{U} \cap \overline{B}\right) \setminus \{\xi\}$.

If a local barrier w at $\xi \in \partial U$ exists on $\overline{U} \cap \overline{B}$, it is easy to see that we can apply the barrier argument in Step 3 on $\overline{U} \cap \overline{B}$; or alternatively we can define $m = \min_{\partial B \cap U} w$ and

$$\tilde{w}(x) = \begin{cases} \min(w(x), m), & \text{for } x \in U \cap B, \\ m, \quad \text{for } x \in U \setminus B, \end{cases}$$

then $\tilde{w}$ is a (global) barrier function at $\xi \in \partial U$.

We now describe briefly how a variant idea involving subsolutions applies to a subclass of (5.36).

Definition. A function $v \in C^2(U) \cap C(\bar{U})$ is called a subsolution of (5.36), if

$$\begin{cases} \Delta v(x) + \displaystyle\sum_{i=1}^{n} b_i(x) v_{x_i}(x) + c(x)v(x) \geq f(x) & x \in U, \\ \qquad\qquad\qquad\qquad\qquad\qquad v(x) \leq g(x) & x \in \partial U. \end{cases}$$

Likewise, a function $w \in C^2(U) \cap C(\bar{U})$ is called a supersolution of (5.36), if

$$\begin{cases} \Delta w(x) + \displaystyle\sum_{i=1}^{n} b_i(x) w_{x_i}(x) + c(x)w(x) \leq f(x) & x \in U, \\ \qquad\qquad\qquad\qquad\qquad\qquad w(x) \geq g(x) & x \in \partial U. \end{cases}$$

The method we discuss now deals with (5.36) in the case $b_i(x) = 0$, and is called the method of sub-and-super solutions. We will first discuss the case of $b_i(x) = 0$ and $c(x) \geq 0$, and assume that there exists a subsolution v to (5.36) in this case, then we examine the iteration scheme with $u_1 = v$, and

$$\begin{cases} \Delta u_{k+1}(x) = -c(x)u_k(x) + f(x) & x \in U, \\ \quad u_{k+1}(x) = g(x) & x \in \partial U. \end{cases}$$

Note that the right hand side is a decreasing function of u_k under our assumption. Assuming appropriate regularity of $f(x)$ and $c(x)$, and boundary regularity of ∂U, u_{k+1} can be constructed from correcting the Newton potential of $-c(x)u_k(x) + f(x)$ in U by an appropriate harmonic function. Other possible approaches include using the framework of the variational method to construct a generalized solution, or using (5.27) to construct a generalized solution.

What is crucial here is that the maximum principle applies, even for generalized solutions, and is used to prove $u_{k+1}(x) \geq u_k(x)$! First $u_2(x) \geq u_1(x)$ in U as a consequence of $\Delta u_2(x) \leq \Delta u_1(x)$ for $x \in U$, and $u_2(x) = g(x) \geq u_1(x)$ for $x \in \partial U$. Next, using $\Delta[u_{k+1}(x) - u_k(x)] = -c(x)[u_k(x) - u_{k-1}(x)] \leq 0$ together with the assumption $c(x) \geq 0$ and the induction hypothesis $u_k(x) - u_{k-1}(x) \geq 0$ in U, it follows that that $u_{k+1}(x) - u_k(x) \geq 0$ in U, and that $\Delta u_{k+1}(x) = -c(x)u_k(x) + f(x) \geq -c(x)u_{k+1}(x) + f(x)$ in U, namely, u_{k+1} continues to be a subsolution.

In order to get a convergence sequence $\{u_k(x)\}$, we need some upper bound on $u_k(x)$. This is guaranteed if we assume that there exists a supersolution $w(x)$ such that $w(x) \geq v(x)$ in U. Then $\Delta u_1(x) \geq -c(x)w(x) + f(x) \geq \Delta w(x)$ in U, again using the sign of $c(x)$. Thus $u_1(x) \leq w(x)$ in U. The same argument works for each $u_k(x)$ inductively to deduce that $u_k(x) \leq w(x)$ in U for all k. Thus we conclude that $\{u_k(x)\}$ converges in U. What remains is to use the derivative estimates to conclude that the limit satisfies the equation and the boundary condition.

Our description of this method relies on our ability to solve the Dirichlet problem for the Laplace operator. The variational approach can be adapted to solve the Dirichlet problem for $\Delta - \lambda$ for any $\lambda > 0$. Using this, we can adapt the approach to solve the Dirichlet problem for $\Delta u(x) = -c(x)u_k(x) + f(x)$ without any assumptions on the sign of $c(x)$ or even for nonlinear $f(x, u(x))$, provided that there exists a pair of subsolution $v(x)$ and supersolution $w(x)$ such that $v(x) \leq w(x)$ in U. We just need to choose $\lambda > 0$ large enough such that $f(x, u) - \lambda u$ is non-increasing for (x, u) such that $v(x) \leq u \leq w(x)$, and modify our scheme into

$$\begin{cases} \Delta u_{k+1}(x) - \lambda u_{k+1}(x) = f(x, u_k(x)) - \lambda u_k(x) & x \in U, \\ u_{k+1}(x) = g(x) & x \in \partial U. \end{cases}$$

Example 5.1. We illustrate this method by showing that the Dirichlet problem

$$\begin{cases} \Delta u(x) = u(x)^3 & x \in U, \\ u(x) = g(x) & x \in \partial U, \end{cases}$$

has a solution for any $g \in C(\partial U)$, where we also assume that all boundary points of ∂U are regular with respect to the Dirichlet problem for the Laplacian.

According to our discussion, the method is applicable if we are able to construct a subsolution $v(x)$ and supersolution $w(x)$ such that $v(x) \leq w(x)$ in U, and $v(x) \leq g(x) \leq w(x)$ on ∂U. The simplest possible candidates for subsolutions or supersolutions are constants. If we take $w(x) = \max\{0, \max_{\partial U} g\}$, we see that $\Delta w(x) \leq w(x)^3$ in U, and $w(x) \geq g(x)$ on ∂U. Likewise, if we take $v(x) = \min\{0, \min_{\partial U} g\}$, we see that $\Delta v(x) \geq v(x)^3$ in U, and $v(x) \leq g(x)$ on ∂U. Furthermore, $v(x) \leq w(x)$ in U, so we can apply the method of subsolutions and supersolutions to conclude that there exists a solution $u(x)$, which satisfies $\min\{0, \min_{\partial U} g\} \leq u(x) \leq \max\{0, \max_{\partial U} g\}$ in U.

Exercises

Exercise 5.8.1. Suppose u is bounded, harmonic in $B \setminus K$, where $K = \{x = (0, 0, x_3, \cdots, x_n) \in B : x_3^2 + \cdots + x_n^2 \leq \frac{1}{2}\}$ (assume $n \geq 3$). Then u

extends to a smooth harmonic function over B. The same conclusion holds when K is a compact submanifold of B with codimension 2 or higher; one could use the Newtonian potential of unit density of K to play the role of $|x|^{2-n}$ above.

Exercise 5.8.2. Prove that the $\tilde{w}(x)$ constructed above is a (global) barrier function at $\xi \in \partial U$.

Exercise 5.8.3. In this exercise, you are asked to construct a barrier at the vertex of a cone K_{θ_0} of opening angle θ_0 for the Dirichlet problem on K_{θ_0}, where $K_{\theta_0} = \{x \in \mathbb{R}^3 : x_3 = |x|\cos\theta, 0 \le \theta \le \theta_0\}$, and $0 < \theta_0 < \pi$. You may construct a barrier in the form of $u(x) = |x|^\alpha \Theta(\theta)$ for appropriate choice of $\alpha > 0$, $\Theta(\theta) : \Theta(\theta) > 0$ for $0 \le \theta < \theta_0$, and $\Theta(\theta_0) = 0$. Note that the condition

$$0 \ge \Delta(|x|^\alpha \Theta(\theta)) = [\alpha(\alpha+1)\Theta(\theta) + \Delta_{\mathbb{S}^2}\Theta(\theta)]\,|x|^{\alpha-2}$$

turns into

$$\Delta_{\mathbb{S}^2}\Theta(\theta) = \sin^{-1}(\theta)(\sin(\theta)\Theta'(\theta))' \le -\alpha(\alpha+1)\Theta(\theta)$$

for some $\alpha > 0$. A positive eigenfunction of $\Delta_{\mathbb{S}^2}$ on $K_{\theta_0} \cap \mathbb{S}^2$ with zero Dirichlet boundary value would do, the existence of which can be obtained by solving the one-dimensional variational problem:

$$\min\{\int_0^{\theta_0} |\Theta'(\theta)|^2 \sin\theta \; d\theta : \int_0^{\theta_0} |\Theta(\theta)|^2 \sin\theta \; d\theta = 1, \Theta(\theta_0) = 0\}.$$

Explain how the argument and conclusion breaks down when $\theta_0 = \pi$.

Exercise 5.8.4. Assume that all boundary points of ∂U are regular with respect to the Dirichlet problem for the Laplacian. Given any $g \in C(\partial U)$. Construct a subsolution $v(x)$ and a supersolution $w(x)$ to

$$\begin{cases} \Delta u(x) = \sin(|x|) + \cos u(x) & \text{in } U, \\ u(x) = g(x) & \text{on } \partial U, \end{cases}$$

such that $v(x) \le w(x)$ in U and prove that this problem has a solution.

We now provide an indirect argument that, in dimension 2 or higher, there are $f \in C(\overline{U})$ for which there is no $C^2(\overline{U})$ solution of $\Delta u(x) = f(x)$ in U! We work with $\overline{U}$ so as to use the Banach space structure of $X = \{u \in C^2(\overline{U}) : u = 0 \text{ on } \partial U\}$, and work with the bounded linear map $T : X \mapsto C(\overline{U})$, where $T(u) = \Delta u$. The norm for $f \in C(\overline{U})$ is the standard $|f|_{0;U} := \max_{\overline{U}} |f|$, while the norm for $u \in X$ is

$|u|_{2,0;U} := \sum_{|\alpha|=0}^{2} |\partial_x^\alpha u|_{0;U}$. Note that $\ker(T) = \{0\}$ by the uniqueness theorem. If for every $f \in C(\overline{U})$, there is a solution $u \in X$ to $\Delta u = f$ in U, then by the Banach theorem, there exists a constant $C > 0$ such that

$$|u|_{2,0;U} \leq C\,|\Delta u|_{0;U}\,, \quad \text{for all } u \in X. \tag{5.37}$$

But we can verify that such an inequality cannot hold for all $u \in X$ by simply testing on a family of functions which are appropriate cut off of some harmonic functions.

Exercise 5.8.5. (i) Assume $0 \in U$, and take $p(x) = x_1^2 - x_2^2$. Construct a family of cut-off functions $\eta \in X$ to show that $\eta(x)p(x)$ can not satisfy (5.37).

(ii) Take $U = B_1(0)$, and define $X_{\text{rad}} = \{u \in C^2(\overline{U}) : u(x) \text{ is radial in } x, \text{ and } u = 0 \text{ on } \partial U\}$. Prove that (5.37) holds for all $u \in X_{\text{rad}}$.

5.9 Additional Problems

Problem 5.9.1. Prove that under the assumption $c(x) \leq 0$ for $x \in U$ and $\alpha(x) \geq 0$ for $x \in \partial U$, there exists at most one solution u to

$$\begin{cases} \Delta u + c(x)u = f, & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} + \alpha(x)u(x) = g(x), & \text{on } \partial U, \end{cases}$$

in the class $C^2(U) \cap C^1(\overline{U})$, unless $c(x) \equiv 0$ and $\alpha(x) \equiv 0$, in which case the uniqueness is up to a constant. Give an example of the failure of the uniqueness when the condition on $c(x)$ or α is not satisfied.

Problem 5.9.2. Suppose that 0 is an interior point of the domain U in $\mathbb{R}^n$, $n \geq 3$, and $u(x)$ is a nonnegative harmonic function on $U \setminus \{0\}$. Prove that there exists a constant $A \geq 0$ and a smooth harmonic function $h(x)$ in U such that

$$u(x) = A|x|^{2-n} + h(x), \quad \text{for all } x \in U.$$

Problem 5.9.3. Suppose U is a bounded domain and $x_0 \in \partial U$. Let $u \in C(\overline{U} \setminus \{x_0\})$ be a bounded harmonic function in U such that $u \equiv 0$ on $\partial U \setminus \{x_0\})$. Prove that $u \equiv 0$ in U.

Problem 5.9.4. Let u be a bounded harmonic function on $U = \{x = (x', x_n) : 0 < x_n < h\}$. Prove that

$$\sup_{\overline{U}} |u| = \sup_{\partial U} |u|.$$

Problem 5.9.5. Let B^+ denote the half disk $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}$. Suppose $u \in C^2(B^+) \cap C(\overline{B}^+)$ is a solution of

$$\begin{cases} \partial_x^2 u + y\partial_y^2 u + c(x,y)u = f(x,y), & \text{in } B^+, \\ u(x,y) = g(x,y), & \text{on } \partial B^+. \end{cases} \tag{*}$$

1. There is at most one solution of (*) under the assumption $c(x,y) \le 0$.
2. Assume $-c_0 \le c(x,y) \le 0$ in B^+. Then there exists a constant $C > 0$ depending only on c_0 such that for any solution u to (*),

$$\max_{B^+} |u| \le C \left[\max_{B^+} |f| + \max_{\partial B^+} |g| \right].$$

Problem 5.9.6. (i). Prove that for a bounded domain U with its boundary being a piecewise C^1 hypersurface and any bounded measurable f defined on ∂U, $\int_{\partial U} \Phi(x-y) f(y)\, d\sigma(y)$ defines a $C^2(U) \cap C^0(\overline{U})$ harmonic function.

(ii). Prove that for any bounded measurable f defined in a bounded domain U, $\int_U \Phi(x-y) f(y)\, dy$ defines a $C^{1,\alpha}(U) \cap C^1(\overline{U})$ function for every $0 < \alpha < 1$, and

$$D_{x_i} \int_U \Phi(x-y) f(y)\, dy = \int_U D_{x_i} \Phi(x-y) f(y)\, dy.$$

Problem 5.9.7. For $x, y \in \partial U$, $x \neq y$, define $K(x,y) = \frac{\partial \Phi(x-y)}{\partial n(y)}$. Prove that, if ∂U is assumed to be a piecewise C^2 surface, then for any continuous function f defined on ∂U, $\int_{\partial U} \frac{\partial \Phi(x-y)}{\partial n(y)} f(y)\, d\sigma(y)$, $x \in U$, extends continuously to $\overline{U}$, and for any $\bar{x} \in \partial U$,

$$\lim_{x \in U, x \to \bar{x}} \int_{\partial U} \frac{\partial \Phi(x-y)}{\partial n(y)} f(y)\, d\sigma(y) = \frac{1}{2} f(\bar{x}) + \int_{\partial U} K(\bar{x}, y) f(y)\, d\sigma(y).$$

Problem 5.9.8. Here is a concrete construction of a $C^2(B)$ function f for which there is no C^2 solution of $\Delta u = f$ in any neighborhood of $(0,0)$. Let η be a radial smooth cut-off function in $C_0^\infty(\{(x,y) \in \mathbb{R}^2 : |(x,y)| < 2\})$ with $\eta \equiv 1$ when $|(x,y)| \le 1$. Let $c_k \to 0$ as $k \to \infty$ and $\sum_k c_k$ divergent, with $\sum_{k=1}^N c_k = o(2^N)$, and let $P(x,y) = x^2 - y^2$,

$$u_N(x,y) = \sum_{k=1}^N c_k \eta(2^k x, 2^k y) P(x,y).$$

Prove that $f(x,y) = \lim_{N\to\infty} \Delta u_N(x,y)$ is in $C(\bar{B})$, that $\widehat{u}(x,y) = \lim_{N\to\infty} u_N(x,y)$ is in $C^1(\bar{B})$, with $\widehat{u}(x,y) \equiv 0$ on ∂B, but not in $C^2(\bar{B})$, and that there is no C^2 solution of $\Delta u = f$ for such an f in any neighborhood of $(0,0)$.

Problem 5.9.9. Consider the Dirichlet problem

$$\begin{cases} \Delta u(x) - m^2 u(x) = 0 & \text{for } x \in \mathbb{R}^n_+, \\ u(x) = g(x') & \text{for } x = (x', 0) \in \partial\mathbb{R}^n_+. \end{cases}$$

(i). Use Fourier's method to derive that a solution $u(x)$ with appropriate behavior at ∞ has the representation

$$u(x) = \int_{\partial\mathbb{R}^n_+} g(\eta) P(x' - \eta, x_n)\, d\eta,$$

where

$$P(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi - \sqrt{|\xi|^2 + m^2}x_n} d\xi.$$

(ii). Use the method of descent to derive that a solution $u(x)$ with appropriate behavior at ∞ has the alternative representation

$$u(x) = \int_{\partial\mathbb{R}^n_+} g(\eta) \widehat{P}(x' - \eta, x_n)\, d\eta,$$

where

$$\widehat{P}(x', x_n) = \frac{2x_n I_{\frac{n+1}{2}}(m\sqrt{|x'|^2 + x_n^2})}{|\mathbb{S}^n| \left(|x'|^2 + x_n^2\right)^{\frac{n}{2}}},$$

and

$$I_{\frac{n+1}{2}}(s) = \int_{\mathbb{R}} \frac{\cos(sz)}{(1 + |z|^2)^{\frac{n+1}{2}}} dz.$$

(iii). The $I_{\frac{n+1}{2}}(s)$ is related to the Fourier transform of $(1 + |z|^2)^{-\frac{n+1}{2}}$. For odd n's, this can be done using the calculus of residues; but in general it is a non-trivial task. Prove that $J(s) = s^{-n/2} I_{\frac{n+1}{2}}(s)$ satisfies the modified Bessel's equation of order $\frac{n}{2}$: $s^2 J''(s) + sJ'(s) + \left[-s^2 - (\frac{n}{2})^2\right] J(s) = 0$. Use this relation to identify $I_{\frac{n+1}{2}}(s)$. Comparison with (5.28) (which corresponds to the $m = 0$ case) reveals that

$$I_{\frac{n+1}{2}}(0) = \int_{\mathbb{R}} \frac{1}{(1 + |z|^2)^{\frac{n+1}{2}}} dz = \frac{|\mathbb{S}^n|}{|\mathbb{S}^{n-1}|}.$$

(iv). Use the method of image to derive an alternative representation for a solution $u(x)$ in terms of the Green's function for this problem, which, in turn, is expressed in terms of solutions to the the modified Bessel's equation.

Part II

Some More General Methods and Results for BVPs, IVPs, or IBVPs

Chapter 6

Adjoint/Transpose Operators and Fundamental Solutions

6.1 Green's Identities and the Fundamental solution of the Heat Equation

We have used the fundamental solutions of the Laplace and heat equations on several occasions. We now introduce two other concepts that will play important roles in the study of PDEs: (i) adjoint/transpose operators and Green's identities, and (ii) fundamental solutions of more general linear differential operators. We will first introduce these concepts in the context of the study of heat equations.

Green's identities are based on the Green/Stokes/Divergence theorems applied to certain bilinear expressions associated to a differential operator. In the case of the heat equation, we have

$$\begin{aligned}&\left(\partial_t - \Delta_x\right) u(x,t) \cdot v(x,t) - u(x,t) \cdot \left(-\partial_t - \Delta_x\right) v(x,t) \\ =&\partial_t \left[u(x,t) \cdot v(x,t)\right] + \nabla \left[u(x,t) \cdot \nabla v(x,t) - v(x,t) \cdot \nabla u(x,t)\right],\end{aligned} \tag{6.1}$$

$-\partial_t - \Delta_x$ is called the adjoint/transpose operator of $\partial_t - \Delta_x$. Based on the divergence structure of the right hand side above, if $u(x,t)$ and $v(x,t)$ are in $C^{2,1}_{x,t}(\overline{U} \times [0,T])$, and U is a bounded domain, we can use Green's identities to derive

$$\begin{aligned}&\int_{t_0}^{t_1}\int_U (\partial_t-\Delta_x)u(x,t)\cdot v(x,t)\,dxdt\\&=\int_{t_0}^{t_1}\int_U u(x,t)\cdot(-\partial_t-\Delta_x)v(x,t)\,dxdt+\int_U u(x,t_1)\cdot v(x,t_1)\,dx-\int_U u(x,t_0)\cdot v(x,t_0)\,dx\\&\quad+\int_{t_0}^{t_1}\int_{\partial U}\left(u(x,t)\cdot\frac{\partial v(x,t)}{\partial n}-v(x,t)\cdot\frac{\partial u(x,t)}{\partial n}\right)\,d\sigma(x)dt\quad\text{for any } 0\le t_0<t_1\le T.\end{aligned}\tag{6.2}$$

We can get useful information by judicious choices of the pair $u(x,t)$ and $v(x,t)$. For instance, if $(\partial_t-\Delta_x)u(x,t)=f(x,t)\in C(\overline{U}\times[0,T])$, and we are interested in representing u in terms of f and the initial/boundary values of u, we take $v(x,t)=K(X-x,T+\epsilon-t)$ for some $\epsilon>0$ in (6.2), where

$$K(x,t)=\begin{cases}\frac{e^{-\frac{|x|^2}{4t}}}{(4\pi t)^{n/2}} & \text{if } t>0\\ 0 & \text{if } t\le 0\end{cases}$$

is the fundamental solution of the heat equation which we encountered earlier. Note that

$$(-\partial_t-\Delta_x)v(x,t)=(\partial_T-\Delta_X)K(X-x,T+\epsilon-t)=0$$

for $(x,t)\in U\times[0,T]$, and $\epsilon>0$ is inserted so as to make sure that $K(X-x,T+\epsilon-t)\in C^2(\overline{U}\times[0,T])$; then (6.2) with $t_0=0$ and $t_1=T$ would imply

$$\begin{aligned}&\int_U u(x,T)\cdot K(X-x,\epsilon)\,dx\\&=\int_U u(x,0)\cdot K(X-x,T+\epsilon)\,dx+\int_0^T\int_U f(x,t)\cdot K(X-x,T+\epsilon-t)\,dxdt\\&\quad-\int_0^T\int_{\partial U}\left(u(x,t)\cdot\frac{\partial K(X-x,T+\epsilon-t)}{\partial n(x)}-K(X-x,T+\epsilon-t)\cdot\frac{\partial u(x,t)}{\partial n(x)}\right)\,d\sigma(x)dt.\end{aligned}$$

Sending $\epsilon\searrow 0$, and noting that $\int_U u(x,T)\cdot K(X-x,\epsilon)\,dx\to u(X,T)$, we obtain

$$\begin{aligned}u(X,T)=&\int_U u(x,0)\cdot K(X-x,T)\,dx+\int_0^T\int_U f(x,t)\cdot K(X-x,T-t)\,dxdt\\&-\int_0^T\int_{\partial U}\left(u(x,t)\cdot\frac{\partial K(X-x,T-t)}{\partial n(x)}-K(X-x,T-t)\cdot\frac{\partial u(x,t)}{\partial n(x)}\right)\,d\sigma(x)dt.\end{aligned}\tag{6.3}$$

This representation can be used to read off the regularity of u in terms of that of f—the integral term $\int_U u(x,0)\cdot K(X-x,T)\,dx$ and the other two boundary integral terms provide C^∞ functions of (X,T) for $X \in U$ and $T>0$ using the smooth dependence of $K(X-x,T-t)$ on (X,T), $X \in U$, $T>0$, when $(x,t)=(x,0)$, or $x \in \partial U$.

Remark 6.1. If $u(x,t)$ satisfies $(\partial_t - \Delta_x)\, u(x,t) \le 0$ for $x \in \mathbb{R}^n$ and $0<t<T$, and has some growth control as $|x| \to \infty$, say, $u(x,t)$ bounded uniformly, then we apply (6.2) to $u(x,t)$ and $v(x,t) = K(X-x, T+\epsilon-t)\eta(x/R)$ on $B_R \times [t_0,t_1]$ for any $0<t_0<t_1<T$, where η is a non-negative cut-off function with compact support such that $\eta(y) \equiv 1$ for $|y| \le 1/2$. Using

$$(-\partial_t - \Delta_x) v(x,t) = -2R^{-1}\nabla K(X-x,T+\epsilon-t)\cdot\nabla\eta(x/R) - R^{-2}K(X-x,T+\epsilon-t)\Delta\eta(x/R)$$

and the assumed bounds on $u(x,t)$, by sending $R\to\infty$, then $\epsilon \to 0+$, we get

$$\int_{\mathbb{R}^n} u(x,t_1)K(X-x,T-t_1)\,dx \le \int_{\mathbb{R}^n} u(x,t_0)K(X-x,T-t_0)\,dx.$$

Namely, $t \mapsto \int_{\mathbb{R}^n} u(x,t)K(X-x,T-t)\,dx$ is monotone non-increasing for $0<t<T$. Such kind of properties are crucial for analyzing behavior of solutions of many nonlinear problems.

If u has compact support in $\mathbb{R}^{n+1}$, then we can apply (6.3) to obtain

$$u(X,T) = \iint_{\mathbb{R}^{n+1}} f(x,t)\cdot K(X-x,T-t)\,dxdt = \int_0^T \int_{\mathbb{R}^n} (\partial_t - \Delta_x)\, u(x,t)\cdot K(X-x,T-t)\,dxdt,$$

thus the integral operator $f \mapsto \iint_{\mathbb{R}^{n+1}} f(x,t)\cdot K(X-x,T-t)\,dxdt$ provides a **left inverse** to the operator $\partial_t - \Delta_x$ on $C_c^\infty(\mathbb{R}^{n+1})$.

As we will see, if f has Hölder continuity in x, then the same integral operator also provides a **right inverse** to the operator $\partial_t - \Delta_x$ in the sense that

$$(\partial_T - \Delta_X) \iint_{\mathbb{R}^{n+1}} f(x,t)\cdot K(X-x,T-t)\,dxdt = f(X,T).$$

We can also use (6.2) to reduce a proof for uniqueness to an IBVP to the existence of solutions to an adjoint IBVP. Suppose that U is a bounded domain in $\mathbb{R}^n$, and $u \in C(\overline{U}\times[0,T]) \cap C^{2,1}_{x,t}(U\times(0,T])$ satisfies

$$\begin{cases} (\partial_t - \Delta_x)u(x,t) = 0 & (x,t) \in U \times (0,T], \\ u(x,t) = 0 & (x,t) \in \partial U \times (0,T], \\ u(x,0) = 0 & x \in U, \end{cases}$$

and we would like to prove that $u \equiv 0$ in $\overline{U}\times[0,T]$. If we can solve for $v(x,t)$

$$\begin{cases} (-\partial_t - \Delta_x)v(x,t) = 0 & (x,t) \in U \times [0,\tau), \\ v(x,t) = 0 & (x,t) \in \partial U \times [0,\tau), \\ v(x,\tau) = g(x) & x \in U, \end{cases}$$

for any $0 < \tau \le T$ and any $g \in C(\overline{U})$, then we use this $v(x,t)$ with $u(x,t)$ in (6.2) with $t_0 = 0$, $t_1 = \tau$, to get

$$\int_U u(x,\tau) \cdot g(x)\, dx = 0.$$

Since $g \in C(\overline{U})$ is arbitrary, this leads to $u(x,\tau) \equiv 0$ for all $0 < \tau \le T$. Note that the adjoint IBVP takes τ as its initial time, and solves for $0 \le t < \tau$. A change of variable $w(x,s) = v(x,\tau - s)$ shows that the adjoint IBVP is equivalent to

$$\begin{cases} (\partial_s - \Delta_x) w(x,s) = 0 & (x,s) \in U \times (0,\tau], \\ w(x,s) = 0 & (x,s) \in \partial U \times (0,\tau], \\ w(x,0) = g(x) & x \in U, \end{cases}$$

This technique of reducing a uniqueness question to a question on the existence of an adjoint problem is called **Holmgren**'s method. When U is a one dimensional interval we have proved that this IBVP always has a solution for any $g \in C(\overline{U})$, thus we have proved uniqueness to the IBVP for the heat equation by the Holmgren's method for this case.

(6.2) can also be used to prove uniqueness of solution of the Cauchy problem for the heat equation on $\mathbb{R}^n \times (0,T]$ among the class of functions with controlled growth.

Theorem 6.1. *Suppose that* $(\partial_t - \Delta_x) u(x,t) = 0$ *on* $\mathbb{R}^n \times (0,T]$, *and there exists* $C > 0$ *such that* $|u(x,t)| + |\nabla u(x,t)| \le C exp(\frac{a|x|^2}{4T})$ *for some* $0 < a < 1$ *and all* $(x,t) \in \mathbb{R}^n \times (0,T]$. *Then for all* $X \in \mathbb{R}^n$ *and* $0 < \tau \le T$,

$$u(X,\tau) = \int_{\mathbb{R}^n} u(x,0) K(X - x, \tau)\, dx. \tag{6.4}$$

In particular, if $u(x,0) \equiv 0$, *then* $u(x,t) = 0$ *in* $\mathbb{R}^n \times (0,T]$.

This is proved by applying (6.2) with $v(x,t) = K(X - x, \tau + \epsilon - t)$ for $\epsilon > 0$ small, on increasingly large domains of the form $B_R \times [0,\tau]$, and by sending $\epsilon \searrow 0$. This representation and its consequence of uniqueness do not necessarily hold if we do not impose the growth restrictions; the first example of non-uniqueness was due to A. N. Tikhonov.

Exercises

Exercise 6.1.1. Supply details for the proof outlined above. Also, modify $v(x,t) = K(X - x, \tau + \epsilon - t)$ in the suggested proof into $v(x,t) = K(X - x, \tau + \epsilon - t)\eta_R(x)$, where $\eta_R(x)$ is a smooth cut-off function supported in B_R and $\eta_R(x) \equiv 1$ in $B_{R/2}$. This will remove the restriction on the growth of $|\nabla u(x,t)|$.

Exercise 6.1.2. Suppose that $f(x,t)$ is bounded and measurable on $\mathbb{R}^n \times [0,T]$. Prove that $\int_0^T \int_{\mathbb{R}^n} f(x,t)K(X-x, T+\epsilon-t)\,dxdt \to \int_0^T \int_{\mathbb{R}^n} f(x,t)K(X-x,T-t)\,dxdt$ as $\epsilon \searrow 0$.

Exercise 6.1.3. This exercise uses (6.1) to establish a mean value property for any solution $u(x,t)$ to $(\partial_t - \Delta_x)u(x,t) = 0$. For any $a > 0$, consider the set

$$E_{(X,T);a} = \{(x,t) : K(X-x, T-t) \geq a\},$$

which includes (X,T) as a boundary point. Integrate (6.1), with $v(x,t) = K(X-x, T-t) - a$ over $E_{(X,T);a} \cap \{t \leq T-\epsilon\}$ for $\epsilon > 0$ small, then let $\epsilon \to 0$ to establish

$$\begin{aligned} &u(X,T) \\ &= -\int_{\partial E_{(X,T);a}} [u\, v\, n_t + u\, \mathbf{n}_x \cdot \nabla_x v - v\, \mathbf{n}_x \cdot \nabla_x u]\; d\sigma(x,t), \end{aligned}$$

where $(n_t(x,t), \mathbf{n}_x(x,t))$ is the exterior unit normal to $\partial E_{(X,T);a}$ at (x,t), thus is the unit vector in the direction of $-(K_t(X-x, T-t), \nabla_x K(X-x, T-t))$. Using $v(x,t) = 0$ for $(x,t) \in \partial E_{(X,T);a}$, and

$$\begin{aligned} -\mathbf{n}_x(x,t)\nabla_x v(x,t) &= \frac{|\nabla_x K(X-x,T-t)|^2}{\sqrt{K_t(X-x,T-t)^2 + |\nabla_x K(X-x,T-t)|^2}} \\ &= \frac{\frac{|X-x|^2}{|T-t|^2} K(X-x,T-t)^2}{4\sqrt{K_t(X-x,T-t)^2 + |\nabla_x K(X-x,T-t)|^2}} \end{aligned}$$

to prove the mean value property

$$u(X,T) = \int_{\partial E_{(X,T);a}} u(x,t) \frac{\frac{|X-x|^2}{|T-t|^2} a^2}{4\sqrt{K_t(X-x,T-t)^2 + |\nabla_x K(X-x,T-t)|^2}}\, d\sigma(x,t).$$

This can be used easily to prove the strong maximum principle for a solution of $(\partial_t - \Delta_x)u(x,t) = 0$, namely, if $u(X,T) \geq u(x,t)$ for all $(x,t) \in \partial E_{(X,T);a}$ then $u(X,T) = u(x,t)$ for all $(x,t) \in \partial E_{(X,T);a}$. Since $E_{(X,T);a}$ can be foliated by the level sets $\partial E_{(X,T);a'}$ for $a' \geq a$, using the co-area formula to the foliation by the level sets $\{K(X-x,T-t) = a'\}$, we get a cleaner form of the mean value property

$$\begin{aligned} &\iint_{E_{(X,T);a}} u(x,t) \frac{|X-x|^2}{|T-t|^2}\, dx\, dt \\ &= \int_a^\infty \int_{E_{(X,T);a'}} u(x,t) \frac{\frac{|X-x|^2}{|T-t|^2}}{\sqrt{K_t(X-x,T-t)^2 + |\nabla_x K(X-x,T-t)|^2}}\, d\sigma(x,t)\, da' \\ &= \int_a^\infty 4u(X,T)/a'^2\, da' \\ &= 4u(X,T)/a. \end{aligned}$$

6.2 Definition of Adjoint/Transpose Operators

I hope that the above discussions illustrate the usefulness of the notion of adjoint/transpose operator and fundamental solutions. We now extend these notions to more general linear differential operators. A linear differential operator $P(\partial_x)$ is defined in terms of a polynomial of degree m whose coefficients are (sufficiently regular) functions of x:

$$P(\partial_x) = \sum_{|\alpha|=0}^{m} a_\alpha(x)\partial_x^\alpha,$$

where we assume that at any x there exists some α_* with $|\alpha_*| = m$ such that $a_{\alpha_*}(x) \neq 0$. m is called the order of the operator $P(\partial_x)$.

To establish a relation for $P(\partial_x)$ similar to (6.2), we would like to find a linear differential operator $P'(\partial_x)$, called **transpose** of $P(\partial_x)$, such that for any pair of sufficiently smooth functions $u(x)$ and $v(x)$, $[P(\partial_x)u(x)]\,v(x) - u(x)P'(\partial_x)v(x)$ is a divergence of a vector field which is built in terms of $u(x)$ and $v(x)$ and their derivatives up to order $m-1$, and bilinear in u and v:

$$[P(\partial_x)u(x)]\,v(x) - u(x)P'(\partial_x)v(x) = \sum_{a=1}^{n} \partial_{x_a}\left(B_a[u,v]\right). \tag{6.5}$$

It turns out that $P'(\partial_x)v(x) = \sum_{|\alpha|=0}^{m}(-1)^{|\alpha|}\partial_x^\alpha\left(a_\alpha(x)v(x)\right)$.

Example 6.1. For $P_1 = \Delta_x = \sum_{a=1}^{n} \partial_{x_a}^2$, $P_1' = \sum_{a=1}^{n} \partial_{x_a}^2 = P_1$, and

$$\Delta_x u(x) \cdot v(x) - u(x) \cdot \Delta_x v(x) = \sum_{a=1}^{n} \partial_{x_a}\left(u_{x_a}(x)v(x) - u(x)v_{x_a}(x)\right).$$

For $P_2 = \Delta_x + \sum_{a=1}^{n} b_a(x)\partial_{x_a}$, $P_2' = \Delta_x - \sum_{a=1}^{n} \partial_{x_a}\left(b_a(x)\cdot\right)$, and

$$[P_2u(x)] \cdot v(x) - u(x) \cdot P_2'v(x) = \sum_{a=1}^{n} \partial_{x_a}\left[u_{x_a}(x)v(x) - u(x)v_{x_a}(x) + b_a(x)u(x)v(x)\right].$$

For $P_3 = \partial_t - \Delta_x$, $P_3' = -\partial_t - \Delta_x$, and (6.1) holds.

For $P_4 = \partial_t^2 + a(x,t)\partial_t + b(x,t)\partial_{tx}^2 - \partial_x^2$, we have

$$P_4'[v] = \partial_t^2 v(x,t) - \partial_t[a(x,t)v(x,t)] + \partial_{tx}^2[b(x,t)v(x,t)] - \partial_x^2 v(x,t).$$

This follows from

$$\begin{aligned} v(x,t)\partial_t^2 u(x,t) =& \partial_t[v(x,t)\partial_t u(x,t)] - \partial_t v(x,t)\partial_t u(x,t) \\ =& \partial_t[v(x,t)\partial_t u(x,t)] - \partial_t[\partial_t v(x,t) u(x,t)] + \partial_t^2 v(x,t) u(x,t), \\ v(x,t)a(x,t)\partial_t u(x,t) =& \partial_t[a(x,t)v(x,t) u(x,t)] - \partial_t[a(x,t)v(x,t)] u(x,t), \\ v(x,t)b(x,t)\partial_{tx}^2 u(x,t) =& \partial_x[v(x,t)b(x,t)\partial_t u(x,t)] - \partial_x[v(x,t)b(x,t)]\partial_t u(x,t) \\ =& \partial_x[v(x,t)b(x,t)\partial_t u(x,t)] - \partial_t\{\partial_x[v(x,t)b(x,t)] u(x,t)\} \\ & + \partial_{xt}[v(x,t)b(x,t)] u(x,t) \\ v(x,t)\partial_x^2 u(x,t) =& \partial_x[v(x,t)\partial_x u(x,t) - u(x,t)\partial_x v(x,t)] + u(x,t)\partial_x^2 v(x,t), \end{aligned}$$

so we see that

$$\begin{aligned} & v(x,t)P_4[u(x,t)] - u(x,t)P_4'[v(x,t)] \\ =& \partial_t[v(x,t)\partial_t u(x,t) - \partial_t v(x,t) u(x,t) + a(x,t)v(x,t) u(x,t) - \partial_x[v(x,t)b(x,t)] u(x,t)] \\ & + \partial_x[v(x,t)b(x,t)\partial_t u(x,t) - v(x,t)\partial_x u(x,t) + u(x,t)\partial_x v(x,t)]. \end{aligned}$$

The bilinear form $B[u,v]$ here looks complicated, but as we will see later, we often pay attention to the terms in $B[u,v]$ which do not involve differentiation in $u(x,t)$.

Remark 6.2. We have confined ourselves to differential operators with real coefficients acting on real valued functions and considered the transpose of a differential operator in the context of treating the linear paring $\int P[u]v\,dx$ as a linear functional on u (or v). In some context, there is a need to use a specific inner product (or Hermitian product) structure of the function space (using complex valued functions in the latter context). In such a setting the transpose should be modified to satisfy

$$\{P(\partial_x)u(x)\}\,\overline{v(x)} - u(x)\overline{P^*(\partial_x)v(x)} = \sum_{a=1}^n \partial_{x_a}\left(B_a[u,\overline{v}]\right).$$

where $P^*v(x) = \overline{P'(\overline{v}(x))}$ and P^* is called the adjoint of P. For operators with real coefficients they are the same.

For example, for $P_2 = \Delta_x + \sum_{a=1}^n b_a(x)\partial_{x_a}$, where $b_a(x)$ may be complex valued,

$$\begin{aligned} & [P_2u(x)]\cdot\overline{v(x)} - u(x)\left\{\overline{\Delta_x v(x) - \sum_{a=1}^n \partial_{x_a}\left(\overline{b_a(x)}v(x)\right)}\right\} \\ =& \sum_{a=1}^n \partial_{x_a}\left[u_{x_a}(x)\overline{v(x)} - u(x)\overline{v_{x_a}(x)} + b_a(x)u(x)\overline{v(x)}\right]. \end{aligned}$$

So $P_2^*v(x) = \Delta_x v(x) - \sum_{a=1}^n \partial_{x_a}\left(\overline{b_a(x)}v(x)\right)$ is the adjoint of P, while the transpose of P_2 is simply given by $P_2'v(x) := \Delta_x v(x) - \sum_{a=1}^n \partial_{x_a}(b_a(x)v(x))$, which satisfies

$$\{P_2(\partial_x)u(x)\}\, v(x) - u(x)P_2'(\partial_x)v(x) = \sum_{a=1}^n \partial_{x_a}(B_a[u,v]).$$

For $P_5 = i\frac{d}{dx}$,

$$\left[\left(i\frac{d}{dx}\right)u(x)\right]\overline{v(x)} - u(x)\overline{\left(i\frac{d}{dx}\right)v(x)} = i\frac{d}{dx}\left(u(x)\overline{v(x)}\right),$$

so $P_5^* = i\frac{d}{dx} = P_5$, while the transpose of P_5 will be $-i\frac{d}{dx}$.

The purpose of (6.5) is to establish its integral consequence, namely,

$$\int_{\mathbb{R}^n} [P(\partial_x)u(x)]\, v(x)\, dx = \int_{\mathbb{R}^n} u(x)P'(\partial_x)v(x)\, dx$$

when at one of $u(x)$ or $v(x)$ has compact support.

For the Fourier transform operator $\mathcal{F}[u](\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}\, dx$, using

$$\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} u(x)e^{-ix\cdot\xi}\, dx\right)v(\xi)\, d\xi = \int_{\mathbb{R}^n} u(x)\left(\int_{\mathbb{R}^n} v(\xi)e^{-ix\cdot\xi}\, d\xi\right)dx$$

we see that $\mathcal{F}' = \mathcal{F}$, while $\mathcal{F}^* = \overline{\mathcal{F}}$, namely,

$$\mathcal{F}^*[u](\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^n} u(x)e^{ix\cdot\xi}\, dx,$$

which is the inverse Fourier transform of u.

A proper discussion of the spectrum theory of a linear operator would need to work with a Hermitian product of the function space involving complex scaler field and complex conjugate.

We already discussed the notion of fundamental solutions for the Laplace and heat operators. The same consideration applies to a general linear differential operator. There are often two perspectives:

(I). Represent $u(x)$ in terms of an integral operator of the form $\int F(y;x)P(\partial_y)[u(y)]\, dy$ and some boundary integrals—for $u(x) \in C_c^\infty(\mathbb{R}^n)$, the boundary integrals will be absent so we expect $u(x) = \int_{\mathbb{R}^n} F(y;x)P(\partial_y)[u(y)]\, dy$.

This is usually used to study the regularity of $u(x)$ in terms of that of $P(\partial_y)[u(y)]$.

(II). Find a function (more properly a distribution) $E(x;y)$ such that

$$P(\partial_x)\int_{\mathbb{R}^n} E(x;y)f(y)\,dy = f(x), \quad \text{at any } x\text{, at least for } f \in C_c^\infty(\mathbb{R}^n).$$

In other words $u(x) := \int_{\mathbb{R}^n} E(x;y)f(y)\,dy$ provides a solution to $P(\partial_x)u(x) = f(x)$.

The proper formulation of (I) also involves the notion of distribution. The idea is that if for any $x \in U$, one can find a family $v_\epsilon(y) \in C^m(U)$ such that $P'(\partial_y)v_\epsilon(y) = \eta_\epsilon(y)$, where the family $\eta_\epsilon(y) \in C_c^\infty(U)$ is an approximation of identity (at x) in the sense that

$$\lim_{\epsilon\to 0}\int_U u(y)\eta_\epsilon(y)\,dy = u(x) \quad \text{for any } u \in C(U),$$

and if one assumes that the family $v_\epsilon(y) \to F(y;x)$ in appropriate sense, then integrating (6.5) for $u \in C_c^\infty(U), v_\epsilon$:

$$\int_U [P(\partial_y)u(y)]\,v_\epsilon(y)\,dy = \int_U u(y)P'(\partial_y)v_\epsilon(y)\,dy,$$

and taking the limit as $\epsilon \to 0$, one obtains the representation in (I). Here, we assumed $u \in C_c^\infty(U)$ for convenience of discussion to avoid dealing with the boundary integrals.

We next give a brief description of the language of distribution to put our discussion on a firmer ground.

6.3 A Brief Introduction of Distribution*

We have used an approximation of identity in several instances, where a family of C^∞ (or $C(U)$) functions $\eta_\epsilon(y)$ converges to a limiting object not in a norm, but considered as a continuous linear functional on $C_c(U) \ni u \mapsto \int u(y)\eta_\epsilon\,dy \to u(x) \in \mathbb{R}$ as $\epsilon \to 0$. This notion of convergence produces objects that form the space of distributions.

6.3.1 Definition of Distribution

The spaces $C_c^\infty(U)$, or $C^\infty(U)$, or $C(U)$ are test function spaces. We often choose to work with $\mathscr{E}(U) := C^\infty(U)$ or $\mathscr{D}(U) := C_c^\infty(U)$. $\mathscr{E}(U)$ is a complete metric space with the metric

$$\rho_{\mathscr{E}(U)}(\phi,\psi) = \sum_{l=1}^\infty \frac{1}{2^l}\frac{p_l(\phi-\psi)}{1+p_l(\phi-\psi)},$$

*The material of this section is only used in a limited fashion in the remaining sections of this chapter; a student can work through the remaining part of these notes without a systematic mastery of the material of this section.

where we choose a sequence of increasing compact subdomains $K_1 \subset K_2 \subset \cdots \subset U$ exhausting U: $U = \cup_{l=1}^{\infty} K_l$, and for any $m \in \mathbb{Z}_{\geq 0}$ and compact $K \subset U$,

$$p_{m,K}(\phi) = \sum_{|\alpha| \leq m} \sup_{x \in K} |\partial^\alpha \phi(x)|, \quad p_l := p_{l,K_l}.$$

A linear functional l on $\mathscr{E}(U)$ is continuous if the set $\{\phi \in \mathscr{E}(U) : |\langle l, \phi \rangle| < 1\}$ contains an open neighborhood of 0 in $\mathscr{E}(U)$ of the form $\{\phi \in \mathscr{E}(U) : p_{m,K}(\phi) < \delta\}$ for some $m \in \mathbb{Z}_{\geq 0}$, compact $K \subset U$, and $\delta > 0$. Then for any ϕ in $\mathscr{E}(U)$ with $p_{m,K}(\phi) \neq 0$, and $0 < \delta' < \delta$, the function $g = \delta' \phi / p_{m,K}(\phi) \in \{\phi \in \mathscr{E}(U) : p_{m,K}(\phi) < \delta\}$, so

$$|\langle l, g \rangle| < 1, \text{ which then implies } |\langle l, \phi \rangle| < (\delta')^{-1} p_{m,K}(\phi).$$

Sending $\delta' \to \delta$ and denoting $C = (\delta)^{-1}$, we obtain

$$|\langle l, \phi \rangle| \leq C p_{m,K}(\phi).$$

If $p_{m,K}(\phi) = 0$, then for any $M > 0$, $p_{m,K}(M\phi) = 0$, so $|\langle l, M\phi \rangle| < 1$ for any $M > 0$, which implies that $|\langle l, \phi \rangle| = 0$ for such a ϕ. Thus we obtain the following characterization of a continuous linear functional l on $\mathscr{E}(U)$.

Theorem 6.2. *Suppose that l is a continuous linear functional l on $\mathscr{E}(U)$. Then there exist some $m \in \mathbb{Z}_{\geq 0}$, compact $K \subset U$, and $C > 0$ such that for all $\phi \in \mathscr{E}(U)$,*

$$|\langle l, \phi \rangle| \leq C p_{m,K}(\phi). \tag{6.6}$$

The set of all continuous linear functionals on $\mathscr{E}(U)$ is denoted as $\mathscr{E}'(U)$.

A sequence of continuous linear functionals $\{l_k\}$ on $\mathscr{E}(U)$ is said to converge if for any $\phi \in \mathscr{E}(U)$, $\lim_{k\to\infty} \langle l_k, \phi \rangle$ exists. Namely, the convergence in $\mathscr{E}'(U)$ is the notion of weak-* convergence.

Using the completeness of $\mathscr{E}(U)$ under the metric $\rho_{\mathscr{E}(U)}$ and uniform boundedness principle, one can prove that if $\{l_k\}$ converges, then there exist $m \in \mathbb{Z}_{\geq 0}$, compact $K \subset U$, and $C > 0$, independent of k, such that (6.6) holds for $l = l_k$ for all k; as a result,

$$\langle l, \phi \rangle := \lim_{k \to \infty} \langle l_k, \phi \rangle$$

defines an element in $\mathscr{E}'(U)$.

There is a need to consider continuous linear functionals on $\mathscr{D}(U) \subset \mathscr{E}(U)$. Here we don't use the topology induced by $\rho_{\mathscr{E}(U)}$ on $\mathscr{D}(U)$, as $\mathscr{D}(U)$ is not complete in this metric. Instead, we use the following notion of sequential convergence in $\mathscr{D}(U)$.

For any compact subdomain $K \subset U$, denote by $\mathscr{D}(K)$ the subspace of functions ϕ in $\mathscr{D}(U)$ whose support is a subset of K.

A sequence of functions $\{\phi_k\} \subset \mathscr{D}(U)$ is said to converge in $\mathscr{D}(U)$ if there exists some compact set $K \subset U$ such that $\phi_k \in \mathscr{D}(K)$ and converges in the metric $\rho_{\mathscr{E}(K)}$.

A linear functional l on $\mathscr{D}(U)$ is said to be continuous if for any sequence of functions $\{\phi_k\}$ converging to ϕ in $\mathscr{D}(U)$, we have $\langle l, \phi_k\rangle \to \langle l, \phi\rangle$ as $k \to \infty$. A continuous linear functional on $\mathscr{D}(U)$ is called a distribution on U. The set of all continuous linear functionals on $\mathscr{D}(U)$ is denoted as $\mathscr{D}'(U)$. Note that $\mathscr{E}'(U) \subset \mathscr{D}'(U)$.

Similar to Theorem 6.2, we have

Theorem 6.3. *Given any continuous linear functional $l \in \mathscr{D}'(U)$, then for any compact set $K \subset U$, there exists $m = m(l, K) \in \mathbb{Z}_{\geq 0}$ and $C = C(l, K)$ such that (6.6) holds for all $\phi \in \mathscr{D}(K)$.*

Note that, here, K is not an attribute of l but needs to be chosen as a compact subset of U first, the inequality (6.6) is for $\phi \in \mathscr{D}(K)$, and the constants m and C generally depend on K.

A sequence of distributions $\{l_k\} \subset \mathscr{D}'(U)$ is said to converge if for any $\phi \in \mathscr{D}(U)$, $\lim_{k\to\infty}\langle l_k, \phi\rangle$ exists. In such a case, for any compact $K \subset U$, there exist $m = m(K) \in \mathbb{Z}_{\geq 0}$, and $C = C(K) > 0$, independent of k, such that (6.6) holds for $l = l_k$ for all k and all $\phi \in \mathscr{D}(K)$, and

$$\langle l, \phi\rangle := \lim_{k\to\infty} \langle l_k, \phi\rangle$$

defines an element in $\mathscr{D}'(U)$.

Any function $f \in L^1_{\text{local}}(U)$ defines an element of $\mathscr{D}'(U)$ by

$$\langle f, \phi\rangle := \int_U f(x)\phi(x)\, dx.$$

This definition does not place any growth restriction on f. For example, e^{x^2} defines a distribution on $\mathbb{R}$ by $\langle e^{x^2}, \phi\rangle = \int_{\mathbb{R}} e^{x^2}\phi(x)\, dx$ for $\phi \in \mathscr{D}(\mathbb{R})$.

The inclusion relations

$$\mathscr{D}(U) \subset \mathscr{E}(U) \subset C(U) \subset L^1_{\text{local}}(U)$$

are not just set-theoretic; convergence in each implies convergence in the subsequent one. Using these relations, any function in the above function spaces naturally defines an element of $\mathscr{D}'(U)$; and if $\{f_k\}$ converges in one of the spaces, $\{f_k\}$ also converges as elements of $\mathscr{D}'(U)$. Note, however, a sequence of functions in any of these function spaces may not converge in these function spaces but converge in $\mathscr{D}'(U)$ to a limit! Note also that a function in $L^1_{\text{local}}(U)$ may not define an element in $\mathscr{E}'(U)$.

In fact, any Radon measure μ on U defines an element of $\mathscr{D}'(U)$ by

$$\langle \mu, \phi \rangle := \int_U \phi(x)\, d\mu(x) \text{ for } \phi \in \mathscr{D}(U).$$

The simplest such a distribution is the Dirac distribution $\delta(x-a)$ for any $a \in U$ defined by

$$\langle \delta(x-a), \phi \rangle := \phi(a), \text{ for } \phi \in \mathscr{D}(U).$$

Another simple example of such a distribution is the arclength measure along any differential curve $\Gamma \subset U$ whose length in any compact subset of U is finite.

Since many of the distributions that we commonly encounter arise from an element in $L^1_{\text{local}}(U)$ or a Radon measure μ on U, we often abuse notation and write the pairing between a distribution $\langle f, \phi \rangle$ or $\langle \mu, \phi \rangle$ in the form of an integral.

The notion of support of a function extends to a distribution naturally. A distribution $l \in \mathscr{D}'(U)$ is equal to 0 on an open subset $V \subset U$, if $\langle l, \phi \rangle = 0$ for any $\phi \in \mathscr{D}(V)$. Using partition of unity it is routine to prove that if $l = 0$ on a collection of $V_\alpha \subset U$ of open subsets of U, then $l = 0$ on $\cup_\alpha V_\alpha$. This leads to the notion of support of l as the complement of the open set in which $l = 0$:

$$\text{supp } l := U \setminus \cup_\alpha V_\alpha, \text{ where } l = 0 \text{ in } V_\alpha \subset U.$$

The support of $\delta(x-a)$ is simply $\{a\}$. For any $l \in \mathscr{E}'(U)$, using (6.6) in Theorem 6.2 for any $\phi \in \mathscr{D}(U \setminus K)$, we see that $l = 0$ on $U \setminus K$. Thus

Corollary 6.4. *Every $l \in \mathscr{E}'(U)$ has a compact support. In other words, $\mathscr{E}'(U)$ consists of those distributions in $\mathscr{D}'(U)$ with compact support.*

The notion of singular support of a distribution is defined as the complement of the union of open sets V_α in which its restriction as an element of $\mathscr{D}'(V_\alpha)$ can be identified as a distribution induced by some element in $\mathscr{E}(V_\alpha)$. The Heaviside function H, defined as

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is a locally integrable function; as a distribution, its support is $[0, \infty)$ and its singular support is $\{0\}$.

6.3.2 Operations On and Between Distributions

A most basic operation on distributions is differentiation. For any $l \in \mathscr{D}'(U)$ and any $i, 1 \le i \le n$, we define a distribution $\partial_{x_i} l$ by

$$\langle \partial_{x_i} l, \phi \rangle := -\langle l, \partial_{x_i} \phi \rangle \text{ for } \phi \in \mathscr{D}(U). \tag{6.7}$$

When l is given by a continuously differentiable function in U, this equality holds in the traditional sense by integration-by-parts, but ϕ having compact support in U plays a role here. This is part of the reason it is easier to work with $\mathscr{D}(U)$ as the space of test functions.

A most convenient feature of this operation is that any distribution can be differentiated any number of times. Thus for any distribution $l \in \mathscr{D}'(U)$ and any multi-index α, $\partial^\alpha l$ is defined. Note that if a sequence of distributions $l_k \to l$ in $\mathscr{D}'(U)$, then for any multi-index α, $\partial^\alpha l_k \to \partial^\alpha l$ in $\mathscr{D}'(U)$.

Example 6.2. $\ln|x|$ defines a distribution in $\mathscr{D}(\mathbb{R})$. Its derivative satisfies

$$\int_{\mathbb{R}} (\partial \ln|x|)\, \phi(x)\, dx = -\int_{\mathbb{R}} (\ln|x|)\, \phi'(x)\, dx.$$

We can integrate by parts in the second integral. However, due to the behavior of $\ln|x|$ near $x = 0$, only specific handling of the boundary terms from the integration by parts can those terms be properly accounted for. More specifically, if we take symmetric approximation in the improper integral $\int_{\mathbb{R}} \ln|x| \phi'(x)\, dx$ as follows,

$$\int_{\mathbb{R}} (\ln|x|)\, \phi'(x)\, dx = \lim_{\epsilon \searrow 0} \int_{|x|>\epsilon} (\ln|x|)\, \phi'(x)\, dx = \lim_{\epsilon \searrow 0} \left\{ (\phi(-\epsilon) - \phi(\epsilon)) \ln \epsilon - \int_{|x|>\epsilon} \frac{1}{x} \phi(x)\, dx \right\}$$

we would get

$$\int_{\mathbb{R}} (\partial \ln|x|)\, \phi(x)\, dx = \lim_{\epsilon \searrow 0} \int_{|x|>\epsilon} \frac{1}{x} \phi(x)\, dx.$$

This limit is called the principal value of the integral of $\frac{1}{x}\phi(x)$ over $\mathbb{R}$, and is usually denoted as PV $\int_{\mathbb{R}} \frac{1}{x} \phi(x)\, dx$.

In addition to the vector space structure of $\mathscr{D}'(U)$, there is also multiplication between any $\eta \in \mathscr{E}(U)$ and any distribution $l \in \mathscr{D}'(U)$ by

$$\langle \eta l, \phi \rangle := \langle l, \phi\eta \rangle \text{ for } \phi \in \mathscr{D}(U).$$

For any $l \in \mathscr{D}'(U)$ and any compact subset K of U, (6.6) involves only the $p_{m,K}$ norm of functions $\eta \in \mathscr{D}(K)$, which allows $\langle l, \phi \rangle$ to extend to $C_c^m(K)$. Then for any

$\eta \in C^m(K)$, ηl also is well defined as a distribution on K. If $l \in \mathscr{E}'(U)$, then the m and K are attributes of l, which allow $\langle l, \phi\rangle$ to extend to $C_c^m(U)$ and allow ηl to be defined for any $\eta \in C^m(U)$.

If $l, L \in \mathscr{D}'(U)$ are such that $L = \eta l$ for some $\eta \in \mathscr{E}(U)$ and η never vanishes, then $l = \eta^{-1}L$. This no longer holds if η vanishes somewhere. For example, $x\delta(x) = 0$ in $\mathscr{D}'(\mathbb{R})$, yet $\delta \neq 0$ in $\mathscr{D}'(\mathbb{R})$.

The product rule of differentiation holds for the multiplication between an $\eta \in \mathscr{E}(U)$ and a distribution $l \in \mathscr{D}'(U)$. However, there is no multiplication between two arbitrary distributions in $\mathscr{D}'(U)$.

In fact, the above two operations are examples of a more general operation. Suppose that $T : \mathscr{D}(U) \mapsto \mathscr{D}(U)$ is linear and continuous in the sense that if $\{\phi_k\} \subset \mathscr{D}(U)$ converges in $\mathscr{D}(U)$, then $\{T(\phi_k)\}$ also converges in $\mathscr{D}(U)$; furthermore, that T has an transpose $T' : \mathscr{D}(U) \mapsto \mathscr{D}(U)$ defined via

$$\int_U T'(\phi)\psi\,dx = \int_U \phi T(\psi)\,dx \text{ for all } \phi, \psi \in \mathscr{D}(U),$$

and that T' is also continuous in the same sense. Then for any $l \in \mathscr{D}'(U)$, we define

$$\langle T(l), \phi\rangle = \langle l, T'(\phi)\rangle \text{ any } \phi \in \mathscr{D}(U).$$

We verify easily that this $T(l) \in \mathscr{D}'(U)$. This $T : \mathscr{D}'(U) \mapsto \mathscr{D}'(U)$ is considered an extension of $T : \mathscr{D}(U) \mapsto \mathscr{D}(U)$.

Example 6.3. (i) When we take $T(\phi) = \partial\phi$, we find that $T'(\phi) = -\partial\phi = -T(\phi)$, which gives rise to the operation ∂l on $\mathscr{D}'(U)$.

(ii) For any $\eta \in \mathscr{E}(U)$, we take $T(\phi) = \eta\phi$, then $T'(\phi) = \eta\phi = T(\phi)$, which gives rise to the operation ηl on $\mathscr{D}'(U)$.

(iii) When $U = \mathbb{R}^n$, and A is an invertible $n \times n$ matrix, $T_A(\phi)(\mathbf{x}) = \sqrt{|\det A|}\phi(A\mathbf{x})$, then $T'_A = T_{A^{-1}}$, so we can define $T_A[l]$ for any $l \in \mathscr{D}'(\mathbb{R}^n)$ via

$$\langle T_A[l], \phi\rangle = \langle l, T_{A^{-1}}\phi\rangle.$$

For example, when $A = \lambda I$ for some $\lambda > 0$, $T_A(\phi) = \lambda^{n/2}\phi(\lambda\mathbf{x})$, so for $l = \delta(\mathbf{x})$, we get $T_A(\delta) = \lambda^{n/2}\delta(\lambda\mathbf{x}) = \lambda^{-n/2}\delta(\mathbf{x})$, which implies $\lambda^n\delta(\lambda\mathbf{x}) = \delta(\mathbf{x})$.

(iv) When $U = \mathbb{R}^n$, and $\mathbf{a} \in \mathbb{R}^n$, $T(\phi)(\mathbf{x}) = \phi(\mathbf{x} - \mathbf{a})$, then $T'(\phi)(\mathbf{x}) = \phi(\mathbf{x} + \mathbf{a})$, and $T(l)$ is then well defined, and is usually denoted as $l(\mathbf{x} - \mathbf{a})$. $\langle l(\mathbf{x} - \mathbf{a}), \phi(\mathbf{x})\rangle = \langle l(\mathbf{x}), \phi(\mathbf{x} + \mathbf{a})\rangle$.

(v) For any $\eta \in \mathscr{D}(\mathbb{R}^n)$, the convolution with η

$$T(\phi)(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(\mathbf{x}-\mathbf{y})\eta(\mathbf{y})\, d\mathbf{y} = \int_{\mathbb{R}^n} \phi(\mathbf{y})\eta(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$$

defines a continuous linear map from $\mathscr{D}(\mathbb{R}^n)$ to $\mathscr{D}(\mathbb{R}^n)$, with a continuous linear transpose

$$T'(\phi)(\mathbf{x}) = \int_{\mathbb{R}^n} \phi(\mathbf{y})\eta(\mathbf{y}-\mathbf{x})\, d\mathbf{y},$$

namely, $T'(\phi)$ is convolution with $\mathbf{x} \mapsto \eta(-\mathbf{x})$. This $T(\phi)$ is usually denoted as $\eta * \phi$. We denote $T'(\phi)$ here by $\breve{\eta} * \phi$. Thus we can define, for any $l \in \mathscr{D}'(\mathbb{R}^n)$,

$$\langle \eta * l, \phi \rangle = \langle l, T'(\phi) \rangle = \langle l, \breve{\eta} * \phi \rangle.$$

Note that using the continuity of l, we have

$$\langle l, \breve{\eta} * \phi \rangle = \langle l, \int_{\mathbb{R}^n} \phi(\mathbf{y})\eta(\mathbf{y}-\mathbf{x})\, d\mathbf{y} \rangle = \int_{\mathbb{R}^n} \phi(\mathbf{y})\langle l, \eta(\mathbf{y}-\mathbf{x})\rangle\, d\mathbf{y},$$

and $\lambda(\mathbf{y}) := \langle l, \eta(\mathbf{y}-\mathbf{x})\rangle$ is in $\mathscr{E}(\mathbb{R}^n)$, with

$$\partial_{y_i}\lambda(\mathbf{y}) = \langle l, \partial_{y_i}\eta(\mathbf{y}-\mathbf{x})\rangle = \langle l, -\partial_{x_i}\left[\eta(\mathbf{y}-\mathbf{x})\right]\rangle = \langle \partial_{x_i} l, \eta(\mathbf{y}-\mathbf{x})\rangle,$$

so we can identify $\eta * l$ with the $C^\infty(\mathbb{R}^n)$ function $\langle l, \eta(\mathbf{y}-\mathbf{x})\rangle$.

If we choose a family of approximation of identity $\eta_\epsilon \in \mathscr{D}(\mathbb{R}^n)$, we find that $\eta_\epsilon * l \to l$ in $\mathscr{D}'(\mathbb{R}^n)$. Thus any distribution $l \in \mathscr{D}'(\mathbb{R}^n)$ can be approximated in the sense of distribution by $C^\infty(\mathbb{R}^n)$ functions.

Exercise 6.3.1. Prove that for any $l \in \mathscr{D}'(U), \eta \in \mathscr{D}(U)$, $\lambda(\mathbf{y}) := \langle l, \eta(\mathbf{y}-\mathbf{x})\rangle$ is in $\mathscr{E}(\mathbb{R}^n)$. HINT: Fix any $\mathbf{y}$ and $1 \le j \le n$, prove that $\left(\eta(\mathbf{y}+h\mathbf{e}_j-\mathbf{x}) - \eta(\mathbf{y}-\mathbf{x})\right)/h \to \partial_{y_j}\eta(\mathbf{y}-\mathbf{x})$ in $\mathscr{D}(U)$ as $h \to 0$.

Exercise 6.3.2. For any $\phi \in \mathscr{D}(\mathbb{R}^n)$, suppose that the support of ϕ is contained in the compact box U and that $\{U_j\}$ is a partition of U, $\mathbf{y}_j \in U_j$. Prove that $\sum_j \phi(\mathbf{y}_j)\eta(\mathbf{y}_j - \mathbf{x})\ \mathrm{vol}(U_j) \to \int_{\mathbb{R}^n} \phi(\mathbf{y})\eta(\mathbf{y}-\mathbf{x})\, d\mathbf{y}$ in $\mathscr{D}(\mathbb{R}^n)$, therefore $\sum_j \phi(\mathbf{y}_j)\langle l, \eta(\mathbf{y}_j-\mathbf{x})\rangle\ \mathrm{vol}(U_j) \to \langle l, \int_{\mathbb{R}^n} \phi(\mathbf{y})\eta(\mathbf{y}-\mathbf{x})\, d\mathbf{y}\rangle$.

Exercise 6.3.3. Suppose that $l \in \mathscr{D}'(\mathbb{R})$ satisfies $\partial_x l = 0$. Prove that there exists some constant c such that $\langle l, \phi \rangle = \int_{\mathbb{R}} c\phi(x)\, dx$.

Exercise 6.3.4. Prove that

$$\lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \frac{1}{x+i\epsilon}\phi(x)\, dx = \text{P. V.} \int_{\mathbb{R}} \frac{1}{x}\phi(x)\, dx - \pi i \phi(0)$$

for any $\phi \in \mathscr{D}(\mathbb{R})$. This limit in $\mathscr{D}'(\mathbb{R})$ is usually denoted as $\dfrac{1}{x+i0}$. Define

$$\langle \frac{1}{x-i0}, \phi \rangle = \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} \frac{1}{x-i\epsilon} \phi(x)\, dx.$$

Identify $\dfrac{1}{x-i0}$ in terms of P. V. $\left(\dfrac{1}{x}\right)$ and $\delta(x)$.

Exercise 6.3.5. Define

$$\langle l, \phi \rangle = \int_{\mathbb{R}^n} \frac{\phi(\mathbf{x})}{|\mathbf{x}|^{n-2}}\, d\mathbf{x} \text{ for all } \phi \in \mathscr{D}(\mathbb{R}^n).$$

Show that, for $n \geq 3$, $\Delta l = -(n-2)|\mathbb{S}^{n-1}|\delta(\mathbf{x})$, and

$$\Phi(\mathbf{x}) := \langle l(\cdot - \mathbf{x}), \phi \rangle = \int_{\mathbb{R}^n} \frac{\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{n-2}}\, d\mathbf{y}$$

satisfies $\Delta\Phi(\mathbf{x}) = -(n-2)|\mathbb{S}^{n-1}|\phi(\mathbf{x})$ in the sense of distribution, namely,

$$\langle \Phi(\mathbf{x}), \Delta\psi(\mathbf{x}) \rangle = -(n-2)|\mathbb{S}^{n-1}| \int_{\mathbb{R}^n} \phi(\mathbf{x})\psi(\mathbf{x})\, d\mathbf{x} \text{ for all } \phi, \psi \in \mathscr{D}(\mathbb{R}^n).$$

6.4 Definition and Construction of A Fundamental Solution

We now give the formal definition of a fundamental solution.

Definition. A distribution $E(\mathbf{x};\boldsymbol{\xi})$ in $\mathbf{x}$ parametrized by $\boldsymbol{\xi}$ is called a fundamental solution of the differential operator $P(\partial_{\mathbf{x}})$ with pole at $\boldsymbol{\xi}$ if $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi})$ in the sense of distribution, namely, for any $\eta \in C_c^\infty(\mathbb{R}^n)$,

$$\eta(\boldsymbol{\xi}) = \langle E(\mathbf{x};\boldsymbol{\xi}), P'(\partial_{\mathbf{x}})\eta(\mathbf{x}) \rangle.$$

When we can identify $E(\mathbf{x};\boldsymbol{\xi})$ as a locally integrable function in $\mathbf{x}$, we can write

$$\eta(\boldsymbol{\xi}) = \langle E(\mathbf{x};\boldsymbol{\xi}), P'(\partial_{\mathbf{x}})\eta(\mathbf{x}) \rangle = \int_{\mathbb{R}^n} E(\mathbf{x};\boldsymbol{\xi}) P'(\partial_{\mathbf{x}})\eta(\mathbf{x})\, d\mathbf{x}. \tag{6.8}$$

Note that (6.8) represents a smooth function η (with compact support) in terms of an integral operator acting on $P'(\partial_{\mathbf{x}})\eta(\mathbf{x})$, and the integral operator plays the role of a left inverse to $P'(\partial_{\mathbf{x}})$. Likewise, if one would like to represent a smooth function

$u(\mathbf{x})$ (with compact support) in terms of an integral operator acting on $P(\partial_{\mathbf{x}})u(\mathbf{x})$ in the form

$$u(\boldsymbol{\xi}) = \langle E'(\mathbf{x};\boldsymbol{\xi}), P(\partial_{\mathbf{x}})u(\mathbf{x})\rangle = \int_{\mathbb{R}^n} E'(\mathbf{x};\boldsymbol{\xi})P(\partial_{\mathbf{x}})u(\mathbf{x})\,d\mathbf{x}, \tag{6.9}$$

for some distribution $E'(\mathbf{x};\boldsymbol{\xi})$ (may think of it for now as locally integrable in $\mathbf{x} \in \mathbb{R}^n$), then $E'(\mathbf{x};\boldsymbol{\xi})$ is a fundamental solution of P' in the sense defined above.

In other words, a left inverse for $P(\partial_{\mathbf{x}})$ is constructed using an integral operator involving a fundamental solution of P'; while a left inverse for $P'(\partial_{\mathbf{x}})$ is constructed using an integral operator involving a fundamental solution of P.

The original motivation for defining a fundamental solution is to use it to produce a right inverse to P in the sense that, under appropriate conditions on a given function $f(\boldsymbol{\xi})$, $\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}$—when it makes sense—would satisfy $P(\partial_{\mathbf{x}})\left[\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}\right] = f(\mathbf{x})$. On a formal level, this appears to be always valid if $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = \delta(\mathbf{x}-\boldsymbol{\xi})$; but its actual verification in the classical sense often requires additional work, and appropriate regularity condition on $f(\boldsymbol{\xi})$—even if one can justify passing the differential operator $P(\partial_{\mathbf{x}})$ into the integral $\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}$, (6.8) asks for a function f of $\mathbf{x}$, not $\boldsymbol{\xi}$.

First, a brief discussion on the meaning of $\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}$. It is meant as the limit in the distribution sense of a finite Riemann sum of distributions. Take any $\eta \in \mathscr{D}(U)$, then for any finite number of $\boldsymbol{\xi}_j, v_j$,

$$\langle \sum_j E(\mathbf{x};\boldsymbol{\xi}_j)f(\boldsymbol{\xi}_j)v_j, \eta\rangle = \sum_j \langle E(\mathbf{x};\boldsymbol{\xi}_j),\eta\rangle f(\boldsymbol{\xi}_j)v_j$$

makes sense. If the integral $\int\langle E(\mathbf{x};\boldsymbol{\xi}),\eta\rangle f(\boldsymbol{\xi})\,d\boldsymbol{\xi}$ is defined for any $\eta \in \mathscr{D}(U)$, we say that $\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}$ is defined, and

$$\langle \int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})\,d\boldsymbol{\xi}, \eta\rangle = \int \langle E(\mathbf{x};\boldsymbol{\xi}),\eta\rangle f(\boldsymbol{\xi})\,d\boldsymbol{\xi}.$$

In such a situation we can verify that $P(\partial_{\mathbf{x}})\left[\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}\right] = f(\mathbf{x})$ holds in the sense of distribution as follows. Take $f, \eta \in \mathscr{D}(U)$,

$$
\begin{aligned}
&\langle P(\partial_{\mathbf{x}})\left[\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}\right],\eta\rangle \\
=&\langle\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi},P'(\partial_{\mathbf{x}})\eta\rangle \\
=&\int\langle E(\mathbf{x};\boldsymbol{\xi}),P'(\partial_{\mathbf{x}})\eta\rangle f(\boldsymbol{\xi})d\boldsymbol{\xi} \\
=&\int\langle P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}),\eta\rangle f(\boldsymbol{\xi})d\boldsymbol{\xi} \\
=&\int\eta(\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi},
\end{aligned}
$$

from which it follows that $P(\partial_{\mathbf{x}})\left[\int E(\mathbf{x};\boldsymbol{\xi})f(\boldsymbol{\xi})d\boldsymbol{\xi}\right]=f(\mathbf{x})$ in the sense of distribution.

Note also that when constructing a right inverse to $P(\partial_{\mathbf{x}})$, we construct an integral operator using $E(\mathbf{x};\boldsymbol{\xi})$ and doing superposition (integration) against the pole parameter $\boldsymbol{\xi}$, while when constructing a left inverse to $P(\partial_{\mathbf{x}})$, we construct an integral operator using $E'(\mathbf{x};\boldsymbol{\xi})$ (a fundamental solution of $P'(\partial\mathbf{x})$) and doing superposition (integration) against $\mathbf{x}$.

Note that if $P(\partial_{\mathbf{x}})$ is an mth order differential operator and u is a $C^m(\mathbb{R}^n)$ solution of $P(\partial_{\mathbf{x}})u=0$, and $\eta\in C_c^m(\mathbb{R}^n)$, then by (6.5)

$$
\int_{\mathbb{R}^n}u(\mathbf{x})P'(\partial_{\mathbf{x}})\eta(\mathbf{x})\,d\mathbf{x}=0,
$$

so (6.8) can't hold if we use such a $u(\mathbf{x})$ as $E(\mathbf{x};\boldsymbol{\xi})$ in (6.8). Thus a fundamental solution of P can't be in C^m, and that if $E(\mathbf{x};\boldsymbol{\xi})$ is a fundamental solution for $P(\partial_{\mathbf{x}})$, satisfying (6.8), so will $E(\mathbf{x};\boldsymbol{\xi})+u(\mathbf{x})$. So when $P(\partial_{\mathbf{x}})$ has a fundamental solution, it may not be unique. Additional considerations in choosing a fundamental solution may include causality for evolution equations—this would be reflected in the support of a fundamental solution.

We now relate the notion of fundamental solution to the setting of (I) and (II) on page 258. Any $F(\boldsymbol{\xi};\mathbf{x})$ for (I) satisfies $P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x})=0$ for $\boldsymbol{\xi}$ in any open set not containing $\mathbf{x}$ in the sense of distribution and usually has some kind of singularity at $\boldsymbol{\xi}=\mathbf{x}$. Take any $\boldsymbol{\xi}^*\neq\mathbf{x}$ and $\epsilon>0$ small so that $\mathbf{x}\notin B_\epsilon(\boldsymbol{\xi}^*)$, then if we apply (6.5) with $v(\boldsymbol{\xi})=F(\boldsymbol{\xi};\mathbf{x})$ and $u(\boldsymbol{\xi})\in C_c^\infty(B_\epsilon(\boldsymbol{\xi}^*))$, we get $0=u(\mathbf{x})=\int_{B_\epsilon(\boldsymbol{\xi}^*)}P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x})u(\boldsymbol{\xi})\,d\boldsymbol{\xi}$, which implies that $P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x})=0$ for $\boldsymbol{\xi}\in B_\epsilon(\boldsymbol{\xi}^*)$ in the sense of distribution. Note that if $F(\boldsymbol{\xi};\mathbf{x})$ is smooth in $\boldsymbol{\xi}$ for $\boldsymbol{\xi}$ in some neighborhood of $\boldsymbol{\xi}^*\neq\mathbf{x}$, then it satisfies $P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x})=0$ in the classical sense for $\boldsymbol{\xi}$ in this neighborhood.

Next, suppose that $F(\boldsymbol{\xi};\mathbf{x})$ is smooth in $\boldsymbol{\xi}$ for $\boldsymbol{\xi}\in\mathbb{R}^n\setminus B_\epsilon(\mathbf{x})$ for some small $\epsilon>0$, then, if we apply (6.5) with $v(\boldsymbol{\xi})=F(\boldsymbol{\xi};\mathbf{x})$ and $u(\boldsymbol{\xi})\in C_c^\infty(\mathbb{R}^n)$ over $\mathbb{R}^n\setminus B_\epsilon(\mathbf{x})$, then we should get

$$\int_{\mathbb{R}^n\setminus B_\epsilon(\mathbf{x})} \left([P(\partial_{\boldsymbol{\xi}})u(\boldsymbol{\xi})]\, F(\boldsymbol{\xi};\mathbf{x}) - u(\boldsymbol{\xi})P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x})\right) d\boldsymbol{\xi}$$
$$= -\int_{\partial B_\epsilon(\mathbf{x})} \left(\sum_{a=1}^{n} \left(B_a[u(\boldsymbol{\xi}), F(\boldsymbol{\xi};\mathbf{x})]\right) \nu_a(\boldsymbol{\xi})\right) d\sigma(\boldsymbol{\xi}).$$

From this we can see that to accomplish (I), we should choose $F(\boldsymbol{\xi};\mathbf{x})$ such that

$$P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x}) = 0 \quad \text{for } \boldsymbol{\xi} \neq \mathbf{x},$$

and

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(\mathbf{x})} \left(\sum_{a=1}^{n} \left(B_a[u(\boldsymbol{\xi}), F(\boldsymbol{\xi};\mathbf{x})]\right) \nu_a(\boldsymbol{\xi})\right) d\sigma(\boldsymbol{\xi}) = -u(\mathbf{x})$$

for any $u \in C_c^\infty(\mathbb{R}^n)$.

The bilinear form $B_a[u(\boldsymbol{\xi}), F(\boldsymbol{\xi};\mathbf{x})]$ involves $u(\boldsymbol{\xi})$ and $F(\boldsymbol{\xi};\mathbf{x})$ such that each term in the summand has their total order of derivatives at most $m-1$. The consideration above implies that any boundary integral terms on $\partial B_\epsilon(\mathbf{x})$ containing derivatives of $u(\boldsymbol{\xi})$ would have 0 as their limits as $\epsilon \to 0$, and by taking $u(\boldsymbol{\xi})$ such that $u(\boldsymbol{\xi}) \equiv 1$ in a neighborhood of $\mathbf{x}$, we see that $\int_{\partial B_\epsilon(\mathbf{x})} \left(\sum_{a=1}^{n} \left(B_a[1, F(\boldsymbol{\xi};\mathbf{x})]\right) \nu_a(\boldsymbol{\xi})\right) d\sigma(\boldsymbol{\xi}) = -1$— these give conditions on the kind of singular behavior $F(\boldsymbol{\xi};\mathbf{x})$ is allowed to have as $\boldsymbol{\xi} \to \mathbf{x}$, just as we did for the Laplace operator.

For (II), for any fixed $\mathbf{x}$, if we take any f such that its support separates from $\mathbf{x}$, then $f(\mathbf{x}) = 0$, and we get $P(\partial_{\mathbf{x}}) \int_{\mathbb{R}^n} E(\mathbf{x};\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\boldsymbol{\xi} = 0$ for such f. This suggests that $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = 0$ for $\mathbf{x} \neq \boldsymbol{\xi}$.

So in the general situation we need to find $F(\boldsymbol{\xi};\mathbf{x})$ such that $P'(\partial_{\boldsymbol{\xi}})F(\boldsymbol{\xi};\mathbf{x}) = 0$ for $\mathbf{x} \neq \boldsymbol{\xi}$ for (I), and $E(\mathbf{x};\boldsymbol{\xi})$ such that $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = 0$ for $\mathbf{x} \neq \boldsymbol{\xi}$ for (II), both with some prescribed singular behavior at $\mathbf{x} = \boldsymbol{\xi}$. But when $P(\partial_{\mathbf{x}})$ has constant coefficients, we may consider $F(\boldsymbol{\xi};\mathbf{x})$ and $E(\mathbf{x};\boldsymbol{\xi})$ in the form of $H(\mathbf{x}-\boldsymbol{\xi})$. Then we note that

$$P(\partial_{\mathbf{x}})H(\mathbf{x}-\boldsymbol{\xi}) = P'(\partial_{\boldsymbol{\xi}})H(\mathbf{x}-\boldsymbol{\xi}),$$

and can try to construct such an $H(\mathbf{x}-\boldsymbol{\xi})$ for both (I) and (II).

The construction of fundamental solutions to a general constant coefficient linear operator was carried out in the 1950's using the tool of Fourier transforms. But that construction does not necessarily give an explicit form of a fundamental solution; and it would still take considerable effort to extract properties of a fundamental solution from that construction. We will first illustrate how to construct fundamental solutions to one dimensional equations, and to our prototype PDEs: the Laplace, the heat, and the wave equations — we will make use of the scaling and symmetry of the operators involved for these situations.

Recall that a fundamental solution $E(\mathbf{x};\boldsymbol{\xi})$ satisfies $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = 0$ away from the pole $\boldsymbol{\xi}$ in the sense that

$$\int E(\mathbf{x};\boldsymbol{\xi})P'(\partial_{\mathbf{x}})\eta(\mathbf{x})d\mathbf{x} = 0 \quad \text{if } \boldsymbol{\xi} \notin \text{support } (\eta).$$

In any region of $\mathbb{R}^n \setminus \{\boldsymbol{\xi}\}$ where $E(\mathbf{x};\boldsymbol{\xi})$ is C^m in $\mathbf{x}$, this would imply $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = 0$ in the classical sense there, as Green's identity would imply $\int P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi})\eta(\mathbf{x})d\mathbf{x} = 0$ for any $\eta \in C_c^\infty$ supported in that region. This is often used in constructing a fundamental solution.

Let

$$\Omega_E = \{\mathbf{x} \in \mathbb{R}^n : E(\mathbf{x};\boldsymbol{\xi}) \text{ can be identified as a } C^m \text{ function in a neighborhood of } \mathbf{x}\}.$$

Then $\mathbb{R}^n \setminus \Omega_E$ is called the **singular support** of $E(\mathbf{x};\boldsymbol{\xi})$. As will be seen, for the Laplace and heat equations, the singular support of their fundamental solutions consist of only their pole; while for the wave equation, the singular support of its fundamental solutions consists of a cone with the pole as its vertex.

The behavior of $E(\mathbf{x};\boldsymbol{\xi})$ can be characterized as follows when $E(\mathbf{x};\boldsymbol{\xi})$ is known to be C^m away from $\boldsymbol{\xi}$ and is locally integrable: we split the integral in $\eta(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} E(\mathbf{x};\boldsymbol{\xi})P'(\partial_{\mathbf{x}})\eta(\mathbf{x})d\mathbf{x}$ as $\int_{|\mathbf{x}-\boldsymbol{\xi}|>\epsilon} E(\mathbf{x};\boldsymbol{\xi})P'(\partial_{\mathbf{x}})\eta(\mathbf{x})d\mathbf{x}$, plus the integral over $B_\epsilon(\boldsymbol{\xi})$, which $\to 0$ as $\epsilon \to 0$ due to the local integrability assumption of $E(\mathbf{x};\boldsymbol{\xi})$. The first integral can be treated by Green's identify, using $P(\partial_{\mathbf{x}})E(\mathbf{x};\boldsymbol{\xi}) = 0$ for $\mathbf{x} \neq \boldsymbol{\xi}$, and we get

$$\int_{|\mathbf{x}-\boldsymbol{\xi}|>\epsilon} E(\mathbf{x};\boldsymbol{\xi})P'(\partial_{\mathbf{x}})\eta(\mathbf{x})d\mathbf{x} = -\int_{|\mathbf{x}-\boldsymbol{\xi}|=\epsilon} \sum_{a=1}^{n} \nu_a B_a[E(\mathbf{x};\boldsymbol{\xi}), \eta(\mathbf{x})]d\sigma(\mathbf{x}),$$

where $B_a[\cdot,\cdot]$ is the bilinear form associated with P, as in (6.5). Thus, in such a situation, we expect that for any $\eta \in C_c^m(\mathbb{R}^n)$,

$$\eta(\boldsymbol{\xi}) = -\lim_{\epsilon\to 0}\int_{|\mathbf{x}-\boldsymbol{\xi}|=\epsilon} \sum_{a=1}^{n} \nu_a B_a[E(\mathbf{x};\boldsymbol{\xi}), \eta(\mathbf{x})]d\sigma(\mathbf{x}).$$

We can often use this property on test functions (such as $\eta \in C_c^m(\mathbb{R}^n)$ with $\eta = 1$ near $\boldsymbol{\xi}$) to extract information on $E(\mathbf{x};\boldsymbol{\xi})$ near $\boldsymbol{\xi}$. It should be cautioned, though, that a fundamental solution may not be everywhere C^m away from its pole, as will be seen by the fundamental solutions of the wave equation.

Example 6.4. A fundamental solution $E(x;\xi)$ to the one dimensional operator $\frac{d}{dx}-\lambda$ would satisfy $\frac{dE(x;\xi)}{dx} - \lambda E(x;\xi) = 0$ for $x \neq \xi$ in the sense of distribution. It is to be proved in the exercises that such an $E(x;\xi)$ can be identified with the classical solution $Ce^{\lambda x}$ for $x > \xi$ and for $x < \xi$ with different choices of C; and the condition $\frac{dE(x;\xi)}{dx} - \lambda E(x;\xi) = \delta(x-\xi)$ at $x = \xi$ implies that $E(x;\xi)$ has a jump of size 1 across $x = \xi$: $E(\xi+0;\xi) - E(\xi-0;\xi) = 1$. If $E(\xi-0;\xi) = C$, then the solution is

$$E(x;\xi) = \begin{cases} (C+1)e^{\lambda(x-\xi)} & x > \xi, \\ Ce^{\lambda(x-\xi)} & x \leq \xi. \end{cases}$$

If we choose $C = 0$, we get a fundamental solution $E(x;\xi)$ supported in $\{x \geq \xi\}$, which can be written as $H(x-\xi)e^{\lambda(x-\xi)}$, with $H(\cdot)$ being the Heaviside function; while if we choose $C = -1$, we get a fundamental solution $E(x;\xi)$ supported in $\{x \leq \xi\}$, written as $-H(\xi - x)e^{\lambda(x-\xi)}$.

Recall that the transpose of $\frac{d}{dx} - \lambda$ is $-\frac{d}{dx} - \lambda$:

$$u(x)(\frac{d}{dx} - \lambda)v(x) - v(x)\left(-\frac{d}{dx} - \lambda\right)u(x) = \frac{d}{dx}[u(x)v(x)]. \qquad (*)$$

$E(x;\xi)$ can be used to construct a left inverse of $-\frac{d}{dx} - \lambda$: for any $u \in C_c^1(\mathbb{R})$,

$$\begin{aligned} u(\xi) = \left\langle \left(\frac{d}{dx} - \lambda\right) E(x;\xi), u(x) \right\rangle &= \left\langle E(x;\xi), \left(-\frac{d}{dx} - \lambda\right) u(x) \right\rangle \\ &= \int_{\mathbb{R}} E(x;\xi) \left(-\frac{d}{dx} - \lambda\right) u(x)\, dx. \end{aligned}$$

For $C = 0$, the integral reduces to $\int_\xi^\infty e^{\lambda(x-\xi)} \left(-\frac{d}{dx} - \lambda\right) u(x)\, dx$. If $\lambda > 0$, then $e^{\lambda(x-\xi)}$ grows exponentially in x, and it seems more convenient to choose $C = -1$ to get $u(\xi) = -\int_{-\infty}^\xi e^{\lambda(x-\xi)} \left(-\frac{d}{dx} - \lambda\right) u(x)\, dx$.

This can also been seen from exploiting (6.5), and it would allow u not in $C_c^1(\mathbb{R})$. Let's say we would like to express $u(\xi)$ in terms of an integral operator on $\left(-\frac{d}{dx} - \lambda\right) u(x)$. First take the case that $u \in C_c^1(\mathbb{R})$. If we choose $v = v(x;\xi)$ such that $(\frac{d}{dx} - \lambda)v(x;\xi) = 0$ for $x < \xi$, then integrating (*) from $[-\infty, \xi]$, we get

$$u(\xi)v(\xi) = -\int_{-\infty}^\xi v(x)\left(-\frac{d}{dx} - \lambda\right) u(x)\, dx.$$

Now if we choose $v(x;\xi)$ such that $v(\xi;\xi) = -1$, then we get

$$u(\xi) = \int_{-\infty}^\xi v(x;\xi)\left(-\frac{d}{dx} - \lambda\right) u(x)\, dx \quad \text{for } u \in C_c^1(\mathbb{R}).$$

The $v(x;\xi)$ is a fundamental solution of $\left(\frac{d}{dx}-\lambda\right)$, characterized by $(\frac{d}{dx}-\lambda)v(x;\xi)=0$ for $x<\xi$, and with the additional property $v(\xi;\xi)=-1$ (and implicitly $v(x;\xi)=0$ for $x>\xi$ so $\int_{\mathbb{R}} v(x)\cdots dx=\int_{-\infty}^{\xi} v(x)\cdots dx$). The jump discontinuity of $v(x;\xi)$ at $x=\xi$ is due to $(\frac{d}{dx}-\lambda)v(x)=\delta(x-\xi)$.

Such a fundamental solution, with appropriate boundary/initial condition adapted to the particular BVP/IVP at hand, is usually called a Green's function.

Suppose that u is not necessarily in $C^1_c(\mathbb{R})$ and we would like to express $u(\xi)$ in terms of an integral operator on $\left(-\frac{d}{dx}-\lambda\right)u(x)$ and the initial data of u at 0. Redo the integration above from, say $[0,\xi]$, we get

$$u(\xi)v(\xi)=u(0)v(0)-\int_0^{\xi} v(x)\left(-\frac{d}{dx}-\lambda\right)u(x)\,dx.$$

To construct a left inverse for $\frac{d}{dx}-\lambda$, we use a fundamental solution $E'(x;\xi)$ for its transpose $-\frac{d}{dx}-\lambda$, which is given by $E(\xi;x)=-H(x-\xi)e^{-\lambda(x-\xi)}$. Here we have chosen one supported in $x>\xi$. We then get, for $u\in C^1_c(\mathbb{R})$,

$$u(\xi)=\int_{\mathbb{R}} E'(x;\xi)\left(\frac{d}{dx}-\lambda\right)u(x)\,dx=-\int_{\xi}^{\infty} e^{-\lambda(x-\xi)}\left(\frac{d}{dx}-\lambda\right)u(x)\,dx.$$

On the other hand, the fundamental solution $E(x;\xi)=H(x-\xi)e^{\lambda(x-\xi)}$ of $\frac{d}{dx}-\lambda$ gives rise to a right inverse $\int_{\mathbb{R}} E(x;\xi)f(\xi)\,d\xi$ in the sense that

$$\left(\frac{d}{dx}-\lambda\right)\int_{\mathbb{R}} E(x;\xi)f(\xi)\,d\xi=\left(\frac{d}{dx}-\lambda\right)\int_{-\infty}^{x} e^{\lambda(x-\xi)}f(\xi)\,d\xi=f(x)$$

for any $f\in C^1_c(\mathbb{R})$ (in fact it suffices here for $f\in C_c(\mathbb{R})$). If we choose to work with $E(x;\xi)=-H(\xi-x)e^{\lambda(x-\xi)}$, we get $-\int_x^{\infty} e^{\lambda(x-\xi)}f(\xi)\,d\xi$, which, for $\lambda>0$, would allow f to have some growth at ∞.

Example 6.5. The last approach in the previous example can be used in formulating conditions on a fundamental solution to a Cauchy problem. Let's use a special case of $P_4=\partial_t^2+a(x,t)\partial_t-\partial_x^2$ to illustrate this process. We will start with an even simpler case: $L[u]=\partial_t^2u(t)+a(t)\partial_tu(t)$ on a function $u(t)$ of one variable. Then the Green's identity takes the form

$$\begin{aligned}v(t)L[u]-u(t)L'[v(t)]&=v(t)[\partial_t^2u(t)+a(t)\partial_tu(t)]-u(t)[\partial_t^2v(t)-\partial_t\left(a(t)v(t)\right)]\\&=\partial_t[v(t)\partial_tu(t)-u(t)\partial_tv(t)+a(t)u(t)v(t)].\end{aligned}$$

To construct a solution to the Cauchy problem

$$\begin{cases}L[u(t)]=f(t),t>0,\\ u(0)=u_0,\\ u'(0)=u_0',\end{cases}$$

we integrate the above Green's identity from $0 < t < \tau$ to get

$$\begin{aligned}&[v(t)\partial_t u(t) - u(t)\partial_t v(t) + a(t)u(t)v(t)]\Big|_{t=\tau}\\ =&v(0)\partial_t u(0) - u(0)\partial_t v(0) + a(0)u(0)v(0) + \int_0^\tau [v(t)L[u(t)] - uL'[v(t)]]\,dt.\end{aligned}$$

We see now that if we choose $v = v(t;\tau)$ such that $-\partial_t v(t;\tau)|_{t=\tau} = 1$, $v(\tau;\tau) = 0$, and $L'[v(t;\tau)] = 0$ for $0 < t < \tau$, we would get

$$u(\tau) = v(0;\tau)u'(0) - u(0)\partial_t v(0;\tau) + a(0)u(0)v(0;\tau) + \int_0^\tau v(t;\tau)L[u(t)]\,dt.$$

In summary, we have reduced the construction of a solution to a (non-homogeneous) Cauchy problem to the construction of a fundamental solution to the transpose operator, which solves a homogeneous Cauchy problem with a special initial data —this is consistent with the the Duhamel principle, as we will see that, treating $v(t;\tau)$ as a function of τ, we have $L(\partial_\tau)[v(t;\tau)] = 0$ for $\tau > t$, and $\partial_\tau v(t;\tau)|_{\tau=t} = 1$.

When we generalize this consideration to the Cauchy problem for the partial differential operator $P_4 = \partial_t^2 + a(x,t)\partial_t - \partial_x^2$, we integrate the Green's identity $v(x,t)P[u(x,t)] - u(x,t)P'[v(x,t)]$ for $0 < t < \tau$ and $x \in \mathbb{R}$ (assuming the boundary terms in x will vanish, which is the case when we work with $u(x,t)$ which vanishes for large $|x|$), we would get

$$\begin{aligned}&\int_{\mathbb{R}}[v(x,t)\partial_t u(x,t) - \partial_t v(x,t)u(x,t) + a(x,t)v(x,t)u(x,t)]\Big|_{t=\tau}\,dx\\ =&\int_{\mathbb{R}}[v(x,0)\partial_t u(x,t)|_{t=0} - \partial_t v(x,t)|_{t=0}u(x,0) + a(x,0)v(x,0)u(x,0)]\,dx\\ &+\int_0^\tau\int_{\mathbb{R}}\Big[v(x,t)P[u(x,t)] - u(x,t)P'[v(x,t)]\Big]\,dx\,dt.\end{aligned}$$

We should choose $v = v(x,t;\xi,\tau)$ such that $P'[v(x,t;\xi,\tau)] = 0$, $v(x,\tau;\xi,\tau) = 0$, and $\int_{\mathbb{R}} -\partial_t v(x,t;\xi,\tau)|_{t=\tau}u(x,\tau)\,dx = u(\xi,\tau)$. But no regular function $-\partial_t v(x,t;\xi,\tau)|_{t=\tau}$ can satisfy $\int_{\mathbb{R}} -\partial_t v(x,t;\xi,\tau)|_{t=\tau}u(x,\tau)\,dx = u(\xi,\tau)$ for all functions $u(x,\tau)$. The way we handle this is to note that we can construct a family of functions $\phi_\epsilon(x;\xi)$ such that $\int_{\mathbb{R}}\phi_\epsilon(x;\xi)u(x,\tau)\,dx \to u(\xi,\tau)$, as $\epsilon \to 0$. We then construct $v_\epsilon(x,t;\xi,\tau)$ such that

$$\begin{cases} P'[v_\epsilon(x,t;\xi,\tau)] = 0, & \text{for } t < \tau,\\ v_\epsilon(x,\tau;\xi,\tau) = 0, & \\ \partial_t v_\epsilon(x,t;\xi,\tau)\Big|_{t=\tau} = -\phi_\epsilon(x;\xi). & \end{cases}$$

What remains is to prove that $v_\epsilon(x,t;\xi,\tau)$ has a limit $v(x,t;\xi,\tau)$ as $\epsilon \to 0$, then this $v(x,t;\xi,\tau)$ will be a fundamental solution to the transpose operator.

Remark 6.3. If we are interested in solving a boundary value problem involving $L[u(t)] = \partial_t^2 u(t) + a(t)\partial_t u(t)$, say,

$$\begin{cases} L[u(t)] = f(t), l > t > 0, \\ \quad u(0) = \text{given}, \\ \quad u(l) = \text{given}, \end{cases}$$

we would like to choose a fundamental solution which is more adapted to the boundary value problem here, namely, which would eliminate the dependence on $u'(0)$ and $u'(l)$ in the integral representation for $u(\tau)$ in terms of $L[u(t)]$. First, we carry out a similar analysis on $[0, \tau]$, and instead of forcing $v(\tau;\tau) = 0$, we should choose $v(0;\tau) = 0$, which would leave us with

$$\begin{aligned} &[v(t;\tau)\partial_t u(t) - u(t)\partial_t v(t;\tau) + a(t)u(t)v(t;\tau)]\Big|_{t=\tau-} \\ =& -u(0)\partial_t v(0;\tau) + \int_0^\tau v(t;\tau)L[u(t)]\,dt. \end{aligned}$$

If we repeat a similar computation on $[\tau, l]$, and make sure that $L'[v(t;\tau)] = 0$ for $\tau < t < l$, and $v(l;\tau) = 0$, we would get

$$\begin{aligned} &-u(l)\partial_t v(l;\tau) - [v(t;\tau)\partial_t u(t) - u(t)\partial_t v(t;\tau) + a(t)u(t)v(t;\tau)]\Big|_{t=\tau+} \\ =& \int_\tau^l v(t;\tau)L[u(t)]\,dt. \end{aligned}$$

If we can make $v(t;\tau)$ be continuous as a function of t, but $\partial_t v(\tau+;\tau) - \partial_t v(\tau-;\tau) = 1$, then by adding the two equalities, we get

$$u(\tau) = u(l)\partial_t v(l;\tau) - u(0)\partial_t v(0;\tau) + \int_0^l v(t;\tau)L[u(t)]\,dt.$$

In summary, we need some $v(t;\tau)$ such that $L'[v(t;\tau)] = 0$ for $t \neq \tau$, $v(t;\tau)$ continuous in t, $\partial_t v(t;\tau)|_{t=\tau+} - \partial_t v(t;\tau)|_{t=\tau-} = 1$, and $v(0;\tau) = v(l;\tau) = 0$. This gives us a Green's function for this boundary value problem. The jump discontinuity in $\partial_t v(t;\tau)$ at $t = \tau$ implies that $L'[v(t;\tau)] = \delta(t-\tau)$. The existence of such a Green's function is not automatically guaranteed, even when a fundamental solution without additional conditions is known to exist.

Example 6.6. When $P = \partial_t^2 - c^2\partial_x^2$ is the wave operator in 1-spatial dimension, and $u(x,t) \in C_c^2(\mathbb{R}^2)$, we have, by the d'Alembert's formula with initial time chosen such that both u and u_t vanish at the initial time

$$u(x,t) = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \left[\partial_s^2 - c^2\partial_y^2\right] u(y,s)\,dyds,$$

which can be written as $\iint H(x-y,t-s)\left[\partial_s^2 - c^2\partial_y^2\right] u(y,s)\,dyds$, where

$$H(x-y,t-s) = \begin{cases} \frac{1}{2c} & \text{if } t > s \text{ and } |y-x| \le c(t-s), \\ 0 & \text{otherwise.} \end{cases}$$

So this identifies $E'(y,s;x,t) = H(x-y,t-s)$ defined above as a fundamental solution of the transpose of one dimensional wave operator. Note that this fundamental solution $E'(y,s;x,t) = H(x-y,t-s)$ is supported in the "backward light cone" $\{(y,s) : |y-x| \le c(t-s), s \le t\}$ and has a jump discontinuity along its boundary; while, considered as a function of (x,t), $H(x-y,t-s)$ is supported in the "forward light cone" $\{(x,t) : |y-x| \le c(t-s), t \ge s\}$.

$H(x-y,t-s)$ can also be found as a limit of solutions $u_\epsilon(x,t)$ to

$$\begin{cases} \left[\partial_t^2 - c^2\partial_x^2\right] u_\epsilon(x,t) = 0 & t > s, \\ u_\epsilon(x,s) = 0, & \\ \partial_t u_\epsilon(x,s) = \phi_\epsilon(x), & \\ u_\epsilon(x,t) = 0, & t < s, \end{cases}$$

where $\phi_\epsilon(x) \to \delta(x-y)$ as $\epsilon \to 0$ in the sense of distribution. This is based on the Duhamel principle for constructing solutions to IVP for $[\partial_t^2 - c^2\partial_x^2]\,u(x,t) = \delta(x-y,t-s)$. Since for $t > s$, $u_\epsilon(x,t) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} \phi_\epsilon(z)dz$, if we take $\phi_\epsilon(z)$ to be supported in $[y-\epsilon, y+\epsilon]$ with $\int_{y-\epsilon}^{y+\epsilon} \phi_\epsilon(z)\,dz = 1$, it is now clear that $u_\epsilon(x,t) = 0$ if either $x - c(t-s) \ge y + \epsilon$ or $x + c(t-s) \le y - \epsilon$, and $u_\epsilon(x,t) = \frac{1}{2c}$ if $-c(t-s) + \epsilon \le x - y \le c(t-s) - \epsilon$. Thus for any $-c(t-s) < x - y < c(t-s)$, $u_\epsilon(x,t) \to \frac{1}{2c}$, as $\epsilon \to 0$, and for any $x < y - c(t-s)$ or $x > y + c(t-s)$, $u_\epsilon(x,t) \to 0$ as $\epsilon \to 0$. This at least identifies

$$E(x,t;y,s) = \begin{cases} \frac{1}{2c} & t > s, -c(t-s) < x - y < c(t-s), \\ 0 & x < y - c(t-s) \text{ or } x > y + c(t-s) \text{ or } t < s. \end{cases}$$

What remains is to determine whether $E(x,t;y,s)$ has any singular behavior as a distribution along $|x-y| = c(t-s), t > s$. Since $E(x,t;y,s)$ is constructed as the limit of $u_\epsilon(x,t)$, which are uniformly bounded in (x,t) and ϵ, and converges point wise to $E(x,t;y,s)$, except not known along $|x-y| = c(t-s), t > s$, we conclude that $E(x,t;y,s)$, as a distribution, is defined by the locally integrable function above.

Remark 6.4. When we work with a fundamental solution of a differential operator P which has a (well-posed) IVP, we can often construct or identify a fundamental solution as the limit of solutions to the IVP, in which the highest order Cauchy data approaches the unit source $\delta(x-\xi)$ in the limit, while the lower order Cauchy data are taken to be zero, as was done above.

Exercises

Exercise 6.4.1. Suppose that $\frac{d}{dx}E(x;\xi) - \lambda E(x;\xi) = \delta(x-\xi)$. Show that the distribution $F(x;\xi) := e^{-\lambda(x-\xi)}E(x;\xi)$ satisfies $\frac{d}{dx}F(x;\xi) = \delta(x-\xi)$, and that $F(x;\xi) = H(x-\xi)+C$ for some constant C, where $H(\cdot)$ is the Heaviside function. Use this to show that $E(x;\xi) = e^{\lambda(x-\xi)}\left(H(x-\xi)+C\right)$.

Exercise 6.4.2. Suppose that $A(x)$ is an $n \times n$-matrix valued smooth function of $x \in \mathbb{R}$. Let $S(x)$ be the $n \times n$-matrix valued solution to

$$S'(x) = A(x)S(x), \quad S(0) = I.$$

Note that $S(x)$ remains invertible for all x. Consider $\mathbb{R}^n$-valued distribution $\mathbf{u}(x)$, namely, $\mathbf{u}(x) = (u_1(x), \cdots, u_n(x))^{\mathrm{T}}$, where each $u_j(x)$ is a distribution in x. Suppose that $\mathbf{u}'(x) - A(x)u(x) = (\delta(x), \cdots, \delta(x))^{\mathrm{T}}$. Define an $\mathbb{R}^n$-valued distribution $\mathbf{v}(x)$ by the equation $\mathbf{u}(x) = S(x)\mathbf{v}(x)$. Show that $\mathbf{v}'(x) = (\delta(x), \cdots, \delta(x))^{\mathrm{T}}$ and use this to show that $\mathbf{u}(x) = S(x)\left(\mathbf{c} + H(x)(1, \cdots, 1)^{\mathrm{T}}\right)$, where $\mathbf{c}$ is some constant vector.

Exercise 6.4.3. Suppose that $a_j(x)$, $j = 0, \cdots, k-1$, are smooth functions of x. Show that a fundamental solution $E(x)$ of

$$\frac{d^k}{dx^k}E(x) + a_{k-1}(x)\frac{d^{k-1}}{dx^{k-1}}E(x) + \cdots + a_0(x)E(x) = \delta(x)$$

can be constructed as $S(x)H(x)$, where $S(x)$ is the solution to

$$\begin{cases} \dfrac{d^k}{dx^k}S(x) + a_{k-1}(x)\dfrac{d^{k-1}}{dx^{k-1}}S(x) + \cdots + a_0(x)S(x) = 0, \\ \dfrac{d^j}{dx^j}S(0) = 0, j = 0, \cdots, k-2,\ \dfrac{d^{k-1}}{dx^{k-1}}S(0) = 1. \end{cases}$$

Exercise 6.4.4. Construct a fundamental solution $E(x;\xi)$ to $\frac{d^2}{dx^2} \pm m^2$ on $\mathbb{R}$ which lies in $\mathscr{S}'(\mathbb{R})$. Here m is a real parameter. Is such an $E(x;\xi)$ uniquely determined?

Exercise 6.4.5. Construct a fundamental solution $E(x;\xi)$ to the operator $\frac{d^2}{dx^2} \pm m^2$ on $\mathbb{R}$ which is supported on $\{x : x \geq \xi\}$. Is such an $E(x;\xi)$ in $\mathscr{S}'(\mathbb{R})$?

Exercise 6.4.6. Consider the differential operator $P[u] = u''(x) + (b(x)u(x))'$ on $\mathbb{R}$, and its transpose $P'[v] = v''(x) - b(x)v'(x)$. Follow the instructions to work out fundamental solutions of P and P'.

(i). Derive that

$$E(x;\xi) = \begin{cases} B^{-1}(x)\int_\xi^x B(s)ds, & \text{if } x > \xi, \\ 0, & \text{if } x \le \xi, \end{cases}$$

is a fundamental solutions of P, where $B(x) = \exp(\int^x b(s)ds)$. Furthermore, verify that, if $f \in C_c(\mathbb{R})$, then

$$u(x) = \int_{\mathbb{R}} E(x;\xi)f(\xi)d\xi$$

is a solution of $P[u](x) = f(x)$.

(ii). Verify that

$$E'(x;\xi) = \begin{cases} B^{-1}(\xi)\int_x^\xi B(s)ds, & \text{if } x < \xi, \\ 0, & \text{if } x \ge \xi, \end{cases}$$

is a fundamental solutions of P', where $B(x) = \exp(\int^x b(s)ds)$. Furthermore, verify that

$$u(x) = \int_{\mathbb{R}} E'(y;x)P[u](y)\,dy.$$

HINT: you could also try setting $P[u](x) = f(x)$ and deriving $u(x)$ in terms of an integral operator of f, and then identifying $E'(y;x)$.

Exercise 6.4.7. This exercise discusses the relation between a fundamental solution E of P and a fundamental solution E' of P'. Let $F_a[u,v]$ be defined through

$$P[u](\mathbf{x})v(\mathbf{x}) - u(\mathbf{x})P'[v](\mathbf{x}) = \sum_{a=1}^n \partial_{x_a}\left\{F_a[u(\mathbf{x}), v(\mathbf{x})]\right\}.$$

Let $E(\mathbf{x};\mathbf{y})$ be a fundamental solution of P with pole at $\mathbf{y}$, and $E'(\mathbf{x};\mathbf{z})$ be a fundamental solution of P' with pole at $\mathbf{z}$. Suppose that both are given by functions that are smooth except at their poles.

(i). Prove that if Ω is a domain with piecewise C^1 boundary, and $\mathbf{y}, \mathbf{z} \in \Omega$, then for any u and v which are sufficiently smooth in a domain which contains the closure of Ω, we have

$$u(\mathbf{z}) = \iint_\Omega E'(\mathbf{x};\mathbf{z})P[u](\mathbf{x})\,d\mathbf{x} - \int_{\partial\Omega}\sum_{a=1}^n \nu_a(\mathbf{x})F_a[u(\mathbf{x}), E'(\mathbf{x};\mathbf{z})]\,d\sigma(\mathbf{x}). \tag{6.10}$$

$$v(\mathbf{y}) = \iint_\Omega E(\mathbf{x};\mathbf{y})P'[v](\mathbf{x})\,d\mathbf{x} + \int_{\partial\Omega}\sum_{a=1}^n \nu_a(\mathbf{x})F_a[E(\mathbf{x};\mathbf{y});v(\mathbf{x})]\,d\sigma(\mathbf{x}). \tag{6.11}$$

(ii). Given $\mathbf{y} \neq \mathbf{z} \in \Omega$. Take $u(\mathbf{x}) = E(\mathbf{x};\mathbf{y})$, and apply (6.10) with $B(\mathbf{z}, r)$ replacing Ω, for $r > 0$ small enough such that $B(\mathbf{y}, r) \cap B(\mathbf{z}, r) = \emptyset$, prove that

$$E(\mathbf{z};\mathbf{y}) = -\int_{\partial B(\mathbf{z},r)}\sum_{a=1}^n \nu_a(\mathbf{x})F_a[E(\mathbf{x};\mathbf{y}), E'(\mathbf{x};\mathbf{z})]\,d\sigma(\mathbf{x}).$$

Note also that the integral above is independent of $r > 0$ for sufficiently small r. Next, take $v(\mathbf{x}) = E'(\mathbf{x};\mathbf{z})$, and apply (6.11) with $B(\mathbf{y}, r)$ replacing Ω, for sufficiently small $r > 0$, prove that

$$E'(\mathbf{y};\mathbf{z}) = \int_{\partial B(\mathbf{y},r)}\sum_{a=1}^n \nu_a(\mathbf{x})F_a[E(\mathbf{x};\mathbf{y}), E'(\mathbf{x};\mathbf{z})]\,d\sigma(\mathbf{x}).$$

(iii). Prove that if $\int_{\partial\Omega}\sum_{a=1}^n \nu_a(\mathbf{x})F_a[E(\mathbf{x};\mathbf{y}), E'(\mathbf{x};\mathbf{z})]\,d\sigma(\mathbf{x}) = 0$ for $\mathbf{y} \neq \mathbf{z} \in \Omega$, then $E(\mathbf{z};\mathbf{y}) = E'(\mathbf{y};\mathbf{z})$.

Exercise 6.4.8. Verify that

$$E(\mathbf{x}) = \begin{cases} \frac{|\mathbf{x}|^{4-n}}{2(n-2)(n-4)|\mathbb{S}^{n-1}|} & \text{if } n > 4 \text{ or } n = 3, \\ \frac{\log|\mathbf{x}|^{-1}}{4|\mathbb{S}^3|} & \text{if } n = 4, \\ \frac{|\mathbf{x}|^2\log|\mathbf{x}|}{8\pi} & \text{if } n = 2, \end{cases}$$

provides a fundamental solution for the bi-Laplace operator Δ^2 on $\mathbb{R}^n$.

Exercise 6.4.9. Follow the guidance to find a fundamental solution of $Lu = \sum_{i,j=1}^n a_{ij}\partial^2_{x_i x_j}u(\mathbf{x}) = -\delta(\mathbf{x})$, where $A = (a_{ij})$ forms a positive definite matrix.

(i). Prove that if a linear change of variables $y_k = \sum_{l=1}^n T_{kl}x_l$ is made, where $T = (T_{kl})$ is non-singular, and $v(\mathbf{y}) = u(\mathbf{x})$, then $Lu = \sum_{k,l=1}^n \widetilde{a}_{kl}\partial^2_{y_k y_l}v(\mathbf{y}) := \widetilde{L}v$, where $\widetilde{a}_{kl} = \sum_{i,j=1}^n T_{ki}a_{ij}T_{lj}$. In terms of matrix operation, with $A = (a_{ij})$ and $\widetilde{A} = (\widetilde{a}_{kl})$, this amounts to $\widetilde{A} = TAT'$. If A is positive definite, choose T such that TAT' is the identity matrix, then $Lu = \Delta v(\mathbf{y})$.

(ii) Prove that under the change of variables $\mathbf{y} = T\mathbf{x}$, the integral relation $\int E(\mathbf{x})Lu(\mathbf{x}) = u(0)$ turns into $\int E(T^{-1}\mathbf{y})\widetilde{L}v(\mathbf{y})|\det T|^{-1}\,d\mathbf{y} = v(0)$. Thus if $\widetilde{L} = -\Delta$, then $E(\mathbf{x}) = |\det T|\Phi(T\mathbf{x})$ is a fundamental solution of L.

(iii). Since $\Phi(T\mathbf{x})$ depends on $T\mathbf{x}$ through $|T\mathbf{x}|$, using the relation $|T\mathbf{x}|^2 = \mathbf{x}'T'T\mathbf{x}$, and $TAT' = I$ to derive that $|T\mathbf{x}|^2 = \mathbf{x}'A^{-1}\mathbf{x}$, and that $|\det T| = 1/\sqrt{\det A}$. Thus $E(\mathbf{x}) = \Phi(\sqrt{\mathbf{x}'A^{-1}\mathbf{x}})/\sqrt{\det A}$.

Exercise 6.4.10. Follow the guidance below to construct a fundamental solution of the Helmholtz equation $(\Delta + c)E(x) = -\delta(x)$, where c is a constant. Note that Δ and c have different scaling, so we can't rely on scaling-invariance to find a fundamental solution; but we can still use the rotational invariance of $\Delta + c$ to look for a fundamental solution $E = E(|x|)$.

(i). It is known that $E(r)$ is smooth for $r > 0$, so $E''(r)+\frac{n-1}{r}E'(r)+cE(r) = 0$ for $r > 0$. Recall that when $n = 2$ and $c = 1$, solutions to this ODE are called Bessel's functions (of order 0). A separable solution $R(r)\Theta(\theta)$ to $\Delta u(x)+cu(x) = 0$ (on $\mathbb{R}^n \setminus \{0\}$) leads to

$$R''(r) + \frac{n-1}{r}R'(r) + \left[-\frac{\lambda}{r^2} + c\right] R(r) = 0, \tag{6.12}$$

and $\Delta_\theta\Theta(\theta) + \lambda\Theta(\theta) = 0$ for $\theta \in \mathbb{S}^{n-1}$, where Δ_θ is the Laplace-Beltrami operator on the round sphere $\mathbb{S}^{n-1}$ (when $n = 2$, $\Delta_\theta = \partial_\theta^2$ and $\Theta(\theta) = e^{im\theta}$ for $m \in \mathbb{Z}$, so $\lambda = m^2$; when $n \geq 3$, $\lambda = m(m + n - 2)$ for $m \in \mathbb{Z}_{\geq 0}$).

(ii). Suppose that $R(r)$ is a solution of (6.12). Let $\widehat{R}(r) = r^{\frac{n-2}{2}}R(r)$. Verify that

$$\widehat{R}''(r) + \frac{\widehat{R}'(r)}{r} + \left[c - \frac{\lambda + \left(\frac{n-2}{2}\right)^2}{r^2}\right] \widehat{R}(r) = 0.$$

If $c > 0$, set $J(r) = \widehat{R}(\frac{r}{\sqrt{c}})$, verify that $J(r)$ satisfies (2.50), Bessel equation of order α with $\alpha^2 = \lambda + \left(\frac{n-2}{2}\right)^2$.

(iii). Prove that, when $c > 0$, $E(r) = -\frac{1}{4}Y_0(\sqrt{c}r)$ is a fundamental solution of $(\Delta + c)E(x) = -\delta(x)$ on $\mathbb{R}^2$ (Hint: use $E'(r) \to -\frac{1}{2\pi r}$ as $r \to 0$); for $n \geq 3$, a fundamental solution can be constructed in the form of $a\, r^{\frac{2-n}{2}} J_{-\frac{n-2}{2}}(\sqrt{c}r)$ for n odd, and $a\, r^{\frac{2-n}{2}} Y_{\frac{n-2}{2}}(\sqrt{c}r)$ for n even, where a is chosen appropriately depending on n and c. When $c < 0$, (6.12) can be reduced to the modified Bessel's equation (2.51).

Note also that for $c > 0$, $R(r) = r^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{c}r)$ provides an entire solution on $\mathbb{R}^n$ to $(\Delta + c)E(x) = 0$, which decays to 0 at a rate of $r^{\frac{1-n}{2}}$ as $r \to \infty$; furthermore, for any $y = (y', y_n) \in \mathbb{R}^n_+$, $E(x) = |x-y|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{c}|x-y|) - |x-$

$\hat{y}|^{\frac{2-n}{2}} J_{\frac{n-2}{2}}(\sqrt{c}|x-\hat{y}|)$, where $\hat{y} = (y', -y_n)$ and $x = (x', x_n)$, defines a solution of $(\Delta + c)E(x) = 0$ on $\mathbb{R}^n_+$, such that $E(x)|_{\partial\mathbb{R}^n_+} = 0$, and $E(x) \to 0$ as $x \to \infty$ in $\mathbb{R}^n_+$ ($\sin(\sqrt{c}x_n)$ is also a solution equal to 0 on $\partial\mathbb{R}^n_+$, but it does not decay to 0 as $x \to \infty$ in $\mathbb{R}^n_+$). These provide additional examples for the failure of uniqueness to the Dirichlet problem for $\Delta + c$ on $\mathbb{R}^n_+$.

Exercise 6.4.11. This exercise continues to explore solutions $E(r) := E_{n,c}(r)$ to $E''(r) + \frac{n-1}{r}E'(r) + cE(r) = 0$.

(i). Verify that when $E''_{n,c}(r) + \frac{n-1}{r}E'_{n,c}(r) + cE_{n,c}(r) = 0$ for $r > 0$, then $w(r) = E'_{n,c}(r)/r$ satisfies $w''(r) + \frac{n+1}{r}w'(r) + cw(r) = 0$ for $r > 0$. Conversely, for any solution $w(r)$ of the latter, $rw'(r) + nw(r)$ is a solution of the former. So we can determine $E_{n+2,c}(r)$ in terms of $E_{n,c}(r)$ in the form of $E_{n+2,c}(r) = AE'_{n,c}(r)/r$ for some constant A.

(ii). Note that, if $E_{n,c}(r)$ is a fundamental solution to $(\Delta + c)E = -\delta(x)$, then for $\phi \in C^2_c(\mathbb{R}^n)$, we have $-\phi(0) = \int_{\mathbb{R}^n} E_{n,c}(x)(\Delta + c)\phi(x)\,dx$, and if $\phi \equiv 1$ in $B_\epsilon(0)$, we would have

$$\begin{aligned} -1 = -\phi(0) &= \int_{\mathbb{R}^n} E_{n,c}(x)(\Delta + c)\phi(x)\,dx \\ &= \lim_{\epsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(0)} E_{n,c}(x)(\Delta + c)\phi(x)\,dx \\ &= -\lim_{\epsilon \searrow 0} \int_{\partial B_\epsilon(0)} \left[E_{n,c}(x)\frac{\partial \phi(x)}{\partial r} - \phi(x)\frac{\partial E_{n,c}(x)}{\partial r} \right] d\sigma(x) \\ &= \lim_{\epsilon \searrow 0} \int_{\partial B_\epsilon(0)} \frac{\partial E_{n,c}(x)}{\partial r} d\sigma(x), \end{aligned}$$

so we expect $r^{n-1}E'_{n,c}(r)|\mathbb{S}^{n-1}| = \int_{\partial B_r(0)} \partial_r E_{n,c}(r)\,d\sigma(x) \to -1$ as $r \to 0$. Use this and (i) to determine $E_{n,c}(r)$ for $n = 1, 2, 3, 4$. What role does the sign of c play?

(iii). Use the recursive relations and the previous exercise to establish relations between $J_{\pm\frac{1}{2}}(r)$ with $\frac{\sin r}{\sqrt{r}}$ and $\frac{\cos r}{\sqrt{r}}$, respectively.

6.5 More on Fundamental Solutions

Notice that for a constant coefficient operator $P(\partial_{\mathbf{x}})$ of order m, if E is a fundamental solution with pole 0, then $E(\mathbf{x} - \boldsymbol{\xi})$ is a fundamental solution of $P(\partial_{\mathbf{x}})$ at pole $\boldsymbol{\xi}$,

$$\eta(\boldsymbol{\xi}) = \langle E(\mathbf{x} - \boldsymbol{\xi}), P'(\partial_{\mathbf{x}})\eta(\mathbf{x}) \rangle\,.$$

Furthermore, for any $u \in C_c^m(\mathbb{R}^n)$, if we set $\eta(\mathbf{x}) = u(\boldsymbol{\xi} - \mathbf{x})$, we find that, using $\partial_{\mathbf{x}}^\alpha[u(\boldsymbol{\xi} - \mathbf{x})] = (-1)^{|\alpha|}\partial_{\boldsymbol{\xi}}^\alpha u(\boldsymbol{\xi} - \mathbf{x})$, we have

$$P'(\partial_{\mathbf{x}})\eta(\mathbf{x}) = P'(\partial_{\mathbf{x}})[u(\boldsymbol{\xi}-\mathbf{x})] = \sum_{|\alpha|=0}^{m}(-1)^{|\alpha|}a_\alpha\partial_{\mathbf{x}}^\alpha\eta(\mathbf{x}) = \sum_{|\alpha|=0}^{m}a_\alpha\partial_{\boldsymbol{\xi}}^\alpha u(\boldsymbol{\xi}-\mathbf{x}) = P(\partial_{\boldsymbol{\xi}})u(\boldsymbol{\xi}-\mathbf{x}).$$

So

$$\begin{aligned} u(\boldsymbol{\xi}) = \eta(0) &= \langle E(\mathbf{x}), P'(\partial_{\mathbf{x}})\eta(\mathbf{x})\rangle \\ &= \langle E(\mathbf{x}), P(\partial_{\boldsymbol{\xi}})u(\boldsymbol{\xi} - \mathbf{x})\ \rangle \\ &= \langle E(\boldsymbol{\xi} - \mathbf{y}), P(\partial_{\mathbf{y}})u(\mathbf{y})\ \rangle\,. \end{aligned}$$

In other words, if $E(\mathbf{x} - \boldsymbol{\xi})$ is a fundamental solution for $P(\partial_{\mathbf{x}})$, then $E'(\mathbf{x};\boldsymbol{\xi}) := E(\boldsymbol{\xi} - \mathbf{x})$ is a fundamental solution for $P'(\partial_{\mathbf{x}})$. Put in a different way, $E(\mathbf{x} - \boldsymbol{\xi})$, regarded as a distribution in $\boldsymbol{\xi}$, is a fundamental solution of the transpose $P'(\partial_{\boldsymbol{\xi}})$ of $P(\partial_{\boldsymbol{\xi}})$ with pole at $\mathbf{x}$.

Theorem 6.5. *For a given linear differential operator with constant coefficients $P(\partial_{\mathbf{x}})$, if E is a fundamental solution with pole 0, then for any distributions u and f with compact support, we have*

$$P(\partial_{\mathbf{x}})(E * f) = f, \tag{6.13}$$

$$E * (P(\partial_{\mathbf{x}})u) = u. \tag{6.14}$$

Proof. (6.13) is established as follows when $f \in C_c^\infty(\mathbb{R}^n)$. Using $E * f = \langle E(\mathbf{z}), f(\mathbf{x} - \mathbf{z})\rangle$,

$$\begin{aligned} P(\partial_{\mathbf{x}})(E * f)(\mathbf{x}) &= P(\partial_{\mathbf{x}})(\langle E(\mathbf{z}), f(\mathbf{x} - \mathbf{z})\rangle \\ &= \langle E(\mathbf{z}), P(\partial_{\mathbf{x}})f(\mathbf{x} - \mathbf{z})\rangle \\ &= \langle E(\mathbf{z}), P'(\partial_{\mathbf{z}})\,[f(\mathbf{x} - \mathbf{z})]\rangle \\ &= f(\mathbf{x} - \mathbf{z})\big|_{\mathbf{z}=0} \quad \text{using (6.8) at } \xi = 0 \text{ with } \eta(\xi) = f(\mathbf{x} - \xi) \\ &= f(\mathbf{x}). \end{aligned}$$

For (6.14), for $u \in C_c^\infty(\mathbb{R}^n)$, we have

$$\begin{aligned} E * (P(\partial_{\mathbf{x}})u) =& \langle E(\mathbf{z}), P(\partial_{\mathbf{x}})u(\mathbf{x} - \mathbf{z})\rangle = \langle E(\mathbf{z}), P'(\partial_{\mathbf{z}})[u(\mathbf{x} - \mathbf{z})]\rangle \\ =& \langle P(\partial_{\mathbf{z}})E(\mathbf{z}), u(\mathbf{x} - \mathbf{z})\rangle = u(\mathbf{x}). \end{aligned}$$

□

(6.13) gives us a way of constructing a solution of (6.13) for a given f (with sufficient regularity and compact support), and (6.14) gives us a way of reconstructing u (with compact support) in terms of $P(\partial_{\mathbf{x}})u$, and can be used to establish regularity and estimates on solutions as below and in the following sections.

Suppose Ω is a subdomain of U and $f = P(\partial_{\mathbf{x}})u$ is known to be smooth in Ω. For any subdomain V of Ω, choose a smooth cut off function η such that $\eta \equiv 1$ on V and is compactly supported in Ω. Then $u\eta = E * [P(\partial_{\mathbf{x}})(u\eta)]$. But $P(\partial_{\mathbf{x}})(u\eta) = (P(\partial_{\mathbf{x}})u))\eta + g$, where g represents the sum of terms involving derivatives of η, which vanish in V. Thus for $\mathbf{x} \in V$, $u = u\eta = E*[(P(\partial_{\mathbf{x}})u)\,\eta]+E*g$, where $(P(\partial_{\mathbf{x}})u)\,\eta$ is a smooth distribution with compact support in Ω, and g is zero in V. So $E*[(P(\partial_{\mathbf{x}})u)\,\eta]$ is a smooth function using $(P(\partial_{\mathbf{x}})u)\,\eta$ being smooth with compact support, while *if E is known to be smooth except at* 0, then for $\mathbf{x} \in V$, $E*g(\mathbf{x}) = \int_{\Omega\setminus V} E(\mathbf{x}-y)g(y)\,dy$ is also smooth in $\mathbf{x} \in V$, proving that u is smooth in V. This argument gives

Theorem 6.6. *If a linear differential operator with constant coefficients $P(\partial_{\mathbf{x}})$ has a fundamental solution E with pole 0 such that it is smooth except at 0, then for any distributional solution u to $P(\partial_{\mathbf{x}})u = f$, if f is smooth in Ω, so is u.*

Exercises

Exercise 6.5.1. Establish a Green's representation for solutions to the Helmholtz equation $\Delta u(x) + cu(x) = 0$ similar to (5.23), and use it to prove that any C^2 solution of this equation is C^∞. Also formulate and establish gradient estimates for its solutions; analyze the dependence of the estimates on the radius R of the ball on which the gradient estimates are to be proved. (HINT: although one can use the newly established Green's representation for solutions to the Helmholtz equation to prove their gradient estimates, it's easier to prove them using (5.23) directly, treating $u(x)$ as a solution of $-\Delta u(x) = f(x)$, with $f(x) = cu(x)$ assumed having control on its C^0 norm.)

6.6 Additional Applications of the Fundamental Solution of the Heat Equation

We now carry out another construction of fundamental solutions to $\partial_t - \Delta_{\mathbf{x}}$. The approach here is different from earlier ones using Fourier's method. Based on an argument similar to that for the Duhamel principle, the condition $(\partial_t - \Delta_{\mathbf{x}})E(\mathbf{x}, t) =$

$\delta(\mathbf{x})\delta(t)$ can be satisfied if

$$\begin{cases} (\partial_t - \Delta_{\mathbf{x}})E(\mathbf{x},t) = 0 & \text{if } \mathbf{x} \in \mathbb{R}^n,\ t > 0, \\ E(\mathbf{x},0) = \delta(\mathbf{x}) & \\ E(\mathbf{x},t) = 0 & \text{if } \mathbf{x} \in \mathbb{R}^n,\ t < 0. \end{cases}$$

Note that if $E_\lambda(\mathbf{x},t) = \lambda^n E(\lambda\mathbf{x}, \lambda^2 t)$ for $\lambda > 0$, then $E_\lambda(\mathbf{x},t)$ would satisfy the same conditions above—including the initial condition $E_\lambda(\mathbf{x},0) = \delta(\mathbf{x})$; other scalings such as $\lambda^m E(\lambda\mathbf{x}, \lambda^2 t)$ for $m \neq n$ also satisfies the equation, but fails to satisfy the initial condition, so it is reasonable to look for $E(\mathbf{x},t)$ such that $E_\lambda(\mathbf{x},t) = E(\mathbf{x},t)$ for all $\lambda > 0$. This leads to $E(\mathbf{x},t) = t^{-n/2}E(\mathbf{x}/\sqrt{t}, 1)$ when we choose $\lambda = 1/\sqrt{t}$. Due to the rotational symmetry of $\Delta_{\mathbf{x}}$, we may expect $E(\mathbf{x}/\sqrt{t}, 1)$ to depend on $\|\mathbf{x}\|/\sqrt{t}$. Thus we look for some $f(\rho)$ for $\rho \geq 0$ such that

$$E(\mathbf{x},t) = \begin{cases} \frac{f(\|\mathbf{x}\|/\sqrt{t})}{t^{n/2}} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

Then the equation $(\partial_t - \Delta_{\mathbf{x}})E(\mathbf{x},t) = 0$ for $t > 0$ turns into the following ODE for $f(\rho)$:

$$f''(\rho) + \left(\frac{\rho}{2} + \frac{n-1}{\rho}\right) f'(\rho) + \frac{n}{2} f(\rho) = 0, \tag{6.15}$$

where we have used

$$E_t(\mathbf{x},t) = -\frac{nf(\rho) + \rho f'(\rho)}{2t^{n/2+1}},$$

$$\Delta_{\mathbf{x}} E(\mathbf{x},t) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) E(r,t) = \frac{f''(\rho) + \frac{n-1}{\rho} f'(\rho)}{t^{n/2+1}}.$$

(6.15) can be rewritten as

$$\left(f'(\rho) + \frac{\rho}{2} f(\rho)\right)' + \frac{n-1}{\rho}\left(f'(\rho) + \frac{\rho}{2} f(\rho)\right) = 0,$$

from which it follows that

$$\left[\rho^{n-1}\left(f'(\rho) + \frac{\rho}{2} f(\rho)\right)\right]' = 0.$$

Thus $\rho^{n-1}\left(f'(\rho) + \frac{\rho}{2} f(\rho)\right) = C$ for some constant C and all $\rho \geq 0$. Examining the behavior near $\rho = 0$ implies that $C = 0$, which leads to $f(\rho) = C' e^{-\rho^2/4}$. C' is determined by the requirement that $\int_{\mathbb{R}^n} E(\mathbf{x},t)\,d\mathbf{x} \to 1$ for $t \to 0$.

$$\begin{aligned}\int_{\mathbb{R}^n} E(\mathbf{x},t)\,d\mathbf{x} &= C' \int_{\mathbb{R}^n} \frac{e^{-\frac{\|\mathbf{x}\|^2}{4t}}}{t^{n/2}}\,d\mathbf{x} \\ &= C' \left(\int_{\mathbb{R}} \frac{e^{-\frac{x_1^2}{4t}}}{t^{1/2}}\,dx_1 \right)^n \\ &= C' \left(2 \int_{\mathbb{R}} e^{-z_1^2}\,dz_1 \right)^n \\ &= C' \left(2\sqrt{\pi}\right)^n\end{aligned}$$

so $C' = (4\pi)^{-n/2}$, and $E(\mathbf{x},t) = (4\pi t)^{-n/2} e^{-\frac{\|\mathbf{x}\|^2}{4t}}$ for $t > 0$, which agrees with the heat kernel $K(\mathbf{x},t)$ we found earlier.

Poisson's representation such as (6.3) can also be used to establish derivative estimates on solutions.

Theorem 6.7. *Suppose that $(\partial_t - \Delta_{\mathbf{x}})u(\mathbf{x},t) = 0$ for $(\mathbf{x},t)$ in a region $U_{2R}(X_0,T_0) = B_{2R}(X_0) \times (T_0 - 4R^2, T_0]$, and $||u||_{L^1(U_{2R}(X_0,T_0))} < \infty$. Then there exists $C = C_{|\alpha|+2j} > 0$ such that*

$$\sup_{U_{R/2}(X_0,T_0)} R^{|\alpha|+2j} |\nabla_{\mathbf{x}}^{\alpha} \nabla_t^j u(X,T)| \le C_{|\alpha|+2j} R^{-n-2} ||u||_{L^1(U_{2R}(X,T))}. \tag{6.16}$$

Proof. We will scale the problem to $U_2(0,0)$ as follows. Fix a smooth cut off function $\eta(\mathbf{x},t)$ such that $\eta \equiv 1$ in $U_1(0,0)$, and is compactly supported in $U_2(0,0)$. It is easiest to use the scale invariance to work with $u_R(\mathbf{x},t) = u(X_0 + R\mathbf{x}, T_0 + R^2 t)$ defined on $U_2(0,0)$. Then

$$(\partial_t - \Delta_{\mathbf{x}})u_R(\mathbf{x},t) = R^2(\partial_t - \Delta_{\mathbf{x}})u(X_0 + R\mathbf{x}, T_0 + R^2 t) = 0$$

in $U_2(0,0)$, if $(\partial_t - \Delta_{\mathbf{x}})u(\mathbf{x},t) = 0$ in $U_{2R}(X_0,T_0)$.

In (6.3), take $U = U_2(0,0)$, $(\xi,\tau) \in U_{1/2}(0,0)$, u to be $u_R(\mathbf{x},t)\eta(\mathbf{x},t)$, $v(\mathbf{x},t) =$

$K(\xi-\mathbf{x},\tau-t)$, $t_1=\tau$, $t_0=-4$. Then

$$\begin{aligned}
u_R(\xi,\tau)=&\int_{-4}^{\tau}\int_{B_2(0)}(\partial_t-\Delta_{\mathbf{x}})\left[u_R(\mathbf{x},t)\eta(\mathbf{x},t)\right]\cdot K(\xi-\mathbf{x},\tau-t)\,d\mathbf{x}dt\\
=&\int_{-4}^{\tau}\int_{B_2(0)}\left[(\partial_t-\Delta_{\mathbf{x}})\eta(\mathbf{x},t)\cdot u_R(\mathbf{x},t)\right]\cdot K(\xi-\mathbf{x},\tau-t)\,d\mathbf{x}dt\\
&-2\int_{-4}^{\tau}\int_{B_2(0)}\left[\nabla_{\mathbf{x}}u_R(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}\left(\eta(\mathbf{x},t)\right)\right]\cdot K(\xi-\mathbf{x},\tau-t)\,d\mathbf{x}dt\\
=&\int_{-4}^{\tau}\int_{B_2(0)}\left[(\partial_t-\Delta_{\mathbf{x}})\eta(\mathbf{x},t)\right]K(\xi-\mathbf{x},\tau-t)u_R(\mathbf{x},t)\,d\mathbf{x}dt\\
&+\int_{-4}^{\tau}\int_{B_2(0)}\left[2\nabla_{\mathbf{x}}\left\{K(\xi-\mathbf{x},\tau-t)\nabla_{\mathbf{x}}\eta(\mathbf{x},t)\right\}\right]u_R(\mathbf{x},t)\,d\mathbf{x}dt\\
:=&\int_{-4}^{\tau}\int_{B_2(0)}\hat{K}(\xi,\tau;\mathbf{x},t)u_R(\mathbf{x},t)\,d\mathbf{x}dt,
\end{aligned}\tag{6.17}$$

where we have applied the divergence theorem and the zero boundary condition of $\nabla_{\mathbf{x}}\left(\eta(\mathbf{x},t)\right)$ on $\partial B_2(0)\times[-4,\tau]$ to deduce

$$\begin{aligned}
&-2\int_{-4}^{\tau}\int_{B_2(0)}\left[\nabla_{\mathbf{x}}u_R(\mathbf{x},t)\cdot\nabla_{\mathbf{x}}\left(\eta(\mathbf{x},t)\right)\right]\cdot K(\xi-\mathbf{x},\tau-t)\,d\mathbf{x}dt\\
=&\int_{-4}^{\tau}\int_{B_2(0)}\left[2\nabla_{\mathbf{x}}\left\{K(\xi-\mathbf{x},\tau-t)\nabla_{\mathbf{x}}\eta(\mathbf{x},t)\right\}\right]u_R(\mathbf{x},t)\,d\mathbf{x}dt.
\end{aligned}$$

Note that

$$\hat{K}(\xi,\tau;\mathbf{x},t)=(\partial_t-\Delta_{\mathbf{x}})\eta(\mathbf{x},t)\cdot K(\xi-\mathbf{x},\tau-t)+2\nabla_{\mathbf{x}}\left\{K(\xi-\mathbf{x},\tau-t)\nabla_{\mathbf{x}}\eta(\mathbf{x},t)\right\}$$

is a smooth function of $(\xi,\tau;\mathbf{x},t)\in U_{1/2}(0,0)\times\mathbb{R}^{n+1}$ supported for $(\mathbf{x},t)$ in $U_2(0,0)\setminus U_1(0,0)$, due to the support of η and $\eta\equiv 1$ in $U_1(0,0)$, with $\hat{K}(\xi,\tau;\mathbf{x},\tau)=0$ for $(\xi,\tau)\in U_{1/2}(0,0)$ and $\mathbf{x}\in B_2(0)\setminus B_1(0)$. Thus, using the observation that

$$\partial_{\mathbf{x}}^{\alpha}\partial_t^m K(\mathbf{x},t)=\frac{1}{t^{\frac{|\alpha|}{2}+m}}p_{|\alpha|+2m}\left(\frac{\mathbf{x}}{\sqrt{t}}\right)K(\mathbf{x},t),$$

where $p_{|\alpha|+2m}$ is a polynomial of degree $|\alpha|+2m$, we have

$$\begin{aligned}
&|\partial_{\xi}^{\alpha}\partial_{\tau}^{j}u_R(\xi,\tau)|\\
\leq&\int_{-4}^{\tau}\int_{B_2(0)}|\partial_{\xi}^{\alpha}\partial_{\tau}^{j}\hat{K}(\xi,\tau;\mathbf{x},t)||u_R(\mathbf{x},t)|\,d\mathbf{x}dt\\
\leq&C\int_{-4}^{\tau}\int_{B_2(0)}M_{\eta}(\mathbf{x},t)\frac{\left(1+\frac{|\xi-\mathbf{x}|}{\sqrt{|\tau-t|}}\right)^{|\alpha|+2j}K(\xi-\mathbf{x},\tau-t)}{|\tau-t|^{|\alpha|/2+j}}|u_R(\mathbf{x},t)|\,d\mathbf{x}dt,
\end{aligned}$$

where $M_\eta(\mathbf{x},t) = |\partial_t \eta(\mathbf{x},t)| + |\Delta_{\mathbf{x}}\eta(\mathbf{x},t)| + |\nabla_{\mathbf{x}}\eta(\mathbf{x},t)|$. The integration is effectively carried out on $(\mathbf{x},t) \in U_2(0,0) \setminus U_1(0,0)$, and when $(\xi,\tau) \in U_{1/2}(0,0)$ and $(\mathbf{x},t) \in U_2(0,0) \setminus U_1(0,0)$, there is $C_{|\alpha|+2j} > 0$ such that

$$|\tau - t|^{-|\alpha|/2-j}|(1 + \frac{|\xi - \mathbf{x}|}{\sqrt{|\tau - t|}})^{|\alpha|+2j} K(\xi - \mathbf{x}, \tau - t) \leq C_{|\alpha|+2j}.$$

Thus

$$\max_{U_{1/2}(0,0)} |\nabla_\xi^\alpha \nabla_\tau^j u_R| \leq C_{|\alpha|+2j}||u_R||_{L^1(U_2(0,0))} \leq C_{|\alpha|+2j} R^{-n-2}||u||_{L^1(U_{2R}(X,T))}.$$

When $(X,T) = (X_0 + R\xi, T_0 + R^2\tau) \in U_{R/2}(X_0,T_0)$, $(\xi,\tau) \in U_{1/2}(0,0)$, so

$$R^{|\alpha|+2j}|\nabla_{\mathbf{x}}^\alpha \nabla_t^j u(X,T)| = |\nabla_\xi^\alpha \nabla_\tau^j u_R(\xi,\tau)| \leq C_{|\alpha|+2j} R^{-n-2}||u||_{L^1(U_{2R}(X,T))}.$$

□

With the derivative estimates, we have the following version of Liouville theorem and convergence properties.

Theorem 6.8. *Suppose that $u(\mathbf{x},t)$ is an ancient solution of $(\partial_t - \Delta_{\mathbf{x}})u(\mathbf{x},t) = 0$ on $\mathbb{R}^n$, namely, it is defined on $\mathbb{R}^n \times (-\infty, T]$ for some finite T. Suppose that $u(\mathbf{x},t)$ is bounded, then it must be a constant.*

Proof. At any (X,τ), with $\tau \leq T$, we can apply the above gradient estimates on $U_R(X,\tau)$ for any $R > 0$ to obtain

$$R|\nabla_{\mathbf{x}} u(X,\tau)| + R^2|\nabla_t u(X,\tau)| \leq CR^{-n-2}||u||_{L^1(U_{2R}(X,\tau))} \leq C' \sup_{\mathbb{R}^n \times (-\infty,T]} |u(\mathbf{x},t)|.$$

Since this holds for any $R > 0$, we conclude that $|\nabla_{\mathbf{x}} u(X,\tau)| = |\nabla_t u(X,\tau)| = 0$. Since these two equations hold for all $(X,\tau) \in \mathbb{R}^n \times (-\infty,T]$, we conclude that u must be a constant on $\mathbb{R}^n \times (-\infty,T]$. □

Theorem 6.9. *(i). Uniform limit of a sequence of solutions to*

$$(\partial_t - \Delta_{\mathbf{x}})u(\mathbf{x},t) = 0 \tag{6.18}$$

in a region U_T is a solution of (6.18).

(ii). A bounded sequence of solutions in U_T must have a subsequence that converges on any compact subset of U_T to a solution of (6.18) in U_T.

(iii) If U is a bounded domain, and for a sequence $g_k \in C(\partial' U_T)$, there exists a (unique) solution u_k to

$$\begin{cases} (\partial_t - \Delta) u_k = 0, & \text{in } U_T, \\ u_k = g_k, & \text{on } \partial' U_T, \end{cases} \tag{6.19}$$

and $g_k \to g$ uniformly on $\partial' U_T$, then (6.19) with g as boundary value has a unique solution.

The proof for (i) and (ii) is similar to that for the Laplace equation, relying on the derivative estimates and Arzela-Ascoli theorem; details will be left as exercises. We sketch here a proof for (iii). Since $g_k \to g$ uniformly on $\partial' U_T$, $\{g_k\}$ is Cauchy in $C(\partial' U_T)$. By the maximum principle, $\max_{\overline{U_T}} |u_j - u_k| \le \max_{\partial' U_T} |g_j - g_k|$, which $\to 0$ as $j, k \to \infty$. Thus there exists $u \in C(\overline{U_T})$ such that $u_k \to u$ uniformly over $\overline{U_T}$, and $u = g$ on $\partial' U_T$. It remains to prove that $(\partial_t - \Delta) u = 0$ in U_T. It suffices to do this for any $B_R(\mathbf{x}_0) \times [t_0 - R^2, t_0] \subset U_T$, for which $B_{4R}(\mathbf{x}_0) \times [t_0 - 16R^2, t_0] \subset U_T$. But the gradient estimates can be applied to $\{u_k\}$ over $B_{4R}(\mathbf{x}_0) \times [t_0 - 16R^2, t_0]$, which would then imply that there is a subsequence $\{u_{k_l}\}$ such that $\{\partial_t u_{k_l}\}$ and $\{\partial^2_{\mathbf{x}} u_{k_l}\}$ converge uniformly over $B_R(\mathbf{x}_0) \times [t_0 - R^2, t_0]$. Since each u_{k_l} satisfies $(\partial_t - \Delta) u_{k_l} = 0$ in U_T, this then implies $(\partial_t - \Delta) u = 0$ in $B_R(\mathbf{x}_0) \times [t_0 - R^2, t_0]$.

Exercise 6.6.1. Verify that for the heat kernel defined for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$,

$$\partial_x^l \partial_t^m K(x, t) = \frac{1}{t^{\frac{l}{2} + m}} p_{l+2m}\left(\frac{x}{\sqrt{t}}\right) K(x, t),$$

where p_{l+2m} is a polynomial of degree $l + 2m$.

Exercise 6.6.2. Verify that

$$K_A(\mathbf{x}, t) = \begin{cases} \frac{e^{-\frac{\mathbf{x}' A^{-1} \mathbf{x}}{4t}}}{\sqrt{\det A} (4\pi t)^{n/2}} & t > 0 \\ 0 & t \le 0 \end{cases}$$

is a fundamental solution of $(\partial_t - L) u = u_t - \sum_{i,j=1}^n a_{ij} \partial^2_{x_i x_j} u(\mathbf{x}, t) = \delta(0, 0)$. Here $A = (a_{ij})$ is assumed to be positive definite. Hint: Make a linear change of variables to transform the equation to the standard heat equation, as was done in Exercise 6.4.9.

Exercise 6.6.3. Verify that $e^{ct} K(\mathbf{x} + bt, t)$ is a fundamental solution of $\partial_t u(\mathbf{x}, t) - \Delta_{\mathbf{x}} u(\mathbf{x}, t) - \sum_{j=1}^n b_j \partial_{x_j} u(\mathbf{x}, t) - c u(\mathbf{x}, t) = \delta(0, 0)$, where $b = (b_1, \cdots, b_n)$.

Exercise 6.6.4. Verify that for $n \ge 3$, $\int_0^\infty K(\mathbf{x}, t) dt = \Phi(\mathbf{x})$, where $\Phi(\mathbf{x})$ is the fundamental solution to $-\Delta_{\mathbf{x}} \Phi(\mathbf{x}) = \delta(\mathbf{x})$ introduced earlier.

6.7 Fourier transform on tempered distributions*

Fourier's method has been very essential in our study of PDE problems. Unfortunately, the Fourier transform, as a linear functional on $\mathscr{D}(\mathbb{R}^n)$, does not map $\mathscr{D}(\mathbb{R}^n)$ into itself. For any $\phi \in \mathscr{D}(\mathbb{R}^n)$, its Fourier transform $\mathscr{F}[\phi]$ is in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$, the function space consisting those functions in $\mathscr{E}(\mathbb{R}^n)$ which, together with any of their derivatives, decay faster than any power of $|\mathbf{x}|^{-1}$ as $\mathbf{x} \to \infty$. For any $\phi \in \mathscr{S}(\mathbb{R}^n)$, define the semi-norm

$$\|\phi\|_{\alpha,\beta} = \sup_{\mathbf{x}\in\mathbb{R}^n} |\mathbf{x}^\alpha \partial_{\mathbf{x}}^\beta \phi(\mathbf{x})|$$

for any multi-indices α, β. Then $\mathscr{S}(\mathbb{R}^n)$ is the subspace of $\mathscr{E}(\mathbb{R}^n)$ consisting of functions for which $\|\phi\|_{\alpha,\beta} < \infty$ for any multi-indices α, β. Furthermore, we can define a metric $\rho_{\mathscr{S}(\mathbb{R}^n)}$ on $\mathscr{S}(\mathbb{R}^n)$ by

$$\rho_{\mathscr{S}(\mathbb{R}^n)}(\phi, \psi) = \sum_{k=0}^{\infty} \sum_{|\alpha|+|\beta|=k} \frac{1}{2^k} \frac{\|\phi - \psi\|_{\alpha,\beta}}{1 + \|\phi - \psi\|_{\alpha,\beta}}.$$

Then $\mathscr{S}(\mathbb{R}^n)$ is a complete metric space, and a converging sequence in $\mathscr{S}(\mathbb{R}^n)$ also converges in $\mathscr{E}(\mathbb{R}^n)$. In addition, $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}(\mathbb{R}^n)$, and a converging sequence in $\mathscr{D}(\mathbb{R}^n)$ also converges in $\mathscr{S}(\mathbb{R}^n)$, although a sequence in $\mathscr{D}(\mathbb{R}^n)$ converging in the metric of $\mathscr{S}(\mathbb{R}^n)$ may not converge in $\mathscr{D}(\mathbb{R}^n)$.

As a consequence, any continuous linear functional of $\mathscr{S}(\mathbb{R}^n)$ is also a continuous linear functional of $\mathscr{D}(\mathbb{R}^n)$. A continuous linear functional of $\mathscr{S}(\mathbb{R}^n)$ is called a tempered distribution. The space of tempered distributions is denoted as $\mathscr{S}'(\mathbb{R}^n)$.

In the following sense a tempered distribution has a finite order of differentiation and a finite order of growth at infinity: If $l \in \mathscr{S}'(\mathbb{R}^n)$, then there exists some $k \in \mathbb{Z}^{\geq 0}$ and a constant $C > 0$ such that

$$|\langle l, \phi \rangle| \leq C \sum_{|\alpha|+|\beta|\leq k} \|\phi\|_{\alpha,\beta} \tag{6.20}$$

for all $\phi \in \mathscr{S}(\mathbb{R}^n)$.

This is seen by noting that there is an open neighborhood $\mathscr{O}$ of $\phi = 0$ in $\mathscr{S}(\mathbb{R}^n)$ such that $|\langle l, \phi \rangle| < 1$ for all $\phi \in \mathscr{O}$. But $\mathscr{O}$ must contain $\{\phi \in \mathscr{S}(\mathbb{R}^n) : \rho_{\mathscr{S}(\mathbb{R}^n)}(\phi, 0) < \epsilon\}$ for some $\epsilon > 0$. Note that there exists some $k \in \mathbb{Z}^{\geq 0}$ such that

$$\sum_{|\alpha|+|\beta|>k} \frac{1}{2^k} \frac{\|\phi\|_{\alpha,\beta}}{1 + \|\phi\|_{\alpha,\beta}} \leq \sum_{|\alpha|+|\beta|>k} \frac{1}{2^k} < \epsilon/2,$$

*The material of this section is not used in a substantial way in the remaining notes.

and whenever $\sum_{|\alpha|+|\beta|\le k} \|\phi\|_{\alpha,\beta} < \epsilon/2$, we have

$$\rho_{\mathscr{S}(\mathbb{R}^n)}(\phi, 0) \le \sum_{|\alpha|+|\beta|\le k} \|\phi\|_{\alpha,\beta} + \epsilon/2 < \epsilon,$$

therefore for such ϕ, we have $|\langle l, \phi\rangle| < 1$. Now for any $\phi \in \mathscr{S}(\mathbb{R}^n)$, if $A := \sum_{|\alpha|+|\beta|\le k} \|\phi\|_{\alpha,\beta} > 0$, then for any $0 < \epsilon' < \epsilon/2$, $\sum_{|\alpha|+|\beta|\le k} \|\epsilon'\phi/A\|_{\alpha,\beta} < \epsilon/2$, therefore $|\langle l, \epsilon'\phi/A\rangle| < 1$, which implies that

$$|\langle l, \phi\rangle| < (\epsilon')^{-1} \sum_{|\alpha|+|\beta|\le k} \|\phi\|_{\alpha,\beta}.$$

This leads to

$$|\langle l, \phi\rangle| \le \epsilon^{-1} \sum_{|\alpha|+|\beta|\le k} \|\phi\|_{\alpha,\beta}.$$

If $\sum_{|\alpha|+|\beta|\le k} \|\phi\|_{\alpha,\beta} = 0$, then $\phi = 0$, and the above inequality holds trivially. This proves (6.20).

Any $L^1_{\text{local}}(\mathbb{R}^n)$ function $p(\mathbf{x})$ with at most polynomial growth at ∞ defines a tempered distribution by $\phi \mapsto \int_{\mathbb{R}^n} p(\mathbf{x})\phi(\mathbf{x})\, d\mathbf{x}$ for $\phi \in \mathscr{S}(\mathbb{R}^n)$. If $l \in \mathscr{S}'(\mathbb{R}^n)$, then $\partial_{x_k} l$ as defined via (6.7) is an element of $\mathscr{S}'(\mathbb{R}^n)$. So is $p(\mathbf{x})l$ for any polynomial $p(\mathbf{x})$ defined by

$$\langle p(\mathbf{x})l, \eta\rangle = \langle l, p(\mathbf{x})\eta\rangle \quad \text{for } \eta \in \mathscr{S}(\mathbb{R}^n).$$

Note, however, e^{kx} for $k \ne 0 \in \mathbb{R}$ is in $\mathscr{D}'(\mathbb{R})$, but not in $\mathscr{S}'(\mathbb{R})$.

Theorem 6.10. *The Fourier transform is a continuous linear function from $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$ and has a continuous inverse. Furthermore*

$$\int_{\mathbb{R}^n} F[\phi](\xi)\psi(\xi)\, d\xi = \int_{\mathbb{R}^n} \phi(\xi)F[\psi](\xi)\, d\xi \quad \textit{for any } \phi, \psi \in \mathscr{S}(\mathbb{R}^n).$$

Using this theorem, we can define the Fourier transform of any tempered distribution $l \in \mathscr{S}'(\mathbb{R}^n)$ via

$$\langle F[l], \phi\rangle = \langle l, F[\phi]\rangle \quad \text{for any } \phi \in \mathscr{S}(\mathbb{R}^n). \tag{6.21}$$

In the above F can be any of the alternative formulation of the Fourier transform. Sometimes we may use the convention $\widehat{\phi}$ of Fourier transform as defined in (2.30), or the convention $\tilde{\phi}$, and denote their respective extension to $l \in \mathscr{S}'(\mathbb{R}^n)$ accordingly. The different conventions may differ by a factor of a power of 2π when dealing with the convolution or computing $F \circ F$ and $F \circ \partial$, or the Plancherel identity. In the following we will take $F[\phi] = \tilde{\phi}$ to avoid carrying a factor of a power of 2π for the first three situations.

The following two fundamental properties of Fourier transform on $\mathscr{S}(\mathbb{R}^n)$ extend to $\mathscr{S}'(\mathbb{R}^n)$ as well:

$$F[\partial_{x_k} l] = i\xi_k F[l], \quad F[x_k l] = i\partial_{\xi_k} F[l], \tag{6.22}$$

except that the distinction between x_k and ξ_k becomes blurred in this context, as one does not necessarily treat the distribution $F[l]$ as a continuous linear functional on functions of $\boldsymbol{\xi}$ in $\mathscr{D}(\mathbb{R}^n)$.

For $l \in \mathscr{S}'(\mathbb{R}^n)$ this can be seen by taking $\eta \in \mathscr{S}(\mathbb{R}^n)$ and evaluating

$$\begin{aligned}
\langle \partial_{x_k} l, F[\eta] \rangle &= -\langle l, \partial_{x_k} F[\eta] \rangle \\
&= \langle l, iF[\xi_k \eta] \rangle \\
&= \langle F[l], i\xi_k \eta \rangle \\
&= \langle i\xi_k F[l], \eta \rangle.
\end{aligned}$$

Also note that

$$F[e^{i\mathbf{x}\cdot\mathbf{a}} l](\boldsymbol{\xi}) = F[l](\boldsymbol{\xi} - \mathbf{a}), \quad F[l(\cdot - \mathbf{a})] = e^{-i\boldsymbol{\xi}\cdot\mathbf{a}} F[l](\boldsymbol{\xi}) \text{ for } \mathbf{a} \in \mathbb{R}^n. \tag{6.23}$$

The relation between the Fourier transform and the linear map $T_A\phi(\mathbf{x}) = \sqrt{|\det A|}\phi(A\mathbf{x})$ is given by

$$F \circ T_A = T_{(A^{-1})^T} \circ F.$$

This property extends to $l \in \mathscr{S}'(\mathbb{R}^n)$. Taking the case of $A = \lambda I$ for $\lambda > 0$, we get

$$F[\phi(\lambda \cdot)](\xi) = \lambda^{-n} F[\phi](\frac{\xi}{\lambda}). \tag{6.24}$$

Another most useful property of Fourier transform on $\mathscr{S}(\mathbb{R}^n)$:

$$F[\phi * \psi] = F[\phi]F[\psi] \quad \phi, \psi \in \mathscr{S}(\mathbb{R}^n), \tag{6.25}$$

also has an extension when $\phi \in \mathscr{S}'(\mathbb{R}^n), \psi \in \mathscr{E}'(\mathbb{R}^n)$ or when $\phi \in \mathscr{E}'(\mathbb{R}^n), \psi \in \mathscr{S}'(\mathbb{R}^n)$, for, when $\psi \in \mathscr{E}'(\mathbb{R}^n)$, $F[\psi] \in \mathscr{E}(\mathbb{R}^n)$ with the property that $F[\psi]$ and any of its derivatives are bounded on $\mathbb{R}^n$, so for any $\eta \in \mathscr{S}(\mathbb{R}^n)$, $F[\psi]\eta \in \mathscr{S}(\mathbb{R}^n)$, and

$$\langle F[\phi]F[\psi], \eta \rangle := \langle F[\phi], F[\psi]\eta \rangle$$

is a well defined element in $\mathscr{S}'(\mathbb{R}^n)$.

Example 6.7. (i) By direct evaluation of the integral, we get

$$F[e^{-ax^2}] = \int_{\mathbb{R}} e^{-ax^2 - i\xi x}\, dx = \sqrt{\frac{\pi}{a}} e^{-\frac{\xi^2}{4a}}$$

for $a > 0$.

(ii) $F[e^{-|x|}] = \frac{2}{1+|\xi|^2}$ and $F[\frac{1}{\pi(1+|\xi|^2)}] = e^{-|x|}$ on $\mathscr{S}'(\mathbb{R})$. The first follows by an easy evaluation of the integral in $F[e^{-|x|}]$, while the second one follows by using $F \circ F[\phi(x)] = 2\pi\phi(-x)$.

(iii) $F[\delta(\mathbf{x} - \mathbf{a})] = e^{-i\xi\cdot\mathbf{a}}$ and $F[e^{-i\mathbf{x}\cdot\mathbf{a}}] = (2\pi)^n \delta(\mathbf{x} + \mathbf{a})$ on $\mathscr{S}'(\mathbb{R}^n)$. The first one follows from

$$\begin{aligned}
\langle F[\delta(\mathbf{x} - \mathbf{a})], \phi\rangle &= \langle \delta(\mathbf{x} - \mathbf{a}), F[\phi]\rangle \\
&= \langle \delta(\mathbf{x} - \mathbf{a}), \int_{\mathbb{R}^n} \phi(\boldsymbol{\xi}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}\rangle \\
&= \int_{\mathbb{R}^n} \phi(\boldsymbol{\xi}) e^{-i\mathbf{a}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi} = \langle e^{-i\boldsymbol{\xi}\cdot\mathbf{a}}, \phi\rangle.
\end{aligned}$$

The second one follows from using $F \circ F[\phi(\mathbf{x})] = (2\pi)^n \phi(-\mathbf{x})$ or

$$\begin{aligned}
\langle F[e^{-i\mathbf{x}\cdot\mathbf{a}}], \phi\rangle &= \langle e^{-i\mathbf{x}\cdot\mathbf{a}}, F[\phi]\rangle \\
&= \langle e^{-i\mathbf{x}\cdot\mathbf{a}}, \int_{\mathbb{R}^n} \phi(\boldsymbol{\xi}) e^{-i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\boldsymbol{\xi}\rangle \\
&= (2\pi)^n \phi(-\mathbf{a}) \quad \text{(by the Fourier inversion formula)} \\
&= \langle (2\pi)^n \delta(\mathbf{x} + \mathbf{a}), \phi\rangle.
\end{aligned}$$

(iv) $F[(-i\mathbf{x})^\alpha e^{-i\mathbf{x}\cdot\mathbf{a}}] = (2\pi)^n \partial_{\mathbf{x}}^\alpha \delta(\mathbf{x} + \mathbf{a})$ on $\mathscr{S}'(\mathbb{R}^n)$ for any multi-index α. In particular, setting $\mathbf{a} = \mathbf{0}$ gives $F[(-i\mathbf{x})^\alpha] = (2\pi)^n \partial_{\mathbf{x}}^\alpha \delta(\mathbf{x})$. This follows from applying the rule (6.22) to $F[e^{-i\mathbf{x}\cdot\mathbf{a}}] = (2\pi)^n \delta(\mathbf{x} + \mathbf{a})$.

Remark 6.5. Sometimes one needs to evaluate certain integrals involving some (tempered) distributions which do not correspond to an integrable function in the traditional sense. In such a situation, one often uses some kind of *regularization* to approximate the given distribution in the sense of $\mathscr{D}'(\mathbb{R}^n)$ or $\mathscr{S}'(\mathbb{R}^n)$.

For example, to evaluate $F[H]$, where H is the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

one can't evaluate directly $\int_{\mathbb{R}} H(x)e^{-i\xi x}\,dx$. One uses $e^{-\epsilon x}H(x)$ for $\epsilon \searrow 0$ to approximate $H(x)$ in $\mathscr{S}'(\mathbb{R})$. Since

$$F[e^{-\epsilon x}H(x)] = \int_{\mathbb{R}} e^{-\epsilon x}H(x)e^{-i\xi x}\,dx = \frac{1}{\epsilon + i\xi},$$

one can use the limit

$$\lim_{\epsilon\searrow 0} F[e^{-\epsilon x}H(x)] = \lim_{\epsilon\searrow 0} \frac{1}{\epsilon + i\xi}$$

as $F[H]$. Here the limit is in the sense of $\mathscr{S}'(\mathbb{R})$:

$$\lim_{\epsilon\searrow 0} \int_{\mathbb{R}} \frac{1}{\epsilon + i\xi}\phi(\xi)\,d\xi \quad \text{for } \phi \in \mathscr{S}(\mathbb{R});$$

this limit exists by noting that

$$\int_{\mathbb{R}} \frac{1}{\epsilon + i\xi}\phi(\xi)\,d\xi = -i\int_{\mathbb{R}} \phi(\xi)\,d\ln(\epsilon + i\xi) = i\int_{\mathbb{R}} \ln(\epsilon + i\xi)\phi'(\xi)\,d\xi,$$

and that $\ln(\epsilon + i\xi) \to \ln|\xi| + \operatorname{sign}(\xi)\frac{\pi}{2}\,i$ as $\epsilon \searrow 0$, resulting in

$$\lim_{\epsilon\searrow 0} \int_{\mathbb{R}} \frac{1}{\epsilon + i\xi}\phi(\xi)\,d\xi = i\left(\int_{\mathbb{R}} \ln|\xi|\,\phi'(\xi)\,d\xi - i\pi\phi(0)\right).$$

We can further identify

$$\begin{aligned} &\int_{\mathbb{R}} \ln|\xi|\,\phi'(\xi)\,d\xi \\ =&\lim_{\epsilon\searrow 0}\int_{|\xi|>\epsilon} \ln|\xi|\,\phi'(\xi)\,d\xi \\ =&\lim_{\epsilon\searrow 0}\left(\ln\epsilon\,[-\phi(\epsilon)+\phi(-\epsilon)] - \int_{|\xi|>\epsilon}\frac{\phi(\xi)}{\xi}\,d\xi\right) \\ =&-\ \operatorname{PV}\int_{\mathbb{R}}\frac{\phi(\xi)}{\xi}\,d\xi. \end{aligned}$$

In summary,

$$F[H] = \lim_{\epsilon\searrow 0}\frac{1}{\epsilon + i\xi} = -i\ \operatorname{PV}\left(\frac{1}{\xi}\right) + \pi\delta(\xi).$$

Likewise, one can compute directly that $F[\chi_{[-1,1]}] = \frac{2\sin(\xi)}{\xi}$, but the right hand side is not in $L^1(\mathbb{R})$ directly, so one can't directly evaluate $F[\frac{2\sin(\xi)}{\xi}] = \int_{\mathbb{R}} \frac{2\sin(\xi)}{\xi}e^{-ix\xi}\,d\xi$. Here one can use $e^{-\epsilon\xi^2}\frac{2\sin(\xi)}{\xi}$ to regularize $\frac{2\sin(\xi)}{\xi}$ as $\epsilon \searrow 0$. Since

$$F[\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}}] = e^{-\epsilon\xi^2},$$

we see that

$$e^{-\epsilon\xi^2}\frac{2\sin(\xi)}{\xi} = F[\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}}]F[\chi_{[-1,1]}] = F[\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} * \chi_{[-1,1]}]$$

and

$$\begin{aligned}&\int_{\mathbb{R}} e^{-\epsilon\xi^2 - ix\xi}\frac{2\sin(\xi)}{\xi}\,d\xi\\ =&F[e^{-\epsilon\xi^2}\frac{2\sin(\xi)}{\xi}]\\ =&F\circ F[\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} * \chi_{[-1,1]}]\\ =&2\pi\left(\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} * \chi_{[-1,1]}\right)\end{aligned}$$

Since, as $\epsilon \searrow 0$,

$$\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} * \chi_{[-1,1]} \to \chi_{[-1,1]}$$

in $L^1(\mathbb{R})$, and therefore in $\mathscr{S}'(\mathbb{R})$, we obtain

$$F[\frac{2\sin(\xi)}{\xi}] = 2\pi\lim_{\epsilon\searrow 0}\frac{e^{-\frac{x^2}{4\epsilon}}}{\sqrt{4\pi\epsilon}} * \chi_{[-1,1]} = 2\pi\chi_{[-1,1]}.$$

In the above we used $F\circ F[\phi](x) = 2\pi\phi(-x)$ only for $\phi \in \mathscr{S}(\mathbb{R})$, not assuming its validity for $\phi \in \mathscr{S}'(\mathbb{R})$.

Fourier transform is a useful tool in constructing a fundamental solution for a constant coefficient operator $P(\partial_{\mathbf{x}})$. Suppose that $P(\partial_{\mathbf{x}})E = \delta(\mathbf{x})$ and $E \in \mathscr{S}'(\mathbb{R}^n)$, then $P(i\boldsymbol{\xi})F[E] = F[P(\partial_{\mathbf{x}})E] = F[\delta(\mathbf{x})] = 1$. If $P(i\boldsymbol{\xi}) \neq 0$ for any $\boldsymbol{\xi} \in \mathbb{R}^n$, then we can conclude that $F[E] = 1/P(i\boldsymbol{\xi})$ and $E = F^{-1}(1/P(i\boldsymbol{\xi}))$. An argument needs to be made that $1/P(i\boldsymbol{\xi}) \in \mathscr{S}'(\mathbb{R}^n)$ and it may not be easy to have more explicit information about E.

If there exists $\boldsymbol{\xi} \in \mathbb{R}^n$ such that $P(i\boldsymbol{\xi}) = 0$, then one can no longer deduce directly that $F[E] = 1/P(i\boldsymbol{\xi})$; in any case, one has to make sense of $1/P(i\boldsymbol{\xi})$ as an element of $\mathscr{S}'(\mathbb{R}^n)$. We sketch out a solution following an argument given by Nirenberg in 1953. One step of the argument uses Cauchy's Theorem on contour integrals of complex analytic functions; students without this knowledge can either assume this property or skip this part.

First, we construct E in the form of $(1-\Delta)^N E_N$ for some $E_N \in \mathscr{S}'(\mathbb{R}^n)$. This leads to $E_N = F^{-1}(\frac{1}{(1+|\boldsymbol{\xi}|^2)^N P(i\boldsymbol{\xi})})$. We choose N sufficiently large to help make $\frac{1}{(1+|\boldsymbol{\xi}|^2)^N P(i\boldsymbol{\xi})}$

decay sufficiently fast as $\boldsymbol{\xi} \to \infty$. Second, we assume that P is a degree k differential operator and a rotation of axes has been made, if necessary, so that the coefficient of ∂_n^k is 1. Thus $P(i\boldsymbol{\xi})$ is a degree k polynomial in ξ_n with coefficients of ξ_n^j a polynomial of $\boldsymbol{\xi}' = (\xi_1, \cdots, \xi_{n-1})$ of degree $\leq k - j$.

For each fixed $\boldsymbol{\xi}'$, $P(i(\boldsymbol{\xi}', \zeta)) = 0$ has k roots in $\zeta = \xi_n + i\eta \in \mathbb{C}$. In the strip $|\mathfrak{Im}(\zeta)| \leq 1/2$ of the complex plane of ζ, there exists a band parallel to the ξ_n axis of width $(k+1)^{-1}$ which contains none of the roots, so there exists a line $\eta = c(\boldsymbol{\xi}')$ whose distance to any of the roots is at least $(2k+2)^{-1}$. This implies that $|P(i(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))| \geq (2k+2)^{-k}$. It is also clear that

$$|1 + |\boldsymbol{\xi}'|^2 + (\xi_n + ic(\boldsymbol{\xi}'))^2| \geq \frac{1}{2}\left(1 + |(\boldsymbol{\xi}', \xi_n)|^2\right).$$

Thus

$$\frac{1}{|\left(1 + |\boldsymbol{\xi}'|^2 + (\xi_n + ic(\boldsymbol{\xi}'))^2\right)^N||P(i(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))|} \leq (2k+2)^{-k} 2^N \left(1 + |(\boldsymbol{\xi}', \xi_n))|^2\right)^{-N}.$$

Then the integral

$$(2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{e^{i\mathbf{x}\cdot(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))}}{\left(1 + |\boldsymbol{\xi}'|^2 + (\xi_n + ic(\boldsymbol{\xi}'))^2\right)^N P(i(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))} \, d\xi_n \, d\boldsymbol{\xi}'$$

is well defined, and we call it $E_N(\mathbf{x})$. $E_N(\mathbf{x})$ has as many desired differentiability in $\mathbf{x}$ as one would like as long as one chooses N sufficiently large.

We now verify that $E_N(\mathbf{x})$ satisfies $P(\partial_{\mathbf{x}})(1-\Delta)^N E_N(\mathbf{x}) = \delta(\mathbf{x})$ in the sense that

$$u(0) = \int_{\mathbb{R}^n} E_N(\mathbf{x})(1-\Delta)^N P'(\partial_{\mathbf{x}}) u(\mathbf{x}) \, d\mathbf{x}$$

for any $u \in \mathscr{D}(\mathbb{R}^n)$. We may suppose that u is supported in $B_R(0)$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^n} E_N(\mathbf{x})(1-\Delta)^N P'(\partial_{\mathbf{x}}) u(\mathbf{x}) \, d\mathbf{x} \\
=&(2\pi)^{-n} \int_{B_R(0)} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{e^{i\mathbf{x}\cdot(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))}(1-\Delta)^N P'(\partial_{\mathbf{x}}) u(\mathbf{x})}{\left(1 + |\boldsymbol{\xi}'|^2 + (\xi_n + ic(\boldsymbol{\xi}'))^2\right)^N P(i(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))} \, d\xi_n \, d\boldsymbol{\xi}' \, d\mathbf{x} \\
=&(2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{B_R(0)} \frac{e^{i\mathbf{x}\cdot(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))}(1-\Delta)^N P'(\partial_{\mathbf{x}}) u(\mathbf{x})}{\left(1 + |\boldsymbol{\xi}'|^2 + (\xi_n + ic(\boldsymbol{\xi}'))^2\right)^N P(i(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))} \, d\mathbf{x} \, d\xi_n \, d\boldsymbol{\xi}'.
\end{aligned}$$

But

$$\begin{aligned}
&\int_{B_R(0)} e^{i\mathbf{x}\cdot(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))}(1-\Delta)^N P'(\partial_{\mathbf{x}}) u(\mathbf{x}) \, d\mathbf{x} \\
=&\int_{B_R(0)} P(\partial_{\mathbf{x}})(1-\Delta)^N e^{i\mathbf{x}\cdot(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))} u(\mathbf{x}) \, d\mathbf{x} \\
=&\int_{B_R(0)} \left(1 + |\boldsymbol{\xi}'|^2 + (\xi_n + ic(\boldsymbol{\xi}'))^2\right)^N P(i(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}')) e^{i\mathbf{x}\cdot(\boldsymbol{\xi}', \xi_n + ic(\boldsymbol{\xi}'))} u(\mathbf{x}) \, d\mathbf{x}
\end{aligned}$$

so

$$\begin{aligned}&\int_{\mathbb{R}^n} E_N(\mathbf{x})(1-\Delta)^N P'(\partial_{\mathbf{x}})u(\mathbf{x})\,d\mathbf{x}\\ =&(2\pi)^{-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\int_{B_R(0)} e^{i\mathbf{x}\cdot(\boldsymbol{\xi}',\xi_n+ic(\boldsymbol{\xi}'))}u(\mathbf{x})\,d\mathbf{x}\,d\xi_n\,d\boldsymbol{\xi}'.\end{aligned}$$

The function

$$\zeta \mapsto \int_{B_R(0)} e^{i\mathbf{x}\cdot(\boldsymbol{\xi}',\zeta)}u(\mathbf{x})\,d\mathbf{x}$$

is complex analytic in $\zeta = \xi_n + i\eta$ and decays sufficiently fast as $\xi_n \to \infty$ for $|\eta| \le 1/2$. Then Cauchy Theorem in complex analysis allows us to shift the line of integral $\mathfrak{Im}\zeta = c(\boldsymbol{\xi}')$ to $\mathfrak{Im}\zeta = 0$ to imply that

$$\int_{\mathbb{R}}\int_{B_R(0)} e^{i\mathbf{x}\cdot(\boldsymbol{\xi}',\xi_n+ic(\boldsymbol{\xi}'))}u(\mathbf{x})\,d\mathbf{x}\,d\xi_n = \int_{\mathbb{R}}\int_{B_R(0)} e^{i\mathbf{x}\cdot(\boldsymbol{\xi}',\xi_n)}u(\mathbf{x})\,d\mathbf{x}\,d\xi_n,$$

which implies that

$$\int_{\mathbb{R}^n} E_N(\mathbf{x})P'(\partial_{\mathbf{x}})(1-\Delta)^N u(\mathbf{x})\,d\mathbf{x} = (2\pi)^{-n}\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\int_{B_R(0)} e^{i\mathbf{x}\cdot(\boldsymbol{\xi}',\xi_n)}u(\mathbf{x})\,d\mathbf{x}\,d\xi_n\,d\boldsymbol{\xi}' = u(0)$$

by the Fourier inversion formula applied to u.

In the case of a constant coefficient operator $P(\partial_x)$ of order k in one variable x, $P(i\xi)$ is a polynomial of one variable ξ. Any root ξ of $P(i\xi) = 0$ gives rise to a solution $e^{i\xi x}$ of $P(\partial_x)e^{i\xi x} = 0$. When ξ is a real root, this solution is smooth and bounded so lies in $\mathscr{S}'(\mathbb{R})$; this also means that $P(\partial_x)$ will not have a unique fundamental solution in $\mathscr{S}'(\mathbb{R})$. When ξ is a non-real root, this solution grows exponentially as $x \to \infty$ or $-\infty$ so is not in $\mathscr{S}'(\mathbb{R})$.

One possible construction of a fundamental solution in this case is to adapt the approach in Example 6.5. Namely, construct a solution $G(x)$ such that $P(\partial_x)G = 0$, $\partial_x^j G(0) = 0$ for $j = 0, \cdots, k-2$, and $\partial_x^{k-1}G(0) = 1$, and take $E(x) = H(x)G(x)$, where $H(x)$ is the Heaviside function. An application of the Cauchy Theorem shows that

$$G(x) = \frac{1}{2\pi}\int_{\xi\in\partial D_R(0)} \frac{e^{ix\xi}}{P(i\xi)}d\xi \tag{6.26}$$

is one such solution, where $R > 0$ is chosen such that $D_R(0)$ contains all the roots of $P(i\xi) = 0$. It is elementary to verify that

$$\partial_x^j\left(\int_{\xi\in\partial D_R(0)} \frac{e^{ix\xi}}{P(i\xi)}d\xi\right) = \int_{\xi\in\partial D_R(0)} \frac{(i\xi)^j e^{ix\xi}}{P(i\xi)}d\xi,$$

so

$$P(\partial_x)G(x) = \frac{1}{2\pi}\int_{\xi\in\partial D_R(0)} e^{ix\xi}\,d\xi,$$

and

$$G^{(j)}(0) = \frac{1}{2\pi}\int_{\xi\in\partial D_R(0)} \frac{(i\xi)^j}{P(i\xi)}d\xi.$$

The evaluation of these integrals to their respective values relies on the Cauchy Theorem.

Another modification of the Fourier transform to deal with such exponentially growing functions is the Laplace transform, used in the context of evolution equations, where one transform a function f defined on $\mathbb{R}^+$ (under some grow control) to $L[f](\xi) = \int_0^\infty f(t)e^{i\xi t}\,dt$, restricting the parameter ξ to be complex parameter such that $\mathfrak{Im}(\xi)$ is sufficiently large.

Exercises

Exercise 6.7.1. Suppose that $l \in \mathscr{D}'(\mathbb{R})$ satisfies $\partial_x l = \delta(x)$. Prove that there exists some constant c such that $l = H(x) + c$, where $H(x)$ is the Heaviside function.

Exercise 6.7.2. Suppose that $m \neq 0$. Find $l \in \mathscr{D}'(\mathbb{R})$ such that $\partial_x l - ml = \delta(x)$. Is it unique? For $m \in \mathbb{R}$, can you find one such that $l \in \mathscr{S}'(\mathbb{R})$?

Exercise 6.7.3. Prove that

$$F[H(-x)] = i\ \mathrm{PV}\left(\frac{1}{\xi}\right) + \pi\delta(\xi) \quad \text{and } F[H(x) - H(-x)] = -2i\ \mathrm{PV}\left(\frac{1}{\xi}\right).$$

Exercise 6.7.4. Suppose that $l \in \mathscr{S}'(\mathbb{R})$ satisfies $\partial_x l = \delta(x)$. Prove that that $i\xi F[l] = 1$. Does it follow that $F[l] = -i\ \mathrm{PV}\left(\frac{1}{\xi}\right)$ and that $l = [H(x) - H(-x)]/2$?

Exercise 6.7.5. Try to use Fourier transform to construct a fundamental solution $E(x;\xi)$ to $\frac{d^2}{dx^2} \pm m^2$ on $\mathbb{R}$ which lies in $\mathscr{S}'(\mathbb{R})$. Here m is a real parameter. Is such an $E(x;\xi)$ uniquely determined? Can the method be adapted to produce a fundamental solution supported on $\{x : x \geq \xi\}$?

Exercise 6.7.6. If one applies the argument as given by Nirenberg to the operator $\frac{d^2}{dx^2} + m^2$ on $\mathbb{R}$, one would get

$$E(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{ix(\xi+ci)}}{m^2 - (\xi+ci)^2}\,d\xi$$

where c is a real parameter with $|c| \leq 1/2$. Can this $E(x)$ be identified? The similarly defined $G(x)$ in (6.26) is infinitely times differentiable. Is this $E(x)$ also infinitely times differentiable? Note: The solution may need knowledge of the calculus of residues.

6.8 Fundamental Solution of the Wave Equation

Here we are looking for a distribution $E(\mathbf{x}, t)$ such that

$$\left[\partial_t^2 - c^2\Delta_{\mathbf{x}}\right] E(\mathbf{x}, t) = \delta(\mathbf{x}, t).$$

Based on an argument similar to that for the Duhamel's principle, if we are looking for an $E(\mathbf{x}, t)$ which is supported in $\{t \geq 0\}$, then such an $E(\mathbf{x}, t)$ can also be characterized by

$$\begin{cases} \left[\partial_t^2 - c^2\Delta_{\mathbf{x}}\right] E(\mathbf{x}, t) = 0 \\ E(\mathbf{x}, 0) = 0 \\ E_t(\mathbf{x}, 0) = \delta(\mathbf{x}) \\ E(\mathbf{x}, t) = 0 \quad \text{for } t < 0. \end{cases}$$

It is possible to look for an $E(\mathbf{x}, t)$ with other constraints on its support. In addition, $E(\mathbf{x}, t)$ has the property that a solution $u(\mathbf{x}, t)$ to

$$\begin{cases} \left[\partial_t^2 - c^2\Delta_{\mathbf{x}}\right] u(\mathbf{x}, t) = 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}) \\ u_t(\mathbf{x}, 0) = h(\mathbf{x}) \end{cases} \tag{6.27}$$

can be represented through $E(\mathbf{x}, t)$ and initial data, via Green's identity, as

$$u(\mathbf{x}, t) = \langle E(\mathbf{x} - \mathbf{y}, t)h(\mathbf{y})\rangle + \frac{\partial}{\partial t}\langle E(\mathbf{x} - \mathbf{y}, t), g(\mathbf{y})\rangle,$$

which we often write informally as

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^n} E(\mathbf{x} - \mathbf{y}, t)h(\mathbf{y})\, d\mathbf{y} + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^n} E(\mathbf{x} - \mathbf{y}, t)g(\mathbf{y})\, d\mathbf{y}\right).$$

We will find $E(\mathbf{x}, t)$ using one of the characterizations here depending on the approach taken.

If we use Fourier's method to construct a solution to (6.27), then we need to solve $\widehat{u}(\xi, t)$ such that

$$\begin{cases} \widehat{u}_{tt}(\boldsymbol{\xi}, t) + c^2\|\boldsymbol{\xi}\|^2\widehat{u}(\boldsymbol{\xi}, t) = 0 \quad t > 0 \\ \widehat{u}(\boldsymbol{\xi}, 0) = \widehat{g}(\boldsymbol{\xi}) \\ \widehat{u}_t(\boldsymbol{\xi}, 0) = \widehat{h}(\boldsymbol{\xi}). \end{cases}$$

The solution is given by $\widehat{u}(\boldsymbol{\xi}, t) = \cos(c\|\boldsymbol{\xi}\|t)\widehat{g}(\boldsymbol{\xi}) + \frac{\sin(c\|\boldsymbol{\xi}\|t)}{c\|\boldsymbol{\xi}\|}\widehat{h}(\boldsymbol{\xi})$. Thus

$$u(\mathbf{x}, t) = (2\pi)^{-n}\int_{\mathbb{R}^n}\left(\cos(c\|\boldsymbol{\xi}\|t)\,\widehat{g}(\boldsymbol{\xi}) + \frac{\sin(c\|\boldsymbol{\xi}\|t)}{c\|\boldsymbol{\xi}\|}\,\widehat{h}(\boldsymbol{\xi})\right)e^{i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi},$$

and we expect to be able to write

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\frac{\sin(c\|\boldsymbol{\xi}\|t)}{c\|\boldsymbol{\xi}\|}\widehat{h}(\boldsymbol{\xi})e^{i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}=E(\cdot,t)*h,$$

and

$$(2\pi)^{-n}\int_{\mathbb{R}^n}\cos(\|\boldsymbol{\xi}\|t)\widehat{g}(\boldsymbol{\xi})e^{i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}=\partial_t\left[E(\cdot,t)*g\right].$$

This amounts to requiring $\widehat{E}(\boldsymbol{\xi},t)=\frac{\sin(c\|\boldsymbol{\xi}\|t)}{c\|\boldsymbol{\xi}\|}$. Since $\frac{\sin(c\|\boldsymbol{\xi}\|t)}{c\|\boldsymbol{\xi}\|}$ is not in $L^1(\mathbb{R}^n)$ or $L^2(\mathbb{R}^n)$ when $n\geq 2$, the identification of such an $E(\mathbf{x},t)$ would require the framework of Fourier transforms on temperate distributions, and is not an easy task anyway. We will present an approach below using more elementary means.

We can still make use the scaling invariance to look for $E(\mathbf{x},t)$ such that $E_\lambda(\mathbf{x},t)=\lambda^{n-1}E(\lambda\mathbf{x},\lambda t)=E(\mathbf{x},t)$ for all $\lambda>0$ (the factor λ^{n-1} is chosen so that $\partial_t E_\lambda(\mathbf{x},t)\big|_{t=0}$ has the same scaling as the point source function $\delta(\mathbf{x})$). We will exploit some invariance property of the wave operator $\partial_t^2-c^2\Delta_{\mathbf{x}}$ in the construction of such an $E(\mathbf{x},t)$.

The wave operator $\partial_t^2-c^2\Delta_{\mathbf{x}}$ is invariant under the **Lorentz transforms**, which are defined as linear transformations on $\mathbb{R}^{n+1}$

$$\begin{bmatrix} s\\ \mathbf{y}\end{bmatrix}=T\begin{bmatrix} t\\ \mathbf{x}\end{bmatrix}\quad\text{such that}\quad T\begin{bmatrix}1&0&\cdots\\0&-c^2&\cdots\\ \vdots&\vdots&\vdots\\0&\cdots&-c^2\end{bmatrix}T'=\begin{bmatrix}1&0&\cdots\\0&-c^2&\cdots\\ \vdots&\vdots&\vdots\\0&\cdots&-c^2\end{bmatrix}. \tag{6.28}$$

This follows from writing $\partial_t^2-c^2\Delta_{\mathbf{x}}$ in matrix form

$$\begin{bmatrix}\partial_t&\partial_{x_1}&\cdots&\partial_{x_n}\end{bmatrix}\begin{bmatrix}1&0&\cdots\\0&-c^2&\cdots\\ \vdots&\vdots&\vdots\\0&\cdots&-c^2\end{bmatrix}\begin{bmatrix}\partial_t\\ \partial_{x_1}\\ \vdots\\ \partial_{x_n}\end{bmatrix},$$

and noting

$$\begin{bmatrix}\partial_t&\partial_{x_1}&\cdots&\partial_{x_n}\end{bmatrix}=\begin{bmatrix}\partial_s&\partial_{y_1}&\cdots&\partial_{y_n}\end{bmatrix}T\text{ and }\begin{bmatrix}\partial_t\\ \partial_{x_1}\\ \vdots\\ \partial_{x_n}\end{bmatrix}=T'\begin{bmatrix}\partial_s\\ \partial_{y_1}\\ \vdots\\ \partial_{y_n}\end{bmatrix}.$$

(6.28) is equivalent to

$$T' \begin{bmatrix} c^2 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & \cdots & -1 \end{bmatrix} T = \begin{bmatrix} c^2 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & \cdots & -1 \end{bmatrix}, \tag{6.29}$$

from which it follows that such transformations preserve the quadratic form $c^2t^2 - \|\mathbf{x}\|^2$:

$$c^2t^2 - \|\mathbf{x}\|^2 = c^2s^2 - \|\mathbf{y}\|^2 \text{ when } \begin{bmatrix} s \\ \mathbf{y} \end{bmatrix} = T \begin{bmatrix} t \\ \mathbf{x} \end{bmatrix}.$$

The verification of these basic algebraic properties is left as an exercise for the reader.

Due to the invariance of $\partial_t^2 - c^2\Delta_{\mathbf{x}}$ and of $c^2t^2 - \|\mathbf{x}\|^2$ under the Lorentz transforms, we can look for an $E(\mathbf{x}, t)$ which is invariant under the Lorentz transforms. This leads us to look for an $E(\mathbf{x}, t) = F(c^2t^2 - \|\mathbf{x}\|^2)$ for some function (or distribution) F. If we combine this with scaling, and want $E(\mathbf{x}, t) = \lambda^{n-1}E(\lambda\mathbf{x}, \lambda t)$ for all $\lambda > 0$, then F will obey $\lambda^{n-1}F(\lambda^2(c^2t^2 - \|\mathbf{x}\|^2)) = F(c^2t^2 - \|\mathbf{x}\|^2)$, namely, F has homogeneity of $\frac{1-n}{2}$. If $E(\mathbf{x}, t)$ is represented by a locally integrable function, we expect $E(\mathbf{x}, t) = A(c^2t^2 - \|\mathbf{x}\|^2)^{\frac{1-n}{2}}$ for some constant A.

Based on our experience with the 1-dimensional wave equation having propagation speed $\le c$, we can look for an $E(\mathbf{x}, t)$ such that it is supported in $\{(\mathbf{x}, t) : \|\mathbf{x}\| \le ct\}$. Thus we may look for an $E(\mathbf{x}, t)$ such that

$$E(\mathbf{x}, t) = \begin{cases} A(c^2t^2 - \|\mathbf{x}\|^2)^{\frac{1-n}{2}} & \|\mathbf{x}\| < ct, \\ 0 & \|\mathbf{x}\| \ge ct, \end{cases}$$

for some constant A. But such an $E(\mathbf{x}, t)$ is not locally integrable in $\mathbf{x}$ when $n \ge 3$, so we have to re-examine our analysis*, or find an alternative approach when $n \ge 3$.

When $n = 1$, this approach gives an answer consistent with our earlier result that

$$E(x, t) = \begin{cases} \frac{1}{2c} & |x| < ct, \\ 0 & |x| \ge ct. \end{cases}$$

*A proper analysis along this line can be done in the theory of distribution; but it requires considerable background preparation, including the theory of homogeneous distributions and of pull-back of a distribution, see section 6.2 of *The Analysis of Linear Partial Differential Operators I* by Lars Hörmander.

When $n = 2$, we get a trial solution of the form

$$E(\mathbf{x}, t) = \begin{cases} A(c^2t^2 - \|\mathbf{x}\|^2)^{-\frac{1}{2}} & \|\mathbf{x}\| < ct, \\ 0 & \|\mathbf{x}\| \geq ct. \end{cases}$$

We expect

$$u(\mathbf{x}, t) = \int_{\mathbb{R}^2} E(\mathbf{x} - \mathbf{y}, t) h(\mathbf{y})\, d\mathbf{y} = A \int_{\|\mathbf{x}-\mathbf{y}\| < ct} (c^2t^2 - \|\mathbf{x} - \mathbf{y}\|^2)^{-\frac{1}{2}} h(\mathbf{y})\, d\mathbf{y}$$

to provide a solution of (6.27) with $g = 0$ and $u_t(\mathbf{x}, 0) = h(\mathbf{x})$. Since $u(\mathbf{x}, t) = t$ is a solution with $g(\mathbf{y}) \equiv 0$ and $h(\mathbf{y}) \equiv 1$, we expect

$$t = A \int_{\|\mathbf{x}-\mathbf{y}\| < ct} (c^2t^2 - \|\mathbf{x} - \mathbf{y}\|^2)^{-\frac{1}{2}}\, d\mathbf{y}.$$

But a change of variables $\mathbf{y} = \mathbf{x} + ct\,\mathbf{z}$ in the right hand side integral gives

$$t = Act \int_{\|\mathbf{z}\| < 1} (1 - \|\mathbf{z}\|^2)^{-\frac{1}{2}}\, d\mathbf{z} = 2\pi Act.$$

Thus $A = 1/(2\pi c)$. One can now verify directly that

$$E(\mathbf{x}, t) = \begin{cases} (2\pi c)^{-1}(c^2t^2 - \|\mathbf{x}\|^2)^{-\frac{1}{2}} & \|\mathbf{x}\| < ct, \\ 0 & \|\mathbf{x}\| \geq ct, \end{cases}$$

provides a fundamental solution of the 2-dimensional wave equation (6.27).

For $n \geq 3$, we will use a recursive structure in the dimension n of radial solutions on $\mathbb{R}^n \times \mathbb{R}$ to $[\partial_t^2 - c^2\Delta_{\mathbf{x}}]\, u(\mathbf{x}, t) = 0$, similar to that in **Exercise** 6.4.11: if $E_n(r, t)$ is a solution of

$$\left[\partial_t^2 - c^2\Delta_{\mathbf{x}}\right] E_n(r, t) = \partial_t^2 E_n(r, t) - c^2 \left[\partial_r^2 + \frac{n-1}{r}\partial_r\right] E_n(r, t) = 0,$$

then $E_{n+2}(r, t) := A\partial_r E_n(r, t)/r$ is a solution of

$$\partial_t^2 E_{n+2}(r, t) - c^2 \left[\partial_r^2 + \frac{n+1}{r}\partial_r\right] E_{n+2}(r, t) = 0.$$

We apply this to our fundamental solution of the 1-dimensional wave equation to produce a candidate for a fundamental solution of the 3-dimensional wave equation. Since for each t, our fundamental solution of the 1-dimensional wave equation is a step function in r, with a jump discontinuity at $r = ct$, so $\partial_r E_1(r, t) = -\frac{\delta(r-ct)}{2c}$, and we have $E_3(r,t) = A\frac{\delta(r-ct)}{r}$ as a candidate for a fundamental solution of the 3-dimensional wave equation. Again A can be determined by examining

$$u(\mathbf{x},t) = E_3(r,t) * h = \frac{A}{ct}\int_{\|\mathbf{x}-\mathbf{y}\|=ct} h(\mathbf{y})\; d\sigma(\mathbf{y})$$

in the case $u(\mathbf{x},t) = t$, which is a solution of $[\partial_t^2 - c^2\Delta_{\mathbf{x}}]\, u(\mathbf{x},t) = 0$ with $u(\mathbf{x},0) = 0$ and $h(\mathbf{x}) = u_t(\mathbf{x},0) = 1$. We thus expect

$$t = \frac{A}{ct}\int_{\|\mathbf{x}-\mathbf{y}\|=ct} d\mathbf{y} = 4\pi ctA.$$

So $A = (4\pi c)^{-1}$.

Note that the scaling behavior of this fundamental solution $E(\mathbf{x},t) = \frac{\delta(\|\mathbf{x}\|-ct)}{4\pi c\|\mathbf{x}\|}$ is consistent with our earlier scaling analysis: it is supported on $\{(\mathbf{x},t) : \|\mathbf{x}\| = ct\}$, and has scaling consistent with $E(\mathbf{x},t) = \lambda^{n-1}E(\lambda\mathbf{x},\lambda t)$ for $n = 3$; it is just that the behavior of $E(\mathbf{x},t)$ on $\{(\mathbf{x},t) : \|\mathbf{x}\| = ct\}$ does not follow directly from the scaling argument.

We can now verify directly that

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2 t}\int_{\|\mathbf{x}-\mathbf{y}\|=ct} h(\mathbf{y})\, d\sigma(\mathbf{y})$$

provides a solution of (6.27) with $u(\mathbf{x},0) \equiv 0$ and $u_t(\mathbf{x},0) = h(\mathbf{x})$, provided that $h \in C^2(\mathbb{R}^3)$. With $\mathbf{y} = \mathbf{x} + ct\boldsymbol{\omega}$, $\boldsymbol{\omega} \in \mathbb{S}^2$, the integral can be written as $u(\mathbf{x},t) = \frac{t}{4\pi}\int_{\mathbb{S}^2} h(\mathbf{x}+ct\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega})$. Thus

$$\begin{aligned} u_{tt}(\mathbf{x},t) &= \frac{1}{2\pi}\int_{\mathbb{S}^2} \frac{\partial}{\partial t}\left[h(\mathbf{x}+ct\boldsymbol{\omega})\right]\, d\sigma(\boldsymbol{\omega}) + \frac{t}{4\pi}\int_{\mathbb{S}^2}\frac{\partial^2}{\partial t^2}\left[h(\mathbf{x}+ct\boldsymbol{\omega})\right]\, d\sigma(\boldsymbol{\omega}) \\ &= \frac{t}{4\pi}\int_{\mathbb{S}^2}\left[\frac{\partial^2}{\partial t^2} + \frac{2}{t}\frac{\partial}{\partial t}\right] h(\mathbf{x}+ct\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}). \end{aligned}$$

With $z = r\boldsymbol{\omega} = ct\boldsymbol{\omega}$, we note that

$$\begin{aligned} &\left[\frac{\partial^2}{\partial t^2} + \frac{2}{t}\frac{\partial}{\partial t}\right] h(\mathbf{x}+ct\boldsymbol{\omega}) \\ =&c^2\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] h(\mathbf{x}+r\boldsymbol{\omega}) \\ =&c^2\left[\Delta_{\mathbf{z}} h(\mathbf{x}+\mathbf{z})\big|_{\mathbf{z}=ct\boldsymbol{\omega}} - \frac{\Delta_{\boldsymbol{\omega}}[h(\mathbf{x}+r\boldsymbol{\omega})]}{r^2}\Big|_{r=ct}\right] \\ =&c^2\left[\Delta_{\mathbf{x}} h(\mathbf{x}+\mathbf{z})\big|_{\mathbf{z}=ct\boldsymbol{\omega}} - \frac{\Delta_{\boldsymbol{\omega}}[h(\mathbf{x}+r\boldsymbol{\omega})]}{r^2}\Big|_{r=ct}\right]. \end{aligned}$$

So

$$u_{tt}(\mathbf{x},t) = \frac{c^2 t}{4\pi}\int_{\mathbb{S}^2}\left[\Delta_{\mathbf{x}} h(\mathbf{x}+ct\boldsymbol{\omega}) - \frac{\Delta_{\boldsymbol{\omega}}[h(\mathbf{x}+r\boldsymbol{\omega})]}{r^2}\right] d\sigma(\boldsymbol{\omega}) = \frac{c^2 t}{4\pi}\int_{\mathbb{S}^2} \Delta_{\mathbf{x}} h(\mathbf{x}+ct\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}),$$

using $\int_{\mathbb{S}^2} \Delta_{\boldsymbol{\omega}}[h(\mathbf{x}+r\boldsymbol{\omega})]\, d\sigma(\boldsymbol{\omega}) = 0$. One key fact above that will also be useful for other purposes is the following three dimensional case of the **Euler-Poisson-Darboux** equation

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right]\int_{\mathbb{S}^2} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}) = \int_{\mathbb{S}^2} \Delta_{\mathbf{x}} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}). \tag{6.30}$$

Next,

$$\Delta_{\mathbf{x}} u(\mathbf{x},t) = \frac{t}{4\pi}\int_{\mathbb{S}^2} \Delta_{\mathbf{x}} h(\mathbf{x}+ct\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}),$$

thus $[\partial_t^2 - c^2\Delta_{\mathbf{x}}]\, u(\mathbf{x},t) = 0$. Finally, $u(\mathbf{x},0) = \frac{t}{4\pi}\int_{\mathbb{S}^2} h(\mathbf{x}+ct\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega})\big|_{t=0} = 0$, and $u_t(\mathbf{x},0) = (4\pi)^{-1}\int_{\mathbb{S}^2} h(\mathbf{x}+ct\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega})\big|_{t=0} = h(\mathbf{x})$.

Note that a C^2 solution $u(\mathbf{x},t)$ of (6.27) gives rise to a C^1 $h(\mathbf{x}) = u_t(\mathbf{x},0)$; but in general a C^2 $h(\mathbf{x})$ is needed to produce a C^2 solution $u(\mathbf{x},t)$ of (6.27). Note also that h propagates at exactly speed c in the $n=3$ case, and that $E_3(\mathbf{x},t)$ is represented by a surface integral along $\|\mathbf{x}\| = ct$, not by a locally integrable function. The procedure for getting $E_3(\mathbf{x},t)$ also suggests that for $n>3$ odd, $E_n(\mathbf{x},t)$ is expected to involve derivatives of the δ function supported on $\|\mathbf{x}\| = ct$, so will be more singular in some sense.

Using Duhamel's principle, we obtain a solution of the non-homogeneous version of the 3-dimensional (6.27) as

$$\begin{aligned} u(\mathbf{x},t) &= \frac{1}{4\pi c^2 t}\int_{\|\mathbf{x}-\mathbf{y}\|=ct} h(\mathbf{y})\ d\sigma(\mathbf{y}) + \frac{\partial}{\partial_t}\left(\frac{1}{4\pi c^2 t}\int_{\|\mathbf{x}-\mathbf{y}\|=ct} g(\mathbf{y})\ d\sigma(\mathbf{y})\right) \\ &\quad + \frac{1}{4\pi c^2}\int_0^t\int_{\|\mathbf{x}-\mathbf{y}\|=c(t-s)} \frac{f(\mathbf{y},s)}{t-s}\ d\sigma(\mathbf{y})ds. \end{aligned}$$

This formula is called the **Kirchhoff**'s formula.

The last term is an integral over a section of the cone $\{(\mathbf{y},s)\in\mathbb{R}^4 : \|\mathbf{x}-\mathbf{y}\| = c(t-s), 0\le s\le t\}$ in $\mathbb{R}^4$ with vertex $(\mathbf{x},t)$, which can be parametrized as a graph over the ball $\{\mathbf{y}\in\mathbb{R}^3 : \|\mathbf{x}-\mathbf{y}\|\le ct\}$, and allows one to rewrite the integral as

$$\int_0^t\int_{\|\mathbf{x}-\mathbf{y}\|=c(t-s)} \frac{f(\mathbf{y},s)}{t-s}\ d\sigma(\mathbf{y})ds = \int_{\|\mathbf{x}-\mathbf{y}\|\le ct} \frac{f(\mathbf{y}, t-\frac{\|\mathbf{x}-\mathbf{y}\|}{c})}{\|\mathbf{x}-\mathbf{y}\|}\ d\mathbf{y}.$$

Note that the integral is a weighted integral of $f(\mathbf{y}, t-\frac{\|\mathbf{x}-\mathbf{y}\|}{c})$ with a retarded time $t-\frac{\|\mathbf{x}-\mathbf{y}\|}{c}$.

The approach we used to obtain the fundamental solution for the wave equation may seem somewhat ad hoc; we choose this approach to illustrate the need to use whatever approach that is effective, instead of insisting on using only one approach. In the exercises we will explore the method of **spherical means** and the method of **descent** to solve (6.27).

Let's record here the Euler-Poisson-Darboux equation in $\mathbb{R}^n$ satisfied by the spherical mean of a function $h(\mathbf{x})$ defined by

$$M_h(r;\mathbf{x}) = \frac{1}{|\partial B_r(\mathbf{x})|}\int_{\partial B_r(\mathbf{x})} h(\mathbf{y})\ d\sigma(\mathbf{y}) = |\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} h(\mathbf{x}+r\boldsymbol{\omega})\ d\sigma(\boldsymbol{\omega}).$$

The Euler-Poisson-Darboux equation is

$$\Delta_{\mathbf{x}} M_h(r;\mathbf{x}) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_h(r;\mathbf{x}). \tag{6.31}$$

The proof for the three dimensional case above generalizes readily. For those who are not comfortable working with the spherical Laplace operator, here is another proof.

Start with

$$\begin{aligned}
&\int_0^r \int_{\mathbb{S}^{n-1}} \Delta_{\mathbf{x}} h(\mathbf{x}+s\boldsymbol{\omega}) s^{n-1}\ d\boldsymbol{\omega} ds \\
=&\int_{\|z\|<r} \Delta_{\mathbf{x}} h(\mathbf{x}+\mathbf{z})\, d\mathbf{z} = \int_{\|\mathbf{z}\|<r} \Delta_{\mathbf{z}} h(\mathbf{x}+\mathbf{z})\, d\mathbf{z} \\
=&\int_{\mathbb{S}^{n-1}} \frac{\partial h(\mathbf{x}+r\boldsymbol{\omega})}{\partial r} r^{n-1}\ d\sigma(\boldsymbol{\omega}) \quad \text{(using the divergence theorem)} \\
=&r^{n-1}\frac{\partial}{\partial r}\left(\int_{\mathbb{S}^{n-1}} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega})\right).
\end{aligned}$$

Then differentiating with respect to r to obtain

$$r^{n-1}\int_{\mathbb{S}^{n-1}} \Delta_{\mathbf{x}} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}) = \frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}\right]\left(\int_{\mathbb{S}^{n-1}} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega})\right).$$

This then gives

$$\begin{aligned}
\Delta_{\mathbf{x}} M_h(r;\mathbf{x}) &= |\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} \Delta_{\mathbf{x}} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}) \\
&= r^{1-n}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}\right] M_h(r;\mathbf{x}) \\
&= \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_h(r;\mathbf{x}).
\end{aligned}$$

Define $\Delta_{\boldsymbol{\omega}}[h(r\boldsymbol{\omega})]$ through the relation

$$\Delta_z h(z) = \left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) h(r\boldsymbol{\omega}) + \frac{\Delta_{\boldsymbol{\omega}}[h(r\boldsymbol{\omega})]}{r^2},$$

then the same relation above can also be recognized as

$$\begin{aligned}
&\int_{\mathbb{S}^{n-1}} \left[\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) h(\mathbf{x}+r\boldsymbol{\omega}) + \frac{\Delta_{\boldsymbol{\omega}}[h(\mathbf{x}+r\boldsymbol{\omega})]}{r^2}\right] \, d\sigma(\boldsymbol{\omega}) \\
=&\int_{\mathbb{S}^{n-1}} \Delta_z h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}) \\
=&r^{1-n}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}\right]\left(\int_{\mathbb{S}^{n-1}} h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega})\right) \\
=&\int_{\mathbb{S}^{n-1}} r^{1-n}\frac{\partial}{\partial r}\left[r^{n-1}\frac{\partial}{\partial r}\right] h(\mathbf{x}+r\boldsymbol{\omega})\, d\sigma(\boldsymbol{\omega}).
\end{aligned}$$

Comparing both sides gives

$$\int_{\mathbb{S}^{n-1}} \Delta_{\boldsymbol{\omega}}[h(\mathbf{x}+r\boldsymbol{\omega})] \, d\sigma(\boldsymbol{\omega}) = 0,$$

which is used in the derivation for the three dimensional Euler-Poisson-Darboux equation.

Exercises

Exercise 6.8.1. Prove that (6.28) is equivalent to (6.29). Furthermore, any matrix satisfying (6.28) must have $\det A = \pm 1$, and

$$T^{-1}_{ij} = \begin{cases} T_{ji} & i, j \geq 1 \text{ or } i = j = 1 \\ -c^{-2}T_{j1} & i = 1, j \geq 2 \\ -c^2 T_{1i} & i \geq 2, j = 1 \end{cases}$$

Exercise 6.8.2. Prove that in $\mathbb{R}^3$

$$\int_{\|\mathbf{x}\|=ct} e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\, d\sigma(\mathbf{x}) = 4\pi ct\frac{\sin(\|\boldsymbol{\xi}\|ct)}{\|\boldsymbol{\xi}\|},$$

and use this to confirm that

$$\mathscr{F}\left[\frac{\delta(\|\mathbf{x}\| - ct)}{4\pi c^2 t}\right] = \frac{\sin(\|\boldsymbol{\xi}\|ct)}{c\|\boldsymbol{\xi}\|} \quad \text{in } \mathbb{R}^3.$$

Exercise 6.8.3. Suppose that $E(r,t)$ is a solution of

$$\partial_t^2 E(r,t) - c^2\left[\partial_r^2 + \frac{m}{r}\partial_r\right] E(r,t) = 0.$$

(i). Verify that $F(r,t) = \partial_r E(r,t)/r$ is a solution of

$$\partial_t^2 F(r,t) - c^2 \left[\partial_r^2 + \frac{m+2}{r}\partial_r\right] F(r,t) = 0.$$

(ii). Verify that $G(r,t) = r^{m-1}E(r,t)$ is a solution of

$$\partial_t^2 G(r,t) - c^2 \left[\partial_r^2 + \frac{-m+2}{r}\partial_r\right] G(r,t) = 0.$$

(iii). Use (i) and (ii) to solve radial solutions to (6.27) for the cases of $n = 3, 5$. ANSWER: Extend $u(r,0) = g(r)$ and $u_t(r,0) = h(r)$ as even functions for $r \in \mathbb{R}$. Then for $n = 3$,

$$u(r,t) = \frac{1}{2cr}\int_{r-ct}^{r+ct} h(s)s\,ds + \frac{\partial}{\partial t}\left(\frac{1}{2cr}\int_{r-ct}^{r+ct} g(s)s\,ds\right);$$

and for $n = 5$,

$$u(r,t) = \frac{1}{4cr^3}\int_{r-ct}^{r+ct}\left[(r^2 - c^2t^2) + s^2\right] h(s)s\,ds + \frac{\partial}{\partial t}\left(\frac{1}{4cr^3}\int_{r-ct}^{r+ct}\left[(r^2 - c^2t^2) + s^2\right] g(s)s\,ds\right).$$

Exercise 6.8.4. Given $u(\mathbf{x},t)$ and $h(\mathbf{x})$, define the spherical mean of h as at the end of this section and that of $u(\mathbf{x},t)$ as

$$M_u(r;\mathbf{x},t) = \frac{1}{|\partial B_r(\mathbf{x})|}\int_{\partial B_r(\mathbf{x})} u(\mathbf{y},t)\ d\sigma(\mathbf{y}) = |\mathbb{S}^{n-1}|^{-1}\int_{\mathbb{S}^{n-1}} u(\mathbf{x}+r\boldsymbol{\omega},t)\ d\sigma(\boldsymbol{\omega}).$$

(i). Verify that if $[\partial_t^2 - c^2\Delta_{\mathbf{x}}]\, u(\mathbf{x},t) = 0$, then

$$\partial_t^2 M_u(r;\mathbf{x},t) - c^2\left(\frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}\right) M_u(r;\mathbf{x},t) = 0.$$

(ii). Verify that $M_u(r;\mathbf{x},0) = M_{u(\cdot,0)}(r;\mathbf{x})$ and $M_{u_t(\cdot,0)}(r;\mathbf{x}) = \partial_t M_u(r;\mathbf{x},t)\big|_{t=0}$.

(iii). Suppose that $u(\mathbf{x},t)$ solves (6.27). Construct a corresponding IVP for $M_u(r;\mathbf{x},t)$ in the variables (r,t), and construct its solution for $n = 3, 5$. HINT: Use the previous exercise.

(iv). Using $u(\mathbf{x},t) = \lim_{r\searrow 0} M_u(r;\mathbf{x},t)$ to construct a solution of (6.27).

Exercise 6.8.5. This exercise works out the method of descent, which is a method for solving (6.27) in dimension $n-1$, if the solution formula is known in dimension n. We will use $n = 3$ to illustrate how this works. For the given g and h in $\mathbb{R}^2$, we extend them to $\mathbb{R}^3$ by $\widetilde{g}(x_1, x_2, x_3) = g(x_1, x_2)$, and $\widetilde{h}(x_1, x_2, x_3) = h(x_1, x_2)$, then note that $\widetilde{u}(x_1, x_2, x_3, t) = u(x_1, x_2, t)$ solves the wave equation in dimension 3, and use the solution formula for the wave equation in dimension 3 and evaluate $\widetilde{u}(x_1, x_2, x_3, t)$ at $(x_1, x_2, 0, t)$ to construct a solution of (6.27) in dimension 2:

$$u(x_1, x_2, t) = \frac{1}{4\pi c^2 t} \iint_{\|(x_1,x_2,0)-\mathbf{y}\|=ct} \widetilde{h}(\mathbf{y})\, d\sigma(\mathbf{y}) + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t} \iint_{\|(x_1,x_2,0)-\mathbf{y}\|=ct} \widetilde{g}(\mathbf{y})\, d\sigma(\mathbf{y})\right).$$

Note that $\{y \in \mathbb{R}^3 : \|(x_1, x_2, 0) - \mathbf{y}\| = ct\}$ is given by two halves of the hemispheres $y_3 = \pm\sqrt{(ct)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}$. Prove that

$$\iint_{\|(x_1,x_2,0)-\mathbf{y}\|=ct} \widetilde{h}(\mathbf{y})\, d\sigma(\mathbf{y}) = \iint_{(y_1-x_1)^2+(y_2-x_2)^2<(ct)^2} \frac{2ct\, h(y_1, y_2)}{\sqrt{(ct)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}}\, d\mathbf{y},$$

which establishes

$$\begin{aligned} u(x_1, x_2, t) = &\frac{1}{2\pi c} \iint_{(y_1-x_1)^2+(y_2-x_2)^2<(ct)^2} \frac{h(y_1, y_2)}{\sqrt{(ct)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}}\, d\mathbf{y} \\ &+ \frac{\partial}{\partial t}\left(\frac{1}{2\pi c} \iint_{(y_1-x_1)^2+(y_2-x_2)^2<(ct)^2} \frac{g(y_1, y_2)}{\sqrt{(ct)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}}\, d\mathbf{y}\right). \end{aligned}$$

Note that making the change of variables $(y_1, y_2) = (x_1, x_2) + r(\cos\theta, \sin\theta)$, then $r = ct\sin s$, the second integral reduces to

$$\frac{1}{2\pi c} \iint_{(y_1-x_1)^2+(y_2-x_2)^2<(ct)^2} \frac{g(y_1, y_2)}{\sqrt{(ct)^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}}\, d\mathbf{y} = t\int_0^{\frac{\pi}{2}} M_g(ct\sin s; x)\sin s\, ds,$$

where $M_g(ct\sin s; x) = (2\pi)^{-1}\int_0^{2\pi} g(x_1 + ct\sin s\cos\theta, x_2 + ct\sin s\sin\theta)\, d\theta$.

Exercise 6.8.6. The method of descent can also be used to solve the Cauchy problem for the Telegraph equation

$$\begin{cases} u_{tt} - c^2 u_{xx} + m^2 u = 0 & x \in \mathbb{R}, t \in \mathbb{R}, \\ u(x, 0) = g(x), & x \in \mathbb{R}, \\ u_t(x, 0) = h(x), & x \in \mathbb{R}, \end{cases}$$

where $m \in \mathbb{R}$ is a real parameter. Construct $u(x, t)$ by considering $v(x, y, t) = \cos(my/c)u(x, t)$, which would solve a 2-dimensional wave equation. Use the method of descent as described in the last Exercise to verify that

$$\begin{aligned} u(x, t) =& \frac{1}{2c}\int_{x-ct}^{x+ct} J_0\left(\frac{m}{c}\sqrt{(ct)^2 - |x - z|^2}\right)h(z)\, dz \\ &+ \frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} J_0\left(\frac{m}{c}\sqrt{(ct)^2 - |x - z|^2}\right)g(z)\, dz\right), \end{aligned}$$

where $J_0(s) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos(s\sin\theta)d\theta$ is the Bessel function of order 0, solving $sJ_0''(s) + J_0'(s) + sJ_0(s) = 0$, with $J_0(0) = 1$. $J_0(s)$ decreases from 1 at $s = 0$ to about -0.4, and then starts to oscillate around 0 as $s \to \infty$, and has an asymptotic $\sqrt{\frac{2}{\pi s}}\cos(s - \frac{\pi}{4})$ as $s \to \infty$, so the above integral representation indicates that solutions to the Telegraph equation still have a propagation speed $\le c$, and the solution in terms of the initial data is weighted more heavily around $|x - z| = ct$.

Exercise 6.8.7. A solution of the Cauchy problem for the Telegraph equation can also be constructed using Fourier'e method. Verify that a solution with $g = 0$ can be constructed as

$$(2\pi)^{-1}\int_{\mathbb{R}} \widehat{h}(\xi)\frac{\sin(\sqrt{\xi^2+m^2}t)}{\sqrt{\xi^2+m^2}}e^{ix\xi}d\xi.$$

Based on this construction, a fundamental solution would be given by $\int_{\mathbb{R}} \frac{\sin(\sqrt{\xi^2+m^2}t)}{\sqrt{\xi^2+m^2}}e^{ix\xi}d\xi$, namely, the inverse Fourier transform of $\frac{\sin(\sqrt{\xi^2+m^2}t)}{\sqrt{\xi^2+m^2}}$. But a direct evaluation of this inverse Fourier transform is not easy; and it would take additional efforts to see the phenomenon of finite speed of propagation from this approach. Also use different approaches to investigate the solution of

$$\begin{cases} u_{tt} - c^2u_{xx} - m^2u = 0 & x \in \mathbb{R}, t \in \mathbb{R}, \\ u(x,0) = g(x), & x \in \mathbb{R}, \\ u_t(x,0) = h(x), & x \in \mathbb{R}, \end{cases}$$

and compare the approaches.

Exercise 6.8.8. Solutions to the Telegraph equation $u_{tt}(x,t) - c^2\Delta_{\mathbf{x}}u(\mathbf{x},t) + \lambda u(\mathbf{x},t) = 0$ for $(\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}$ which are invariant under the Lorentz transformations can be expressed as $u(\mathbf{x},t) = U(s)$, where $s = \sqrt{c^2t^2 - \|\mathbf{x}\|^2}$. Verify that $U(s)$ would satisfy $U''(s) + \frac{n}{s}U'(s) + \frac{\lambda}{c^2}U(s) = 0$, and that, if $\lambda = m^2 > 0$, then $V(s) = s^{\frac{n-1}{2}}U(\frac{cs}{m})$ would satisfy Bessel's equation of order $\frac{n-1}{2}$: $s^2V''(s) + sV'(s) + \left[s^2 - \left(\frac{n-1}{2}\right)^2\right]V(s) = 0$. Use this relation to establish that, for $n = 1$,

$$E(\mathbf{x},t) = \begin{cases} \frac{1}{2c}J_0(\frac{m}{c}\sqrt{c^2t^2 - |x|^2}) & \text{if } |x| < ct, \\ 0 & \text{otherwise,} \end{cases}$$

is a fundamental solution of $u_{tt} - c^2u_{xx} + m^2u = 0$.

Exercise 6.8.9. Use the method of descent to construct a solution of the boundary value problem for the Helmholtz equation on the upper half plane

$$\begin{cases} u_{xx} + u_{yy} - m^2u = 0 & (x,y) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) = g(x), & x \in \mathbb{R}, \end{cases}$$

Does the method work if the equation $u_{xx} + u_{yy} - m^2 u = 0$ is replaced by $u_{xx} + u_{yy} + m^2 u = 0$? Try also the Fourier's method.

Exercise 6.8.10. Use a similar approach as in the text to prove that

$$E_5(\mathbf{x}, t) = \begin{cases} \frac{\delta(\|\mathbf{x}\| - ct) - \|x\| \partial_{\|\mathbf{x}\|} \delta(\|\mathbf{x}\| - ct)}{3c|\mathbb{S}^4| r^3} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

defines a fundamental solution of the wave operator $\partial_t^2 - c^2 \Delta_{\mathbf{x}}$ in dimension 5. Here for a test function $\phi \in C_c^2(\mathbb{R}^5)$, and $t > 0$,

$$\begin{aligned}
\langle E_5(\cdot, t), \phi \rangle =& (3c|\mathbb{S}^4|)^{-1} \left[\int_{\|\mathbf{x}\|=ct} \frac{\phi(\mathbf{x})}{\|\mathbf{x}\|^3} \, d\sigma(\mathbf{x}) - \int_{\mathbb{S}^4} \int_0^\infty \partial_r \delta(\|\mathbf{x}\| - ct) \left[r^2 \phi(r\boldsymbol{\omega}) \right] \, dr d\boldsymbol{\omega} \right] \\
=& (3c|\mathbb{S}^4|)^{-1} \left[\int_{\|\mathbf{x}\|=ct} \frac{\phi(\mathbf{x})}{\|\mathbf{x}\|^3} \, d\sigma(\mathbf{x}) + \int_{\mathbb{S}^4} \int_0^\infty \delta(\|\mathbf{x}\| - ct) \partial_r \left[r^2 \phi(r\boldsymbol{\omega}) \right] \, dr d\boldsymbol{\omega} \right] \\
=& (3c|\mathbb{S}^4|)^{-1} \left[\int_{\|\mathbf{x}\|=ct} \frac{\phi(\mathbf{x})}{\|\mathbf{x}\|^3} \, d\sigma(\mathbf{x}) + \int_{\mathbb{S}^4} \partial_r \left[r^2 \phi(r\boldsymbol{\omega}) \right] \Big|_{r=ct} d\boldsymbol{\omega} \right] \\
=& (3c|\mathbb{S}^4|)^{-1} \left[3ct \int_{\mathbb{S}^4} \phi(ct\boldsymbol{\omega}) \, d\boldsymbol{\omega} + (ct)^2 \partial_r \left(\int_{\mathbb{S}^4} \phi(r\boldsymbol{\omega}) \, d\boldsymbol{\omega} \right) \Big|_{r=ct} \right].
\end{aligned}$$

Thus we expect $u(\mathbf{x}, t) = (3c|\mathbb{S}^4|)^{-1} \left[3ct \int_{\mathbb{S}^4} h(\mathbf{x} + ct\boldsymbol{\omega}) \, d\boldsymbol{\omega} + (ct)^2 \partial_r \left(\int_{\mathbb{S}^4} h(\mathbf{x} + r\boldsymbol{\omega}) \, d\boldsymbol{\omega} \right) \Big|_{r=ct} \right]$ to define a solution of (6.27) in dimension 5 with $u(\mathbf{x}, 0) = 0$, and $u_t(\mathbf{x}, 0) = h(\mathbf{x})$.

Chapter 7

Hadamard-Petrowsky Wellposedness Condition for Cauchy Problems

In chapter 1, we made a preliminary discussion on the well-posedness issue. In discussing the prototype heat and wave equations, we found that the Cauchy problem for the heat equation is well-posed for forward time, but not well-posed for backward time; while the wave equation is well-posed for both forward and backward time. In this chapter we study the Hadamard-Petrowsky well-posedness criteria for the Cauchy problem of a general constant coefficient differential operator. We will only outline the main ideas here; please refer to chapter 5 of Fritz John's text [J] and chapter 3 of Jeffrey Rauch's text [R] for full details.

7.1 Hadamard-Petrowsky Wellposedness Condition

We will discuss the Cauchy problem of a general constant coefficient differential operator such that the initial value is posed on a hypersurface of the type $\Sigma = \{(\mathbf{x}, t) : \mathbf{x} \in \mathbb{R}^n, t = t_0\}$. Here t is singled out as a distinguished variable, and we then write the m-th order differential operator $P = \sum_{j+|\alpha|\leq m} c_{j\alpha}\partial_{\mathbf{x}}^{\alpha}\partial_t^j$ in descending order of differentiation in t:

$$P = \sum_{j=0}^{l} c_j(\partial_{\mathbf{x}})\partial_t^j,$$

where $c_j(\partial_{\mathbf{x}})$ is a differential operator of order m_j in $\mathbf{x}$ such that $m_j + j \leq m$.

If $l = m$, then c_m is a constant; if we further assume that $c_m \neq 0$, then the hyperplane Σ is said to be **non-characteristic** with respect to the operator P. In general we will assume that the leading term $c_l(i\boldsymbol{\xi}) \neq 0$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$ (recall that $c_l(\partial_{\mathbf{x}})e^{i\mathbf{x}\cdot\boldsymbol{\xi}} = c_l(i\boldsymbol{\xi})e^{i\mathbf{x}\cdot\boldsymbol{\xi}}$); in fact, we will assume that $c_l(i\boldsymbol{\xi})$ is a non-zero constant, normalized to be 1, so that

$$P = \partial_t^l + \sum_{j=0}^{l-1} c_j(\partial_{\mathbf{x}})\partial_t^j.$$

The Cauchy problem for P with Σ as the hypersurface for initial data is formulated as

$$\begin{cases} Pu(\mathbf{x},t) = f(\mathbf{x},t) & \text{for } (\mathbf{x},t) \in \mathbb{R}^n \times (t_1,t_2), \\ \partial_t^j u(\mathbf{x},t_0) = g_j(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^n,\ 0 \le j \le l-1, \end{cases} \tag{7.1}$$

where $t_0 \in [t_1, t_2]$, and $g_j(\mathbf{x})$ are prescribed functions with sufficient regularity (and compact support or fast enough decay at spatial infinity, if needed); we will also assume $f(\mathbf{x},t)$ to have sufficient regularity, and compact support or fast enough decay at spatial infinity, if needed. We will often take $t_0 = 0$ and $[t_1, t_2]$ to be $[0, T]$, or $[-T, 0]$, or $[-T, T]$ for some $T > 0$.

Example 7.1. The Cauchy problem for $\partial_t^2 - c^2\Delta_{\mathbf{x}}$ with $\{t = 0\}$ as initial surface would be the same as (3.3); $\{t = 0\}$ is non-characteristic with respect to $\partial_t^2 - c^2\Delta_{\mathbf{x}}$.

$\{t = 0\}$ as an initial surface for the Cauchy problem of $\partial_t - \Delta_{\mathbf{x}}$ is characteristic, as the highest order differentiation in t is 1, while this operator is second order.

The Cauchy problem for $\partial_x^2 + \partial_y^2$ with $\{y = 0\}$ as an initial surface is

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 & x \in \mathbb{R}, y \in [-T,T], \\ u(x,0) = g(x) & x \in \mathbb{R}, \\ u_y(x,0) = h(x) & x \in \mathbb{R}. \end{cases}$$

$\{y = 0\}$ is non-characteristic for this Cauchy problem. Note the difference between this Cauchy problem and the boundary value problem

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 & x \in \mathbb{R}, y \in (0,\infty), \\ u(x,0) = g(x) & x \in \mathbb{R}, \end{cases}$$

which has a bounded continuous solution for any bounded continuous g on $\mathbb{R}$. This indicates that the Cauchy problem for the Laplace equation is not well-posed, as there would be no freedom to prescribe $u_y(x,0)$.

We now formulate a more definite form of the well-posedness for (7.1). Recall that for the wave equation—when $P = \partial_t^2 - c^2\partial_x^2$, we proved that for any $g_j \in C^\infty(\mathbb{R})$, $j = 0, 1$, $f \in C^\infty(\mathbb{R}^2)$, there is a unique solution $u(x,t)$ defined on $\mathbb{R} \times [t_1, t_2]$, which depends on g_j's and f in a continuous fashion. In this case we can keep track of the differentiability of u in relation to that of f and g_j's, but we decide not to keep an exact account here, as we would like to leave some flexibility when dealing with more general operators.

Continuity in $C^\infty(\mathbb{R} \times [t_1, t_2])$ or $C^\infty(\mathbb{R})$ can not be measured in terms of a single norm. However, either of the above spaces carries a countable number of natural norms of the form $\max_K\{|\partial_x^k\partial_t^j u(x,t)| : k \le m, j \le l\}$ or $\max_K\{|\partial_x^k g(x)| : k \le m\}$, where K is compact, and these can be used to define a metric on these spaces so that they are complete metric spaces.

Furthermore, in the case of wave equation, we know that for any compact set K of $\mathbb{R} \times [t_1, t_2]$, there is a compact set V such that the value of u on K is determined by that of f on V and of g_j's on $V \cap (\mathbb{R}^n \times \{t_0\})$. In addition, using the metric on $C^\infty(\mathbb{R}^n \times [t_1, t_2])$ the continuous dependence of solution on data in the well-posedness of (7.1) can be formulated as follows:

> For any compact $K \subset \mathbb{R}^n \times [t_1, t_2]$, for any $l_1, m_1 \in \mathbb{Z}_{\ge 0}$, there exist a compact set $V \subset \mathbb{R}^n \times [t_1, t_2]$, $l_2, m_2, m_3 \in \mathbb{Z}_{\ge 0}$, and a constant $C > 0$ depending on P, K, l_1, and m_1 such that for any $u \in C^\infty(\mathbb{R}^n \times [t_1, t_2])$, we have

$$\begin{aligned}&\max_K\{|\partial_{\mathbf{x}}^k\partial_t^j u(\mathbf{x},t)| : k \le m_1, j \le l_1\}\\ \le&C\left[\max_V\{|\partial_{\mathbf{x}}^k\partial_t^j Pu(\mathbf{x},t)| : k \le m_2, j \le l_2\} + \max_{V\cap\Sigma}\{|\partial_{\mathbf{x}}^k\partial_t^j u(\mathbf{x},t_0)| : k \le m_3, j \le l-1\}\right]\end{aligned} \tag{7.2}$$

(7.2) says that the map

$$u \mapsto (Pu, u(\mathbf{x},t_0), \partial_t u(\mathbf{x},t_0), \cdots, \partial_t^{l-1}u(\mathbf{x},t_0))$$

as a map from $C^\infty(\mathbb{R}^n \times [t_1, t_2]) \mapsto C^\infty(\mathbb{R}^n \times [t_1, t_2]) \times \Pi_{j=0}^{l-1} C^\infty(\mathbb{R}^n)$

has a continuous inverse.

For the Cauchy problem of the heat equation, we learned that in order to have a well-posed Cauchy problem, some growth restriction on g, f, and u is needed; in addition, solutions to the heat equation have infinite speed of propagation, so using (7.2) as well-posedness for the Cauchy problem for equations such as the heat equation would not be appropriate, as (7.2) would imply a finite speed of propagation. One way to deal with these issues is to work in the space $\mathscr{S}(\mathbb{R}^n)$ of Schwartz functions. which are defined as smooth functions in $\mathbb{R}^n$ For simplicity, we will not provide details of the well-posedness formulation in this framework. We will instead use an L^2 based framework.

We now reformulate the well-posedness of (7.1) on $[0,T]$ in the L^2 framework as follows. Define $X_k = \{u \in C([0,T], H^k(\mathbb{R}^n)) : \partial_t^j u \in C([0,T], H^{k-j}(\mathbb{R}^n)), 1 \le j \le k\}$, with $||u||_{X_k} := \sum_{j=0}^k \max_{0\le t\le T} ||\partial_t^j u(\cdot,t)||_{H^{k-j}(\mathbb{R}^n)}$.

> (7.1) is well-posed on $[0,T]$, if there exist $k \ge m, k_1, k_2 \in \mathbb{Z}_{\ge 0}$ such that for any $f \in X_{k_1}$, $g_j \in H^{k_2-j}(\mathbb{R}^n)$, there is a unique solution $u \in X_k$ to (7.1). Furthermore, there exists $C > 0$ depending only on P, T, and on k, k_1, k_2, such that
>
> $$||u||_{X_k} \le C\left[||f||_{X_{k_1}} + \sum_{j=0}^{l-1} ||g_j||_{H^{k_2-j}(\mathbb{R}^n)}\right], \tag{7.3}$$
>
> for all $f \in X_{k_1}$, $g_j \in H^{k_2-j}(\mathbb{R}^n)$.

This is a first attempt to formulate the well-posedness in the L^2 framework, as we are not being careful in differentiating the possibly different orders of differentiability in t and $\mathbf{x}$, such as appeared in the heat equation; but that can be accommodated easily by modifying the above formulation accordingly.

Formulation (7.2) is easier to work with. We will first use (7.2) to derive a necessary condition for the well-posedness of (7.1). We take $t_0 = 0$, $T > 0$, K to be the set of a single point $\{(0,T)\} \subset \mathbb{R}^n \times [0,T]$, $l_1 = m_1 = 0$, and apply it to $u(\mathbf{x},t) = e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$, where $\boldsymbol{\xi} \in \mathbb{R}^n$, and τ is chosen so that $Pe^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)} = \sum_{j=0}^l c_j(i\boldsymbol{\xi})(i\tau)^j = 0$, then, since $P[e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}] = 0$, the condition (7.2) becomes

$$\begin{aligned} e^{-\Im\mathrm{m}(\tau)T} = |u(0,T)| \le & C\left[\max_{V\cap\Sigma}\{|\partial_{\mathbf{x}}^\alpha \partial_t^j [e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}]| : |\alpha| \le m_3, j \le l-1\}\right] \\ \le & C\max\{|\boldsymbol{\xi}|^k |\tau|^j : k \le m_3, j \le l-1\}. \end{aligned} \tag{7.4}$$

Since τ is determined through $\sum_{j=0}^l c_j(i\boldsymbol{\xi})(i\tau)^j = 0$, and we have normalized $c_l = 1$, we see that there exists $C' > 0$ depending on P such that any root τ of this equation satisfies

$$|\tau| \le C'(1+|\boldsymbol{\xi}|)^{m-l+1}. \tag{7.5}$$

(7.5) follows from

$$\begin{aligned}|\tau|^l &\le \sum_{j=0}^{l-1} |c_j(i\boldsymbol{\xi})||\tau|^j \\ &\le \sum_{j=0}^{l-1} C''(1+|\boldsymbol{\xi}|)^{m-j}|\tau|^j \\ &\le \sum_{j=0}^{l-1} C''\left[\epsilon|\tau|^l + \epsilon^{-\frac{j}{l-j}}(1+|\boldsymbol{\xi}|)^{\frac{m-j}{l-j}l}\right],\end{aligned}$$

and by choosing $\epsilon > 0$ sufficiently small, we obtain the inequality

$$|\tau|^l \le 2C'' \sum_{j=0}^{l-1} \epsilon^{-\frac{j}{l-j}}(1+|\boldsymbol{\xi}|)^{\frac{m-j}{l-j}l} \le C'(1+|\boldsymbol{\xi}|)^{(m-l+1)l},$$

for some $C' > 0$, from which (7.5) follows. (7.4) and (7.5) then imply that any root τ of $\sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$ would satisfy

$$e^{-\mathfrak{Im}(\tau)T} \le C''(1+|\boldsymbol{\xi}|)^b \quad \text{for some } C'' > 0,\ b > 0, \text{ and all } \boldsymbol{\xi} \in \mathbb{R}^n. \tag{7.6}$$

(7.6) is a condition on the roots of the polynomial associated with the operator P, and is a necessary condition for the well-posedness of (7.1) on $\mathbb{R}^n \times [0, T]$. It is referred to as the **Hadamard-Petrowsky** well-posedness condition; and in some sources it is also referred to as the **Gårding**'s condition.

(7.6) is equivalent to

$$-\mathfrak{Im}(\tau) \le b' \ln(1+|\boldsymbol{\xi}|) + c \quad \text{for some } b' > 0, c, \text{ and all } \boldsymbol{\xi} \in \mathbb{R}^n.$$

Based on a theorem of Seidenberg-Taski on the roots of polynomials, the above condition is equivalent to

$$-\mathfrak{Im}(\tau) \le M \quad \text{for some } M, \text{ and all } \boldsymbol{\xi} \in \mathbb{R}^n. \tag{7.7}$$

If we use the formulation (7.3), we can still arrive at (7.6) and (7.7) as follows. We would take $u(\mathbf{x}, t) = e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}\eta(\mathbf{x})$, where $\boldsymbol{\xi} \in \mathbb{R}^n$, τ is still chosen so that $Pe^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)} = \sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$, $\eta \in C_c^\infty(\mathbb{R}^n)$ is a cut-off function such that $\eta \equiv 1$ in B_r and support$(u) \subset B_{2r}$ for r large, with $|\nabla^k \eta| \le C_k/R^k$ for some $C_k > 0$ independent of R. Note that

$$\max_{0\le t\le T} ||u(\cdot, t)||_{H^k(\mathbb{R}^n)} \ge ||u(\cdot, T)||_{L^2(\mathbb{R}^n)} \ge e^{-\mathfrak{Im}(\tau)T}|B_R|^{1/2},$$

while

$$P[e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}\eta(\mathbf{x})] = \eta(\mathbf{x})P[e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}] + Q(e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}, \partial_{\mathbf{x}}\eta) = Q(e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}, \partial_{\mathbf{x}}\eta)$$

where $Q(e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}, \partial_{\mathbf{x}}\eta)$ represents certain differential operator acting on $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ and $\partial_{\mathbf{x}}\eta$, is bilinear in $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ and $\partial_{\mathbf{x}}\eta$, and contains derivatives of η of order 1 or higher, so for $0 \le t \le T$,

$$\begin{aligned} &||\partial_t^j Pu(\cdot, t)||_{H^{k_1-j}(\mathbb{R}^n)} \\ \le &||\partial_t^j Q(e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}, \partial_{\mathbf{x}}\eta)||_{H^{k_1-j}(\mathbb{R}^n)} \\ \le &C(1+|\boldsymbol{\xi}|)^{k_1-j}|\tau|^j e^{-\mathfrak{Im}(\tau)T}|B_{2R}(0)\setminus B_R(0)|^{1/2}/R. \end{aligned}$$

Similarly

$$||\partial_t^j \left(e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}\eta(\mathbf{x})\right)\big|_{t=0}||_{H^{k_2-j}(\mathbb{R}^n)} \le C(1+|\boldsymbol{\xi}|)^{k_2-j}|\tau|^j|B_{2R}(0)|^{1/2}.$$

Putting these in (7.3), and noting that $|B_R(0)|$, $|B_{2R}(0)\setminus B_R(0)|$, and $|B_{2R}(0)|$ are comparable — all a constant multiple of R^n, so after dividing through by R^n and noting the additional R^{-1} factor in front of the first term on the right hand side, by sending $R\to\infty$, we arrive at (7.4) again.

The main theorem of this chapter is the following

Theorem 7.1. *Let $P = \sum_{j=0}^{l} c_j(\partial_{\mathbf{x}})\partial_t^j$ be a constant coefficient differential operator of order m such that $c_l(i\boldsymbol{\xi}) \ne 0$ for all $\boldsymbol{\xi}\in\mathbb{R}^n$. Then (7.1) is well-posed on $[0,T]$ for some $T>0$ iff (7.7) is satisfied.*

(7.1) is well-posed on $[-T,0]$ for some $T>0$ iff

$$\text{there exists some } M>0 \text{ such that for all } \boldsymbol{\xi}\in\mathbb{R}^n, \text{ and all roots } \tau \text{ of } \sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0, \quad \mathfrak{Im}(\tau)\le M. \tag{7.8}$$

As a consequence, (7.1) is well-posed on $[-T,T]$ for some $T>0$ iff there exists some $M>0$ such that for all $\boldsymbol{\xi}\in\mathbb{R}^n$, and all roots τ of $\sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$, $|\mathfrak{Im}(\tau)|\le M$.

Definition. When $\{t=t_0\}$ is non-characteristic with respect to P, and both (7.7) and (7.8) are satisfied, we say that P is **hyperbolic** with respect to t.

Remark 7.1. If $\{t=t_0\}$ is non-characteristic with respect to P, then (7.7) and (7.8) imply each other. This is because the sum of the roots τ is a degree 1 polynomial in ξ by Vieta's theorem for roots of polynomials. Either (7.7) or (7.8) would imply that the imaginary part of this degree 1 polynomial must be a constant.

In checking (7.7) or (7.8), what matters is the behavior of $\mathfrak{Im}(\tau)$ when $|\boldsymbol{\xi}|$ is large, or more importantly, whether *all* solutions of the form $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ for $\boldsymbol{\xi} \in \mathbb{R}^n$ have bounded or at most polynomial growth in $|\boldsymbol{\xi}|$ at some relevant $t \neq 0$. If this is the case, we can at least choose initial data to be in the Schwartz class so that $\widehat{g}_j(\boldsymbol{\xi})$ decays faster than any negative power of $|\boldsymbol{\xi}|$ as $\boldsymbol{\xi} \to \infty$, and construct solutions to the homogeneous case of (7.1) in the form of linear combination of $\int_{\mathbb{R}^n} a_j(\boldsymbol{\xi}) e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau_j(\boldsymbol{\xi})t)}\, d\boldsymbol{\xi}$.

Counting multiplicity, $\sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$ has l solutions for each $\boldsymbol{\xi} \in \mathbb{R}^n$; and we need *all* l solutions to satisfy (7.7) or (7.8) so as to use all of them in the construction of solutions to (7.1) to satisfy the l initial conditions in (7.1). For boundary value problems on the half space or $\mathbb{R}^n \times [t_1, t_2]$, such as the one for the Laplace equation, we only need *some* of the roots of $\sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$ to behave in a favorable way, such as in (7.7), so as to be used in constructing solutions. This is a heuristic explanation for the difference between the well-posedness for a Cauchy problem and that for a boundary value problem—we will not have space to discuss general boundary value problems and their well-posedness—the most studied boundary value problems are elliptic boundary value problems, and the boundary conditions that correspond to well-posedness are referred to as the Shapiro-Lopatinski conditions; we just mention that, although the technical results on these boundary value problems may look very different, their analysis is rooted in similar considerations as sketched here using Fourier analysis and information on the roots to $\sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$ to construct solutions to the boundary value problem with the prescribed boundary conditions.

When $\{t = t_0\}$ is non-characteristic with respect to P, the behavior of the roots to $\sum_{j=0}^{l} c_j(i\boldsymbol{\xi})(i\tau)^j = 0$ is reflected in the behavior of the roots to the **principal part** $P_m := \sum_{j+|\alpha|=m} c_{j\alpha}\partial_{\mathbf{x}}^{\alpha}\partial_t^j$ of P.

Proposition 7.2. *Suppose that $\{t = t_0\}$ is non-characteristic with respect to P, and* (7.7) *or* (7.8) *are satisfied, then all roots to $\sum_{j+|\alpha|=m} c_{j\alpha}(i\boldsymbol{\xi})^{\alpha}(i\tau)^j = 0$ must be real.*

Definition. If $\{t = t_0\}$ is non-characteristic with respect to P, and for any $\boldsymbol{\xi} \neq 0 \in \mathbb{R}^n$, the roots τ to the principal part of P, $P_m(i\boldsymbol{\xi}, i\tau) = 0$ are real and <u>distinct</u>, then we say P is **strictly hyperbolic** with respect to $\{t = t_0\}$.

Proposition 7.3. *Suppose that P is strictly hyperbolic with respect to $\{t = t_0\}$, then P satisfies* (7.7) *and* (7.8).

Example 7.2. For $P_1 = \partial_t^2 - c^2\Delta_{\mathbf{x}}$, we need to examine the growth in $\boldsymbol{\xi}$ of solutions of the form $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ for $t \neq 0$, which are determined by the roots to $(i\tau)^2 + c^2|\boldsymbol{\xi}|^2 = 0$. The roots are $\tau = \pm c|\boldsymbol{\xi}| \in \mathbb{R}$, so $\mathfrak{Im}(\tau) = 0$, $|e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}| = 1$, and both (7.7) and (7.8) are satisfied. By Theorem 7.1, the Cauchy problem for $P_1 = \partial_t^2 - c^2\Delta_{\mathbf{x}}$ with $t = t_0$ as hypersurface for initial data is well-posed for both forward and backward evolution. P_1 is strictly hyperbolic with respect to $\{t = t_0\}$.

For $P_2 = \partial_t - \Delta_{\mathbf{x}}$, we need to examine the growth in $\boldsymbol{\xi} \in \mathbb{R}^n$ of solutions of the form $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ for $t \neq 0$, which are determined by the roots to $i\tau + |\boldsymbol{\xi}|^2 = 0$. $\tau = i|\boldsymbol{\xi}|^2$. (7.7) is satisfied, but (7.8) is not satisfied: $|e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}| = e^{-|\boldsymbol{\xi}|^2 t}$, so the Cauchy problem for $P_2 = \partial_t - \Delta_{\mathbf{x}}$ with $t = t_0$ as hypersurface for initial data is well-posed for forward evolution, but not well-posed for backward evolution. However, if we choose $\{(\mathbf{x}, t) : \mathbf{x} = 0\}$ as the surface for initial data (assuming $\mathbf{x} \in \mathbb{R}$), then we need to examine the growth in $\tau \in \mathbb{R}$ of solutions of the form $e^{i(x\xi+\tau t)}$ for $\tau \in \mathbb{R}$: we still have $i\tau + \xi^2 = 0$, from which we have $\xi = \pm\sqrt{-i\tau}$, and $|e^{i(x\xi+\tau t)}| = e^{\pm x\sqrt{|\tau|/2}}$, one of which would grow exponentially with $|\tau|$ for $x \neq 0$. So the Cauchy problem for P_2 with $\{(x, t) : x = 0\}$ as the surface for initial data is not well-posed.

For $P_3 = \partial_t^2 + \Delta_{\mathbf{x}}$ (the Laplace operator in $\mathbb{R}^{n+1}$), we need to examine the the growth in $\boldsymbol{\xi} \in \mathbb{R}^n$ of solutions of the form $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ for $t \neq 0$, which are determined by the roots to $(i\tau)^2 - |\boldsymbol{\xi}|^2 = 0$. $\tau = \pm i|\boldsymbol{\xi}|$. So neither (7.7) nor (7.8) is satisfied: $|e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}| = e^{\pm|\boldsymbol{\xi}|t}$; at any $t \neq 0$, one solution grows exponentially in $|\boldsymbol{\xi}|$. The Cauchy problem for $P_3 = \partial_t^2 + \Delta_{\mathbf{x}}$ with $t = t_0$ as hypersurface for initial data is not well-posed for either forward or backward evolution. However, we can use the well behaved root to establish the well-posedness of a boundary value problem (prescribing either $u(\mathbf{x}, 0)$ or $u_t(\mathbf{x}, 0)$) on $\mathbb{R}^n \times \mathbb{R}^+$ for P_3.

For $P_4 = i\partial_t - \Delta_{\mathbf{x}}$, we need to examine the growth in $\boldsymbol{\xi} \in \mathbb{R}^n$ of solutions of the form $e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}$ for $t \neq 0$, which are determined by the roots to $-\tau + |\boldsymbol{\xi}|^2 = 0$. $\tau = |\boldsymbol{\xi}|^2$. Both (7.7) and (7.8) are satisfied: $|e^{i(\mathbf{x}\cdot\boldsymbol{\xi}+\tau t)}| = 1$. By Theorem 7.1, the Cauchy problem for $P_4 = i\partial_t - \Delta_{\mathbf{x}}$ with $t = t_0$ as hypersurface for initial data is well-posed for both forward and backward evolution.

Consider the following perturbation of P_1: $P_5 = \partial_t^2 - c^2\partial_x^2 + a\partial_t + b\partial_x + d$ for some constants a, b, and d. We need to examine the growth in $\xi \in \mathbb{R}$ of solutions of the form $e^{i(x\xi+\tau t)}$ for $t \neq 0$, which are determined by the roots to $(i\tau)^2 + c^2|\xi|^2 + ia\tau + ib\xi + d = 0$. The roots are given by

$$\tau = \frac{ia}{2} \pm \sqrt{c^2|\xi|^2 + ib\xi + d - \frac{a^2}{4}}.$$

What matters in verifying (7.7) or (7.8) is the behavior of $\mathfrak{Im}(\tau)$ when $|\xi|$ is large.

But when $|\xi| \to \infty$,

$$
\begin{aligned}
\tau &= \frac{ia}{2} \pm c|\xi|\sqrt{1 + \frac{ib\xi + d - \frac{a^2}{4}}{c^2|\xi|^2}} \\
&= \frac{ia}{2} \pm c|\xi| \left[1 + \frac{ib\xi + d - \frac{a^2}{4}}{2c^2|\xi|^2} + o(\frac{1}{|\xi|}) \right] \\
&= \frac{ia}{2} \pm \left[c|\xi| + \frac{ib\xi + d - \frac{a^2}{4}}{2c|\xi|} + o(1) \right].
\end{aligned}
$$

From this it is clear that $\mathfrak{Im}(\tau)$ remains bounded for all $\xi \in \mathbb{R}$, so the Cauchy problem for P_5 with $t = t_0$ as hypersurface for initial data is well-posed for both forward and backward evolution.

Consider the following perturbation of P_4: $P_6 = i\partial_t - \partial_x^2 + b\partial_x$. We need to examine the roots to $-\tau + |\xi|^2 + ib\xi = 0$. If $b \neq 0$ is real, we see that $\mathfrak{Im}(\tau) = b\xi$, which does not satisfy either (7.7) or (7.8). Thus the Cauchy problem for P_6 with $t = t_0$ as hypersurface for initial data is not well-posed for either forward or backward evolution.

This difference of behavior of the well-posedness under lower order term perturbations is due to the Cauchy problem for P_5 with $t = t_0$ as hypersurface for initial data is *non-characteristic* and strictly hyperbolic, while that for P_6 is characteristic.

Proposition 7.2 is proved using the Implicit Function Theorem. Heuristically, if $P_m(i\boldsymbol{\xi}, i\tau) = 0$ had a root τ_0 with $\mathfrak{Im}(\tau_0) < 0$ for some $\boldsymbol{\xi}_0$. Then by homogeneity, $P_m(i\lambda\boldsymbol{\xi}_0, i\lambda\tau_0) = 0$ for all $\lambda \in \mathbb{R}$. Note that for large $\lambda \in \mathbb{R}^+$, $\mathfrak{Im}(\lambda\tau_0) = \lambda\mathfrak{Im}(\tau_0)$ would be a large negative number. One can use the Implicit Function Theorem to prove that for sufficiently large $\lambda \in \mathbb{R}^+$, one would find a root τ to $P(i\lambda\boldsymbol{\xi}, i\lambda\tau) = 0$ near $(i\lambda\boldsymbol{\xi}_0, i\lambda\tau_0)$, which would then violate (7.7). Proposition 7.3 is also proved using the Implicit Function Theorem.

Exercises

Exercise 7.1.1. For each of the following operators, determine whether the Hadamard-Petrowsky condition for forward or backward evolution with respect to t holds.

(i). $\partial_t + (\Delta_x)^2$;

(ii). $\partial_t - (\Delta_x)^2$;

(iii). $\partial_t^2 + (\Delta_x)^2$;

(iv). $\partial_t^2 - (\Delta_x)^2$;

(v). $\partial_t - (\partial_x)^m$;

(vi). $\partial_t^2 - (\partial_x)^2 + \partial_x$;

(vii). $\partial_t^2 + (\partial_x)^4 + (\partial_x)^3$.

Exercise 7.1.2. Determine whether the Hadamard-Petrowsky condition for forward or backward evolution with respect to y holds for the operator $\partial_y - i\partial_x$. Furthermore, determine necessary and sufficient conditions on $u_0(x)$ for the existence of a C^1 solution of

$$\begin{cases} (\partial_y - i\partial_x)\, u(x,y) = 0 & x^2 + y^2 < 1 \\ u(x,0) = u_0(x) & |x| < 1 \end{cases}$$

7.2 Outline of Proof for Sufficiency of the Hadamard-Petrowsky Condition

We will first do two reductions. First, by Duhamel's principle, it suffices to establish the solvability of (7.1) for $f(\mathbf{x},t) \equiv 0$. Second, we only need to establish the solvability of (7.1) for $f(\mathbf{x},t) \equiv 0$, $g_j \equiv 0$ for $j = 0, \cdots, l-2$, and $g_{l-1}(\mathbf{x}) = g(\mathbf{x})$, a prescribed function with sufficient regularity and decay. The second reduction is based on the following. Let $u_g(\mathbf{x},t)$ stand for the solution of

$$\begin{cases} Pu(\mathbf{x},t) = 0 & \text{for } (\mathbf{x},t) \in \mathbb{R}^n \times (0,T), \\ \partial_t^j u(\mathbf{x},0) = 0 & \text{for } \mathbf{x} \in \mathbb{R}^n,\ 0 \le j \le l-2, \\ \partial_t^{l-1} u(\mathbf{x},0) = g(\mathbf{x}) & \text{for } \mathbf{x} \in \mathbb{R}^n. \end{cases} \tag{7.9}$$

Then note that $v_{l-2}(\mathbf{x},t) = [\partial_t + c_{l-1}(\partial_\mathbf{x})]\, u_{g_{l-2}}$ satisfies

$$\begin{cases} Pv_{l-2}(\mathbf{x},t) = [\partial_t + c_{l-1}(\partial_\mathbf{x})]\, Pu_{g_{l-2}}(\mathbf{x},t) = 0, \\ \partial_t^j v_{l-2}(\mathbf{x},0) = \left[\partial_t^{j+1} + c_{l-1}(\partial_\mathbf{x})\partial_t^j\right] u_{g_{l-2}}(\mathbf{x},0) = 0 \text{ if } j \le l-3, \\ \partial_t^{l-2} v_{l-2}(\mathbf{x},0) = \left[\partial_t^{l-1} + c_{l-1}(\partial_\mathbf{x})\partial_t^{l-2}\right] u_{g_{l-2}}(\mathbf{x},0) = g_{l-2}(\mathbf{x}), \\ \partial_t^{l-1} v_{l-2}(\mathbf{x},0) = \left[\partial_t^{l} + c_{l-1}(\partial_\mathbf{x})\partial_t^{l-1}\right] u_{g_{l-2}}(\mathbf{x},0) = -\displaystyle\sum_{j=0}^{l-2} c_j(\partial_\mathbf{x})\partial_t^j u_{g_{l-2}}(\mathbf{x},0) = 0. \end{cases}$$

By similar reasoning, we know that

$$u_{g_{l-1}} + [\partial_t + c_{l-1}(\partial_\mathbf{x})]\, u_{g_{l-2}} + \left[\partial_t^2 + c_{l-1}(\partial_\mathbf{x})\partial_t + c_{l-2}(\partial_\mathbf{x})\right] u_{g_{l-3}} + \cdots + \left[\partial_t^{l-1} + c_{l-1}(\partial_\mathbf{x})\partial_t^{l-2} + \cdots + c_1(\partial_\mathbf{x})\right] u_{g_0}$$

would provide a solution of (7.1) with $f \equiv 0$. Thus we can focus on constructing the solution u_g to (7.9).

u_g will be constructed using Fourier's method

$$u_g(\mathbf{x},t) = \int_{\mathbb{R}^n} e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\,\widehat{u}(\boldsymbol{\xi},t)\,d\boldsymbol{\xi},$$

where $\widehat{u}(\boldsymbol{\xi},t)$ needs to satisfy

$$\begin{cases} P(\partial_\mathbf{x},\partial_t)\left[e^{i\mathbf{x}\cdot\boldsymbol{\xi}}\widehat{u}(\boldsymbol{\xi},t)\right] = e^{i\mathbf{x}\cdot\boldsymbol{\xi}} P(i\boldsymbol{\xi},\partial_t)\widehat{u}(\boldsymbol{\xi},t) = 0, \\ \partial_t^j \widehat{u}(\boldsymbol{\xi},0) = 0, \text{ for } 0 \le j \le l-2, \\ \partial_t^{l-1}\widehat{u}(\boldsymbol{\xi},0) = \widehat{g}(\boldsymbol{\xi}) \end{cases}$$

This is an IVP for an l-th order linear ODE. Solutions to $P(i\boldsymbol{\xi}, \partial_t)\widehat{u}(\boldsymbol{\xi}, t) = 0$ can be constructed in the form $\sum c_j e^{i\tau_j t}$, where τ_j are roots of $P(i\boldsymbol{\xi}, i\tau) = 0$. When $P(i\boldsymbol{\xi}, i\tau) = 0$ has roots of multiplicity great than 1, additional solutions of the form $t^k e^{i\tau_j t}$ need to be included.

The following formula provides a way to solve the above IVP without having to directly address the complications caused by the possibility of higher multiplicity roots:

$$\widehat{u}(\boldsymbol{\xi}, t) = \frac{\widehat{g}(\boldsymbol{\xi})}{2\pi} \int_\Gamma \frac{e^{i\lambda t}}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda, \tag{7.10}$$

where Γ is a closed contour in the complex plane consisting of boundary of the union of unit disks with each root of $P(i\boldsymbol{\xi}, i\tau) = 0$ as center, with counterclockwise orientation.

We verify that

$$P(i\boldsymbol{\xi}, \partial_t)\widehat{u}(\boldsymbol{\xi}, t) = \frac{\widehat{g}(\boldsymbol{\xi})}{2\pi} \int_\Gamma \frac{e^{i\lambda t} P(i\boldsymbol{\xi}, i\lambda)}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda = 0,$$

and

$$\partial_t^j \widehat{u}(\boldsymbol{\xi}, 0) = \frac{\widehat{g}(\boldsymbol{\xi})}{2\pi} \int_\Gamma \frac{(i\lambda)^j}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda.$$

Γ has enclosed all the roots of $P(i\boldsymbol{\xi}, i\tau) = 0$, so we can deform Γ to a large circle $|\lambda| = R$ in evaluating the above integral. When $0 \le j \le l-2$, $\left|\frac{(i\lambda)^j}{P(i\boldsymbol{\xi}, i\lambda)}\right| = O(R^{-2})$ when $|\lambda| = R >> 1$, thus

$$\partial_t^j \widehat{u}(\boldsymbol{\xi}, 0) = \frac{\widehat{g}(\boldsymbol{\xi})}{2\pi} \int_\Gamma \frac{(i\lambda)^j}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda = 0 \quad \text{if } 0 \le j \le l-2.$$

When $j = l-1$,

$$\left|\frac{(i\lambda)^{l-1}}{P(i\boldsymbol{\xi}, i\lambda)} - \frac{1}{i\lambda}\right| = O(\frac{1}{R^2}) \text{ when } |\lambda| = R >> 1,$$

thus

$$\partial_t^{l-1} \widehat{u}(\boldsymbol{\xi}, 0) = \frac{\widehat{g}(\boldsymbol{\xi})}{2\pi} \int_\Gamma \frac{(i\lambda)^{l-1}}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda = \widehat{g}(\boldsymbol{\xi}).$$

Finally, under (7.6), we will use (7.10) to estimate that $|\widehat{u}(\boldsymbol{\xi}, t)|$ has rapid decay in $|\boldsymbol{\xi}|$, for $0 \le t \le T$, if we choose g such that $|\widehat{g}(\boldsymbol{\xi})|$ has sufficiently rapid decay in $|\boldsymbol{\xi}|$.

Using (7.6), we see that $|e^{i\tau t}| = e^{-\mathfrak{Im}\tau t} \le e^{MT}$ for τ being a root of $P(i\boldsymbol{\xi}, i\tau) = 0$ and $0 \le t \le T$. In fact, even when $\lambda \in \Gamma$, we know that $-\mathfrak{Im}\lambda \le M+1$, so we still have $|e^{i\lambda t}| \le e^{(M+1)T}$ for $0 \le t \le T$. In addition, using the factorization

$P(i\boldsymbol{\xi}, i\lambda) = \Pi_{j=1}^{l}(i\lambda - i\tau_j(\boldsymbol{\xi}))$, we see that $|P(i\boldsymbol{\xi}, i\lambda)| \geq 1$ when $\lambda \in \Gamma$. Using (7.5) and the observation that the total length of Γ is at most $2l\pi$, it now follows that

$$\begin{aligned}&\left|\partial_t^j \widehat{u}(\boldsymbol{\xi}, t)\right| \\ \leq &\frac{|\widehat{g}(\boldsymbol{\xi})|}{2\pi} \int_{\Gamma} \left|\frac{(i\lambda)^j e^{i\lambda t}}{P(i\boldsymbol{\xi}, i\lambda)}\right| |d\lambda| \\ \leq &\frac{|\widehat{g}(\boldsymbol{\xi})|}{2\pi} \int_{\Gamma} \left||\lambda|^j e^{(M+1)T}\right| |d\lambda| \\ \leq &C'|\widehat{g}(\boldsymbol{\xi})|(1+|\boldsymbol{\xi}|)^{j(m-l+1)} e^{(M+1)T}.\end{aligned} \tag{7.11}$$

Based on (7.10) and (7.11), we have

$$\partial_{\mathbf{x}}^{\beta} \partial_t^j u_g(\mathbf{x}, t) = \int_{\mathbb{R}^n} \partial_t^j \widehat{u}(\boldsymbol{\xi}, t)(i\boldsymbol{\xi})^{\beta} e^{i\mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi},$$

provided $|\partial_t^j \widehat{u}(\boldsymbol{\xi}, t)||\boldsymbol{\xi}|^{|\beta|} \in L^1(\mathbb{R}^n)$; and a sufficient condition for $|\partial_t^j \widehat{u}(\boldsymbol{\xi}, t)||\boldsymbol{\xi}|^{|\beta|} \in L^1(\mathbb{R}^n)$ is that

$$|\partial_t^j \widehat{u}(\boldsymbol{\xi}, t)||\boldsymbol{\xi}|^{|\beta|}| \leq C'|\widehat{g}(\boldsymbol{\xi})|(1+|\boldsymbol{\xi}|)^{|\beta|+j(m-l+1)} e^{(M+1)T} \leq C(1+|\boldsymbol{\xi}|)^{-n-1} \quad \text{for all } \boldsymbol{\xi} \in \mathbb{R}^n.$$

This can be satisfied if $|\widehat{g}(\boldsymbol{\xi})| \leq C(1+|\boldsymbol{\xi}|)^{-n-1-|\beta|-j(m-l+1)}$. Since it's natural to ask that $u \in C_{\mathbf{x},t}^{m,l}$, we will take $|\beta| = m$ and $j = l$ in the above, then the condition becomes $|\widehat{g}(\boldsymbol{\xi})| \leq C(1+|\boldsymbol{\xi}|)^{-n-1-m-l(m-l+1)}$. Using the relation that $i^{|\beta|}\boldsymbol{\xi}^{\beta}\widehat{g}(\boldsymbol{\xi}) = \widehat{\partial_{\mathbf{x}}^{\beta} g}(\boldsymbol{\xi})$, we see that this condition is satisfied if $g \in C_c^{n+1+m+l(m-l+1)}(\mathbb{R}^n)$.

The above argument shows that if $g \in C_c^{n+1+m+l(m-l+1)}(\mathbb{R}^n)$, then $u_g(\mathbf{x}, t) = \int_{\mathbb{R}^n} \widehat{u}(\boldsymbol{\xi}, t) e^{i\mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$ provides a $C_{\mathbf{x},t}^{m,l}(\mathbb{R}^n \times [0, T])$ solution. To finish our proof for Theorem 7.1, as formulated there, we estimate $||\partial_t^j u_g(\mathbf{x}, t)||_{H^{k-j}(\mathbb{R}^n)}$: for $|\beta| \leq k - j$, by using Plancherel theorem

$$\begin{aligned}||\partial_t^j u_g(\mathbf{x}, t)||_{H^{k-j}(\mathbb{R}^n)} \leq &||\partial_t^j \widehat{u}(\boldsymbol{\xi}, t)(i\boldsymbol{\xi})^{\beta}||_{L^2(\mathbb{R}^n)} \\ \leq &C|||\widehat{g}(\boldsymbol{\xi})|(1+|\boldsymbol{\xi}|)^{j(m-l+1)+|\beta|}||_{L^2(\mathbb{R}^n)} e^{(M+1)T} \\ \leq &Ce^{(M+1)T}||g||_{H^{j(m-l)+k}(\mathbb{R}^n)}.\end{aligned}$$

Exercises

Exercise 7.2.1. Provide a detailed justification for $\int_{\Gamma} \frac{(i\lambda)^{l-1}}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda = 2\pi$.

Exercise 7.2.2. If τ is a root of $P(i\boldsymbol{\xi}, i\tau) = 0$ with multiplicity k, and γ is a circle with τ as center and τ is the only root of $P(i\boldsymbol{\xi}, i\tau) = 0$ inside γ, prove that $\int_{\gamma} \frac{e^{i\lambda t}}{P(i\boldsymbol{\xi}, i\lambda)} d\lambda = Ct^{k-1} e^{i\tau t}$ for some constant C.

Exercise 7.2.3. Using Fourier's method to prove that if $g \in C_c^5(\mathbb{R})$, then the Cauchy problem

$$\begin{cases} u_t - u_{xxx} = 0 & (x,t) \in \mathbb{R}^2 \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

has a classical solution in $C_{x,t}^{3,1}(\mathbb{R} \times \mathbb{R})$. Also prove that this Cauchy problem has a (generalized) solution in $C([-T,T], H^1(\mathbb{R}))$ for $g \in H^1(\mathbb{R})$ and $T > 0$.

Exercise 7.2.4. Prove that the Cauchy problem

$$\begin{cases} u_t - u_{xxx} - u_{xx} = 0 & (x,t) \in \mathbb{R}^2 \\ u(x,0) = g(x) & x \in \mathbb{R} \end{cases}$$

is well-posed for forward evolution, but not well-posed for backward evolution. Furthermore, prove that the above problem has a classical solution in $C_{x,t}^{3,1}(\mathbb{R} \times (0,T])$ for any $g \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$ and $T > 0$, and that the solution is in $C(\mathbb{R} \times [0,T])$ if $g \in C_c^2(\mathbb{R})$ or $g \in H^1(\mathbb{R})$.

7.3 More General Formulation of Non-characteristic and HP Conditions

Suppose we would like to solve a Cauchy problem for a differential operator $P = \sum_{|\alpha| \le m} a_\alpha \partial_{\mathbf{x}}^\alpha$ of constant coefficients with respect to a hyperplane $H = \{\mathbf{x} \in \mathbb{R}^n : \sum_{j=1}^n \nu_j x_j = t_0\}$, where $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_n)$ is a unit normal vector to H. How do we formulate the non-characteristic and HP conditions with respect to H?

This can be done via a linear change of variables to reduce the problem to one studied in the previous section. If $\nu_n \neq 0$, then we can make the change of variables

$$\mathbf{x} = (x_1, \cdots, x_n) \mapsto \mathbf{y} = (y_1, \cdots, y_{n-1}, t) = (x_1, \cdots, x_{n-1}, \sum_{j=1}^n \nu_j x_j),$$

and introduce $v(\mathbf{y}) = u(\mathbf{x})$. Then

$$\begin{cases} \partial_{x_j} u(\mathbf{x}) = \partial_{y_j} v(\mathbf{y}) + \nu_j \partial_t v(\mathbf{y}), & \text{for } 1 \le j \le n-1, \\ \partial_{x_n} u(\mathbf{x}) = \nu_n \partial_t v(\mathbf{y}). \end{cases}$$

Then $\partial_{x_j}^{\alpha_j} \partial_{x_n}^{\alpha_n} u(\mathbf{x}) = \left(\partial_{y_j} + \nu_j \partial_t\right)^{\alpha_j} (\nu_n \partial_t)^{\alpha_n} v(\mathbf{y})$, and the principal part of P is now

$$P_m u = \sum_{|\alpha|=m} a_\alpha \partial_{\mathbf{x}}^\alpha u(\mathbf{x}) = \sum_{|\alpha|=m} a_\alpha \left(\partial_{y_j} + \nu_j \partial_t\right)^{\alpha_j} (\nu_n \partial_t)^{\alpha_n} v(\mathbf{y}).$$

H is non-characteristic with respect to P if the coefficient of ∂_t^m is non-zero. But the coefficient of ∂_t^m is $\sum_{|\alpha|=m} a_\alpha (\nu_j)^{\alpha_j} (\nu_n)^{\alpha_n} = \sum_{|\alpha|=m} a_\alpha \nu^\alpha$, so H is non-characteristic with respect to P if $\sum_{|\alpha|=m} a_\alpha \nu^\alpha \neq 0$.

The Hadamard-Petrowsky (HP) condition is formulated in terms of $\boldsymbol{\xi}$ and τ such that

$$e^{i\mathbf{y}\cdot\boldsymbol{\xi}+i\tau t} = e^{i\mathbf{x}\cdot\boldsymbol{\xi}+i\tau(\sum_{j=1}^n \nu_j x_j)} = e^{i\mathbf{x}\cdot(\boldsymbol{\xi}+\tau\boldsymbol{\nu})},$$

is a solution of $Pu = 0$ — the $\boldsymbol{\xi}$ is initially chosen to be of the form $(\xi_1, \cdots, \xi_{n-1}, 0)$; but we can allow arbitrary $\boldsymbol{\xi}$, as this may only affect the real, but not the imaginary part of τ. Since

$$Pe^{i\mathbf{x}\cdot(\boldsymbol{\xi}+\tau\boldsymbol{\nu})} = e^{i\mathbf{x}\cdot(\boldsymbol{\xi}+\tau\boldsymbol{\nu})} P(i(\boldsymbol{\xi}+\tau\boldsymbol{\nu})),$$

condition (7.7) is now formulated as

$$\begin{gathered} \exists 1 \leq l \leq m \text{ such that } \sum_{|\alpha|=l} a_\alpha \boldsymbol{\nu}^\alpha \neq 0, \text{ and} \\ \text{there exists a constant } C > 0 \text{ such that for any } \boldsymbol{\xi} \in \mathbb{R}^n, \\ -\mathfrak{Im}(\tau) \leq C \text{ holds for any root } \tau \text{ to } P(i(\boldsymbol{\xi}+\tau\boldsymbol{\nu})) = 0. \end{gathered} \tag{7.12}$$

Exercises

Exercise 7.3.1. Prove that the Cauchy problem

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) = 0 \quad (x,t) \in \mathbb{R}^2, \\ \qquad u(x,t) = u_0(x,t) \quad (x,t) \in H = \{(x,t) : \nu_1 x + \nu_0 t = h_0\} \\ \nu_1 u_x(x,t) + \nu_0 u_t(x,t) = u_1(x,t) \quad (x,t) \in H \end{cases}$$

is well-posed if $c^2\nu_1^2 - \nu_0^2 \neq 0$.

Exercise 7.3.2. Let $H = \{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \sum_{j=1}^n \nu_j x_j + \nu_0 t = h_0\}$, where $n \geq 2$, and $(\nu_1, \cdots, \nu_n, \nu_0)$ is a unit vector in $\mathbb{R}^{n+1}$. Prove that the Cauchy problem

$$\begin{cases} u_{tt}(\mathbf{x},t) - c^2 \Delta_{\mathbf{x}} u(\mathbf{x},t) = 0 \quad (\mathbf{x},t) \in \mathbb{R}^n \times \mathbb{R}, \\ \qquad u(\mathbf{x},t) = u_0(\mathbf{x},t) \quad (\mathbf{x},t) \in H \\ \sum_{j=1}^n \nu_j u_{x_j}(\mathbf{x},t) + \nu_0 u_t(\mathbf{x},t) = u_1(\mathbf{x},t) \quad (\mathbf{x},t) \in H \end{cases}$$

is well-posed iff $c^2 \sum_{j=1}^n \nu_j^2 - \nu_0^2 < 0$.

Chapter 8

First Order Scalar PDEs and Notion of Shock Wave Solutions

8.1 The Method of Characteristic Curves for Solving Linear and Quasilinear First Order Scaler PDEs

The method of characteristic curves that we used for solving the one-dimensional wave equation can be easily extended to solve the local Cauchy problem for a *quasilinear* first order scaler PDE near a given initial hypersurface Σ:

$$\begin{cases} \sum_{i=1}^{n} a_i(x,u)u_{x_i}(x) = c(x,u), & \text{near } \Sigma, \\ u(x) = g(x), & \text{on } \Sigma. \end{cases} \tag{8.1}$$

Here we assume $a_i(x,u)$, $i = 1, \cdots, n$, are locally Lipschitz functions of their arguments. In the case that $a_i(x,u)$, $i = 1, \cdots, n$, and c do not depend on u, (8.1) reduces to a linear first order PDE:

$$\begin{cases} \sum_{i=1}^{n} a_i(x)u_{x_i}(x) = c(x), & \text{near } \Sigma, \\ u(x) = g(x), & \text{on } \Sigma. \end{cases} \tag{8.2}$$

In such cases the geometric meaning of (8.2) is clear: $\sum_{i=1}^{n} a_i(x)u_{x_i}(x)$ is the directional derivative of u along the vector field $X(x) = (a_1(x), \cdots, a_n(x))$. $X(x)$ is called the *characteristic direction* at x for the PDE in (8.2), The integral curves of $X(x)$, namely, curves $\{x(\tau) : \tau \in \text{an interval}\}$ that satisfy

$$\frac{dx_i(\tau)}{d\tau} = a_i(x(\tau)), i = 1, \cdots, n, \tag{8.3}$$

are called *characteristic curves of the PDE* in (8.2).

If a solution $u = u(x)$ exists, then along an integral curve $\{x(\tau)\}$, we have

$$\frac{du(x(\tau))}{d\tau} = \sum_{i=1}^{n} a_i(x(\tau))u_{x_i}(x(\tau)) = c(x(\tau)).$$

This last equation provides the rate of change of $u(x(\tau))$ along the integral curve $\{x(\tau)\}$, and we can use this to solve for u: for any given x, we look for an integral curve $\{x(\tau)\}$ to (8.3) subject to the initial condition that $x(\tau = \tau_0) = x$, where τ_0 is a reference parameter; suppose that this integral curve exists for a long enough interval containing τ_0, and that there exists τ_x depending on x such that $x_n(\tau_x) = 0$, namely, this integral curve hits the hyperplane $x_n = 0$ at $\tau = \tau_x$, then we obtain

$$u(x) = u(x_1(\tau_x), \cdots, x_{n-1}(\tau_x), 0) + \int_{\tau_x}^{\tau_0} c(x(\tau))d\tau.$$

Here $u(x_1(\tau_x), \cdots, x_{n-1}(\tau_x), 0)$ is considered given initial data on the hyperplane $x_n = 0$. The justification for this solution requires some work: conditions need to be imposed so that τ_x is defined properly; and its differentiability needs to be studied. It's often easier to modify the procedure—to start the characteristic curves on the initial data hyper surface at $\tau = 0$, and rely on the Implicit Function Theorem (IFT) to justify the solution as follows.

Let's first illustrate the method when $\Sigma = \{x = (x_1, \cdots, x_{n-1}, x_n) : x_n = 0\}$. Then $u(x_1, \cdots, x_{n-1}, 0) = g(x_1, \cdots, x_{n-1})$ is prescribed, so $u_{x_i}(x_1, \cdots, x_{n-1}, 0) = g_{x_i}(x_1, \cdots, x_{n-1})$, for $i = 1, \cdots, n-1$, are known. In order for $u(x)$ to solve (8.2) at $(x_1, \cdots, x_{n-1}, 0)$, We need to determine $u_{x_n}(x_1, \cdots, x_{n-1}, 0)$ from the equation (8.2). For that purpose, we need $a_n(x_1, \cdots, x_{n-1}, 0) \neq 0$. It turns out that our argument will rely on the local existence of solution of the ODE system (8.3) and the continuous dependence the solution on initial data; our first result will establish solvability of (8.2) locally near $(x_1, \cdots, x_{n-1}, 0)$. Since we will use $x = (x_1, \cdots, x_{n-1}, x_n)$ as coordinates for points in this neighborhood, and determine $u(x)$ in terms of g at the point of intersection with the hyperplane Σ of the characteristic curve through x, we will use $(s, 0)$ as a parameter for points on Σ, where $s \in \mathbb{R}^{n-1}$ takes values near some $s^* \in \mathbb{R}^{n-1}$.

Thus we assume $a_n(s^*,0) \neq 0$ for some fixed $(s^*,0) \in \Sigma$, so $a_n(s,0) \neq 0$ for $s \in \mathbb{R}^{n-1}$ near s^*, and we solve for the above characteristic system (8.3) with initial data

$$x_i(\tau = 0) = s_i, \quad i = 1, \cdots, n-1; \quad x_n(\tau = 0) = 0;$$

$$\text{supplemented by } \frac{dU(\tau)}{d\tau} = c(x(\tau)), \text{ and } \quad U(\tau = 0) = g(s, 0).$$

By the ODE theory there exists $\delta > 0$ such that a unique solution exists for $|\tau| < \delta$. Since the solution depends on s as well, we denote the solution by $x_i(\tau; s), U(\tau; s)$. We next verify that the map

$$\Phi : (s, \tau) \mapsto (x_1(\tau; s), \cdots, x_n(\tau; s))$$

defines a local diffeomorphism from a neighborhood of $(s^*, 0)$ in $\mathbb{R}^n$ to a neighborhood of $(s^*, 0)$. This is due to the condition that $a_n(s^*, 0) \neq 0$ and the IFT, as the Jacobian matrix of Φ at $(s^*, 0)$ is

$$\begin{bmatrix} e_1 & \cdots & e_{n-1} & a \end{bmatrix},$$

where a is the column vector $[a_1(s^*, 0), \cdots, a_n(s^*, 0)]^T$, so the Jacobian determinant of Φ at $(s^*, 0)$ is $a_n(s^*, 0)$, which is $\neq 0$. Thus there is a neighborhood W of $(s^*, 0)$ in which Φ^{-1} is well defined and C^1. For any $x \in W$, let $(s, \tau) = \Phi^{-1}(x)$, then the characteristic curve which starts at $(s, 0)$ will pass through x at τ. Define $u(x) = U(\tau; s)$ for $x \in W$; in other words $u(\Phi(s, \tau)) = U(\tau; s)$. Then $u(x)$ is a C^1 function of x in W. We claim that $u(x)$ solves (8.2) in W and satisfies the initial condition in $W \cap \Sigma$. Based on the defining relation for u, $u \circ \Phi(\tau; s) = U(\tau; s)$, it follows that

$$\frac{dU}{d\tau} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{dx_i}{d\tau} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} (\Phi(\tau; s)) a_i(\Phi(\tau; s)).$$

But $\frac{dU}{d\tau} = c(x(\tau; s))$. Thus we have

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} (\Phi(\tau; s)) a_i(\Phi(\tau; s)) = c(x(\tau; s)).$$

We can now conclude that at each $x = \Phi(\tau; s) \in W$, $\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x) a_i(x) = c(x)$, namely $u(x)$ solves (8.2) in W. When $x \in W \cap \Sigma$, $\Phi^{-1}(x) = (s, 0)$ for some s, so $u(x) = U(0; s) = g(s, 0)$.

Next we need to extend the method of characteristic curves to solve (8.2) for a more general *non-characteristic* initial surface; but the discussion for a non-characteristic initial surface for (8.2) can be done at the same time as for the notion of *non-characteristic initial data* for the quasilinear PDE (8.1), so we will combine the discussion in one place. We define an associated vector field in the extended space R^{n+1}

whose value at $(x, u) \in \mathbb{R}^{n+1}$ is the vector $(a_1(x,u), \cdots, a_n(x,u), c(x,u))$ and define its *characteristic curves* to be the integral curves of the system of ODEs

$$\begin{cases} \dfrac{dx_i(\tau)}{d\tau} = a_i(x(\tau), u(\tau)), i = 1, \cdots, n, \\ \dfrac{du(\tau)}{d\tau} = c(x(\tau), u(\tau)). \end{cases} \tag{8.4}$$

Then the quasilinear PDE (8.1) is interpreted geometrically as looking for a graph $(x, u(x))$ in $\mathbb{R}^{n+1}$ which is tangential to the associated characteristic curves everywhere on it, as (8.1) can be interpreted as

$$(a_1(x,u), \cdots, a_n(x,u), c(x,u)) \perp (u_{x_1}(x), \cdots, u_{x_n}(x), -1).$$

The latter is a normal to the graph of $u = u(x)$ at $(x, u(x))$, so the above condition means that $(a_1(x,u), \cdots, a_n(x,u), c(x,u))$ is tangent to the graph of u.

To solve (8.2) or (8.1) near a general initial surface, we describe the initial surface Σ and the initial data parametrically: locally Σ is given by

$$x_i = \phi_i(s_1, \cdots, s_{n-1}), \; i = 1, \cdots, n, \quad \text{or symbolically } x = \phi(s), \tag{8.5}$$

where ϕ_i are regular (say C^1) functions of $(s_1, \cdots, s_{n-1})$ and satisfy the full rank condition:

$$\begin{pmatrix} \partial_1\phi_1 & \partial_1\phi_2 & \cdots & \partial_1\phi_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{n-1}\phi_1 & \partial_{n-1}\phi_2 & \cdots & \partial_{n-1}\phi_n \end{pmatrix} \tag{8.6}$$

has rank $n-1$ at $s^* = (s_1^*, \cdots, s_{n-1}^*)$. Set $x^* = \phi(s^*) \in \Sigma$. The Cauchy data is described as $u(\phi(s)) = g(\phi(s)) = \tilde{g}(s)$.

Let's discuss the compatibility conditions along Σ and the notion of non-characteristic initial data. The main idea is that if u is a C^1 function in a neighborhood of Σ satisfying (8.1) and

$$u(\phi_1(s), \cdots, \phi_n(s)) = \tilde{g}(s),$$

for s in an open neighborhood of s^*, then we ought to be able to determine $u_{x_i}(\phi_1(s), \cdots, \phi_n(s))$ for s near s^* from these relations. Differentiating in the s_j direction of the above equation, $j = 1, \cdots, n-1$, we obtain

$$\sum_{i=1}^{n} u_{x_i}(\phi_1(s), \cdots, \phi_n(s)) \partial_{s_j}\phi_i(s) = \tilde{g}_{s_j}(s), \quad j = 1, \cdots, n-1. \tag{8.7}$$

These $n-1$ linear equations in the n quantities $u_{x_i}(\phi_1(s), \cdots, \phi_n(s))$ are the compatibility conditions for u and g along Σ. The equation in (8.2), or (8.1), provides one more condition for these n quantities.

We say the Cauchy problem (8.2), or (8.1), is *non-characteristic*, if the Cauchy data and equation together can determine $\nabla u(x)$ for points on Σ. In our case of (8.2), or (8.1), we can determine $u_{x_i}(\phi_1(s), \cdots, \phi_n(s))$ completely from the Cauchy data and the PDE, if the joint linear system (8.7) and (8.2) (or (8.1)) is uniquely solvable. This amounts to the condition that

$$\begin{pmatrix} \partial_1\phi_1 & \partial_1\phi_2 & \cdots & \partial_1\phi_n \\ \vdots & \vdots & \vdots & \vdots \\ \partial_{n-1}\phi_1 & \partial_{n-1}\phi_2 & \cdots & \partial_{n-1}\phi_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \quad \text{is non-degenerate along } \Sigma. \tag{8.8}$$

Here the a_j's are evaluated at $(\phi(s), \tilde{g}(s))$. Note that each $(\partial_j\phi_1(s), \partial_j\phi_2(s), \cdots, \partial_j\phi_n(s))$ is a tangent to Σ at $\phi(s)$, so another way to describe this non-characteristic condition is that

$$(a_1(\phi(s), \tilde{g}(s)), \cdots, a_n(\phi(s), \tilde{g}(s))) \text{ is transversal to } \Sigma \text{ at } \phi(s).$$

Note that for (8.2), this condition depends only on $(a_1(\phi(s)), \cdots, a_n(\phi(s)))$ and Σ; while for (8.1), this condition also depends on the initial data $\tilde{g}(s)$.

We say Σ is a characteristic surface for (8.2) if $(a_1(x), \cdots, a_n(x))$ is tangent to Σ for every $x \in \Sigma$. This implies that every characteristic curve of (8.2) through a point on Σ will stay on Σ; in other words, a characteristic surface for (8.2) consists of union of characteristic curves.

Theorem 8.1. *Assume that $a_i(x, u)$, $i = 1, \cdots, n$, and $c(x, u)$ are C^1 functions of (x, u) near (x^*, u^*), where $x^* = \phi(s^*) \in \Sigma$ and $u^* = g(\phi(s^*)) = \tilde{g}(s^*)$, with g a C^1 function of x near x^*. Assume the non-characteristic condition (8.8) holds at $s = s^*$. Then there exists a neighborhood W of x^* such that a unique C^1 solution $u = u(x)$ exists to (8.1).*

Proof. For s near s^*, we solve for the characteristic system (8.4) with initial data

$$x_i(\tau = 0) = \phi_i(s), \quad i = 1, \cdots, n, \quad U(\tau = 0) = g(\phi(s)).$$

By ODE theory there exists $\delta > 0$ such that a unique solution exists for $|\tau| < \delta$. Since the solution depends on s as well, we denote the solution by $x_i(\tau; s)$, and $U(\tau; s)$. We next verify that the map

$$\Phi : (s, \tau) \mapsto (x_1(\tau; s), \cdots, x_n(\tau; s))$$

defines a local diffeomorphism from a neighborhood of $(s^*, 0)$ in $\mathbb{R}^n$ to a neighborhood of x^*. This is due to the non-characteristic condition (8.8) and the IFT, as the non-characteristic condition (8.8) implies that the Jacobian matrix of Φ is non-degenerate at s^*, therefore near s^* as well. Thus there is a neighborhood W of x^* in which Φ^{-1} is well defined and C^1. Define $u(x) = U \circ \Phi^{-1}(x)$ for $x \in W$. Then $u(x)$ is a C^1 function of x in W. We claim that $u(x)$ solves (8.1) in W and satisfies the initial condition in $W \cap \Sigma$. Based on the defining relation for u, we have $u \circ \Phi(s, \tau) = U(\tau; s)$, from which it follows that

$$\frac{dU}{d\tau} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{dx_i}{d\tau} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(\Phi(s,\tau)) a_i(\Phi(s,\tau), U(\tau; s)).$$

But $\frac{dU}{d\tau} = c(x(\tau; s); U(\tau; s)) = c(\Phi(s, \tau); U(\tau; s))$. Thus we have

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(\Phi(s,\tau)) a_i(\Phi(s,\tau), U(\tau; s)) = c(\Phi(s,\tau), U(\tau; s)).$$

We can now conclude that at each $x = \Phi(s, \tau) \in W$, $\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x) a_i(x, u(x)) = c(x, u(x))$, namely $u(x)$ solves (8.1) in W. When $x \in W \cap \Sigma$, $\Phi^{-1}(x) = (s, 0)$ for some s, so $u(x) = U(0; s) = \tilde{g}(s)$.

The uniqueness follows from the uniqueness of the ODE system as follows. Suppose that $v(x)$ is another solution of (8.1) in W. We will prove $u(x) = v(x)$ in W by considering an ODE in τ of

$$V(\tau; s) = u(\Phi(s,\tau)) - v(\Phi(s,\tau)) = U(\tau; s) - v \circ \Phi(s,\tau),$$

where $U(\tau; s)$ and $\Phi(s, \tau)$ are as above. Note that $V(0; s) = \tilde{g}(s) - \tilde{g}(s) = 0$, and

$$\begin{aligned}
\frac{dV}{d\tau} &= \frac{dU}{d\tau} - \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \frac{dx_i}{d\tau} \\
&= c(x(\tau; s), U(\tau; s)) - \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(x(\tau; s)) a_i(x(\tau; s), U(\tau; s)) \\
&= c(x(\tau; s), V(\tau; s) + v \circ \Phi(s,\tau)) - \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(x(\tau; s)) a_i(x(\tau; s), V(\tau; s) + v \circ \Phi(s,\tau)).
\end{aligned}$$

In the ODE system above for $V(\tau; s)$, $V \equiv 0$ is a solution due to $v(x)$ being a solution of (8.1). By the uniqueness of the ODE system, $V(\tau; s) \equiv 0$, namely, $U(\tau; s) = v \circ \Phi(s, \tau)$, proving that $v(x) = u(x)$ in W. □

Remark 8.1. The best we can get here is local existence. Even though no two characteristic curves in the extended (x, u) space can intersect by the uniqueness theorem for ODE systems, it may happen that for some $(s_1,\tau_1) \neq (s_2,\tau_2)$, we have $(x_1(s_1,\tau_1),\cdots,x_n(s_1,\tau_1)) = (x_1(s_2,\tau_2),\cdots,x_n(s_2,\tau_2))$, yet $U(s_1,\tau_1) \neq U(s_2,\tau_2)$; if this happens, then there will be no well defined u at $x = (x_1(s_1,\tau_1),\cdots,x_n(s_1,\tau_1))$. Geometrically, this means that, if we project the characteristic curves $(x(\tau;s),U(\tau;s))$ into the lower dimensional subspace $x(\tau;s) \in \mathbb{R}^n$, it is possible for two such projected characteristic curves to cross each other.

Example 8.1. Solve $xu_y - yu_x = u$, $u(x,0) = h(x)$.

The characteristic curves are

$$\begin{cases} \dfrac{dx}{d\tau} = -y, \\ \dfrac{dy}{d\tau} = x, \\ \dfrac{dU}{d\tau} = U. \end{cases}$$

Note that the initial curve $s \mapsto (s,0)$ is characteristic at $(0,0)$, so we will take an interval on the real x-axis not containing $(0,0)$ as the initial curve, say the positive real axis. Thus, $x(0;s) = s, y(0;s) = 0, U(0;s) = h(x(0;s),y(0;s)) = h(s)$ for $s \in \mathbb{R}^+$, then $x(\tau;s) = s\cos\tau, y(\tau;s) = s\sin\tau, U(\tau;s) = U(0;s)e^\tau = h(s)e^\tau$. To obtain u as a function of (x,y), we determine s and τ in terms of (x,y) from $x(\tau;s) = s\cos\tau, y(\tau;s) = s\sin\tau$: $s = \sqrt{x^2+y^2}$, and $\tau = \arg(x+iy)$ for $(x,y) \neq (0,0)$. Here $\arg(x+iy)$ is the argument of $x+iy$, which can be defined as a continuous (in fact, smooth) function of (x,y) in $\mathbb{R}^2$ with a slit from $(0,0)$ to ∞. We will take the slit to be the negative real axis, then

$$u(x,y) = h(\sqrt{x^2+y^2})e^{\arg(x+iy)} \quad \text{for } (x,y) \in \mathbb{R}^2 \setminus \mathbb{R}^-.$$

One can see that, for $x < 0$, $\lim_{y\to 0+} u(x,y) = e^\pi h(|x|)$, and $\lim_{y\to 0-} u(x,y) = e^{-\pi}h(|x|)$, so $u(x,0)$ can not be freely prescribed here, once $u(x,0)$ for $x > 0$ is prescribed. This example illustrates how the map Φ from the parameter space (s,τ) to the physical coordinate space (x,y) can fail to be one-to-one, therefore causing difficulty to define a solution $u(x,y)$ in the large; it also clarifies how two characteristic curves can intersect.

Exercise 8.1.1. Construct a solution of the following Cauchy problem

$$\begin{cases} u_t(x,t) + \displaystyle\sum_{j=1}^n b_j u_{x_j}(x,t) + cu(x,t) = 0 \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \\ \qquad\qquad u(x,0) = g(x), \quad x \in \mathbb{R}^n, \end{cases}$$

where b_j and c are constants.

Exercise 8.1.2. Construct a solution of the following Cauchy problem

$$\begin{cases} tu_t(x,t) + xu_x(x,t) + cu(x,t) = 0 \quad (x,t) \in \mathbb{R}^2, \\ \qquad\qquad\qquad\qquad u(x,-1) = g(x), \quad x \in \mathbb{R}. \end{cases}$$

Describe the domain of existence of the solution.

Exercise 8.1.3. Let $\Gamma = \{(\xi(t), t) : t \in I\}$ be a C^1 curve in a region U which separates U into two connected subregions U_1 and U_2. Suppose that $u(x,t)$ is C^1 in the closure of U_j, and satisfies $u_t(x,t) + u(x,t)u_x(x,t) = 0$ in each of U_j, $j = 1, 2$. Furthermore u is in $C(U)$, but $u_x(x,t)$ has a jump discontinuity across Γ. Prove that Γ is the projection of a characteristic curve for the equation $u_t(x,t)+u(x,t)u_x(x,t) = 0$.

Exercise 8.1.4. Show that the solution u to $u_t(x,t) + a(u(x,t))u_x(x,t) = 0$ with initial data $u(x,0) = g(x)$ becomes singular for some positive t, unless $a(g(x))$ is a non-decreasing function of x.

8.2 Notion of Shock Wave Solution

Next, we will discuss the possible failure of existence of smooth solutions in large domains. This is due to the crossing of the "projected" characteristic curves. We first examine an example.

Example 8.2. Solve $u_y + uu_x = 0$, with $u(x,0) = g(x)$.

The characteristic curves are

$$\begin{cases} \dfrac{dx}{d\tau} = U, \\ \dfrac{dy}{d\tau} = 1, \\ \dfrac{dU}{d\tau} = 0. \end{cases}$$

Thus, along any characteristic curve, $U \equiv u_0$ for some constant u_0, and $x = x_0 + u_0\tau$, $y = y_0 + \tau$. The initial data can be described as $s \mapsto (s, 0, g(s))$. So the characteristic curve through $(s, 0, g(s))$ at $\tau = 0$ is $x = s + g(s)\tau$, $y = \tau$, and $U = g(s)$. If we can solve for s in terms of $(x, y = \tau)$, then we find the solution u in terms of (x,y). The projected characteristic $\tau \mapsto (s + g(s)\tau, \tau)$ is a straight line through $(s, 0)$. Two such projected characteristics will intersect if $g(s_1) \neq g(s_2)$: they will meet at τ such that

$s_1 + g(s_1)\tau = s_2 + g(s_2)\tau$, namely, $\tau = -\frac{s_2 - s_1}{g(s_2) - g(s_1)}$. So if there exist $s_1 < s_2$ such that $g(s_1) > g(s_2)$, then the projected characteristics through $(s_1, 0)$ and $(s_2, 0)$ will meet at a positive $y = \tau$, at which U along one projected characteristic is $g(s_1)$, but along the other projected characteristic is $g(s_2)$. Thus, although U can be defined as a smooth function of (s, τ), u can not be defined as a smooth function of (x, y) for $y \geq -\frac{s_2 - s_1}{g(s_2) - g(s_1)}$.

For the particular case of

$$g(x) = \begin{cases} 1 & \text{if } x < 0, \\ 1 - x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x \geq 1, \end{cases}$$

the characteristic curve through $(s, 0, g(s))$ at $\tau = 0$ is, for $s < 0$, $x = s + \tau$, $y = \tau$, and $U = 1$; for $0 \leq s < 1$, $x = s + (1 - s)\tau$, $y = \tau$, and $U = 1 - s$; and for $s \geq 1$, $x = s$, $y = \tau$, and $U = 0$.

A notion of weak solution is needed to define the solution for large time. In this context, solutions may arise which are piecewise smooth, but have a discontinuous jump across an interface. Solutions with such discontinuities are called *shock wave solutions*.

The equation in **Example** 8.2 is a special case of a first order PDE of the form $\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0$. Suppose that $u(x, y)$ is a C^1 solution on either side of the curve $\Gamma : x = \xi(y)$. To define a weak solution, we reformulate the PDE as follows. When $u(x, y)$ is a smooth solution across Γ, for any $a < \xi(y) < b$, we have

$$\frac{\partial}{\partial y} \int_a^b R(u(x, y))\, dx + S(u(b, y)) - S(u(a, y)) = 0. \tag{8.9}$$

But (8.9) makes sense for a piecewise smooth function $u(x, y)$, with discontinuity across Γ. Suppose that $u(x, y)$ satisfies $\frac{\partial R(u)}{\partial y} + \frac{\partial S(u)}{\partial x} = 0$ on either side of Γ, and satisfies (8.9). Let $u_-(\xi(y), y) = \lim_{x \nearrow \xi(y)} u(x, y)$, and $u_+(\xi(y), y) = \lim_{x \searrow \xi(y)} u(x, y)$. Then

$$\begin{aligned}
&\frac{\partial}{\partial y} \int_a^b R(u(x, y))\, dx \\
=&\frac{\partial}{\partial y} \left(\int_a^{\xi(y)} R(u(x, y))\, dx + \int_{\xi(y)}^b R(u(x, y))\, dx \right) \\
=&\xi'(y) R(u_-(\xi(y), y) - \xi'(y) R(u_+(\xi(y), y) + \left(\int_a^{\xi(y)} + \int_{\xi(y)}^b \right) \frac{\partial R(u(x, y))}{\partial y}\, dx,
\end{aligned}$$

while

$$S(u(b,y))-S(u(a,y)) = S(u_+(\xi(y),y))-S(u_-(\xi(y),y)+\left(\int_a^{\xi(y)} + \int_{\xi(y)}^b\right) \frac{\partial S(u(x,y))}{\partial x}\, dx.$$

So we must have

$$\xi'(y)\,[R(u_-(\xi(y),y) - R(u_+(\xi(y),y)] + S(u_+(\xi(y),y)) - S(u_-(\xi(y),y) = 0,$$

from which we obtain the Rankine-Hugoniot shock condition

$$\xi'(y) = \frac{S(u_+(\xi(y),y)) - S(u_-(\xi(y),y)}{R(u_+(\xi(y),y) - R(u_-(\xi(y),y)} \tag{8.10}$$

In **Example** 8.2, $R(u) = u$, $S(u) = u^2/2$, $u_+(\xi(y),y) = 0$, and $u_-(\xi(y),y) = 1$, so $\xi'(y) = \frac{1}{2}$. Since the shock starts at $(1,1)$, we define Γ by $x = 1 + \frac{1}{2}(y-1)$, and extend the definition of $u(x,y)$ to $y \geq 1$ by

$$u(x,y) = \begin{cases} 1 & \text{if } x < 1 + \frac{1}{2}(y-1), \\ 0 & \text{if } x > 1 + \frac{1}{2}(y-1). \end{cases}$$

Then the extended u is a weak solution of the Burger's equation with the particular initial data as in **Example** 8.2.

Note that for a C^1 solution u to the Burger's equation, the equation is essentially equivalent to $uu_t + u^2u_x = 0$, which can be cast in the form of a conservation law, with $R(u) = u^2/2$, $S(u) = u^3/3$. But if we use this formulation, with the initial data as in **Example** 8.2, then $\xi'(y) = \frac{2}{3}$. This examples shows that in dealing with conservations laws, it's important to work with the correct choise of $R(u)$ and $S(u)$.

Remark 8.2. If one denotes by $[R(u)]$ and $[S(u)]$ the jump of $R(u)$ and $S(u)$ respectively across Γ: $[R(u)] = R(u_+(\xi(y),y) - R(u_-(\xi(y),y,$ $[S(u)] = S(u_+(\xi(y),y)) - S(u_-(\xi(y),y)$, then (8.10) has a geometric interpreation: $([R(u)],[S(u)])\cdot(\xi'(y),-1) = 0$, namely, the vector of jump $([R(u)],[S(u)])$ must be tangential to Γ.

In order to justify that this is an appropriate notion of weak solution, one needs to establish some kind of uniqueness and stability result for the weak solution. This topic will be pursued later.

8.3 Fully Nonlinear First Order Scalar PDEs

We encountered first order linear and quasilinear scalar PDEs and used the method of integrating along the characteristic ODEs to construct their solutions locally. We now extend the method to deal with fully nonlinear first order scalar PDEs.

A general nonlinear first order scalar PDE can be written in the form of

$$F(\nabla u(x), u(x), x) = 0,$$

where $F(p, u, x)$ is some sufficiently differentiable functions in its arguments (p, u, x). There seems no direct interpretation of the equation as describing the directional derivative of u. However, if one differentiates the equation in the x_i direction, the resulting equation for $p_i(x) = u_{x_i}(x)$,

$$0 = F_{x_i} + F_u u_{x_i}(x) + \sum_{j=1}^{n} F_{p_j} \partial_{x_i} p_j = F_{x_i} + F_u p_i + \sum_{j=1}^{n} F_{p_j} \partial_{x_j} p_i,$$

using $\partial_{x_i} p_j = \partial_{x_j} p_i = u_{x_i x_j}$, has the geometric interpretation: along

$$\frac{dx_j}{dt} = F_{p_j}(x(t), u(x(t)), \nabla u(x(t))),$$

we would have

$$\begin{aligned} \frac{dp_i(x(t))}{dt} &= \sum_{j=1}^{n} F_{p_j} \partial_{x_j} p_i \\ &= -F_u(\nabla u(x(t)), u(x(t)), x(t)) u_{x_i}(x) - F_{x_i}(\nabla u(x(t)), u(x(t)), x(t)) \\ &= -F_u(\nabla u(x(t)), u(x(t)), x(t)) p_i(x(t)) - F_{x_i}(\nabla u(x(t)), u(x(t)), x(t)). \end{aligned}$$

We now have $2n$ ODEs for $(x(t), p(t))$, where $p(t) = \nabla u(x(t))$; but the system also involves $u(x(t))$. To get a closed system, we derive an ODE for $u(x(t))$:

$$\frac{du(x(t))}{dt} = \sum_{j=1}^{n} u_{x_j}(x(t)) \frac{dx_j}{dt} = \sum_{j=1}^{n} p_j(t) F_{p_j}(\nabla u(x(t), u(x(t)), x(t)).$$

We have now arrived at a closed system of ODEs for $(x(t), u(t), p(t))$:

$$\begin{cases} \dfrac{dx_j}{dt} = F_{p_j}(p(t), u(t), x(t)), \quad j = 1, \cdots, n, \\ \dfrac{dp_j}{dt} = -F_u(p(t), u(t), x(t)) p_j(t) - F_{x_j}(p(t), u(t), x(t)), \quad j = 1, \cdots, n, \\ \dfrac{du}{dt} = \displaystyle\sum_{j=1}^{n} p_j(t) F_{p_j}(p(t), u(t), x(t)). \end{cases} \tag{8.11}$$

(8.11) is called the *characteristic ODEs* for the nonlinear equation $F(\nabla u(x), u(x), x) = 0$.

To use the characteristic ODEs (8.11) to construct a local solution of the Cauchy problem

$$\begin{cases} F(\nabla u(x), u(x), x) = 0 \quad \text{near } \Sigma, \\ \qquad\qquad u(x) = g(x) \quad \text{for } x \in \Sigma, \end{cases} \tag{8.12}$$

we first need to use a parametric representation $s \in \Omega \subset \mathbb{R}^{n-1} \mapsto \phi(s) \in \Sigma$ for Σ near $x_* = \phi(s_*)$ and the initial data $g(x)$, together with the given PDE, to determine the initial values $u(0;s) = g(\phi(s)) = \tilde{g}(s)$, $p_j(0;s) = u_{x_j}(\phi(s))$ for s near a given parameter s_* in terms of $\tilde{g}(s)$ and $\phi(s)$. The $p_j(0;s)$ must first satisfy the compatibility conditions:

$$\frac{\partial \tilde{g}(s)}{\partial s_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(\phi(s)) \frac{\partial \phi_i(s)}{\partial s_j} = \sum_{i=1}^{n} p_i(0;s) \frac{\partial \phi_i(s)}{\partial s_j}, \quad j = 1, \cdots, n-1.$$

They also need to satisfy $F(\phi(s), \tilde{g}(s), p(0;s)) = 0$. Thus we need to find local, differentiable solutions $p_i(0;s), i = 1, \cdots, n$, to

$$\begin{cases} \dfrac{\partial \tilde{g}(s)}{\partial s_j} = \displaystyle\sum_{i=1}^{n} p_i(0;s) \frac{\partial \phi_i(s)}{\partial s_j}, \quad j = 1, \cdots, n-1 \\ F(p(0;s), \tilde{g}(s), \phi(s)) = 0, \end{cases} \tag{8.13}$$

for s near s_*.

Suppose that, at s_*, there is p_* satisfying (8.13), and

$$F_p(p_*, \tilde{g}(s_*), \phi(s_*)) \text{ is transversal to } T_{\phi(s_*)}(\Sigma), \tag{8.14}$$

then, by the implicit function theorem, there is a local differentiable solution $p = p(0;s)$ to (8.13) for s near s_*, and $p(0;s)$ near p_*. (8.14) is precisely the condition that Σ is non-characteristic with respect to (8.12) on the initial data g at $x = \phi(s_*)$.

Under the same condition (8.14), we can construct the local solution $x(\tau;s), p(\tau;s)$, $u(\tau;s)$ to (8.11) subject to the initial conditions $x(0;s) = \phi(s), p(0;s)$ as given above, and $u(0;s) = \tilde{g}(s)$, and prove that the map $\Phi : (\tau;s) \mapsto x(\tau;s)$ is a local diffeomorphism near $(0;s_*)$. This is because the differential of Φ at $(0;s_*)$ is

$$\begin{bmatrix} \Phi_{s_1}(0;s_*) \\ \vdots \\ \Phi_{s_{n-1}}(0;s_*) \\ \Phi_\tau(0;s_*) \end{bmatrix} = \begin{bmatrix} \phi_{s_1}(s_*) \\ \vdots \\ \phi_{s_{n-1}}(s_*) \\ F_p(p_*, \tilde{g}(s_*), \phi(s_*)) \end{bmatrix},$$

which is non-degenerate when (8.14) is assumed. We will follow the same proof as in the proof for the linear and quasilinear case to prove that $U(x) = u \circ \Phi^{-1}(x)$ is a local solution of (8.12).

Theorem 8.2. *Suppose that $F(p, u, x)$ is a C^1 functions of its arguments, that at $x_* = \phi(s_*)$, $u_* = g(x_*)$, there exists p_* satisfying (8.13), and that (8.14) holds. Then there exists a neighborhood U of x_* and a C^1 solution $U(x)$ to (8.12).*

Proof. The proof will consist of two steps: (a) Prove that $F(p(\tau; s), u(\tau; s), x(\tau; s)) = 0$ for (τ, s) in a neighborhood of $(0, s_*)$; (b) Prove that $U_{x_i}(x(\tau; s)) = p_i(\tau; s)$ for (τ, s) in a neighborhood of $(0, s_*)$, therefore $F(\nabla U(x), U(x), x) = 0$ for x in a neighborhood of $x_* = \phi(s_*)$.

For (a), first note that $F(p(\tau; s), u(\tau; s), x(\tau; s)) = 0$ for $\tau = 0$ and s near s_* by the construction of the initial data. Next,

$$
\begin{aligned}
&\frac{\partial}{\partial \tau} F(p(\tau; s), u(\tau; s), x(\tau; s)) \\
=&\sum_{i=1}^{n} \left\{ F_{x_i}(p(\tau; s), u(\tau; s), x(\tau; s)) \frac{dx_i}{d\tau} + F_{p_i}(p(\tau; s), u(\tau; s), x(\tau; s)) \frac{dp_i}{d\tau} \right\} \\
&+ F_u(p(\tau; s), u(\tau; s), x(\tau; s)) \frac{du}{d\tau} \\
=&\sum_{i=1}^{n} \{ F_{x_i}(p(\tau; s), u(\tau; s), x(\tau; s)) F_{p_i}(p(\tau; s), u(\tau; s), x(\tau; s)) \\
&- F_{p_i}(p(\tau; s), u(\tau; s), x(\tau; s)) \left[F_{x_i}(p(\tau; s), u(\tau; s), x(\tau; s)) + F_u(p(\tau; s), u(\tau; s), x(\tau; s)) p_i(t) \right] \} \\
&+ F_u(p(\tau; s), u(\tau; s), x(\tau; s)) \sum_{i=1}^{n} p_i(\tau, s) F_{p_i}(p(\tau; s), u(\tau; s), x(\tau; s)) \\
=&0.
\end{aligned}
$$

This proves (a).

For (b), we will prove that

$$
\begin{cases}
\dfrac{\partial u(\tau; s)}{\partial s_j} = \displaystyle\sum_{i=1}^{n} p_i(\tau; s) \frac{\partial x_i(\tau; s)}{\partial s_j} \quad \text{for } j = 1, \cdots, n-1, \\
\dfrac{\partial u(\tau; s)}{\partial \tau} = \displaystyle\sum_{i=1}^{n} p_i(\tau; s) \frac{\partial x_i(\tau; s)}{\partial \tau}.
\end{cases}
\tag{8.15}
$$

The last equation above follows from (8.11) directly. For the first $n-1$ equations, let $G_j(\tau, s) = \frac{\partial u(\tau; s)}{\partial s_j} - \sum_{i=1}^{n} p_i(\tau; s) \frac{\partial x_i(\tau; s)}{\partial s_j}$, for $j = 1, \cdots, n-1$. Then $G_j(0, s) = 0$ by the compatibility of the initial data. We next compute

$$
\begin{aligned}
&\frac{\partial G_j(\tau,s)}{\partial\tau}\\
=&\frac{\partial}{\partial s_j}\left(\frac{\partial u(\tau;s)}{\partial\tau}\right)-\sum_{i=1}^n\left\{p_i(\tau;s)\frac{\partial}{\partial s_j}\left(\frac{\partial x_i(\tau;s)}{\partial\tau}\right)+\frac{\partial p_i(\tau;s)}{\partial\tau}\frac{\partial x_i(\tau;s)}{\partial s_j}\right\}\\
=&\frac{\partial}{\partial s_j}\left(\sum_{i=1}^n p_i(\tau,s)F_{p_i}(p(\tau;s),u(\tau;s),x(\tau;s))\right)\\
&-\sum_{i=1}^n\left\{p_i(\tau;s)\frac{\partial F_{p_i}(p(\tau;s),u(\tau;s),x(\tau;s))}{\partial s_j}+\frac{\partial p_i(\tau;s)}{\partial\tau}\frac{\partial x_i(\tau;s)}{\partial s_j}\right\}\\
=&\sum_{i=1}^n\left\{\frac{\partial p_i(\tau,s)}{\partial s_j}F_{p_i}(p(\tau;s),u(\tau;s),x(\tau;s))-\frac{\partial p_i(\tau;s)}{\partial\tau}\frac{\partial x_i(\tau;s)}{\partial s_j}\right\}.
\end{aligned}
$$

Since $F(p(\tau;s),u(\tau;s),x(\tau;s))=0$, we have, after differentiation in s_j,

$$
\begin{aligned}
0=&\sum_{i=1}^n\left\{F_{x_i}(p(\tau;s),u(\tau;s),x(\tau;s))\frac{\partial x_i(\tau;s)}{\partial s_j}+F_{p_i}(p(\tau;s),u(\tau;s),x(\tau;s))\frac{\partial p_i(\tau,s)}{\partial s_j}\right\}\\
&+F_u(p(\tau;s),u(\tau;s),x(\tau;s))\frac{\partial u(\tau;s)}{\partial s_j},
\end{aligned}
$$

from which we get

$$
\begin{aligned}
&\frac{\partial G_j(\tau,s)}{\partial\tau}\\
=&-\sum_{i=1}^n F_{x_i}(p(\tau;s),u(\tau;s),x(\tau;s))\frac{\partial x_i(\tau;s)}{\partial s_j}-F_u(p(\tau;s),u(\tau;s),x(\tau;s))\frac{\partial u(\tau;s)}{\partial s_j}\\
&-\sum_{i=1}^n\left(-F_u(p(\tau;s),u(\tau;s),x(\tau;s))p_i(\tau,s)-F_{x_i}(p(\tau;s),u(\tau;s),x(\tau;s))\right)\frac{\partial x_i(\tau;s)}{\partial s_j}\\
=&-F_u(p(\tau;s),u(\tau;s),x(\tau;s))\left(\frac{\partial u(\tau;s)}{\partial s_j}-\sum_{i=1}^n p_i(\tau,s)\frac{\partial x_i(\tau;s)}{\partial s_j}\right)\\
=&-F_u(p(\tau;s),u(\tau;s),x(\tau;s))G_j(\tau,s),
\end{aligned}
$$

from which we conclude (8.15).

Since by the chain rule, we also have

$$
\begin{cases}
\dfrac{\partial u(\tau;s)}{\partial s_j}=\displaystyle\sum_{i=1}^n\frac{\partial U(x(\tau;s))}{\partial x_i}\frac{\partial x_i(\tau;s)}{\partial s_j} & \text{for } j=1,\cdots,n-1,\\
\dfrac{\partial u(\tau;s)}{\partial\tau}=\displaystyle\sum_{i=1}^n\frac{\partial U(x(\tau;s))}{\partial x_i}\frac{\partial x_i(\tau;s)}{\partial\tau},
\end{cases}
\tag{8.16}
$$

now noting that the coefficient matrix

$$\begin{pmatrix} \frac{\partial x_1(\tau;s)}{\partial s_1} & \cdots & \frac{\partial x_n(\tau;s)}{\partial s_1} \\ \frac{\partial x_1(\tau;s)}{\partial s_2} & \cdots & \frac{\partial x_n(\tau;s)}{\partial s_2} \\ \vdots & \cdots & \vdots \\ \frac{\partial x_1(\tau;s)}{\partial s_{n-1}} & \cdots & \frac{\partial x_n(\tau;s)}{\partial s_{n-1}} \\ \\ \frac{\partial x_1(\tau;s)}{\partial \tau} & \cdots & \frac{\partial x_n(\tau;s)}{\partial \tau} \end{pmatrix}$$

is non-degenerate near $(\tau, s) = (0, s_*)$, we conclude (b) by using (8.15) and (8.16). □

Example 8.3. The initial curve $s \mapsto (s, \phi(s))$ is non-characteristic with respect to the equation $u_t^2(x,t) + u_x^2(x,t) = 1$ on the initial data $u(s, \phi(s)) = h(s)$, iff we can determine $u_x(s, \phi(s))$ and $u_t(s, \phi(s))$ as differentiable functions of s from the equations

$$\begin{cases} u_t^2(s, \phi(s)) + u_x^2(s, \phi(s)) = 1, \\ u_x(s, \phi(s)) + u_t(s, \phi(s))\phi'(s) = h'(s). \end{cases}$$

This is equivalent to $|h'(s)| < \sqrt{1 + |\phi'(s)|^2}$ — this is the non-characteristic condition for the initial data. In fact, we can determine $u_x(s, \phi(s))$ and $u_t(s, \phi(s))$ in terms of $\phi(s)$ and $h(s)$ as

$$u_x(s, \phi(s)) = \frac{h'(s) - \phi'(s)\sqrt{1 + |\phi'(s)|^2 - |h'(s)|^2}}{1 + |\phi'(s)|^2} \stackrel{\text{def}}{=} p_x(0, s),$$

$$u_t(s, \phi(s)) = \frac{h'(s)\phi'(s) + \sqrt{1 + |\phi'(s)|^2 - |h'(s)|^2}}{1 + |\phi'(s)|^2} \stackrel{\text{def}}{=} p_t(0, s).$$

To use the characteristic ODEs to construct a local solution of the Cauchy problem

$$\begin{cases} u_t^2(x,t) + u_x^2(x,t) = 1, \\ u(s, \phi(s)) = h(s), \end{cases} \tag{8.17}$$

we first write down the characteristic ODEs for the variables $x(\tau; s), t(\tau; s), p_x(\tau; s), p_t(\tau; s)$, and $u(\tau; s)$: with $F = p_t^2 + p_x^2 - 1$,

$$\begin{cases} \frac{dx}{d\tau} = F_{p_x} = 2p_x, \\ \frac{dt}{d\tau} = F_{p_t} = 2p_t, \\ \frac{dp_x}{d\tau} = -F_x = 0, \\ \frac{dp_t}{d\tau} = -F_t = 0, \\ \frac{du}{d\tau} = p_x F_{p_x} + p_t F_{p_t} = 2(p_x^2 + p_t^2). \end{cases}$$

The initial conditions at $\tau = 0$ are determined as above, so $x(0; s) = s$, $t(0; s) = \phi(s)$, $p_x(0; s)$ and $p_t(0; s) = p$ as given above, and $u(0; s) = h(s)$. Thus $p_x(\tau; s) = p_x(0; s)$, $p_t(\tau; s) = p_t(0; s)$, $x(\tau; s) = s+2p_x(0, s)\tau$, $t(\tau; s) = \phi(s)+2p_t(0, s)\tau$. Since $p_x(\tau, s)^2 + p_t(\tau, s)^2 = p_x(0, s)^2 + p_t(0, s)^2 = 1$, we have $u(\tau; s) = u(0; s) + 2\tau = h(s) + 2\tau$. Finally, we need to determine (τ, s) in terms of (x, t), at least when (x, t) is close to $(s, \phi(s))$. This is possible because the Jacobian determinant of the map $(\tau, s) \mapsto (x, t) = (s + 2p_x(0, s)\tau, \phi(s) + 2p_t(0, s)\tau)$ at $(0, s)$ is

$$\begin{pmatrix} 2p_x(0, s) & 2p_t(0, s) \\ 1 & \phi'(s) \end{pmatrix} = -\sqrt{1 + |\phi'(s)|^2 - |h'(s)|^2},$$

which is non-zero when the initial data is assumed to be non-characteristic. Two special cases can be understood with more details: $h'(s) \equiv 0$ or $\phi'(s) \equiv 0$. In the former case, the non-characteristic condition is always satisfied, and the transformation Φ is $(\tau, s) \mapsto (x, t) = (s - \frac{2\phi'(s)}{\sqrt{1+|\phi'(s)|^2}}\tau, \phi(s) + \frac{2}{\sqrt{1+|\phi'(s)|^2}}\tau)$, which has the geometric interpretation that the straight line segment from (x, t) to $(s, \phi(s))$ is normal to the initial curve at $(s, \phi(s))$, with a distance equal to 2τ. In other words, the initial curve can be considered as a wave front with $h \equiv$ a constant h_0, and the level curves of the solution u, $u = h_0 + 2\tau$, are at a distance 2τ away from the initial curve.

Example 8.4. When $F = F(\nabla u)$ does not have explicit dependence on x or u, (8.11) implies that $\frac{dp_j}{dt} = 0$, so $p_j(t) = p_j(0)$, and the system for $x_j(t)$ implies that $\frac{dx_j}{dt} = F_{p_j}(p(0))$, so the projected characteristic curves in the x-space are straight lines $x(t) = x(0) + F_p(p(0))t$. A smooth solution can not exist on a region in which two projected characteristic curves cross each other.

Example 8.5. When $F = H(\nabla u, x)$ does not have explicit dependence on u, the first $2n$ equations in (8.11) form a closed system of ODEs, which are the Hamilton's ODEs:

$$\begin{cases} \dfrac{dx_j}{dt} = H_{p_j}(p(t), x(t)), \quad j = 1, \cdots, n, \\ \dfrac{dp_j}{dt} = -H_{x_j}(p(t), x(t)), \quad j = 1, \cdots, n. \end{cases} \tag{8.18}$$

A property of the integral curves of Hamilton's ODEs is that $H(p(t), x(t)) \equiv$ a constant along any integral curve.

When $F(p, x)$ has the form $\frac{1}{2}\sum_{i,j=1}^{n} a^{ij}(x)p_i p_j$, where $(a^{ij}(x))$ is the inverse of the positive definite $(a_{ij}(x))$ which defines a Riemannian metric g, then the integral curves of the corresponding Hamilton's ODEs are associated with the geodesics of the metric g, as the characteristic ODEs

$$\begin{cases} \dfrac{dx_j}{dt} = \displaystyle\sum_{i=1}^{n} a^{ij}(x)p_i, \\ \dfrac{dp_j}{dt} = -\dfrac{1}{2}\displaystyle\sum_{l,m=1}^{n} \dfrac{\partial a^{lm}}{\partial x_j} p_l p_m. \end{cases}$$

imply

$$\begin{aligned} \frac{d^2x_j}{dt^2} &= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} \frac{\partial a^{ij}(x)}{\partial x_k}\frac{dx_k}{dt}p_i + a^{ij}(x)\frac{dp_i}{dt}\right) \\ &= \sum_{i=1}^{n}\left(\sum_{k=1}^{n} \frac{\partial a^{ij}(x)}{\partial x_k}\frac{dx_k}{dt}p_i - \sum_{l,m=1}^{n}\frac{a^{ij}(x)}{2}\frac{\partial a^{lm}}{\partial x_i}p_l p_m\right) \end{aligned}$$

Using $\frac{\partial a^{lm}}{\partial x_j} = -\sum_{k,h=1}^{n} a^{lk}\frac{\partial a_{kh}(x)}{\partial x_i}a^{hm}$, and $\sum_{m=1}^{n} a^{hm}p_m = \frac{dx_h}{dt}$, $\sum_{l=1}^{n} a^{lk}p_l = \frac{dx_k}{dt}$, we see that

$$\begin{aligned} \frac{d^2x_j}{dt^2} &= \sum_{i=1}^{n}\left(-\sum_{k,h,l=1}^{n} a^{il}\frac{\partial a_{lh}(x)}{\partial x_k}a^{hj}p_i\frac{dx_k}{dt} + \sum_{k,h=1}^{n}\frac{a^{ij}(x)}{2}\frac{\partial a_{kh}(x)}{\partial x_i}\frac{dx_h}{dt}\frac{dx_k}{dt}\right) \\ &= \sum_{k,h,l=1}^{n}\left(-\frac{\partial a_{lh}(x)}{\partial x_k}a^{lj} + \frac{a^{lj}(x)}{2}\frac{\partial a_{kh}(x)}{\partial x_l}\right)\frac{dx_k}{dt}\frac{dx_h}{dt} \\ &= -\sum_{k,h=1}^{n}\Gamma^j_{kh}\frac{dx_k}{dt}\frac{dx_h}{dt}, \end{aligned}$$

which are the ODEs for the geodesics in the metric g.

If we are to construct a solution $u(x)$ to $\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}(x)u_{x_j}(x) = c$, then we need to use an additional ODE in the characteristic ODEs: $\frac{du(t)}{dt} = \sum_i^n p_i(t)F_{p_i}(p(t), x(t)) = \sum_{i,j=1}^{n} a^{ij}(x(t))p_i(t)p_j(t)$. But along the solutions to the characteristic ODEs, $F(p(t), x(t)) = 0$, so $\frac{du(t)}{dt} = c$. The solution $u(x)$ here can represent the wave front in wave optics.

Exercises

Exercise 8.3.1. Construct a local solution of the Cauchy problem: $u_y(x,y) - u_x^3(x,y) = 0, u(x,0) = x^{3/2}$. Also show that if u is a C^1 global solution of $u_y(x,y) - u_x^3(x,y) = 0$ on $\mathbb{R}^2$, then $u(x,y)$ must be a linear function in x and y.

Exercise 8.3.2. Work out the equations for the characteristic curves of $u_t(x,t) + |\nabla_x u(x,t)|^2 = 0$ for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, and $u(x,0) = g(x)$. Show that, if $g \in C^1(\mathbb{R}^n)$, then for any $(x_0, 0)$, a local solution exists for (x,t) in a neighborhood of $(x_0, 0)$, and that $u(x,t) = g(y) + |\nabla_y g(y)|^2 t$, where y attains the (local) minimum of $\frac{|x-\xi|^2}{4} + g(\xi)$ for ξ near x_0.

Exercise 8.3.3. Prove that the Cauchy problem

$$\begin{cases} a(x,y)(u_x(x,y))^2 + 2b(x,y)u_x(x,y)u_y(x,y) + c(x,y)(u_y(x,y))^2 = d(x,y) \\ u(x,\phi(x)) = h(x) \end{cases}$$

is non-characteristic at $(x,\phi(x))$ iff at $(x,y) = (x,\phi(x))$,

$$(h'(x))^2 \left[b^2 - ac\right] + d\left[a(\phi'(x))^2 - 2b\phi'(x) + c\right] > 0.$$

Note that $(1,\phi'(x))$ is a tangent to the initial curve at $(x,y) = (x,\phi(x))$, so $(-\phi'(x),1)$ is a normal to the curve there, and $a(\phi'(x))^2 - 2b\phi'(x) + c = a\nu_1^2 + 2b\nu_1\nu_2 + c\nu_2^2$, with $(\nu_1,\nu_2) = (-\phi'(x),1)$. Examine special cases when $d(x,\phi(x)) = 0$, or $c = 0$, or $h'(x) = 0$, or $a(\phi'(x))^2 - 2b\phi'(x) + c = 0$ at $(x,y) = (x,\phi(x))$.

Chapter 9

CAUCHY-KOWALEVSKAYA THEOREM

Cauchy-Kowalevskaya theorem gives local existence of analytic solutions to a non-characteristic Cauchy problem of a partial differential equation (or system) that is analytic in its arguments. Its predecessor is Cauchy's 1842 theorem on the local existence of analytic solutions to the initial value problem for ordinary differential equations

$$\frac{du}{dt} = f(u(t), t), \quad u(0) = u_0,$$

when $f(u,t)$ is assumed to be analytic in a neighborhood of $(u_0, 0)$. A natural approach is to look for an analytic solution of the form $u(t) = \sum_{j=0}^{\infty} a_j t^j$ and determine the coefficients a_j through the initial condition and repeatedly differentiating the equation: $a_0 = u(0) = u_0$, $a_1 = u'(0) = f(u_0, 0)$, $a_2 = 2u''(0) = 2[f_u(u_0, 0)u'(0) + f_t(u_0, 0)]$, etc. Cauchy was able to show the convergence of the obtained series through his *method of majorants*. This theorem was extended by Cauchy, and later by Kowalevskaya, to the initial value problem for partial differential equations for the form:

$$\frac{\partial u(x,t)}{\partial t} = f(\partial_x u(x,t), u(x,t), x, t), \quad u(x,0) = g(x), \tag{9.1}$$

for (x,t) near $(x_0, 0)$, where $\partial_x u(x,t)$ stands for the gradient vector of $u(x,t)$ in the x-variables, and $f(p, u, x, t)$ is analytic in (p, u, x, t) near $(\partial_x g(x_0), g(x_0), x_0, 0)$.

For the initial value problem for higher order partial differential equations, Cauchy discussed a procedure to reduce the problem to a (larger) system of first order partial differential equations of the form above. Kowalevskaya clarified the type of equations for which the method initiated by Cauchy would work (Kowalevskaya in fact was not aware of Cauchy's work).

Kowalevskaya pointed out that although a formal power series solution can be determined for the initial value problem of the heat equation

$$\partial_t u(x,t) = \partial_x^2 u(x,t), \quad u(x,0) = g(x),$$

the power series does not need to converge. In fact, for $g(x) = \frac{1}{1-x}$, the formal power series for a solution $u(x,t)$ at $(x,0)$ would be

$$\sum_{j=0}^{\infty} \frac{g^{(2j)}(x)}{j!} t^j = \sum_{j=0}^{\infty} \frac{(2j)!}{j!(1-x)^{2j+1}} t^j,$$

which is not convergent for any $t \neq 0$! Thus Cauchy's theorem is not valid if one allows terms of the kind $\partial_x^\alpha u(x,t)$ for $|\alpha| > 1$ in the right hand side of (9.1).

In the following we will formulate several versions of the Cauchy-Kowalevskaya's theorem so that we can conclude the existence of local analytic solutions without having to go through the reduction process to check whether the given problem can be reduced to one of the form (9.1). We will explain Cauchy-Kowalevskaya's theorem first in the context of the initial value problem for linear partial differential equations and the initial value is prescribed with respect to a distinguished variable t; then we will discuss how to formulate the Cauchy-Kowalevskaya's theorem when the initial surface is a general *non-characteristic* surface; and finally we describe the theorem for nonlinear partial differential equations.

9.1 Cauchy-Kowalevskaya Theorem: Linear Case with Special Non-characteristic Initial Surface

Let's first examine the case of a linear differential operator in the form

$$P = \partial_t^m + \sum_{j<m,\, j+|\alpha|\leq m} c_{j,\alpha}(x,t) \partial_t^j \partial_x^\alpha,$$

where the coefficients $c_{j,\alpha}(x,t)$ are analytic in (x,t) around a point $(x_0, 0)$ on the initial surface $t = 0$. We seek to solve

$$\begin{cases} Pu = f(x,t), \quad \text{near } (x_0,0), \\ \partial_t^j u(x,0) = g_j(x), \ j = 0, \cdots, m-1, \quad \text{near } x_0, \end{cases} \tag{9.2}$$

where $f(x,t)$, and $g_j(x)$ are analytic functions around $(x_0, 0)$ and x_0 respectively.

Theorem 9.1 (Linear case with special non-characteristic initial surface)**.** *Suppose that $c_{j,\alpha}(x,t)$ are analytic in a neighborhood V around $(x_0,0)$. Then there is a neighborhood $U \subset V$ of $(x_0,0)$, such that for any $f(x,t)$ analytic in U_1 around $(x_0,0)$, and any $g_j(x)$ analytic in W around x_0, there is a unique analytic solution to (9.2) in $U \cap U_1 \cap (W \times \mathbb{R})$.*

Remark 9.1. This local existence of analytic solution does not imply the wellposedness of the Cauchy problem in the usual sense. For example, the above theorem applies to both $P_1 = \partial_t^2 - \partial_x^2$ and $P_2 = \partial_t^2 + \partial_x^2$ with $\{t=0\}$ as initial surface, yet the Cauchy problem (with respect to t) is wellposed for P_1, but not for P_2.

Remark 9.2. Although the Cauchy-Kowalevskaya theorem can not be applied to the initial value problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & \text{for } (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) = g(x) & \text{for } x \in V \subset \mathbb{R}, \end{cases}$$

it can be applied to

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = 0 & \text{for } (x,t) \in V \subset \mathbb{R} \times \mathbb{R}, \\ u(0,t) = h_0(t) & \text{for } t \in W \subset \mathbb{R}, \\ u_x(0,t) = h_1(t) & \text{for } t \in W \subset \mathbb{R}, \end{cases}$$

in a neighborhood V of $(0,t_0)$ for any $t_0 \in \mathbb{R}$, even though this Cauchy problem is not well-posed. In fact, one can choose analytic h_0 and $h_1 \equiv 0$ such that the formal solution $u(x,t) = \sum_{k=0}^{\infty} \frac{h_0^{(k)}(t)}{(2k)!} x^{2k}$ is convergent for all $(x,t) \in \mathbb{R} \times \mathbb{R}$ and smooth in $\mathbb{R} \times \mathbb{R}$; in addition, $u(x,t) \equiv 0$ for $(x,t) \in \mathbb{R} \times (-\infty, 0]$! — see the text of either F. John or J. Rauch for details.

Remark 9.3. There is no general local existence result when the analyticity assumptions are dropped. In 1956 H. Lewy constructed the first example of a linear differential equation that has smooth coefficients but has no solution *anywhere*.

Example 9.1 (Application to local existence of isothermal coordinates on a surface with analytic metric)**.** Given a local Riemannian metric on a surface M near a point $z_0 \in M$:

$$ds^2 = E(x,y)dx^2 + 2F(x,y)dxdy + G(x,y)dy^2,$$

one is interested in knowing whether there exists a local change of coordinates, $(x,y) \mapsto (u,v)$ and a conformal factor $\Lambda(x,y) > 0$ such that in the new local coordinates (u,v)

$$\Lambda(x,y)ds^2 = du^2 + dv^2. \tag{9.3}$$

When the metric coefficients $E(x,y), F(x,y), G(x,y)$ are real analytic functions of (x,y) near z_0, the answer is affirmative and can be proved via the help of Cauchy-Kowalevskaya Theorem as follows.

ds^2 being a Riemannian metric implies that $E(x,y), G(x,y) > 0$ and
$J = E(x,y)G(x,y) - F(x,y)^2 > 0$, so we can "complete squre" and write

$$ds^2 = \frac{\left(E(x,y)dx + (F(x,y) + i\sqrt{J(x,y)})dy\right)\left(E(x,y)dx + (F(x,y) - i\sqrt{J(x,y)})dy\right)}{E(x,y)}.$$

As a consequence, if suffices to find a $W(x,y)$ such that

$$W(x,y)\left(E(x,y)dx + (F(x,y) + i\sqrt{J(x,y)})dy\right) = dw = du + idv \tag{9.4}$$

for some function $w(x,y) = u(x,y) + iv(x,y)$, for then,

$$\overline{W(x,y)}\left(E(x,y)dx + (F(x,y) - i\sqrt{J(x,y)})dy\right) = d\bar{w} = du - idv,$$

and

$$E(x,y)|W(x,y)|^2 ds^2 = (du + idv)(du - idv) = du^2 + dv^2.$$

(9.4) amounts to

$$W(x,y)E(x,y) = w_x(x,y) \quad \text{and} \quad W(x,y)(F(x,y) + i\sqrt{J(x,y)}) = w_y(x,y),$$

which has a local solution $(W(x,y), w(x,y))$ iff there is a solution $W(x,y)$ to

$$[W(x,y)E(x,y)]_y = \left[W(x,y)(F(x,y) + i\sqrt{J(x,y)})\right]_x \tag{9.5}$$

When ds^2 is an analytic metric, we can apply the Cauchy-Kowalevskaya Theorem to (9.5) near any given point, with a segment parallel to either the x-axis or y-axis as initial curve, to obtain the existence of a local analytic solution $W(x,y)$, which would then provide the corresponding $w(x,y) = (u(x,y), v(x,y))$ as local isothermal coordinates.

$W(x,y)$ in the context of (9.4) would be called an integrating factor. If the coefficients of the differential $E(x,y)dx + (F(x,y) + i\sqrt{J(x,y)})dy$ were real, the existence of an integral factor would be available from elementary ODE even when the coefficients are only C^1 functions—we will review this below.

In the context of (9.5), if we write out (9.5) in terms of the real and imaginary parts of $W(x,y) = U(x,y) + iV(x,y)$, we would have

$$E\begin{pmatrix} U \\ V \end{pmatrix}_y = \begin{pmatrix} F & -\sqrt{J} \\ \sqrt{J} & F \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} F_x - E_y & -(\sqrt{J})_x \\ (\sqrt{J})_x & F_x - E_y \end{pmatrix}\begin{pmatrix} U \\ V \end{pmatrix}. \tag{9.6}$$

When the analyticity assumption on the coefficients $E(x,y), F(x,y), G(x,y)$ is dropped, none of the theorems we have learned so far can be applied to prove the existence of a local solution to (9.6).

The first ingredient of Cauchy's method is the (formal) determination of the Taylor series of a possible analytic solution $u(x,t)$ around $(x_0,0)$. This is relatively straight forward in this set up: first note that any derivatives of u at $(x_0,0)$ whose order in t is $m-1$ or less along $t=0$ can be determined by the initial data alone: $\partial_x^\alpha u(x,0) = \partial_x^\alpha g_0(x)$ (in fact for all α), and for any $1 \le j < m$,

$$\partial_t^j \partial_x^\alpha u(x,0) = \partial_x^\alpha g_j(x).$$

Also

$$\begin{aligned} \partial_t^m \partial_x^\beta u(x,0) &= \partial_x^\beta \partial_t^m u(x,0) \\ =& \partial_x^\beta \left(- \sum_{j<m,\, j+|\alpha|\le m} c_{j,\alpha}(x,0) \partial_t^j \partial_x^\alpha u(x,0) + f(x,0) \right) \\ =& \partial_x^\beta \left(- \sum_{j<m,\, j+|\alpha|\le m} c_{j,\alpha}(x,0) \partial_x^\alpha g_j(x) + f(x,0) \right). \end{aligned}$$

Next we simply differentiate the equation in t and differentiate the initial conditions in x inductively to represent all $\partial_t^j \partial_x^\alpha u(x_0,0)$ in terms of given Cauchy data, the right hand side, and lower order derivatives.

The second ingredient is to prove the convergence of the constructed power series. This was done by Cauchy and Kowalevskaya by the so called **majorant** method for power series, and it is here that the non-characteristic assumption of $\{t=0\}$ is crucially used.

Here we will not provide a full proof as given by Cauchy-Kowalevskaya. Instead, we will describe a reduction procedure which is often used to reduce the general case to a first order system. This is done by introducing new variables and using the compatibility conditions as new equations: set $U = (u, U_{j,\alpha} = \partial_t^j \partial_x^\alpha u \mid j + |\alpha| < m)$. Then for any $j + |\alpha| < m-1$,

$$\partial_t U_{j,\alpha} = \partial_t^{j+1} \partial_x^\alpha u = U_{j+1,\alpha},$$

and for any $j + |\alpha| = m-1$ with $\alpha \ne 0$,

$$\partial_t U_{j,\alpha} = \partial_t^{1+j} \partial_x^\alpha u = \partial_{x_{\alpha_1}} U_{1+j,\alpha-\alpha_1},$$

where α_1 is the first component of α that is not zero. Finally

$$\partial_t U_{m-1,0} = \partial_t^m u = -\sum_{j<m,\, j+|\alpha|\le m} c_{j,\alpha}(x,t)\partial_t^j \partial_x^\alpha u(x,t) + f(x,t),$$

where terms with $j + |\alpha| \le m-1$ are linear combinations of $U_{j,\alpha}$, and terms with $j+|\alpha| = m$ and $j < m$ can be written as linear combinations of $\partial_{x_{\alpha_1}} U_{j,\alpha-\alpha_1}$. So we are led to studying a system of the form

$$\begin{cases} \partial_t u_i = \sum_{j=0}^{N}\sum_{\alpha=1}^{n} c_{ij}^\alpha(x,t)\partial_\alpha u_j(x,t) + \sum_{j=0}^{N} d_{ij}(x,t)u_j(x,t) + f_i(x,t), & \text{for } i = 0,\cdots,N \\ u_i(x,0) = g_i(x), & \text{for } i=0,\cdots,N, \end{cases}$$

or using vector notation $u = (u_0, u_1, \cdots, u_N)^T$,

$$\begin{cases} \partial_t u = \sum_{\alpha=1}^{n} C^\alpha(x,t)\partial_\alpha u(x,t) + D(x,t)u(x,t) + F(x,t), \\ u(x,0) = G(x). \end{cases} \tag{9.7}$$

9.2 Notion of a Non-characteristic Initial Manifold

Recall that the notion of a characteristic initial manifold (hypersurface) arose in the discussion for the Hadamard-Petrovsky well-posedness for the Cauchy problems, and was formulated as follows.

We represent an initial surface Σ as the level set of a defining functions σ: $\Sigma = \{x : \sigma(x) = 0\}$, where $\nabla_x \sigma(x) \ne 0$ along Σ. One idea in defining the the notion of a non-characteristic initial surface with respect to a linear differential operator $P = \sum_{|\alpha|\le m} c_\alpha(x)\partial_x^\alpha$ is to make a change of variables, locally, to "flatten" Σ. For instance, if $x_0 \in \Sigma$ is such that $\partial_{x_n}\sigma(x_0) \ne 0$, then in a neighborhood of x_0, Σ can be represented as a graph x_n in terms of $x_1, \cdots, x_{n-1}$. In fact, $x = (x', x_n) \mapsto (y', \tau)$, with $y' = x'$, and $\tau = \sigma(x)$ is a local diffeomorphism. If we adopt (y',τ) as new coordinates and set $v(y',\tau) = u(x',x_n)$, then Pu is expressed as a linear differential operator $\tilde{P}v$ of the same order m, and the coefficient of $\partial_\tau^m v$ at $(y',0)$ is given by $\sum_{|\alpha|=m} c_\alpha(x)(\nabla\sigma(x))^\alpha$, where $x = (x',x_n) \mapsto (y',0)$. This is from the chain rule

$$\text{for } 1 \le j \le n-1, \partial_{x_j} = \partial_{y_j} + \sigma_{x_j}\partial_\tau, \quad \partial_{x_n} = \sigma_{x_n}\partial_\tau.$$

So for any $|\alpha| = k$,

$$\partial_x^\alpha u(x) = (\nabla\sigma(x))^\alpha \partial_\tau^k v + R \tag{9.8}$$

where R stands for terms of differentiation order not higher than k and with $k-1$ or fewer derivatives in τ.

Definition. $\Sigma = \{x : \sigma(x) = 0\}$ is called non-characteristic with respect to P at $x_0 \in \Sigma$ if $\sum_{|\alpha|=m} c_\alpha(x_0)(\nabla\sigma(x_0))^\alpha$ is non-degenerate. $\Sigma = \{x : \sigma(x) = 0\}$ is called non-characteristic with respect to P if it is non-characteristic at every point on it.

The level surface $\{\sigma = 0\}$ is called characteristic with respect to P if $\sum_{|\alpha|=m} c_\alpha(x)(\nabla\sigma(x))^\alpha$ is degenerate and $\nabla\sigma \neq 0$ for every point on the surface.

Remark 9.4. The definition above used non-degeneracy of $\sum_{|\alpha|=m} c_\alpha(x_0)(\nabla\sigma(x_0))^\alpha$ as a definition for non-characteristic, as this notion also works if we are dealing with an $N \times N$ system, in which each $c_\alpha(x_0)$ would be an $N \times N$ matrix.

Since σ is not uniquely determined from Σ, this freedom is also reflected in $\sum_{|\alpha|=m} c_\alpha(x)(\nabla\sigma(x))^\alpha$, which is homogeneous in $\nabla\sigma$. Thus whether a hypersurface is non-characteristic with respect to P is independent of parametrization for the surface.

Definition. A (co-)vector $\xi \neq 0 \in \mathbb{R}^n$ is called a characteristic direction for P at x_0 if

$$\sum_{|\alpha|=m} c_\alpha(x_0)\xi^\alpha = 0 (\text{or degenerate when dealing with an } N \times N \text{ system.})$$

(the covector nature of ξ becomes clear when the differential operator is given on a manifold, and one needs to work with its representations in different coordinate patches.)

P is called elliptic at x_0 if it has no characteristic direction at x_0, *i.e.*, for any $\xi \in \mathbb{R}^n$,

$$\sum_{|\alpha|=m} c_\alpha(x_0)\xi^\alpha = 0 \Longrightarrow \xi = 0.$$

P is called elliptic in a region if it is elliptic at every point in this region.

Remark 9.5. When dealing with a system, the coefficients c_α are interpreted as matrices, a (co-)vector $\xi \neq 0 \in \mathbb{R}^n$ is called a characteristic direction for P at x_0 if the matrix

$$\sum_{|\alpha|=m} c_\alpha(x_0)\xi^\alpha$$

is singular. Thus a level surface of σ is called characteristic with respect to P if $\nabla\sigma \neq 0$ and

$$\det\left(\sum_{|\alpha|=m} c_\alpha(x)(\nabla\sigma(x))^\alpha\right) = 0,$$

for every point on it.

Since the equation for a characteristic direction ξ is a homogeneous equation in ξ, only in dimension 2, do we expect to get a finite number of characteristic directions, up to the homogenuity scaling,

Exercise 9.2.1. Prove that the system (9.6) is elliptic at every point.

Example 9.2. For a first order linear partial differential operator

$$P = a_0(x,t)\partial_t + \sum_{i=1}^{n} a_i(x,t)\partial_{x_i},$$

the initial surface $\sigma(x,t) = 0$ is characteristic if

$$a_0(x,t)\partial_t\sigma(x,t) + \sum_{i=1}^{n} a_i(x,t)\partial_{x_i}\sigma(x,t) = 0, \quad \text{for } (x,t) \text{ on } \sigma(x,t) = 0,$$

which means geometrically, when $\sigma(x,t) = 0$ is a non-degenerate surface, that the vector field

$$(x,t) \mapsto (a_0(x,t), a_1(x,t), \cdots, a_n(x,t))$$

is tangential to the surface $\sigma(x,t) = 0$, when (x,t) is on $\sigma(x,t) = 0$.

In the case $n = 1$, the initial surface is simply a curve. It is characteristic iff it is an integral curve of the vector field $(x,t) \mapsto (a_0(x,t), a_1(x,t))$. But when $n > 1$, the notion of a characteristic (initial) surface is different from that of characteristic curves for a first order scalar PDE; they are, however, related: an initial surface is characteristic with respect to P iff it consists of union of characteristic curves of P.

Example 9.3. For a second order linear partial differential operator

$$P = \sum_{i,j=0}^{n} a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=0}^{n} b_i(x)\partial_{x_i} + c(x),$$

the level surfaces of σ are characteristic with respect to P if σ is a non-trivial solution ($\nabla\sigma \neq 0$) to

$$\sum_{i,j=0}^{n} a_{ij}(x)\partial_{x_i}\sigma(x)\partial_{x_j}\sigma(x) = 0,$$

which is a nonlinear first order PDE for $\sigma(x)$—again, compare with the notion of characteristic curves and of a non-characteristic Cauchy problem for a scalar first order PDE.

For $P = \partial_t^2 - \Delta_x$, this equation becomes $|\partial_t\sigma(t,x)|^2 - |\nabla_x\sigma(t,x)|^2 = 0$. If we further assume that $\sigma(t,x)$ has the form $t - \phi(x)$, then $\phi(x)$ must satisfy $|\nabla_x\phi(x)| = 1$. At the same time the surface $t = \phi(x)$ is non-characteristic with respect to this P, if $|\nabla_x\phi(x)| \neq 1$ on $\{(x,t) : t = \phi(x)\}$.

In the case when the initial surface Σ is a hyperplane given by $\nu_0 t + \sum_{j=1}^{n} \nu_j x_j = 0$, Σ is non-characteristic with respect to $P = \partial_t^2 - \Delta_x$, iff $\nu_0^2 - \sum_{j=1}^{n} \nu_j^2 \neq 0$. If one needs the examine the well-posedness of the Cauchy problem corresponding to the operator $P = \partial_t^2 - \Delta_x$ with Σ as the initial surface, then the Hadamard-Petrovsky condition amounts to examining dependence of $Im(\tau)$ on ξ, where

$$(\xi_0 + \nu_0\tau)^2 - \sum_{j=1}^{n}(\xi_j + \nu_j\tau)^2 = 0,$$

which comes from examining solutions to $Pu = 0$ of the form
$u(x,t) = e^{i[(\xi_0+\nu_0\tau)t+\sum_{j=1}^{n}(\xi_j+\nu_j\tau)x_j]}$.

Example 9.4. For the same operator P as in the previous example, recall that if we make a change of variables $x = (x_1, \ldots, x_n) \mapsto u = (y_1, \ldots, y_n)$, and define $w(y) = z(x)$ under this change of variables, then $Pz = \tilde{P}w$, where the second order differentiation terms on w is given by $\sum_{k,l=1}^{n} \tilde{a}_{kl}\partial_{kl}^2 w(y)$, where $\tilde{a}_{kl}(y) = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i} y_k(x)\partial_{x_j} y_l(x)$. When $n = 2$, it is reasonable to impose two conditions on the three coefficients $\tilde{a}_{kl}(y)$ to simplify them — this amounts to two conditions on the two unknowns $y_1(x), y_2(x)$. One possibility to consider is to impose $\tilde{a}_{11} = \tilde{a}_{22} = 0$, which amounts to

$$\sum_{i,j=1}^{2} a_{ij}(x)\partial_{x_i} y_1(x)\partial_{x_j} y_1(x) = 0 \quad \text{and} \quad \sum_{i,j=1}^{2} a_{ij}(x)\partial_{x_i} y_2(x)\partial_{x_j} y_2(x) = 0.$$

This means that both $y_1(x)$ and $y_2(x)$ need to solve the characteristic equation for P! In order for $x = (x_1, x_2) \mapsto y = (y_1, y_2)$ to form a (local) change of variables, we need $\{\nabla y_1(x), \nabla y_2(x)\}$ be linearly independent, i.e., we need to have a pair of linearly independent characteristic directions for P near x. This can be done iff the quadratic characteristic equation $a_{11}(x)\lambda^2 + 2a_{12}(x)\lambda + a_{22}(x) = 0$ has a pair of *real* distinct solutions (assuming $a_{11}(x) \neq 0$ for simplicity). This can be characterized algebraically: when the matrix

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{12}(x) & a_{22}(x) \end{pmatrix}$$

has a pair of *real* distinct eigenvalues $\lambda_i(x)$, $i = 1, 2$, we can find a pair $u_i(x)$ solving $\partial_{x_1} u_i(x) - \lambda_i(x)\partial_{x_2} u_i(x) = 0$, and make the local change of variables $x = (x_1, x_2) \mapsto y = (y_1, y_2)$ so that in the (y_1, y_2) coordinates, $\tilde{P}w$ has the form $\tilde{w}_{y_1 y_2}+$ lower order terms $= 0$.

When P fails to have any characteristic direction at x—this means that P is elliptic, we won't be able to find any nontrivial solution to its characteristic equation, but we could look for $(y_1(x), y_2(x))$ such that

$$\begin{cases} \sum_{i,j=1}^{2} a_{ij}(x)\partial_{x_i}y_1(x)\partial_{x_j}y_1(x) = \sum_{i,j=1}^{2} a_{ij}(x)\partial_{x_i}y_2(x)\partial_{x_j}y_2(x) \\ \sum_{i,j=1}^{2} a_{ij}(x)\partial_{x_i}y_1(x)\partial_{x_j}y_2(x) = 0. \end{cases} \tag{9.9}$$

If such a pair of solutions can be found, let $\Lambda(x) = \sum_{i,j=1}^{2} a_{ij}(x)\partial_{x_i}y_1(x)\partial_{x_j}y_1(x)$, which is non-zero, then $\tilde{P}w$ would have the form $\Lambda(x)\left(w_{y_1y_1} + w_{y_2y_2}\right) +$ lower order terms $= 0$.

The system of quadratic equations (9.9) for $(y_1(x), y_2(x))$ can be reduced to a simplified form as follows. From the second equation, written as

$$\left(a_{11}\partial_{x_1}y_1 + a_{21}\partial_{x_2}y_1\right)\partial_{x_1}y_2 + \left(a_{12}\partial_{x_1}y_1 + a_{22}\partial_{x_2}y_1\right)\partial_{x_2}y_2 = 0,$$

we see that

$$\begin{cases} \partial_{x_1}y_2 = -W\left(a_{12}\partial_{x_1}y_1 + a_{22}\partial_{x_2}y_1\right) \\ \partial_{x_2}y_2 = W\left(a_{11}\partial_{x_1}y_1 + a_{21}\partial_{x_2}y_1\right). \end{cases} \tag{9.10}$$

for some W. Combining these with the first equation in (9.9), we see that

$$W = 1/\sqrt{a_{11}(x)a_{22}(x) - a_{12}^2(x)} \text{— } W \text{ is real in this case.}$$

Thus (9.10) becomes

$$\begin{cases} \partial_{x_1}y_2 = -\dfrac{a_{12}\partial_{x_1}y_1 + a_{22}\partial_{x_2}y_1}{\sqrt{a_{11}(x)a_{22}(x) - a_{12}^2(x)}} \\ \partial_{x_2}y_2 = \dfrac{a_{11}\partial_{x_1}y_1 + a_{21}\partial_{x_2}y_1}{\sqrt{a_{11}(x)a_{22}(x) - a_{12}^2(x)}} \end{cases} \tag{9.11}$$

When the coefficients $a_{ij}(x)$ are analytic functions of x, one could use Cauchy-Kowalevskaya Theorem to find local solutions to (9.11). Without the analyticity assumptions, one has to find other ways to solve (9.11). One can eliminate y_2 from above to obtain an equation for $y_1(x)$:

$$\left(\frac{a_{11}\partial_{x_1}y_1 + a_{12}\partial_{x_2}y_1}{\sqrt{a_{11}(x)a_{22}(x) - a_{12}^2(x)}}\right)_{x_1} + \left(\frac{a_{12}\partial_{x_1}y_1 + a_{22}\partial_{x_2}y_1}{\sqrt{a_{11}(x)a_{22}(x) - a_{12}^2(x)}}\right)_{x_2} = 0. \tag{9.12}$$

A similar equation can be obtained for $y_2(x)$. It turns out that the operator on the left hand side of (9.12) is the Beltrami-Laplace operator associated with the Riemannian metric constructed using $(a_{ij}(x))$. The variational theory we learned earlier this semester on divergence form elliptic equations can be applied to show existence of solutions with appropriate regularities to (9.12) with prescribed boundary values under appropriate regularity assumptions on $(a_{ij}(x))$.

In **Example 1**, we proved local existence of isothermal coordiantes for an analytic Riemannian metric on a surface. When the analyticity assumption is dropped, the local existence of isothermal coordiantes can be formulated in terms of existence of local solutions to a Beltrami-Laplace equation similar to (9.12). We identify $E(x,y) = a_{11}(x,y)$, $F(x,y) = a_{12}(x,y)$, and $G(x,y) = a_{22}(x,y)$, and look for $u = u(x,y)$, $v = v(x,y)$ such that

$$a_{11}(x,y)dx^2 + 2a_{12}(x,y)dxdy + a_{22}(x,y)dy^2 = \Lambda(x,y)(du^2 + dv^2)$$

holds, which, using $du = u_x dx + u_y dy$ and $dv = v_x dx + v_y dy$, is equivalent to the system

$$\begin{cases} \Lambda(x,y)\left(u_x^2(x,y) + v_x^2(x,y)\right) = a_{11}(x,y), \\ \Lambda(x,y)\left(u_x(x,y)u_y(x,y) + v_x(x,y)v_y(x,y)\right) = a_{12}(x,y), \\ \Lambda(x,y)\left(u_y^2(x,y) + v_y^2(x,y)\right) = a_{22}(x,y), \end{cases} \tag{9.13}$$

The system (9.13) can be simplified as follows.

Exercise 9.2.2. (i). Prove that $\sqrt{J(x,y)} = \Lambda(x,y)\left[u_x(x,y)v_y(x,y) - u_y(x,y)v_x(x,y)\right]$, where $J(x,y) = a_{11}(x,y)a_{22}(x,y) - a_{12}(x,y)^2$.

(ii). Prove that

$$\begin{aligned} v_x(x,y) &= -\frac{a_{11}(x,y)u_y(x,y) - a_{12}(x,y)u_x(x,y)}{\sqrt{J(x,y)}}, \\ v_y(x,y) &= \frac{a_{22}(x,y)u_x(x,y) - a_{12}(x,y)u_y(x,y)}{\sqrt{J(x,y)}}. \end{aligned}$$

HINT: Interpret $u_x(x,y)u_y(x,y)+v_x(x,y)v_y(x,y) = (u_x(x,y), v_x(x,y))\cdot(u_y(x,y), v_y(x,y))$, and $u_x(x,y)v_y(x,y) - u_y(x,y)v_x(x,y) = (u_x(x,y), v_x(x,y)) \cdot (v_y(x,y), -u_y(x,y))$ geometrically as related to orthogonal projections.

(iii). Prove that

$$\left(\frac{a_{11}(x,y)u_y(x,y) - a_{12}(x,y)u_x(x,y)}{\sqrt{J(x,y)}}\right)_y + \left(\frac{a_{22}(x,y)u_x(x,y) - a_{12}(x,y)u_y(x,y)}{\sqrt{J(x,y)}}\right)_x = 0,$$

and

$$\left(\frac{a_{11}(x,y)v_y(x,y) - a_{12}(x,y)v_x(x,y)}{\sqrt{J(x,y)}}\right)_y + \left(\frac{a_{22}(x,y)v_x(x,y) - a_{12}(x,y)v_y(x,y)}{\sqrt{J(x,y)}}\right)_x = 0.$$

Compare with (9.12).

In setting up an initial value problem along a general surface Σ, one can prescribe the initial data in two ways. One way is to prescribe $u(x)$ restricted to Σ as $g_0(x)$, and normal derivatives of $u(x)$ along Σ up to order $m-1$:

$$\frac{\partial^j u(x)}{\partial \nu^j} = g_j(x), \qquad \text{for } x \in \Sigma \text{ and } j = 1, \dots, m-1.$$

Here $\frac{\partial^j u(x)}{\partial \nu^j}$ can be defined through

$$\frac{\partial^j u(x)}{\partial \nu^j} = \frac{d^j u(x + s\nu(x))}{ds^j}\Big|_{s=0} = \sum_{|\alpha|=j} \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n} \frac{\partial^j u(x)}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}.$$

So this can also be interpreted as prescribing these specific linear combinations of the mixed partial derivatives $\frac{\partial^j u(x)}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}}$ of u. J. Rauch has some criticism for this formulation. But the main issue seems the difficulty of using this formulation on a manifold setting. Given $u(x) = g_0(x)$ for $x \in \Sigma$, tangential derivatives along Σ of $u(x)$ of any order can be computed through those of $g_0(x)$ and normal derivatives of $u(x)$ along Σ up to order one less. Together with the prescribed normal derivatives of $u(x)$ along Σ up to order $m-1$, one can determine all partial derivatives of $u(x)$ of order $m-1$ or lower along Σ. These partial derivatives satisfy the compatibility conditions of mixed derivatives along Σ. Σ is non-characteristic for P iff all partial derivatives of $u(x)$ of order m (and therefore higher order derivatives as well) restricted to Σ can be determined through the equation and the initial data.

Here are some more details on the implementation of this formulation. According to (9.8) in locally flattening Σ under the change of variables $x = (x', x_n) \mapsto y = (y', \tau)$, $v(y', \tau) = u(x', x_n)$, where $\tau = \sigma(x)$, and Σ is locally described by $\sigma(x) = 0$, for each $x \in \Sigma$, $\partial_x^\alpha u(x)$ for $|\alpha| \le k$ is determined by $\partial_y^\beta \partial_\tau^j v(y', 0)$ for $0 \le |\beta| + j \le k$. It suffices to verify that each $\partial_\tau^j v(y', 0)$ is determined by $\frac{\partial^l u(x)}{\partial \nu^l}$ for $0 \le l \le j$. But this follows from (9.8):

$$\begin{aligned}\frac{\partial^l u(x)}{\partial \nu^l} &= \sum_{|\alpha|=l} \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n} \frac{\partial^l u(x)}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}} \\ &= \sum_{|\alpha|=l} \nu^\alpha (\nabla\sigma(x))^\alpha \partial_\tau^l v(y', 0) + R \\ &= |\nabla\sigma(x)|^l \partial_\tau^l v(y', 0) + R\end{aligned}$$

as $\nabla\sigma(x) = |\nabla\sigma(x)|\nu(x)$, $\sum_{|\alpha|=l} \nu^{2\alpha} = (\nu_1^2 + \cdots + \nu_n^2)^l = 1$, so $\sum_{|\alpha|=l} \nu^\alpha (\nabla\sigma(x))^\alpha = |\nabla\sigma(x)|^l$, where R stands for terms of differentiation order l or less and with at most $l-1$ derivatives in τ. So we can see inductively that $\partial_\tau^j v(y', 0)$ is determined by $\frac{\partial^l u(x)}{\partial \nu^l}$ for $0 \le l \le j$.

Another way to prescribe the initial data is to prescribe all partial derivatives of $u(x)$ of order $m-1$ or lower along Σ, subject to the natural compatibility conditions. An easy way to do this is to give a function $g(x)$, C^m or analytic near Σ, such that $\partial_\beta u = \partial_\beta g(x)$, along Σ for all $|\beta| \le m-1$.

Remark 9.6. The notion of non-characteristic initial surface implies the following consequence: Suppose P is an m-th order linear differential operator, and u, v are two C^m functions in a neighborhood U such that $Pu \equiv Pv$ in U. If Σ is a hypersurface which is non-characteristic with respect to P and $\partial^\alpha u = \partial^\alpha v$ in $U \cap \Sigma$, for all $|\alpha| \le m-1$, then $\partial^\alpha u = \partial^\alpha v$ in $U \cap \Sigma$, for all $|\alpha| = m$; and if the coefficients of P and the u and v here are assumed to be C^∞ to begin with, then $\partial^\alpha u = \partial^\alpha v$ in $U \cap \Sigma$, for all α. Thus if a piecewise smooth solution to an m-th order linear differential equation has continuous derivatives of order up to and including $m-1$ across the hypersurface, but has discontinuity across a hypersurface in its m-th order derivatives, then the hypersurface must be characteristic.

Remark 9.7. When the initial surface is characteristic, one may still be able to determine the formal power series expansion of a potential solution from the equation and initial data, as in the case for the heat equation $u_t - u_{xx} = 0$ along the characteristic initial surface $\{t = 0\}$, but the convergence of this constructed series is not guaranteed. Furthermore, one may not be able to prescribe freely all Cauchy data along the initial surface. For the case of the heat equation here, Cauchy data in the general sense would mean $g(x) = u(x, 0)$ and $h(x) = u_t(x, 0)$, but we can't prescribe $h(x)$ freely, as it has to satisfy $h(x) = g''(x)$.

Here is another example. Consider the Cauchy problem for $Pu = \partial^2_{xy}u + a\partial_x u + b\partial_y u + cu = 0$, where the initial data is given on $\Sigma = \{(x, y) : y = 0\}$. Presumably we should prescribe $u(x, 0) = g(x)$ and $u_y(x, 0) = h(x)$ as Cauchy data. However, if a C^2 solution exists whose domain includes the line on which the initial data is given, then $h'(x) + ag'(x) + bh(x) + cg(x) = 0$. This means that we can't prescribe $h(x)$ freely, and this is due the Σ being characteristic with respect to P. Furthermore, we will face the same difficulty when trying to determine higher order derivatives of $u(x, y)$ along $y = 0$.

9.3 Notion of Non-characteristic Initial Data for Quasilinear Equations

An operator of the form $P = \sum_{|\alpha|=m} c_\alpha(x, \partial_x^\beta u)\partial_x^\alpha u(x) + D(x, \partial_x^\beta u)$ is called a quasilinear operator if the β's in $c_\alpha(x, \partial_x^\beta u)$ and $D(x, \partial_x^\beta u)$ satisfy $|\beta| \le m-1$. In considering a Cauchy problem for a quasilinear operator P, the notion of non-characteristic initial data depends not only on the operator and the initial surface, but also on the prescribed initial data.

Definition. $\Sigma = \{x : \sigma(x) = 0\}$ is called non-characteristic with respect to P at $x_0 \in \Sigma$ *on the initial data* g if $\sum_{|\alpha|=m} c_\alpha(x_0, \partial_x^\beta g(x_0))(\nabla\sigma(x_0))^\alpha$ is non-degenerate. $\Sigma = \{x : \sigma(x) = 0\}$ is called non-characteristic with respect to P and the initial data g if it is non-characteristic at every point on it.

Example 9.5. Consider the quasilinear problem

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = 0, \\ \qquad\qquad\qquad\quad u(x,0) = g(x). \end{cases}$$

The initial curve $\{(x,t) : t = 0\}$ is non-characteristic for any initial data g. However, for the quasilinear problem

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) = 0, \\ \qquad\qquad\qquad\quad u(0,t) = h(t), \end{cases}$$

the initial curve $\{(x,t) : x = 0\}$ is non-characteristic at $(0,t)$ for the given data h iff $h(t) \neq 0$.

Example 9.6. Recall that for the quasilinear problem (8.1),

$$\begin{cases} \displaystyle\sum_{i=1}^n a_i(x, u(x))\partial_{x_i} u(x) = c(x, u(x)), \\ \qquad\qquad\qquad u(x) = g(x), \quad \text{for } x \in \Sigma, \end{cases}$$

The initial data Σ and g are non-characteristic at $\bar{x} \in \Sigma$, iff

$$\text{the vector } (a_1(\bar{x}, g(\bar{x})), \cdots, a_n(\bar{x}, g(\bar{x}))) \text{ is transversal to } \Sigma \text{ at } \bar{x}.$$

(8.1) is a first order PDE for a *scalar unknown* u, for which we have a different, but related concept of *characteristic curves*, discussed in the precious chapter.

Remark 9.8. Since the notion of non-characteristic also depends on the initial data, it does not make sense to seek a (non-)characteristic initial surface in the absence of providing initial data. However, certain equations have no characteristic directions for any initial data. E.g. the mean curvature equation in the Euclidean space $\mathbb{R}^n$:

$$\nabla\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H(x).$$

At any point $x \in \mathbb{R}^n$, for any hypersurface $\Sigma = \{x \in \mathbb{R}^n : \sigma(x) = 0\}$, and any prescribed initial data on Σ, $u(x), \nabla u(x)$ would be given. A characteristic direction ξ at x would have to satisfy

$$\frac{|\xi|^2}{\sqrt{1+|\nabla u|^2}} - \frac{(\xi\cdot\nabla u(x))^2}{(1+|\nabla u|^2)^{3/2}} = 0.$$

But by the Cauchy-Schwarz inequality, $\frac{(\xi\cdot\nabla u(x))^2}{(1+|\nabla u|^2)^{3/2}} \leq \frac{|\nabla u(x)|^2}{(1+|\nabla u|^2)^{3/2}}|\xi|^2 < \frac{|\xi|^2}{\sqrt{1+|\nabla u|^2}}$, so no characteristic direction exists.

A quasilinear operator is called elliptic at u at x if it has no characteristic direction at x.

Exercise 9.3.1. The velocity potential $\phi(x,y)$ of a steady, isentropic, irrotational 2-dimensional flow satisfies the quasilinear PDE

$$(c^2-\phi_x^2)\phi_{xx} - 2\phi_x\phi_y\phi_{xy} + (c^2-\phi_y^2)\phi_{yy} = 0. \tag{9.14}$$

Here c is the speed of sound, and $(\phi_x(x,y), \phi_y(x,y))$ is the velocity at (x,y). Prove that (9.14) is ellptic at $\phi(x,y)$ iff $\phi_x^2+\phi_y^2 < c^2$.

9.4 Cauchy-Kowalevskaya Theorem: Linear and Quasilinear Case with General Non-characteristic Initial Data

We first still take P to be the linear differential operator as in **Section 2**.

Theorem 9.2 (Linear case with general non-characteristic initial surface)**.** *Suppose that there exists a neighborhood V of $x_0 \in \Sigma$ such that $c_\alpha(x)$ are analytic in V, that Σ is analytic and non-characteristic with respect to P in V. Then there is a neighborhood*

U of x_0, such that for any analytic functions $f(x)$ and $g(x)$ in a neighborhood U_1 around x_0, there is a unique analytic solution in $U \cap U_1$ to

$$\begin{cases} Pu = f(x), & \textit{near } x_0, \\ \partial_\beta u = \partial_\beta g(x), & \textit{along } \Sigma \textit{ for all } |\beta| \le m-1. \end{cases}$$

Example 9.7. For $P_1 = \partial_t + i\partial_x$ in $\mathbb{R}^2$, a characteristic direction $\xi = (\zeta, \eta) \in \mathbb{R}^2$ would have to satisfy $\zeta + i\eta = 0$, which would force $\zeta = \eta = 0$. Thus any regular curve in $\mathbb{R}^2$ in non-characteristic for P_1, and for any such analytic curve γ and analytic initial data g along γ, Cauchy-Kowalevskaya theorem can be applied.

For $P_2 = \partial_t + \partial_x$ in $\mathbb{R}^2$, a characteristic direction $\xi = (\zeta, \eta) \in \mathbb{R}^2$ would have to satisfy $\zeta + \eta = 0$. Thus a regular curve of the form $t - \phi(x) = 0$ is non-characteristic for P_2 iff $1 - \phi'(x) \neq 0$; while $\{(x,t) : t = x\}$ would be a characteristic curve for P_2. One can not apply the Cauchy-Kowalevskaya theorem to

$$\begin{cases} (\partial_t + \partial_x)\, u(x,t) = 0, \\ \qquad\qquad u(x,x) = g(x). \end{cases}$$

For one thing, if a solution existed, it would have forced g to be a constant; secondly, no derivative of u in the direction transversal to the initial curve $\{(x,t) : t = x\}$ can be determined from the equation and the initial condition.

Example 9.8. The initial curve $\{(x,t) : t = x^3\}$ is non-characteristic with respect to the operator $P = \partial_x$ everywhere except at $(0,0)$. The initial value problem

$$\begin{cases} \partial_x u(x,t) = 0, \\ u(x, x^3) = x \end{cases}$$

has no analytic solution near $(0,0)$, for, a solution would have to be a function of t which takes value $x = t^{1/3}$ along $\{(x,t) : t = x^3\}$, thus would have to equal to $t^{1/3}$.

One important application of the Cauchy-Kowalevskaya theorem is its use in Holmgren's proof for the uniqueness of a solution to the Cauchy problem for a *linear* equation with analytic coefficients and for data (not necessarily analytic) prescribed on an analytic non-characteristic initial surface Σ — recall that the uniqueness of an analytic solution to the same Cauchy problem would follow from the Cauchy-Kowalevskaya theorem, if the initial data is also analytic. Another ingredient in Holmgren's proof is the use of the Lagrange-Green's identity, which is used to reduce the proof for the uniqueness of a solution to the existence of a solution to a *dual* problem with a dense set of initial data. See either F. John or J. Rauch's text for details.

The reduction process described above works almost identically for quasilinear operator

$$P = \sum_{|\alpha|=m} c_\alpha(x, \partial_x^\beta u)\partial_x^\alpha u(x) + D(x, \partial_x^\beta u),$$

where $\partial_x^\beta u$ denotes generic terms of differentiation of order $|\beta| \le m-1$. By this we mean the "flattening" process and the determination of the power series expansion based on the Cauchy data (of order $m-1$ or less) and the equation. We may add that the process of reducing a higher order equation to a system of first order equation also works almost identically for higher order quasilinear operator, with one difference: the reduced system is quasilinear, instead of linear.

Theorem 9.3 (Quasilinear case with general non-characteristic initial surface)**.** *Suppose that $c_\alpha(x, \partial_x^\beta u)$ are analytic in its arguments around $(x_0, \partial_x^\beta g(x_0))$, that the initial data g is analytic around $x_0 \in \Sigma$, and that Σ is analytic around x_0 and non-characteristic with respect to P on the initial data g. Then there is a neighborhood of x_0, with a unique analytic solution to*

$$\begin{cases} Pu = 0, & \text{near } x_0, \\ \partial_\beta u = \partial_\beta g(x), & \text{along } \Sigma \text{ for all } |\beta| \le m-1. \end{cases}$$

9.5 Cauchy-Kowalevskaya Theorem: Fully Nonlinear Case

To understand the fully nonlinear version better, it is instructive to first examine the case with the special initial surface $\{t = 0\}$. Given a fully nonlinear operator

$$F = F(x, t, \partial_t^j \partial_x^\alpha u(x,t) |\, j + |\alpha| \le m),$$

where F is analytic in its arguments. Also given is initial data in the form of

$$\partial_t^j u(x, 0) = g_j(x), \quad \text{for } 0 \le j \le m-1.$$

Then all the terms $\partial_t^j \partial_x^\alpha u(x,0) |\, j + |\alpha| \le m$ in F along $(x, 0)$ are determined by $g_j(x)$'s and their derivatives in x with the exception of one term: $\partial_t^m u(x, 0)$. We require that (i) algebraically we can solve for $\partial_t^m u(x_0, 0) = \tilde{u}_m$ from $F(x_0, 0, \partial_x^\alpha g_j(x_0) |\, (j + |\alpha| \le$

$m, j < m), \partial_t^m u(x_0, 0)) = 0$ *at this one point* x_0, and (ii) we can locally solve for $\partial_t^m u(x,t)$ in terms of the other arguments, from

$$F(x, t, \partial_t^j \partial_x^\alpha u(x,t)|_{j+|\alpha|\le m, j<m}, \partial_t^m u(x,t)) = 0.$$

Here, locally means that $\partial_t^m u(x,t)$ is near $\tilde{u}_m$, $\partial_t^j \partial_x^\alpha u(x,t)$, for $j + |\alpha| \le m, j < m$, are near $\partial_x^\alpha g_j(x_0)$, and (x,t) is near $(x_0, 0)$.

By the implicit function theorem, this can be done if we assume

$$\frac{\partial}{\partial\left(\partial_t^m u\right)} F(x_0, 0, \partial_x^\alpha g_j(x_0) |\, (j + |\alpha| \le m, j < m), \tilde{u}_m) \neq 0. \tag{9.15}$$

Theorem 9.4 (Fully nonlinear case with special non-characteristic initial surface)**.** *Suppose that the initial data* $g_j(x)$ *are analytic around* x_0*, that* $F(x_0, 0, \partial_x^\alpha g_j(x_0) |\, (j + |\alpha| \le m, j < m), \tilde{u}_m) = 0$ *has a solution* $\tilde{u}_m$*, that* $F(x, t, \partial_t^j \partial_x^\alpha u(x,t) |\, j + |\alpha| \le m)$ *is analytic in its arguments around* $(x_0, 0, \partial_x^\alpha g_j(x_0) |\, (j + |\alpha| \le m, j < m), \tilde{u}_m)$*, and that* (9.15) *holds. Then there is a neighborhood of* $(x_0, 0)$*, with a unique analytic solution to*

$$\begin{cases} F = 0, & \text{near } (x_0, 0), \\ \partial_\beta u = \partial_\beta g(x), & \text{along } \Sigma \text{ for all } |\beta| \le m-1. \end{cases}$$

Example 9.9. Consider the problem

$$\begin{cases} u_t^2(x,t) + u_x^2(x,t) = 1, \\ \qquad\qquad u(x,0) = g(x). \end{cases}$$

In the case here, we need to be able to (i) solve $u_t(x_0, 0)$ algebraically from the initial condition and equation at $(x_0, 0)$ — this is possible as $u_t(x_0, 0) = \pm\sqrt{1 - |g'(x_0)|^2}$ provided $|g'(x_0)| \le 1$; and (ii) solve $u_t(x,t)$ as an analytic function in terms of other arguments such as $u(x,t)$ and $u_x(x,t)$ for (x,t) near $(x_0, 0)$ and $u_t(x,t)$ near $\pm\sqrt{1 - |g'(x_0)|^2}$ — whichever choice one makes. This can be done if $|g'(x_0)| < 1$. In fact we can solve $u_t(x,t)$ explicitly to recast the equation as

$$u_t(x,t) = \pm\sqrt{1 - u_x^2(x,t)},$$

when $|u_x(x,t)| < 1$. We can choose to work with either branch of the square root. This corresponds to two possible choices for $u_t(x_0, 0) = \pm\sqrt{1 - |g'(x_0)|^2}$. In such cases, the initial data along the curve $\{(x, 0)\}$ is non-characteristic near $(x_0, 0)$.

With this version in hand, the fully nonlinear case with general non-characteristic initial surface can be formulated in the same spirit as we did in the case for linear/quasilinear case. In fact, the easiest approach is to solve the differentiated equation $\partial F = 0$ first. This is a quasilinear equation of order $m + 1$, the coefficient in front of the highest order term $\partial_x^\gamma \partial u$ with $|\gamma| = m$ is

$$\frac{\partial}{\partial\left(\partial_x^\gamma u\right)} F(x, \partial_x^\alpha u(x) |\, |\alpha| \leq m).$$

There are some issues to be dealt with. First, from the given Cauchy data

$$\partial_x^\beta u(x) = \partial_x^\beta g(x), \quad \text{along } \Sigma \text{ for all } |\beta| \leq m-1,$$

we still need to determine the terms $\partial_x^\alpha u$ along Σ for $|\alpha| = m$. They have to satisfy some compatibility conditions along Σ. One way to resolve this issue is to assume that *at the point* $x_0 \in \Sigma$, we can find $\partial_x^\alpha u(x_0) = \tilde{u}_\alpha$ for $|\alpha| = m$ satisfying the compatibility conditions (will be illustrated later in a simple case) and $F(x_0, \partial_x^\alpha u(x_0)) = 0$. We may assume $\sigma_{x_n}(x_0) \neq 0$, so can take the ∂ above to be ∂_{x_n}. We have discussed that $\Sigma = \{\sigma = 0\}$ is non-characteristic with respect to the equation $\partial_{x_n} F = 0$ on the initial data g (and $\tilde{u}_\alpha$) if

$$\sum_{|\gamma|=m} \frac{\partial}{\partial\left(\partial_x^\gamma u\right)} F(x_0, \partial_x^\alpha u(x_0) |\, |\alpha| \leq m) \left(\nabla_x \sigma(x_0)\right)^\gamma \sigma_{x_n}(x_0) \text{ is non-degenerate,}$$

which is now equivalent to

$$\sum_{|\gamma|=m} \frac{\partial}{\partial\left(\partial_x^\gamma u\right)} F(x_0, \partial_x^\alpha u(x_0) |\, |\alpha| \leq m) \left(\nabla_x \sigma(x_0)\right)^\gamma \text{ is non-degenerate.} \tag{9.16}$$

The remaining issue to solve the Cauchy problem for the quasilinear system $\partial_{x_n} F = 0$ is the the appropriate determination of the Cauchy data along Σ—we have data up to order $m-1$ prescribed along Σ and have assumed the determination of data of order m at x_0. It turns out the this, together with (9.16), allows to extend the Cauchy data to a neighborhood of x_0 along Σ by the implicit function theorem.

Theorem 9.5 (Fully nonlinear case with general non-characteristic initial surface)**.** *Suppose that the initial surface Σ and the initial data $g(x)$ are analytic around x_0, that, with $\tilde{u}_\alpha = \partial_x^\alpha g(x_0)$ for $|\alpha| \leq m-1$, $F(x_0, \tilde{u}_\alpha) = 0$ has a solution $\tilde{u}_\alpha$ for $|\alpha| = m$ which also satisfies the compatibility conditions for m-th order partial derivatives along Σ at x_0 with the partial derivatives of order up to m of the $g(x)$ at x_0, that $F(x, \partial_x^\alpha u(x) |\, |\alpha| \leq m)$ is analytic in its arguments around $(x_0, \tilde{u}_\alpha)$, and that (9.16) holds. Then there is a neighborhood of x_0, with a unique analytic solution to*

$$\begin{cases} F(x, \partial_x^\alpha u(x) |\, |\alpha| \leq m) = 0, & \textit{near } x_0, \\ \qquad\qquad \partial_\beta u = \partial_\beta g(x), & \textit{along } \Sigma \textit{ for all } |\beta| \leq m-1. \end{cases}$$

Example 9.10. For the problem

$$\begin{cases} u_t^2(x,t) + u_x^2(x,t) = 1, \\ u(x, \phi(x)) = h(x), \end{cases} \tag{9.17}$$

the initial curve is $\gamma : t = \phi(x)$. Here, the initial data is given in terms of the function $h(x)$ along γ, rather than in terms of a local function in (x,t) near γ. It follows that the compatibility equation

$$u_t(x, \phi(x))\phi'(x) + u_x(x, \phi(x)) = h'(x)$$

must hold. We must first determine $u_t(x, \phi(x))$ and $u_x(x, \phi(x))$ from

$$\begin{cases} u_t^2(x,t) + u_x^2(x,t) = 1, \\ u_t(x, \phi(x))\phi'(x) + u_x(x, \phi(x)) = h'(x). \end{cases} \tag{9.18}$$

This system of algebraic equations in $u_t(x, \phi(x))$ and $u_x(x, \phi(x))$ has a solution iff

$$|h'(x)| \le \sqrt{1 + |\phi'(x)|^2}.$$

The initial curve $\gamma : t = \phi(x)$ is non-characteristic at $(x, \phi(x))$ iff it is non-characteristic with respect to the differentiated problem:

$$u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t) = 0,$$

where the initial data $u_t(x, \phi(x))$ and $u_x(x, \phi(x))$ are determined from the steps above. In other words, at $(x, \phi(x))$, if $u_t(x, \phi(x))$ and $u_x(x, \phi(x))$ are determined from (9.18), then $\gamma : t = \phi(x)$ is non-characteristic at $(x, \phi(x))$ iff $u_t(x,t) - u_x(x,t)\phi'(x) \neq 0$—by taking $\sigma(x,t) = t - \phi(x)$ in the characteristic equation for the differentiated problem.

When $|h'(x)| \le \sqrt{1 + |\phi'(x)|^2}$ holds, (9.18) does have a solution $u_t(x, \phi(x))$ and $u_x(x, \phi(x))$, and we will see that $u_t(x,t) - u_x(x,t)\phi'(x) \neq 0$ holds iff $|h'(x)| < \sqrt{1 + |\phi'(x)|^2}$. This can be seen by studying the joint solution to

$$\begin{cases} u_t(x, \phi(x)) - u_x(x, \phi(x))\phi'(x) = 0, \\ u_t^2(x, \phi(x)) + u_x^2(x, \phi(x)) = 1, \\ u_t(x, \phi(x))\phi'(x) + u_x(x, \phi(x)) = h'(x). \end{cases}$$

Substituting the first equation into the remaining two, we would find

$$|h'(x)|^2 = 1 + |\phi'(x)|^2.$$

Thus, if h and ϕ are analytic near x_0 and $|h'(x_0))|^2 < 1 + |\phi'(x_0)|^2$, then (9.17) is non-characteristic at x_0 on the initial data, and thus has a unique local analytic solution.

Remark 9.9. In **Examples 9** and **10**, we used two scalar first order PDEs to illustrate Cauchy-Kowalevskaya Theorem. In fact, these equations can be studied using the method of characteristic curves even in the absence of any analyticity assumption.

Exercise 9.5.1. The Gaussian curvature of a two dimensional graph $u = u(x, y)$ is given by

$$\frac{u_{xx}u_{yy} - u_{xy}^2}{(1 + |\nabla u|^2)^2} = K(x, y). \tag{9.19}$$

(i). Derive a differentiated equation for (9.19) and use it to confirm that any characteristic direction (ζ, η) at (x, y) would have to satisfy $u_{xx}\eta^2 - 2u_{xy}\zeta\eta + u_{yy}\zeta^2 = 0$.

(ii). Prove that (9.19) has no characteristic direction at (x, y) iff $K(x, y) > 0$.

(iii). Derive the linearization of (9.19) at $u(x, y)$, and prove that the linearization is elliptic at $u(x, y)$ iff $K(x, y) > 0$.

Part III

Further Study of Second Order Elliptic and Parabolic Equations

Chapter 10

Maximum Principle and Applications

From our earlier discussions we have seen the power of the maximum principle in establishing the uniqueness, estimation, convergence theorems, and existence of solutions. It turns out that the maximum principle has extensions to second order variable coefficient elliptic and parabolic operators, and the proofs can be given by fairly elementary means–along similar lines as the proofs for the Laplace operator. So we will present some of these generalizations, together with some applications.

10.1 Maximum Principle for Second Order Elliptic Equations

Definition. $L[u] = -\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}^2 u(x) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x)$ is called elliptic at $x \in U$ if the symmetric matrix $(a_{ij}(x))$ is positive definite at x; L is called elliptic in U if it is elliptic at every $x \in U$. L is called uniformly elliptic in U if

$$\sup_U \left[\Lambda(x)\lambda^{-1}(x)\right] < \infty \quad \text{in } U,$$

where $\Lambda(x)$ and $\lambda(x)$ are the largest and smallest eigenvalue of $(a_{ij}(x))$, respectively.

We will often need to assume

$$|b_i(x)|\lambda^{-1}(x) \quad \text{to be bounded in } U \text{ or in any compact subdomain of } U, \qquad (10.1)$$

where $\lambda(x)$ is the smallest eigenvalue of $(a_{ij}(x))$.

We will often employ the summation convention, and will denote $-a_{ij}(x)\partial^2_{ij}u(x)+b_i(x)\partial_i u(x)$ by $M[u]$. Unless otherwise noted, U always stands for a bounded domain.

Theorem 10.1 (Weak Maximum Principle)**.** *(i) Suppose M is elliptic in U and (10.1) holds on any compact subset of U. Assume $u \in C^2(U)\cap C(\overline{U})$ satisfies $M[u] \le 0$ in U. Then $\max_{\overline{U}} u = \max_{\partial U} u$.*

(ii) Suppose L is elliptic in U, (10.1) holds on any compact subset of U and $c(x) \ge 0$. If $u \in C^2(U)\cap C(\overline{U})$ satisfies $L[u] \le 0$ in U, then $\max_{\overline{U}} u \le \max\left(\max_{\partial U} u, 0\right) := \max_{\partial U} u^+$.

Proof. (i) First, we assume that $M[u] < 0$ in U. Then if $\max_{\overline{U}} u > \max_{\partial U} u$, $\max_{\overline{U}} u$ must be attained at some interior point $x_0 \in U$. This implies that $\nabla u(x_0) = 0$ and $\left(\nabla^2_{ij}u(x_0)\right)$ is a non-positive definite matrix. Since $(a_{ij}(x_0))$ is assumed to be positive definite, we see that $\sum_{i,j} a_{ij}(x_0)\nabla^2_{ij}u(x_0) \le 0$, which implies that $M[u](x_0) \ge 0$, contradicting our assumption that $M[u] < 0$ in U. Thus we have proved $\max_{\overline{U}} u = \max_{\partial U} u$ under the assumption $M[u] < 0$ in U.

For the general case, for any compact domain $V \subset\subset U$, we will construct a function v on V such that $M[v] < 0$ in V; and then apply the above argument to $u + \epsilon v$ on V for any $\epsilon > 0$ to conclude that

$$\max_{\overline{V}}(u+\epsilon v) = \max_{\partial V}(u+\epsilon v).$$

Since this equality holds for any $\epsilon > 0$, by sending $\epsilon \to 0$, we obtain

$$\max_{\overline{V}} u = \max_{\partial V} u.$$

Finally, if $\max_{\overline{U}} u > \max_{\partial U} u$, then we can easily construct a compact domain $V \subset\subset U$ such that $\max_{\overline{V}} u = \max_{\overline{U}} u > \max_{\partial V} u$, contradicting our argument in the paragraph above. This would conclude that $\max_{\overline{U}} u = \max_{\partial U} u$.

The construction of v can be made in the simple form of $v(x) = e^{\gamma x_1}$ for some $\gamma > 0$ large, as $M[e^{\gamma x_1}] = \left(-a_{11}(x)\gamma^2 + b_1(x)\gamma\right)e^{\gamma x_1}$, and $a_{11}(x) \ge \lambda(x)$, thus

$$M[e^{\gamma x_1}] \le \gamma\lambda(x)\left(-\gamma + \frac{b_1(x)}{\lambda(x)}\right)e^{\gamma x_1} < 0$$

if γ is chosen to be larger than the bound of $|\frac{b_1(x)}{\lambda(x)}|$ on V.

For (ii), since $L[u] \le 0$ in U, it follows that $M[u] = L[u] - c(x)u(x) \le -c(x)u(x) \le 0$ in the subdomain $U_+ := \{x \in U : u(x) > 0\}$. Applying the argument in (i) to u on U_+, we have $\max_{\overline{U}_+} u = \max_{\partial U_+} u$. But $\max_{\overline{U}} u \le \max_{\overline{U}_+} u$, and $\max_{\partial U_+} u = \max_{\partial U} u^+$, thus we have proved (ii). □

Theorem 10.2 (Uniqueness)**.** *Suppose L is elliptic in U, (10.1) holds on any compact subset of U and $c(x) \geq 0$. If $u \in C^2(U) \cap C(\overline{U})$ satisfies $L[u] = 0$ in U, $u = 0$ on ∂U, then $u \equiv 0$ in U.*

Remark 10.1. The conclusion $\max_{\overline{U}} u \leq \max\left(\max_{\partial U} u, 0\right)$ in (ii) of Theorem 1 can not be improved, as one can see from the case of $U = (-1, 1)$, $u(x) = -\cosh(x)$ satisfying $-u'' + u = 0$, $u(-1) = u(1) = -\cosh(1) = \max_{\partial U} u < u(0) = \max_{\overline{U}} u \leq 0$. The condition $c(x) \geq 0$ in both (ii) of Theorem 1 and Theorem 2 can not be dropped. For $U = (0, \pi)$, $u(x) = \sin x$ is a nonzero solution of $u'' + u = 0$ in U and $u(0) = u(\pi) = 0$. However, we have the following

Theorem 10.3. *Suppose L is uniformly elliptic in U, (10.1) holds on any compact subset of U and there exists $w \in C^2(U) \cap C(\overline{U})$ satisfies $L[w] \geq 0$ in U, $w > 0$ in $\overline{U}$. Let $u \in C^2(U) \cap C(\overline{U})$ satisfy $L[u] \leq 0$ in U, and $u \leq 0$ on ∂U. Then $u \leq 0$ in U.*

Proof. Set $u(x) = w(x)v(x)$. Then $L[u] = w(x)\widetilde{M}[v] + v(x)L[w]$, where

$$\widetilde{M}[v] = -\sum_{i,j=1}^{n} a_{ij}(x)\partial^2_{ij} v + \sum_{i=1}^{n}\left(b_i(x) + 2\sum_{j=1}^{n} a_{ij}(x)\partial_j w(x)/w(x)\right)\partial_i v(x).$$

So $\widetilde{M}[v] + v(x)L[w]/w \leq 0$, with $L[w]/w \geq 0$ in U, and $v \leq 0$ on ∂U. We can apply (ii) of the Weak Maximum Principle to conclude $v \leq 0$ in U. Therefore $u \leq 0$ in U. □

Corollary 10.4. *Under the assumptions on L and w as in the Theorem above, if $u \in C^2(U) \cap C(\overline{U})$ satisfy $L[u] = 0$ in U, and $u = 0$ on ∂U. Then $u = 0$ in U.*

Example 10.1. Consider $L[u] = -u''(x) - u(x)$ over $U = (0, l)$, $0 < l < \pi$, then for $0 < \delta < \pi - l$, $w(x) = \sin(x + \delta)$ satisfies $L[w] = 0$, and $w(x) > 0$ for $x \in U$, so, if $-u''(x) - u(x) \leq 0$ on $(0, l)$, and $u(0), u(l) \leq 0$, then $u(x) \leq 0$ in $(0, l)$. As a consequence of the uniqueness, the problem $-u''(x) - u(x) = f(x)$ for $x \in (0, l)$ and with $u(0)$, $u(l)$ prescribed, has at most one solution in such a case. The existence of a solution can be established using elementary means such as the variation of parameters method for constructing solutions to linear ODEs. Both conclusions fail when $l = \pi$.

The maximum principle does not hold on unbounded domains without requiring some conditions on the solution's behavior at infinity. For example, if $U = \mathbb{R}^n_+$, then $u(x) = x_n$ is a harmonic function in U such that $u = 0$ on ∂U, yet u is not $\equiv 0$ in U. Some extension of the maximum principle to unbounded domains appear in the problems.

Maximum principle can also be used to estimate the solution.

Theorem 10.5 (Estimation)**.** *Suppose L is elliptic in U, (10.1) is satisfied in U, and $c(x) \geq 0$ in U. Suppose $u \in C^2(U) \cap C(\overline{U})$ satisfies*

$$\begin{cases} L[u] = f(x) & \text{in } U, \\ u = g(x) & \text{on } \partial U. \end{cases}$$

Then

$$\max_{\overline{U}} |u| \leq \max_{\partial U} |g| + C \max_{\overline{U}} \left[|f(x)|/\lambda(x) \right],$$

where $C > 0$ depends only the diameter of U and the bound $\max_U [|b_i(x)|\lambda^{-1}(x)]$, and $\lambda(x)$ is the smallest eigenvalue of $(a_{ij}(x))$.

Proof. The key is to construct a function $v > 0$ in U satisfying $M[v] \geq \lambda(x)$ in U. Then $w = \left(\sup_U \frac{|f(x)|}{\lambda(x)} \right) v(x) + \sup_{\partial U} |g|$ satisfies $L[w] \geq M[w] \geq |f(x)|$ in U. So $L[w \pm u] \geq 0$ in U, and $w \pm u \geq 0$ on ∂U. By (ii) of the Weak Maximum Principle, $w \pm u \geq 0$ in U. Thus

$$|u| \leq w \leq \left(\sup_U \frac{|f(x)|}{\lambda(x)} \right) \max_U v(x) + \sup_{\partial U} |g|,$$

in U. Such a desired v can be found in the form of $v(x) = e^{\gamma d} - e^{\gamma x_1}$ for some $\gamma > 0$ depending on $\max_U [|b_i(x)|\lambda^{-1}(x)]$, where we assume U lie in the slab $0 < x_1 < d$. □

Remark 10.2. A small modification of the above proof can be used to give one-sided estimate. For example, if

$$\begin{cases} L[u] \leq f^+(x) & \text{in } U, \\ u \leq g^+(x) & \text{on } \partial U. \end{cases}$$

for some $f^+(x), g^+(x) \geq 0$, then $\max_{\overline{U}} u \leq \max_{\partial U} g^+ + C \max_{\overline{U}} [f^+(x)/\lambda(x)]$.

Notice also that the weak maximum principle and the uniqueness statements do not require any quantitative bound on the coefficients of L, but the estimation does require quantitative bound on the coefficients of L.

For some purposes the following Strong Maximum Principle is very useful.

Theorem 10.6 (Strong Maximum Principle)**.** *(i) Suppose M is uniformly elliptic on any compact subset of U, (10.1) holds, and $u \in C^2(U) \cap C(\overline{U})$ satisfies $M[u] \leq 0$ in U. Suppose U is connected and u attains its maximum $\max_{\overline{U}} u$ at a point in U, then $u \equiv$ a constant in U.*

(ii) Suppose L is uniformly elliptic on any compact subset of U, $c(x) \geq 0$ in U and

$$|b_i(x)|/\lambda(x), \qquad |c(x)|/\lambda(x) \quad \text{are bounded on any compact subset of } U. \tag{10.2}$$

Suppose U is connected and $u \in C^2(U) \cap C(\overline{U})$ satisfies $L[u] \le 0$ in U, and has a nonnegative maximum in U, then $u \equiv$ a constant in U.

(iii) Suppose L is uniformly elliptic on any compact subset of U and satisfies (10.2). Suppose U is connected and $u \in C^2(U) \cap C(\overline{U})$ satisfies $L[u] \le 0$ in U. If $u \le 0$ in U with $u(\bar{x}) = 0$ for some $\bar{x} \in U$. Then $u \equiv 0$ in U—this is no sign condition on $c(x)$ in this situation.

Remark 10.3. The condition in (ii) that u has a nonnegative maximum in U can not be dropped, as one can see through the example $U = (-1, 1)$, $u(x) = -\cosh(x)$ satisfying $-u'' + u = 0$, $u(-1) = u(1) = -\cosh(1) < u(0) = \max_{\overline{U}} u$.

Corollary 10.7. *Suppose $u \le v$ in a connected domain U and*

$$F(x, u, u_i, u_{jk}) \ge F(x, v, v_i, v_{jk}) \quad \text{in } U,$$

where F is of class C^1 in its argument and is elliptic everywhere, i.e.,

$$\frac{\partial F}{\partial z_{jk}}(x, z, z_i, z_{jk}) \quad \text{is positive definite for any } C^2 \text{ function } z \text{ in } U.$$

Then either $u < v$ in U or $u \equiv v$ in U.

Proof. Note that

$$\begin{aligned}
& - F(x, u, u_i, u_{jk}) + F(x, v, v_i, v_{jk}) \\
&= \int_0^1 \frac{d}{dt} F(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k})\, dt \\
&= - \int_0^1 \frac{\partial F}{\partial z_{jk}}(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k}) w_{x_j x_k}(x)\, dt \\
&\quad - \int_0^1 \frac{\partial F}{\partial z_i}(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k}) w_{x_i}(x)\, dt \\
&\quad - \int_0^1 \frac{\partial F}{\partial z}(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k}) w(x)\, dt
\end{aligned}$$

where $w(x) = u(x) - v(x)$. Setting

$$\begin{cases}
a_{jk}(x) = \displaystyle\int_0^1 \frac{\partial F}{\partial z_{jk}}(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k})\, dt, \\
b_i(x) = \displaystyle\int_0^1 \frac{\partial F}{\partial z_i}(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k})\, dt, \\
c(x) = \displaystyle\int_0^1 \frac{\partial F}{\partial z}(x, tv + (1-t)u, (tv + (1-t)u)_{x_i}, (tv + (1-t)u)_{x_j x_k})\, dt,
\end{cases}$$

we see that $w(x) = u(x) - v(x)$ satisfies

$$\begin{cases} -\sum_{j,k} a_{jk}(x) w_{x_j x_k} - \sum_i b_i(x) w_{x_i} - c(x) w(x) \le 0, & \text{in } U, \\ w(x) \le 0 & \text{on } \partial U, \end{cases}$$

and the condition on F implies that the strong maximum principle can be applied to $w(x)$, so we can conclude that either $u < v$ in U or $u \equiv v$ in U. □

As an application of this corollary, two minimal surfaces can never touch each other, unless they are identical. The proof of the Strong Maximum Principle depends on the Hopf boundary Lemma as given below.

Lemma 10.8 (Hopf Lemma). *Suppose L is uniformly elliptic in a closed ball $\overline{B}$ and satisfies (10.1) in B. Let $x_0 \in \partial B$ and $u \in C^2(B) \cap C(\overline{B})$ satisfies $u(x) < u(x_0)$ for all $x \in B$.*

(i) If $M[u] \le 0$ in B, then

$$\frac{\partial u}{\partial \nu}(x_0) > 0 \quad \text{in the sense} \quad \liminf_{\epsilon \to 0+} \frac{u(x_0) - u(x_0 - \epsilon \nu)}{\epsilon} > 0. \tag{10.3}$$

(ii) If $L[u] \le 0$ in B, $c(x) \ge 0$ in B and satisfies (10.2), and $u(x_0) \ge 0$, then (10.3) also holds; furthermore, if $u(x_0) = 0$, then (10.3) continues to hold regardless of the sign on $c(x)$.

Remark 10.4. The assumption $u(x_0) \ge 0$ in (ii) above can not be dropped, as in the case $U = (0, 1)$, $u(x) = -\cosh(x)$ satisfies $-u''(x) + u(x) = 0$, yet $u'(0) = 0$.

The Strong Maximum Principle and the Hopf Lemma give the uniqueness to the Neumann boundary value problem.

Theorem 10.9. *Assume L is uniformly elliptic in U and satisfies (10.2), and $c(x) \ge 0$ in U. Assume U is connected and ∂U is C^2, and $u \in C^2(U) \cap C^1(\overline{U})$ satisfies*

$$\begin{cases} L[u] = 0 & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U. \end{cases}$$

Then $u \equiv$ a constant in U (in fact $u \equiv 0$ unless $c(x) \equiv 0$).

Proof. Suppose u is not identically a constant, then by considering $-u$ if necessary, we may assume that $\max_{\overline{U}} u > 0$. By (ii) of Strong Maximum Principle, since u is not a constant in U, $\max_{\overline{U}} u$ can not be attained in the interior of U, thus must be attained at a boundary point x_0, and $u(x) < u(x_0)$ for all $x \in U$. But we can apply the Hopf Lemma on a small ball contained in U and touching U at x_0 to conclude that $\frac{\partial u}{\partial \nu}(x_0) > 0$, contradicting the boundary condition at x_0. Therefore, u must be a constant in U. □

Remark 10.5. In the simple case $L[u] = -\Delta u + c(x)u$, one can prove the uniqueness (up to a constant) to the Neumann boundary value problem by the energy method. But that method is not very suitable for general variable coefficient case.

Proof of Strong Maximum Principle. The set $V = \{x \in U : u(x) = \max_{\overline{U}} u\}$ is (relatively) closed in U. We will prove that, under the assumptions for the Strong Maximum Principle, it is also open, therefore conclude that $V = U$, and $u \equiv$ a constant in U. Suppose V is not open, then there exists a point $\bar{x} \in V$, and a sequence of points $x_i \in U$ such that $x_i \to \bar{x}$ as $i \to \infty$, and $u(x_i) < u(\bar{x})$. For i sufficiently large, the distance from x_i to ∂U is obviously greater than its distance to V, thus we can construct a ball B centered at x_i such that $\bar{B} \subset U$ and $\bar{B} \cap \bar{V}$ is a non empty subset of ∂B. We can now apply the appropriate form of the Hopf lemma on a perhaps smaller ball B' tangent to B and $\bar{B}' \cap \bar{V} = \{x'\}$ to conclude that $\frac{\partial u}{\partial \nu}(x') \neq 0$. But x' is an interior maximum point of u, so we are supposed to have $\nabla u(x') = 0$. This contradiction shows that the Strong Maximum Principle holds. □

Proof of the Hopf Lemma. The key idea is to construct a function v on the annulus region $A := B \setminus B'$, where B' is a strictly smaller concentric ball to B, satisfying

$$\begin{cases} L[v] \leq 0 & \text{in } A, \\ v = 0 & \text{on } \partial B, \\ v \geq 0 & \text{in } A, \\ \dfrac{\partial v}{\partial \nu} < 0 & \text{on } \partial B, \end{cases}$$

Then for $\epsilon > 0$ small, $\max_{\partial B'}(u + \epsilon v) \leq u(x_0) = \max_{\partial B}(u + \epsilon v)$, which is ≥ 0, and

$$\begin{cases} L[u + \epsilon v] \leq 0 & \text{in } A, \\ u + \epsilon v \leq u(x_0) & \text{on } \partial A. \end{cases}$$

In the case (i) or case (ii) with $c(x) \geq 0$, we can apply the weak maximum principle directly to conclude that $\max_A(u+\epsilon v) \leq \max_{\partial A}(u+\epsilon v) = u(x_0)$, and in fact, $x_0 \in \partial B$ must be a maximum point. Thus $\frac{\partial(u+\epsilon v)}{\partial \nu}(x_0) \geq 0$. It follows now $\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x_0) > 0$. In the case (ii) where $u(x_0) = 0$ and no sign condition is imposed on $c(x)$, note that

$$0 \geq L[u] = M[u] + c_+(x)(u) - c_-(x)(u) \geq M[u] + c_+(x)(u) \quad \text{in } A,$$

as $u \le 0$ in this case. So u is a subsolution of $M + c_+(x)$, and we can repeat the argument replacing L by $M + c_+(x)$ to draw the same conclusion.. A choice of v satisfying all the requirements can be found in the form of $v = e^{-\alpha|x|^2} - e^{-\alpha R^2}$ for sufficiently large α, if $B = B_R(0)$. We only need to check $L[v] \le 0$ in $B_R(0) \setminus B_(R')(0)$ if $0 < R' < R$ and $\alpha > 0$ is sufficiently large. Since $v_{x_i} = -2\alpha x_i e^{-\alpha|x|^2}$ and $v_{x_i x_j} = [-2\alpha\delta_{ij} + 4\alpha^2 x_i x_j]e^{-\alpha|x|^2}$, we see that

$$\begin{aligned} M[v] &= \left\{ -\sum_{i,j} a_{ij}(x)[-2\alpha\delta_{ij} + 4\alpha^2 x_i x_j] - \sum_i 2\alpha b_i(x) x_i \right\} e^{-\alpha|x|^2} \\ &\le \left\{ -4\alpha^2\lambda(x)|x|^2 + 2\alpha n\Lambda(x) + 2\alpha(\sum_i |b_i(x)|^2)^{\frac{1}{2}}|x| \right\} e^{-\alpha|x|^2} \\ &\le -2\alpha\lambda(x) \left\{ 2\alpha R'^2 - n\frac{\Lambda(x)}{\lambda(x)} + \left[\sum_i \left(\frac{b_i(x)}{\lambda(x)} \right)^2 \right]^{\frac{1}{2}} R \right\} e^{-\alpha|x|^2} < 0, \end{aligned}$$

for $R' \le |x| \le R$ if $\alpha > 0$ is chosen to make $2\alpha R'^2 - n\frac{\Lambda(x)}{\lambda(x)} + \left[\sum_i \left(\frac{b_i(x)}{\lambda(x)}\right)^2\right]^{\frac{1}{2}} R > 0$. □

Maximum principle is not valid on unbounded domains without some growth restrictions on the solution, as illustrated by the harmonic function $u(x) = x_n$ over $U = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, which satisfies $u(x) = 0$ for $u \in \partial U$. However, maximum principle is still valid for solutions on unbounded domains with appropriate growth restrictions which depend on the domain, as illustrated by the Phragmen-Linderlöf Theorem for holomorphic functions on sectors or strips. Similar comments apply to solutions which may be singular on a subset of the domain. We will next formulate and prove such a theorem for subharmonic functions on a sector.

The strategy is to first understand how the maximum principle may fail and to try to see whether there is a threshold growth rate for the failure. Let's start with considering harmonic functions u in the two-dimensional sector $\Sigma_{\theta_0} = \{z \in \mathbb{C} : 0 < \arg(z) < \theta_0\}$ which vanish on $\partial\Sigma_{\theta_0}$. Since both the sector Σ_{θ_0} and the Laplace operator Δ have scaling invariance $z = (x, y) \mapsto tz$ for $t > 0$, it's natural to try to understand how separable solutions in polar coordinates $u = \Phi(r)\Psi(\theta)$ behave—we could have relied on knowledge of holomorphic functions and their relations to harmonic functions in this setting, but we want to illustrate the general approach, which is applicable in other settings. Since the Laplace operator $\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$ in polar coordinates dimension 2, we have

$$\begin{cases} \left[\Phi''(r) + r^{-1}\Phi'(r)\right]\Psi(\theta) + r^{-2}\Phi(r)\Psi''(\theta) = 0 & r > 0, 0 < \theta < \theta_0 \\ \Phi(r)\Psi(\theta) = 0 & \text{when } \theta = 0, \theta_0. \end{cases}$$

Separating variables, we obtain, for some constant λ,

$$\begin{cases} \Psi''(\theta) + \lambda\Psi(\theta) = 0, & 0 < \theta < \theta_0 \\ \Psi(0) = \Psi(\theta_0) = 0, \end{cases} \tag{10.4}$$

and

$$r^2\Phi''(r) + r\Phi'(r) - \lambda\Phi(r) = 0 \quad \text{for } r > 0. \tag{10.5}$$

We know that (10.4) has non-trivial solutions only when $\lambda = \left(\frac{k\pi}{\theta_0}\right)^2$ for $k \in \mathbb{N}$, with $\Psi_k(\theta) = \sin\left(\frac{k\pi\theta}{\theta_0}\right)$. Setting $\lambda_1 = \left(\frac{\pi}{\theta_0}\right)^2$, we see that the corresponding solutions to (10.5) are $\Phi_k(r) = r^{\pm k\lambda_1}$. Thus both $r^{k\lambda_1}\sin\left(\frac{k\pi\theta}{\theta_0}\right)$ and $r^{-k\lambda_1}\sin\left(\frac{k\pi\theta}{\theta_0}\right)$ are harmonic functions in Σ_{θ_0} with vanishing boundary value on $\partial\Sigma_{\theta_0}$, which are non-trivial and exhibit some growth either at ∞ or near 0; note also that the threshold growth rate seems to be $r^{\pm\lambda_1}$, and that the harmonic function in such a case, $r^{\pm\lambda_1}\sin(\lambda_1\theta)$, which is the imaginary part of $z^{\pm\lambda_1}$ respectively, does not change sign in Σ_{θ_0}. This feature—a positive solution with homogeneous boundary data which fails the maximum principle—is often a hallmark of solutions failing the maximum principle at a threshold rate. We now formulate

Theorem 10.10. *Suppose that $u \in C(\overline{\Sigma_{\theta_0}}) \cap C^2(\Sigma_{\theta_0})$ satisfies, for some constants $0 < \lambda < \lambda_1$ and $C > 0$,*

$$\begin{cases} \Delta u \geq 0 & \text{in } \Sigma_{\theta_0}, \\ u \leq 0 & \text{on } \partial\Sigma_{\theta_0}, \\ u(z) \leq C|z|^\lambda & \text{in } \Sigma_{\theta_0}, \end{cases}$$

then $u \leq 0$ in Σ_{θ_0}

The generalization of this kind of maximum principle to a higher dimensional sector will appear in a problem.

Proof. The idea is to prove that for any $\epsilon > 0$, $u(z) \leq \epsilon r^{\lambda_1'}\sin(\lambda_1'\theta + \delta)$ for any $z \in \Sigma_{\theta_0}$, where $\lambda < \lambda_1' < \lambda_1$ and $\delta > 0$ have been chosen such that $\lambda_1'\theta_0 + \delta < \pi$. We made the adjustment from $r^{\lambda_1}\sin(\lambda_1\theta)$ to $r^{\lambda_1'}\sin(\lambda_1'\theta + \delta)$ so as to obtain a uniform lower bound of growth $r^{\lambda_1'}\sin(\lambda_1'\theta + \delta) \geq cr^{\lambda_1'}$ for some $c > 0$ and all $z \in \Sigma_{\theta_0}$. Once we have established $u(z) \leq \epsilon r^{\lambda_1'}\sin(\lambda_1'\theta + \delta)$ for any $z \in \Sigma_{\theta_0}$, since $\epsilon > 0$ is arbitrary in this inequality, we conclude that $u(z) \leq 0$.

Note that $v(z) = r^{\lambda_1'}\sin(\lambda_1'\theta + \delta)$ satisfies $\Delta v = 0$ in Σ_{θ_0}, and is positive in $\overline{\Sigma_{\theta_0}}$; moreover, due to $u(z) \leq C|z|^\lambda$ in Σ_{θ_0}, there exists $R > 0$ depending on $\epsilon > 0$ such that for $|z| \geq R$, $u(z) \leq \epsilon v(z)$. We can now apply the maximum principle to

$u(z) - \epsilon v(z)$ on the bounded domain $\Sigma_{\theta_0} \cap B(0,R)$ to conclude that $u(z) - \epsilon v(z) \leq 0$ in $\Sigma_{\theta_0} \cap B(0,R)$.

Note that the choice of R may depend on ϵ, but a fixed $z_0 \in \Sigma_{\theta_0} \cap B(0,R)$ for *all* small $\epsilon > 0$, so $u(z_0) \leq \epsilon v(z_0)$ for all small $\epsilon > 0$, which is what we set out to prove.

□

Exercises

Exercise 10.1.1. Prove that under the assumption that U satisfies the interior sphere condition, and that $c(x) \leq 0$ for $x \in U$ and $\alpha(x) \geq 0$ for $x \in \partial U$, there exists at most one solution (up to a constant) u to

$$\begin{cases} \Delta u + c(x)u = f & \text{in } U, \\ \dfrac{\partial u}{\partial \nu} + \alpha(x)u(x) = g(x) & \text{on } \partial U, \end{cases}$$

in the class $C^2(U) \cap C^1(\overline{U})$. Give an example of the failure of the uniqueness when the condition on $c(x)$ or α is not satisfied.

Exercise 10.1.2. Suppose that in the two-dimensional truncated sector $\Sigma_{\theta_0} \cap B(0,R)$, $u \in C(\overline{\Sigma_{\theta_0} \cap B(0,R)} \setminus \{0\}) \cap C^2(\Sigma_{\theta_0} \cap B(0,R))$ satisfies, for some constants $0 < \lambda < \lambda_1 = \frac{\pi}{\theta_1}$ and $C > 0$,

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma_{\theta_0} \cap B(0,R), \\ u = 0 & \text{on } \partial\left(\Sigma_{\theta_0} \cap B(0,R)\right), \\ |u(z)| \leq \dfrac{C}{|z|^\lambda} & \text{in } \Sigma_{\theta_0} \cap B(0,R), \end{cases}$$

then $u = 0$ in $\Sigma_{\theta_0} \cap B(0,R)$.

Exercise 10.1.3. In this problem we extend the maximum principle to higher dimensional sectors. Let $\Omega \subset \mathbb{S}^{n-1}$ be an open domain whose boundary $\partial\Omega$ consists of piecewise C^1 hypersurfaces, and in a neighborhood of any boundary point, Ω stays on one side of $\partial\Omega$. A differentiable function $\Psi(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \Omega$ has a naturally defined gradient $\nabla\Psi(\boldsymbol{\omega})$, $|\nabla\Psi(\boldsymbol{\omega})|^2$, and the associated (spherical) Laplacian $\Delta_{\boldsymbol{\omega}}\Psi(\boldsymbol{\omega})$. Recall that

$$\Delta_{\boldsymbol{\omega}}\Psi(\boldsymbol{\omega}) = \frac{1}{\sin(\theta)}\left(\sin(\theta)\Psi_\theta(\theta,\phi)\right)_\theta + \frac{1}{\sin^2(\theta)}\Delta_\phi\Psi(\theta,\phi),$$

where $\boldsymbol{\omega} = (\cos\theta, \sin\theta\ \phi)$, $\phi \in \mathbb{S}^{n-2}$ are geodesic polar coordinates for $\mathbb{S}^{n-1}$, and for a function $u = u(r\boldsymbol{\omega})$ defined in the sector $\Sigma_\Omega = \{x = r\boldsymbol{\omega} : r > 0, \boldsymbol{\omega} \in \Omega\}$,

$$\Delta u(r\boldsymbol{\omega}) = u_{rr} + \frac{n-1}{r}u_r + \frac{1}{r^2}\Delta_{\boldsymbol{\omega}} u.$$

(i). Look for a separable solution $u = \Phi(r)\Psi(\boldsymbol{\omega})$ in the sector Σ_Ω to

$$\begin{cases} \Delta\left(\Phi(r)\Psi(\boldsymbol{\omega})\right) = 0 & \text{in } \Sigma_\Omega, \\ \Phi(r)\Psi(\boldsymbol{\omega}) = 0 & \text{on } \partial\Sigma_\Omega. \end{cases}$$

Deduce the eigenvalue problem that Ψ has to satisfy on Ω and verify that the corresponding equation for Φ is $r^2\Phi_{rr}+(n-1)r\Phi_r+\lambda\Phi = 0$, and that its solution space is spanned by $\{r^{\alpha_+}, r^{\alpha_-}\}$, where $\alpha_\pm$ are the roots to $\alpha(\alpha+n-2) = \lambda$.

(ii). It is known that the eigenvalues in the part above are all real and ordered as $\lambda_1 < \lambda_2 \leq \cdots$, with $\lambda_k \to \infty$ as $k \to \infty$, and that the eigenspace associated with λ_1 is one dimensional and is spanned by an eigenfunction which is positive in Ω. It is further known that λ_1 depends on Ω in a continuous way, and that $\lambda_1(\Omega) > \lambda_1(\Omega')$ when $\Omega \subset\subset \Omega'$. Use such information to prove that if $u \in C(\overline{\Sigma_\Omega}) \cap C^2(\Sigma_\Omega)$ satisfies, for some $0 \leq \alpha < \alpha_+$ and $C > 0$, where α_+ is the positive root to $\alpha(\alpha+n-2) = \lambda_1(\Omega)$,

$$\begin{cases} \Delta u \geq 0 & \text{in } \Sigma_\Omega, \\ u \leq 0 & \text{on } \partial\Sigma_\Omega, \\ u(x) \leq C|x|^\alpha & \text{in } \Sigma_\Omega, \end{cases}$$

then $u \leq 0$ in Σ_Ω. (No explicit information on the eigenfunction Ψ associated with $\lambda_1(\Omega)$, other than those summarized above, is needed for this part.)

Exercise 10.1.4. Suppose that 0 is an interior point of the domain U in $\mathbb{R}^n$, $n \geq 3$, and $u(x)$ is a nonnegative harmonic function on $U \setminus \{0\}$. Prove that there exists a constant $A \geq 0$ and a smooth harmonic function $h(x)$ in U such that

$$u(x) = A|x|^{2-n} + h(x), \quad \text{for all } x \in U.$$

(Hint: Let $\bar{u}(r)$ denote the average of u over the sphere $|x| = r$. First establish that $\bar{u}''(r) + \frac{n-1}{r}\bar{u}'(r) = 0$ for small $r > 0$. Thus $\bar{u}(r) = Ar^{2-n} + B$ for some $A \geq 0$ and B. Next try to use Harnack/Green's identify or Maximum principle.)

Exercise 10.1.5. Suppose U is a bounded domain and $x_0 \in \partial U$. Let $u \in C(\overline{U} \setminus \{x_0\})$ be a bounded harmonic function in U such that $u \equiv 0$ on $\partial U \setminus \{x_0\}$. Prove that $u \equiv 0$ in U.

Exercise 10.1.6. Suppose U is a bounded domain in $\mathbb{R}^2$ with C^1 boundary, g is a C^0 function on ∂U that is locally Hölder at $x_0 \in \partial U$: $|g(x) - g(x_0)| \leq A|x - x_0|^\alpha$ for

$x \in \partial U$ in a neighborhood of x_0 and some $0 < \alpha < 1$, $A > 0$. Let u be the harmonic function in U with g as boundary value. Prove that u is locally Hölder at x_0, i.e., for some $B > 0$, $|u(x) - u(x_0)| \leq B|x - x_0|^\alpha$ for $x \in U$ in a neighborhood of x_0 . (Hint: Try to modify the construction of the barrier function in the barrier argument in the form of $r^\beta f(\theta)$, where $r = |x - x_0|$, and θ is the polar angle with respect to x_0.)

Exercise 10.1.7. (a). Let u be a bounded harmonic function on $U = \{x = (x', x_n) : 0 < x_n < h\}$. Prove that

$$\sup_{\overline{U}} |u| = \sup_{\partial U} |u|.$$

(b). Let u be a bounded harmonic function on $\mathbb{R}^n_+ = \{x = (x', x_n) : x_n > 0\}$. Prove that

$$\sup_{\overline{\mathbb{R}^n_+}} |u| = \sup_{\partial \mathbb{R}^n_+} |u|.$$

Exercise 10.1.8. Let B^+ denote the half disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1, y > 0\}$. Suppose $u \in C^2(B^+) \cap C(\overline{B}^+)$ is a solution of

$$\begin{cases} \partial_x^2 u + y\partial_y^2 u + c(x,y)u = f(x,y) & \text{in } B^+, \\ u(x,y) = g(x,y) & \text{on } \partial B^+. \end{cases} \tag{*}$$

(a). There is at most one solution of (*) under the assumption $c(x, y) \leq 0$.

(b). Assume $-c_0 \leq c(x, y) \leq 0$ in B^+. Then there exists a constant $C > 0$ depending only on c_0 such that for any solution u to (*)

$$\max_{B^+} |u| \leq C \left[\max_{B^+} |f| + \max_{\partial B^+} |g| \right].$$

10.2 Maximum Principle for Second Order Parabolic Equations

All of these maximum principles have their counterparts for second order parabolic operators. Many of such extensions have less strict requirements on the coefficients of the operator.

Definition. When $L = -a_{ij}(x,t)\partial^2_{x_i x_j} + b_i(x,t)\partial_{x_i} + c(x,t)$ is elliptic (uniformly elliptic), we say $\partial_t + L$ is parabolic(uniformly parabolic).

For considerations in parabolic problems, it is often convenient to consider domains of the form $U_T = U \times (0, T]$ in spacetime. The *parabolic boundary* of U_T is defined to be $\partial' U_T = \partial_t U_T \cup \partial_x U_T$, where $\partial_t U_T = \{(x, 0) : x \in \overline{U}\}$, and $\partial_x U_T = \{(x, t) : x \in \partial U, 0 < t \leq T\}$. Because solutions to parabolic equations have different degrees of differentiability in t and x, we define $C^{2,1}_{x,t}(U_T)$ to consist of those functions $u(x, t)$ that have continuous derivatives in x up to order 2 and continuous derivative u_t in U_T.

Theorem 10.11 (Weak Maximum Principle for Parabolic Operators)**.** *(i) Suppose $\partial_t + L$ is parabolic in U_T and satisfies*

$$c(x, t) \geq -\gamma \quad \textit{in } U_T. \tag{10.6}$$

Suppose $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} (\partial_t + L)[u] \leq 0 & \textit{in } U_T, \\ u \leq 0 & \textit{on } \partial' U_T. \end{cases}$$

Then $u \leq 0$ in U_T.

(ii) Suppose $\partial_t + L$ is parabolic in U_T and $c(x, t) \geq 0$. If $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies $(\partial_t + L)[u] \leq 0$ in U_T, then $\max_{\overline{U}_T} u \leq \max\left(\max_{\partial' U_T} u, 0\right) := \max_{\partial' U_T} u^+$.

Note that (i) above does not require the nonpositive sign condition on $c(x)$. As a consequence, neither does the uniqueness to the mixed Dirichlet-Cauchy problem require the sign condition on $c(x)$. The reason is because if we introduce a new variable $v(x, t) = e^{-\gamma t} u(x, t)$, then

$$\begin{cases} \partial_t v + (L + \gamma) v = e^{-\gamma t}(\partial_t u + L[u]) & \leq 0 \quad \text{in } U_T, \\ v = e^{-\gamma t} u(x, t) & \leq 0 \quad \text{on } \partial' U_T. \end{cases}$$

$(L + \gamma)v$ would have nonnegative coefficient in front of v, so we can apply maximum principle on v.

Proof of (i). Because of the above reduction, we may assume $\gamma = 0$ in (10.6). For any $\epsilon > 0$, we note that $(\partial_t + L)[u - \epsilon t] = -\epsilon + (\partial_t + L)[u] - \epsilon c(x, t) t < 0$ in U_T, so $u - \epsilon t$ can not take a positive maximum in U_T, for, if $w(x, t) := u(x, t) - \epsilon t$ attains a positive maximum at $(x_0, t_0) \in U_T$, then $w_t(x_0, t_0) \geq 0$, $w_{x_i}(x_0, t_0) = 0$, and $(\partial^2_{x_i x_j} w(x_0, t_0))$ is non-positive definite, which would imply $(\partial_t + L)\, w(x_0, t_0) \geq 0$, contradicting our set up of $(\partial_t + L)[u(x, t) - \epsilon t] < 0$ in U_T. Since $u - \epsilon t \leq 0$ on $\partial' U_T$, it follows now that $u - \epsilon t \leq 0$ in U_T. Since $\epsilon > 0$ is arbitrary, we conclude that $u \leq 0$ in U_T. $\square$

Proof of (ii). $(\partial_t + L)\,[u - \max_{\partial' U_T} u^+] \leq -c(x,t) \max_{\partial' U_T} u^+ \leq 0$, and $u - \max_{\partial' U_T} u^+ \leq 0$ on $\partial' U_T$. So $u - \max_{\partial' U_T} u^+ \leq 0$ in U_T by (i). □

Theorem 10.12 (Uniqueness)**.** *Suppose $\partial_t + L$ is parabolic in U_T and satisfies (10.6). Suppose $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} (\partial_t + L)[u] = 0 & \text{in } U_T, \\ \qquad\qquad u = 0 & \text{on } \partial' U_T. \end{cases}$$

Then $u \equiv 0$ in U_T.

Proof. We can apply the weak maximum principle to u and $-u$ to conclude that $u \equiv 0$ in U_T. □

Estimation on the solution of parabolic equation also follows routinely.

Theorem 10.13 (Estimation)**.** *Suppose $\partial_t + L$ is parabolic in U_T and satisfies (10.6). Suppose $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies*

$$\begin{cases} (\partial_t + L)[u] = f(x,t) & \text{in } U_T, \\ \qquad\qquad u = g(x,t) & \text{on } \partial' U_T. \end{cases}$$

Then

$$\max_{U_T} |u| \leq e^{\gamma T} \left[T \max_{U_T} |f| + \max_{\partial' U_T} |g| \right]. \tag{10.7}$$

Proof. By our trick above, we may work with $v(x,t) = e^{-\gamma t} u(x,t)$ to get $(\partial_t + [L + \gamma])\, v = e^{-\gamma t} f(x,t)$. Note that $(\partial_t + [L + \gamma])[v - t \max_{U_T} |f| - \max_{\partial' U_T} |g|] \leq 0$ in U_T, and $v - t \max_{U_T} |f| - \max_{\partial' U_T} |g| \leq 0$ on $\partial' U_T$. Thus by the maximum principle,

$$v \leq t \max_{U_T} |f| + \max_{\partial' U_T} |g|, \quad \text{in } U_T.$$

Similarly

$$-v \leq t \max_{U_T} |f| + \max_{\partial' U_T} |g|, \quad \text{in } U_T.$$

Thus (10.7) holds. □

Uniqueness to the mixed Dirichlet-Cauchy problem for fully nonlinear parabolic equations follow in a similar way.

Theorem 10.14. *Suppose $F = F(x, t, u, u_{x_i}, u_{x_i x_j})$ is of class C^1 in its argument and is elliptic everywhere with respect to $u_{x_i x_j}$. Then there exists at most one solution u in the class $C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ to*

$$\begin{cases} \partial_t u - F(x, t, u, u_{x_i}, u_{x_i x_j}) = 0, & \text{in } U_T, \\ \qquad\qquad\qquad\qquad u = g(x,t), & \text{on } \partial' U_T. \end{cases}$$

Proof. Suppose u_1 and u_2 are two solutions. Then $v = u_1 - u_2$ satisfies a linear parabolic equation in U_T with zero boundary data on $\partial' U_T$. By the uniqueness for the linear problem, $v \equiv 0$ in U_T. □

Again the validity of the maximum principle for solutions on unbounded domains requires some growth restrictions on the solutions. Tikhnov constructed a smooth solution of the standard heat equation $u_t(x,t) - \Delta u(x) = 0$ for $(x,t) \in \mathbb{R} \times \mathbb{R}$ such that $u(x,t) = 0$ for any $t \le 0$, but $\{0\} \times (0,1] \subset \text{supp} u$.

It turns out that the growth bound needed for making the maximum principle work for solutions to the Cauchy problem is given by the following positive solution of the heat equation,

$$(\partial_t - \Delta) \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T-t)}} = 0, \quad t < T, \tag{10.8}$$

which, for each fixed $t < T$, grows in x at the rate of $e^{\frac{|x|^2}{4(T-t)}}$. This can be verified directly, but can also be seen easily from noticing that

$$t \mapsto \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} \quad \text{is holomorphic in } t \in \mathbb{C} \setminus \{0\}, removingaslitifneeded,$$

and

$$(\partial_t - \Delta) \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} = 0, \quad t \in \mathbb{R}^+,$$

so

$$(\partial_t - \Delta) \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}} = 0, \quad t \in \mathbb{C} \setminus \{0\},$$

in particular for $t \in \mathbb{R}^-$; we can obviously replace $(4\pi t)^{\frac{n}{2}}$ by $(4\pi|t|)^{\frac{n}{2}}$ when $t \in \mathbb{R}^-$, and then replace t by $t - T$ to obtain (10.8).

Theorem 10.15. *Suppose that $u \in C^{2,1}_{x,t}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ satisfies*

$$\begin{cases} (\partial_t - \Delta) u \le 0 & (x,t) \in \mathbb{R}^n \times [0,T) \\ \quad u(x,0) \le 0, \end{cases}$$

and that there exists $A, a \ge 0$ such that $u(x,t) \le Ae^{a|x|^2}$ for $(x,t) \in \mathbb{R}^n \times [0,T]$. Then $u(x,t) \le 0$ for $(x,t) \in \mathbb{R}^n \times [0,T]$.

Theorem 10.16. *There exists at most one solution $u \in C^{2,1}_{x,t}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ to*

$$\begin{cases} (\partial_t - \Delta) u = f(x,t) & (x,t) \in \mathbb{R}^n \times [0,T) \\ \quad u(x,0) = g(x), \end{cases} \tag{10.9}$$

satisfying the bound $|u(x,t)| \le Ae^{a|x|^2}$ for $(x,t) \in \mathbb{R}^n \times [0,T]$ for some $A, a \ge 0$.

Proof for Theorem 10.15. Fix $T_1 > 0$ such that $a < \frac{1}{4T_1}$. For any $\epsilon > 0$, we will prove that

$$u(x,t) \le \epsilon \frac{1}{(4\pi(T_1 - t))^{\frac{n}{2}}} e^{\frac{|x|^2}{4(T_1-t)}} \tag{10.10}$$

for $(x,t) \in \mathbb{R}^n \times [0, T_1]$. Since (10.10) is valid for arbitrary $\epsilon > 0$, we conclude $u(x,t) \le 0$ for $(x,t) \in \mathbb{R}^n \times [0, T_1]$. Since $T_1 > 0$ can be determined in terms of a, we can repeat this argument a finite number of times over $\mathbb{R}^n \times [0, T_1]$, $\mathbb{R}^n \times [T_1, 2T_1]$, etc. to reach our desired conclusion over $\mathbb{R}^n \times [0, T]$.

(10.10) is established by applying the maximum principle over $B_R \times (0, T_1)$, where $R > 0$ is chosen such that (10.10) is valid on $\partial B_R \times (0, T_1)$, in light of the assumption $u(x,t) \le Ae^{a|x|^2}$ for $(x,t) \in \mathbb{R}^n \times [0,T]$.

□

There are also versions of the strong maximum principle and Hopf boundary point lemma. These are useful for proving the uniqueness to the mixed Neumann-Cauchy problems. They are formulated and proved in similar but slightly more sophisticated ways as for the elliptic versions. We will omit the details.

Exercises

Exercise 10.2.1. Suppose $\partial_t + L$ is parabolic in U_T and satisfies (10.6), and $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} (\partial_t + L)[u] = f & \text{in } U_T, \\ u = 0 & \text{on } \partial' U_T. \end{cases}$$

Suppose f and the coefficients of L are independent of t, and $f \ge 0$ in U_T. Prove that $u_t \ge 0$ in U_T.

Exercise 10.2.2. Suppose f is a locally Lipschitz function and $u, v \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfy

$$\begin{cases} u_t - \Delta u - f(u) \ge v_t - \Delta v - f(v) & \text{in } U_T, \\ u(x,0) \ge v(x,0) & \text{for } x \in U, \\ u(x,t) \ge v(x,t) & \text{for } x \in \partial U \text{ and } 0 < t < T. \end{cases}$$

Prove that $u(x,t) \ge v(x,t)$ in U_T.

Exercise 10.2.3. (Maximum principle for boundary value problem of the heat equation with Neumann or Robin type boundary condition.)

(a). Suppose $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} u_t(x,t) - \Delta u(x,t) \geq 0 & \text{for } (x,t) \in U_T, \\ u(x,0) \geq 0 & \text{for } x \in U, \\ \dfrac{\partial u(x,t)}{\partial \nu(x)} + h(x,t)u(x,t) \geq 0 & \text{for } (x,t) \in \partial U \times (0,T], \end{cases}$$

where U is a bounded convex domain with C^1 boundary (you may take U to be a bounded interval in $\mathbb{R}^1$) and $h(x,t) \geq 0$ for $(x,t) \in \partial U \times (0,T]$. Then $u(x,t) \geq 0$ in U_T.

(b). Under the same assumptions on U and $h(x,t)$, prove that if $u \in C^{2,1}_{x,t}(U_T) \cap C(\overline{U}_T)$ satisfies

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{for } (x,t) \in U_T, \\ u(x,0) = g(x) & \text{for } x \in U, \\ \dfrac{\partial u(x,t)}{\partial \nu(x)} + h(x,t)u(x,t) = b(x,t) & \text{for } (x,t) \in \partial U \times (0,T]. \end{cases}$$

Then

$$\max_{U_T} |u| \leq C \left[\max_{U_T} |f| + \max_{\overline{U}} |g| + \max_{\partial U \times [0,T]} |b| \right].$$

where C depends only on U and T.

Exercise 10.2.4. Consider the parabolic operator $P[u] = u_t - \sum_{i,j=1}^n a_{ij}(x,t)\partial^2_{x_i x_j} u(x,t)$ in $Q_r := \{(x,t) : |x| < r, 0 < t < r^2\}$, where we assume that for some $0 < \lambda \leq \Lambda$, $\lambda I \leq (a_{ij}(x,t)) \leq \Lambda I$ for all $(x,t) \in Q_r$. Assume, in addition, that $a_{ij}(x,t) \in C^1_x(Q_r)$, and there exists $M > 0$ such that

$$\frac{r|\partial_x a_{ij}(x,t)|}{\lambda} \leq M$$

for all $(x,t) \in Q_r$. Suppose that $u(x,t)$ is a solution of $P[u] = 0$ in Q_r and $\partial^3_x u$, $\partial^2_{xt} u \in C(Q_r)$. Modify Bernstein's method to prove that there exists some $A > 0$ depending on M and Λ/λ, such that

$$\max\{|\partial_x u(x,t)| : |x| \leq r/2, \frac{3}{4}r^2 \leq t \leq r^2\} \leq \frac{A}{r} \max_{\overline{Q_r}} |u(x,t)|.$$

Exercise 10.2.5. Let the functions $a_{ij}(x,t), i,j = 1,2,\cdots,n$, be defined for $t > 0$, $x \in \mathbb{R}^n$, and suppose that for all $x \in \mathbb{R}^n$, $t > 0$,

$$a_{ij} = a_{ji}, \sum_{i,j=1}^n a_{ij}^2 \leq \nu^{-2}, \text{ and } \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \nu|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n$$

with a constant $\nu \in (0, 1]$. Consider the function

$$K_{\alpha,\beta}(x,t) = t^{-\alpha} e^{-\frac{|x|^2}{\beta t}}, \quad x \in \mathbb{R}^n,\ t > 0.$$

Show that there exist positive constants $\alpha_1, \alpha_2, \beta_1, \beta_2$, depending only on n and ν, such that for all $x \in \mathbb{R}^n$, $t > 0$,

$$PK_{\alpha_1,\beta_1}(x,t) := \left(\frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t) D_{x_i x_j} \right) K_{\alpha_1,\beta_1}(x,t) \geq 0, \quad PK_{\alpha_2,\beta_2}(x,t) \leq 0.$$

Exercise 10.2.6. Use the previous result to show that

(a). the problem

$$Pu = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T], \quad u(x, 0) \equiv 0$$

has at most one solution in the class $C(\overline{\mathbb{R}^n \times (0,T]}) \cap C^{2,1}_{x,t}(\mathbb{R}^n \times (0,T])$, satisfying the inequality $|u(x,t)| \leq N \exp(a|x|^2)$ in $\mathbb{R}^n \times (0,T]$ with some positive constants N and a. (HINT: *some modification of the computation from the previous problem is needed: verify that one can also choose* $\alpha, \beta > 0$ *such that*

$$\widehat{K}_{\alpha,\beta}(x,t) = (T-t)^{-\alpha} e^{-\frac{|x|^2}{\beta(t-T)}}$$

satisfies $P\widehat{K} \geq 0$ *for* $(x,t) \in \mathbb{R}^n \times [0,T)$. $\alpha, \beta > 0$ *can be chosen independently of* T. *Then one fixes* $T > 0$ *small so that* $\beta T < 1/a$ *and fixes any* $\epsilon > 0$, *and applies the maximum principle between* u *and* $\epsilon \widehat{K}_{\alpha,\beta}(x,t)$ *on* $B_R \times [0,T)$ *for sufficiently large* R.)

(b). if $u \in C(\overline{\mathbb{R}^n \times (0,T]}) \cap C^{2,1}_{x,t}(\mathbb{R}^n \times (0,T])$ satisfies $Pu \leq 0$ and $u(x,t) \leq M e^{a|x|^2}$ for some $M, a \geq 0$ in $\mathbb{R}^n \times (0,T]$, then $u(x,t) \leq \max_{y \in \mathbb{R}^n} u(y,0)$ for all $(x,t) \in \mathbb{R}^n \times (0,T]$.

Exercise 10.2.7. This exercise formulates the strong maximum principle for solutions of parabolic equations and outlines the main steps for a proof. We continue to use the notation $P[u]$ for a parabolic operator as in the step up of the previous exercises and assume the bounds on its coefficients in Q_1. Then the strong maximum principle says that, if $u(x,t)$ satisfies $P[u] \leq 0$ in Q_1, and there exists some $(x_*, t_*) \in Q_1$ such that $u(x,t) \leq u(x_*, t_*)$ for all $(x,t) \in Q_1$ with $t \leq t_*$, then $u(x,t) = u(x_*, t_*)$ for all $(x,t) \in Q_1 \cap \{t \leq t_*\}$. Note that for $0 < t_* < 1$, $u(x,t) = -K(x - x_0, t - t_*)$ satisfies $(\partial_t - \Delta_x) u(x,t) = 0$ and $u(x,t) \leq 0$ in Q_1, if $|x_0| > 1$, $u(x,t) = 0$ for all $(x,t) \in Q_1 \cap \{t \leq t_*\}$, yet $u(x,t) < 0$ for $t > t_*$. Below are two ingredients in a proof of the strong maximum principle.

(i). Suppose that $\overline{B_r(x_0,t_0)} = \overline{\{(x,t) : |x-x_0|^2 + |t-t_0|^2 < r^2\}} \subset Q_1$ is such that $u(x,t) < 0$ in $\overline{B_r(x_0,t_0)}$, and there exists $P' = (x',t') \in \partial B_r(x_0,t_0)$, with $x' \neq x_0$ and $u(x',t') = 0$, then $\nabla_\nu u(x',t') > 0$ for any vector ν pointing outward of $\partial B_r(x_0,t_0)$ at (x',t').

(ii). The following can't occur: there exists some $(x_*,t_*) \in Q_1$, $r > 0$ such that $u(x_*,t_*) = 0$ and $u(x,t) < 0$ for all (x,t) such that $|x - x_*| \leq r$ and $t_* - r^2 \leq t < t_*$. Note that $u(x,t) = \frac{x^2}{2} + t - 1$ satisfies $(\partial_t - \partial_x^2)u = 0$, $u(x,t) < 0 = u(0,1)$ in $D = \{(x,t) : \frac{x^2}{2} + t < 1\}$, which has $(0,1)$ as a boundary point. This example shows that in the strong maximum principle the assumption $u(x,t) \leq u(x_*,t_*)$ for all $(x,t) \in Q_1$ with $t \leq t_*$ can't not freely relaxed.

10.3 A Maximum Principle for Weak Solutions to Divergence Form Second Order Elliptic PDEs

There is also a need for maximum principle for weak solutions to $Lu = (\leq, \geq)0$. Here we take L to be of divergence form

$$Lu = -\sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_{ij}(x) u_{x_i}(x) + d_j(x) u(x) \right)_{x_j} + \sum_{j=1}^{n} b_j(x) u_{x_j}(x) + c(x) u(x) \quad (10.11)$$

with the usual condition

$$\lambda |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \leq \lambda^{-1} |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ and } x \in U. \quad (10.12)$$

For the maximum principle it's natural (and necessary) to impose on L the condition that $-\sum_{j=1}^{n} (d_j(x))_{x_j} + c(x) \geq 0$ in the distribution sense, namely,

$$\int_U \left(\sum_{j=1}^{n} d_j(x) \eta_{x_j}(x) + c(x) \eta(x) \right) dx \geq 0, \quad \text{for all } \eta \in C_c^1(U) \text{ with } \eta(x) \geq 0 \text{ in } U. \quad (10.13)$$

By a density argument, (10.13) holds for η which is an $H^1(U)$ limit of nonnegative functions in $C_c^1(U)$.

Theorem 10.17. *Suppose that (10.13) holds for L and that $u \in H^1(U)$ is a weak* subsolution *to $Lu = 0$, namely,*

$$B_L[u,\eta] \leq 0 \quad \text{for all } \eta \in C_c^1(U) \text{ with } \eta(x) \geq 0 \text{ in } U.$$

Then $sup_U u \le \sup_{\partial U} u^+$. Here $\sup_U u$ is the essential supremum of u over U, defined as $\inf\{l : u(x) \le l,\ a.e.\, x \in U\}$, and $\sup_{\partial U} u^+$ is defined in a similar way.

Corollary 10.18. *Suppose that (10.13) holds for L and that $u \in H_0^1(U)$ is a weak solution of $Lu = 0$, then $u = 0$ in U.*

There is also a need to bound $||u||_{L^\infty(U)}$ in terms of Lu, $||u||_{L^\infty(\partial U)}$, and perhaps $||u||_{L^2(U)}$.

Theorem 10.19. *Suppose $u \in H^1(U)$ is a solution of $Lu = \mathbf{f}$, where, for some $q > n$, $f_0, c(x) \in L^{q/2}(U)$, and $b_j(x), d_j(x), f_j(x) \in L^q(U)$, then there exists $C > 0$ depending on $n, \lambda, q, |U|$ and upper bound for $||b_j, d_j||_{L^q(U)}$ and $||c||_{L^{q/2}(U)}$, such that*

$$\sup_U |u| \le C\left\{||u||_{L^2(U)} + \lambda^{-1}\left[||f_j||_{L^q(U)} + ||f_0||_{L^{q/2}(U)} + ||u||_{L^\infty(\partial U)}\right]\right\}. \qquad (10.14)$$

When (10.13) holds, the $||u||_{L^2(U)}$ term in (10.14) can be removed.

Proof of Theorem 10.17. Suppose that $l = \sup_{\partial U} u^+$ is finite, and $sup_U u > l$ (for, otherwise, we already have our desired conclusion). We will first prove that $sup_U u < \infty$ under the conditions in the Theorem. For any $k \ge l \ge 0$, we can use $\eta = (u-k)^+ \stackrel{\text{def}}{=} v$ as a test function in $B_L[u,\eta] \le 0$. Let $A(k) = \{x \in U : u(x) \ge k\}$. Using $u_{x_i}(x) = v_{x_i}(x)$ when $v(x) > 0$, we obtain

$$\begin{aligned}
&B_L[u,\eta] \\
=&\int_U \Bigg\{\sum_{i,j=1}^n a_{ij}(x)u_{x_i}(x)v_{x_j}(x) \\
&+\sum_{j=1}^n \big(d_j(x)u(x)v_{x_j}(x) + b_j(x)u_{x_j}(x)v(x)\big) + c(x)u(x)v(x)\Bigg\}\,dx \\
\ge&\int_U \Bigg\{\sum_{i,j=1}^n a_{ij}(x)v_{x_i}(x)v_{x_j}(x) + \sum_{j=1}^n \left(d_j(x)+b_j(x)\right)v_{x_j}(x)v(x) + \\
&+c(x)v(x)^2 + k\left(\sum_{j=1}^n d_j(x)v_{x_j}(x) + c(x)v(x)\right)\Bigg\}\,dx \\
\ge&\lambda||\nabla v||^2_{L^2(U)} - \sum_{j=1}^n ||\left(d_j(x)+b_j(x)\right)v||_{L^2(U)}||v_{x_j}||_{L^2(U)} - ||c_-||_{L^{\frac{q}{2}}(A(k))}||v||^2_{L^{\frac{2q}{q-2}}(A(k))} \\
\ge&\lambda||\nabla v||^2_{L^2(U)} - \sum_{j=1}^n ||d_j(x)+b_j(x)||_{L^q(U)}||v||_{L^{\frac{2q}{q-2}}(U)}||v_{x_j}||_{L^2(U)} \\
&-||c_-||_{L^{\frac{q}{2}}(A(k))}||v||^2_{L^{\frac{2q}{q-2}}(A(k))},
\end{aligned}$$

so that

$$\lambda||\nabla v||^2_{L^2(U)} \leq \left(\sum_{j=1}^{n} ||d_j(x)+b_j(x)||^2_{L^q(U)}\right)^{1/2} ||v||_{L^{\frac{2q}{q-2}}(U)}||\nabla v||_{L^2(U)} + ||c_-||_{L^{\frac{q}{2}}(A(k))}||v||^2_{L^{\frac{2q}{q-2}}(A(k))}. \tag{10.15}$$

Then

$$||v||_{L^{\frac{2q}{q-2}}(U)} = ||v||_{L^{\frac{2q}{q-2}}(A(k))} \leq ||v||_{L^{\frac{2n}{n-2}}(A(k))}|A(k)|^{\frac{1}{n}-\frac{1}{q}} \leq C(q,n)||\nabla v||_{L^2(U)}|A(k)|^{\frac{1}{n}-\frac{1}{q}},$$

where we have used Sobolev's inequality in the last estimate. (10.15) now becomes

$$\lambda||\nabla v||^2_{L^2(U)} \leq C(q,n)|A(k)|^{\frac{1}{n}-\frac{1}{q}}\left\{D+|A(k)|^{\frac{1}{n}-\frac{1}{q}}||c_-||_{L^{\frac{q}{2}}(A(k))}\right\}||\nabla v||^2_{L^2(U)}, \tag{10.16}$$

where $D=\left(\sum_{j=1}^{n}||d_j(x)+b_j(x)||^2_{L^q(U)}\right)^{1/2}$. (10.16) can be used in two ways. First, for any $k<\sup_U u$, $|A(k)|>0$ and $||\nabla v||^2_{L^2(U)}>0$, so it follows from (10.16) that

$$\lambda \leq C(q,n)|A(k)|^{\frac{1}{n}-\frac{1}{q}}\left\{\left(\sum_{j=1}^{n}||d_j(x)+b_j(x)||^2_{L^q(U)}\right)^{1/2}+|A(k)|^{\frac{1}{n}-\frac{1}{q}}||c_-||_{L^{\frac{q}{2}}(A(k))}\right\},$$

which implies a positive lower bound for $|A(k)|$ independent of $k<\sup_U u$. As a consequence

$$|\{x\in U: u(x)=\sup_U u\}| = \lim_{k\nearrow\sup_U u}|A(k)|>0. \tag{10.17}$$

We will come back to (10.17) in a moment. Let's point out another consequence of (10.16). It follows from (10.16) that if

$$C(q,n)|A(k)|^{\frac{1}{n}-\frac{1}{q}}\left\{\left(\sum_{j=1}^{n}||\left(d_j(x)+b_j(x)\right)||^2_{L^q(U)}\right)^{1/2}+|A(k)|^{\frac{1}{n}-\frac{1}{q}}||c_-||_{L^{\frac{q}{2}}(U)}\right\}<\lambda \tag{10.18}$$

then $||\nabla v||_{L^2(U)}=0$, which implies that $u\leq k$ for *a.e.* $x\in U$.

We can estimate k in terms of $||u||_{L^2(U)}$ for which (10.18) would hold. $||u||^2_{L^2(U)}\geq k^2|A(k)|$, so it suffices to choose k such that

$$\lambda^{-1}C(q,n)\left\{\left(\sum_{j=1}^{n}||d_j(x)+b_j(x)||^2_{L^q(U)}\right)^{1/2}+|U|^{\frac{1}{n}-\frac{1}{q}}||c_-||_{L^{\frac{q}{2}}(U)}\right\}||u||^{2(\frac{1}{n}-\frac{1}{q})}_{L^2(U)}<k^{2(\frac{1}{n}-\frac{1}{q})}.$$

Thus for any

$$k>\left\{\lambda^{-1}C(q,n)\left[\left(\sum_{j=1}^{n}||d_j(x)+b_j(x)||^2_{L^q(U)}\right)^{1/2}+|U|^{\frac{1}{n}-\frac{1}{q}}||c_-||_{L^{\frac{q}{2}}(U)}\right]\right\}^{\frac{nq}{2(q-n)}}||u||_{L^2(U)}$$
$$:=C||u||_{L^2(U)}.$$

we have $u \leq k$ for $a.e.$ $x \in U$. Recall that we also required $k > l = \sup_{\partial U} u^+$, so we conclude that

$$\sup_U u \leq \max\{\sup_{\partial U} u^+, C||u||_{L^2(U)}\}. \tag{10.19}$$

We next use $\eta = \frac{(u-l)^+}{M+\epsilon-(u-l)^+} \in H_0^1(U)$ as a test function, where $0 < M = \sup_U u - l < \infty$ under our assumption, and $\epsilon > 0$ is arbitrary. Then, with $v = (u-l)^+$, and using $\nabla\eta = \frac{(M+\epsilon)\nabla v}{(M+\epsilon-v)^2}$, we have

$$\begin{aligned}
&B_L[u,\eta] \\
=&\int_U \left\{ \sum_{i,j=1}^n a_{ij}(x)u_{x_i}(x)\eta_{x_j}(x) \right. \\
&\left. + \sum_{j=1}^n \left(d_j(x)u(x)\eta_{x_j}(x) + b_j(x)u_{x_j}(x)\eta(x)\right) + c(x)u(x)\eta(x) \right\} dx \\
=&\int_U \left\{ \sum_{i,j=1}^n a_{ij}(x)v_{x_i}(x)\eta_{x_j}(x) \right. \\
&\left. + \sum_{j=1}^n \left(-d_j(x) + b_j(x)\right)\eta(x)v_{x_j}(x) + \left(\sum_{j=1}^n d_j(x)\left[u(x)\eta(x)\right]_{x_j} + c(x)u(x)\eta(x) \right) \right\} dx \\
\geq&\lambda(M+\epsilon)\int_U \frac{|\nabla v|^2}{(M+\epsilon-v)^2}\,dx - M||-d_j(x)+b_j(x)||_{L^2(U)} \left\{ \int_U \frac{|\nabla v|^2}{(M+\epsilon-v)^2}\,dx \right\}^{1/2},
\end{aligned}$$

where we have used $u(x)\eta(x) = v(x)\eta(x) + l\eta(x) \geq 0$ and is in $H_0^1(U)$ due to (10.19), so is an admissible test function for using (10.13). So we now have, with $w(x) = \ln\left(\frac{M+\epsilon}{M+\epsilon-v(x)}\right)$,

$$||w||_{L^{2^*}(U)} \leq C(2,n)||\nabla w||_{L^2(U)} \leq \lambda^{-1}C(2,n)||-d_j(x)+b_j(x)||_{L^2(U)}. \tag{10.20}$$

The right hand side of (10.20) is independent of ϵ, so we can send $\epsilon \to 0$ to obtain

$$||\ln\left(\frac{M}{M-v(x)}\right)||_{L^{2^*}(U)} \leq \lambda^{-1}C(2,n)||-d_j(x)+b_j(x)||_{L^2(U)},$$

which contradicts (10.17). □

Remark 10.6. Estimate (10.19) is valid without assuming (10.13), as stated in (10.14) and proved below.

The proof above and the one to follow appear to be complicated. But one main point is to estimate the term $||\nabla v||^2_{L^2(U)}$ in terms of integrals that are quadratic in v, but involving at most a linear factor of ∇v, and of integrals that are linear in ∇v or v

only (in the proof below). For the latter, one can use the Cauchy-Schwarz inequality to estimate them in terms of a small factor times $||\nabla v||^2_{L^2(U)}$ plus the square of the $L^2(A(k))$ integrals of f_i and f_0 respectively, which can then be estimated in terms of the L^q (or $L^{\frac{q}{2}}$) norms of f_i (or f_0) multiplied by an appropriate power of $|A(k)|$! While for the former, one can not automatically estimate them in terms of a small multiple $||\nabla v||^2_{L^2(U)}$, but each such term will have a factor which is a positive power of $|A(k)|$ so that for large enough k (which can be estimated in terms of data) these terms are estimated by a small multiple $||\nabla v||^2_{L^2(U)}$!

We next describe a proof for Theorem 10.19 in the special case $b_i = d_i = c = 0$, to illustrate De Giorgi's iteration method.

Proof of Theorem10.19 in a special case. We still use $v = (u-l)^+$ as a test function. Then the same computation as done in the previous proof leads to

$$\lambda||\nabla v||^2_{L^2(U)} \leq \sum_{i=1}^{n} ||f_i||_{L^2(A(k))}||v_{x_i}||_{L^2(U)} + ||f_0||_{L^{\frac{2n}{n+2}}(A(k))}||v||_{L^{\frac{2n}{n-2}}(A(k))},$$

so we have

$$\lambda||\nabla v||^2_{L^2(U)} \leq \lambda^{-1}\sum_{i=1}^{n} ||f_i||^2_{L^2(A(k))} + 2||f_0||_{L^{\frac{2n}{n+2}}(A(k))}||v||_{L^{\frac{2n}{n-2}}(A(k))}.$$

Applying the Sobolev inequality

$$||v||^2_{L^{\frac{2n}{n-2}}(A(k))} = ||v||^2_{L^{\frac{2n}{n-2}}(U)} \leq C(2,n)||\nabla v||^2_{L^2(U)}$$

we obtain

$$||v||^2_{L^{\frac{2n}{n-2}}(A(k))} \leq \lambda^{-2}C(n)\left\{\sum_{i=1}^{n} ||f_i||^2_{L^2(A(k))} + ||f_0||^2_{L^{\frac{2n}{n+2}}(A(k))}\right\}. \tag{10.21}$$

Note that for $h > k$,

$$||v||_{L^{\frac{2n}{n-2}}(A(k))} \geq (h-k)|A(h)|^{\frac{1}{2}-\frac{1}{n}}.$$

Furthermore,

$$||f_i||^2_{L^2(A(k))} \leq ||f_i||^2_{L^q(A(k))}|A(k)|^{2(\frac{1}{2}-\frac{1}{q})}, \text{ and } ||f_0||^2_{L^{\frac{2n}{n+2}}(A(k))} \leq ||f_0||^2_{L^{\frac{nq}{q+n}}(A(k))}|A(k)|^{2(\frac{1}{2}-\frac{1}{q})}.$$

which leads to the iteration scheme

$$(h-k)^2|A(h)|^{2(\frac{1}{2}-\frac{1}{n})} \leq \lambda^{-2}C(n)\left\{\sum_{i=1}^{n} ||f_i||^2_{L^q(A(k))} + ||f_0||^2_{L^{\frac{nq}{q+n}}(A(k))}\right\}|A(k)|^{2(\frac{1}{2}-\frac{1}{q})}. \tag{10.22}$$

Since $q > n$, we have $\frac{1}{2} - \frac{1}{n} < \frac{1}{2} - \frac{1}{q}$, and $\frac{nq}{q+n} < \frac{q}{2}$, so we have

$$|A(h)| \leq \left(\frac{F}{h-k}\right)^{\frac{2n}{n-2}} |A(k)|^{\beta} \quad \text{for } h > k, \tag{10.23}$$

where $\beta = \frac{2n}{n-2}/\frac{2q}{q-2} > 1$, and

$$F^2 = \lambda^{-2} C(n) \left(\sum_{i=1}^{n} ||f_i||^2_{L^q(U)} + ||f_0||^2_{L^{\frac{nq}{q+n}}(U)} \right).$$

It now follows from the Lemma below that

$$|A(l+I)| = 0, \quad \text{where } I = 2^{\frac{\beta}{\beta-1}} F |A(l)|^{\frac{1}{n}-\frac{1}{q}} \leq 2^{\frac{\beta}{\beta-1}} F |U|^{\frac{1}{n}-\frac{1}{q}}.$$

which implies that $u \leq l + 2^{\frac{\beta}{\beta-1}} F |U|^{\frac{1}{n}-\frac{1}{q}}$. Working with $-u$ would provide the remaining bound in (10.14). □

Lemma 10.20. *Suppose that $\Phi(t)$ is defined on $[l, \infty)$, non-negative, and non-increasing, and that for $h > k \geq l$, we have*

$$\Phi(h) \leq \left(\frac{F}{h-k}\right)^{\alpha} \Phi(k)^{\beta} \tag{10.24}$$

where $\alpha > 0$ and $\beta > 1$. Then

$$\Phi(l+I) = 0, \text{where } I = 2^{\frac{\beta}{\beta-1}} F \Phi(l)^{\frac{\beta-1}{\alpha}}. \tag{10.25}$$

Remark 10.7. When lower order terms involving $b_j(x), d_j(x)$ and $c(x)$ are present, one would need to replace $v = (u-l)^+$ by $v = (u-k)^+$ for $k > l$ to be adjusted, and add to the right hand side of (10.21) terms of the kind $\int_U b_i(x) u_{x_i}(x) v(x)\, dx$, $\int_U d_i(x) u(x) v_{x_i}(x)\, dx$ and $\int_U c(x) u(x) v(x)\, dx$ and estimate them. $\int_U b_i(x) u_{x_i}(x) v(x)\, dx = \int_U b_i(x) v_{x_i}(x) v(x)\, dx$ and can be treated as in the proof for Theorem 10.19.

$$\begin{aligned}
&\left| \int_U c(x) u(x) v(x)\, dx \right| \\
\leq &\int_U |c(x)| v^2(x)\, dx + k \int_U |c(x)| v(x)\, dx \\
\leq &||c||_{L^{\frac{q}{2}}(A(k))} ||v||^2_{L^{\frac{2q}{q-2}}(A(k))} + k ||c||_{L^{\frac{q}{2}}(A(k))} ||v||_{L^{\frac{q}{q-2}}(A(k))} \\
\leq &||c||_{L^{\frac{q}{2}}(A(k))} ||v||^2_{L^{\frac{2q}{q-2}}(A(k))} + k ||c||_{L^{\frac{q}{2}}(A(k))} ||v||_{L^{\frac{2n}{n-2}}(A(k))} |A(k)|^{\frac{q-2}{q} - \frac{n-2}{2n}} \\
\leq &||c||_{L^{\frac{q}{2}}(A(k))} ||v||^2_{L^{\frac{2q}{q-2}}(A(k))} + \frac{\epsilon}{2} ||v||^2_{L^{\frac{2n}{n-2}}(A(k))} + \frac{k^2}{2\epsilon} ||c||^2_{L^{\frac{q}{2}}(A(k))} |A(k)|^{1-\frac{2}{q}+2(\frac{1}{n}-\frac{1}{q})}.
\end{aligned}$$

Similarly,

$$\left|\int_U d_i(x)u(x)v_{x_i}(x)\,dx\right|$$
$$\leq \int_U |d_i(x)v_{x_i}(x)|v(x)\,dx + k\int_U |d_i(x)v_{x_i}(x)|\,dx,$$

where the first term can be estimated as in the proof for Theorem 10.19, while the second term can be estimated as above to obtain

$$k\int_U |d_i(x)v_{x_i}(x)|\,dx \leq k||d_i(x)||_{L^2(A(k))}||v_{x_i}||_{L^2(A(k)}$$
$$\leq \frac{\epsilon}{2}||\nabla v||^2_{L^2(A(k)} + \frac{k^2}{2\epsilon}||d_i||^2_{L^q(A(k))}|A(k)|^{1-\frac{2}{q}}.$$

So in place of (10.22), we have an estimate of the form

$$||v||^2_{L^{\frac{2n}{n-2}}(A(k))} \leq G||v||^2_{L^{\frac{2q}{q-2}}(A(k))} + \left(F^2 + k^2G^2\right)|A(k)|^{1-\frac{2}{q}}, \tag{10.26}$$

where $G = \lambda^{-1}\left(||c||_{L^{\frac{q}{2}}(U)}|U|^{\frac{1}{n}-\frac{1}{q}} + ||b_i, d_i||_{L^q(U)}\right)$. One now chooses $k > l$ (depending on G and $||u||_{L^2(U)}$ as in the proof for (10.18)) such that $G||v||^2_{L^{\frac{2q}{q-2}}(A(k))} \leq \frac{1}{2}||v||^2_{L^{\frac{2n}{n-2}}(A(k))}$ to reach the estimate

$$\left(\int_{A(k)} v(x)\,dx\right)^2 \leq ||v||^2_{L^{\frac{2n}{n-2}}(A(k))}|A(k)|^{1+\frac{2}{n}} \leq |A(k)|^{2+\frac{2}{n}-\frac{2}{q}}\left(F^2 + k^2G^2\right). \tag{10.27}$$

The complication here, in comparison to (10.23), is the power of k multipled to $|A(k)|^{2+\frac{2}{n}-\frac{2}{q}}$. The following iteration lemma, due to Ladyzhenskaya and Ural'tceva, which is a modified version of the iteration lemma above, concludes the L^∞ estimate.

Lemma 10.21. *Suppose that $u \in L^1(U)$ satisfies for some $\epsilon > 0$, $0 \leq \alpha \leq 1+\epsilon$, $\gamma > 0$, $k_0 \geq 0$, and for all $k \geq k_0$*

$$\int_U (u-k)^+\,dx \leq \gamma k^\alpha |\{x \in U : u(x) > k\}|^{1+\epsilon}. \tag{10.28}$$

Then the essential maximum of u is bounded above in terms of γ, α, ϵ, and $||u - k_0||_{L^1(A_{k_0})}$, where $A_{k_0} = \{x \in U : u(x) > k_0\}$.

10.4 Eigenfunction Expansion of Sturm-Liouville Problems

The eigenfunction expansion method rests on the involved operator L having a "complete" set of eigenfunctions which spans $L^2(\Omega)$. However, even in the finite dimensional setting, a linear operator may not have a complete set of eigenvectors spanning the entire underlying vector space. On the other hand, a group of linear operators, including the real symmetric and Hermitian operators, do have a complete set of eigenvectors spanning the entire underlying vector space, and have the additional property that one can choose such a set of eigenvectors to be orthonormal. Additionally, the eigenvalues and eigenvectors have a variational characterization.

Many boundary value problems of differential equations share a formal symmetry property with the real symmetric and Hermitian operators. The simplest example is $L = -\frac{d^2}{dx^2}$. Using $(\cdot, \cdot)$ to denote the L^2 inner product, L has the symmetry $(Lu, v) = (u, Lv)$ for u, v in various classes of functions. This is based on

$$Lu \cdot v - u \cdot Lv = u \cdot v'' - v \cdot u'' = (u \cdot v' - v \cdot u')',$$

so

$$\int_a^b [Lu \cdot v - u \cdot Lv]\, dx = (u{\cdot}v' - v{\cdot}u')\big|_a^b = [u(b)v'(b) - v(b)u'(b)] - [u(a)v'(a) - v(a)u'(a)]\,.$$

If u and v both satisfy either the homogeneous Dirichlet or the homogeneous Neumann boundary conditions at a and b, we obviously have $(Lu, v) = (u, Lv)$. In fact, this symmetry continues to hold for more general linear, homogeneous boundary conditions on the boundary points. We can normalize the linear, homogeneous boundary conditions at the ends in (3.14) as

$$\begin{aligned} \cos\alpha\, u(a) + \sin\alpha\, u'(a) = 0, \\ \cos\beta\, u(b) + \sin\beta\, u'(b) = 0, \end{aligned} \tag{3.14'}$$

for some real parameters α and β. Then $(Lu, v) = (u, Lv)$ continues to hold for u, v satisfying these boundary conditions. If we consider L as acting on $u, v \in C_c^2(\mathbb{R})$, instead of as a problem on a finite interval $[a, b]$, then $(Lu, v) = (u, Lv)$ also holds for functions in this class. It turns out that we can extend the spectral properties of real symmetric and Hermitian operators to a large class of boundary value problems of differential equations having the formal symmetry mentioned above, but certain appropriate boundary conditions need to be imposed, and the results and proofs need to take into account of the infinite dimensional nature of the function spaces involved, and may exhibit *continuous spectrum* when the problem loses certain compactness feature.

The class of boundary value problems of differential operators having the closest resemblance of spectral properties as the real symmetric and Hermitian operators have *resolvents* which are *compact operators*. They include the regular Sturm-Liouville problems (3.14) and boundary value problems for second order elliptic operators discussed in this chapter on *bounded domains*.

If $L : X \mapsto Y$ is a linear operator, where X is a subspace of Y, then for those scalars λ such that $(L - \lambda I_Y)$ has a bounded linear inverse $(L - \lambda I_Y)^{-1}$ defined on Y (namely, for every $y \in Y$, there is a unique $x \in X$ satisfying $(L - \lambda I_Y)x = y$, and there exists $C > 0$ such that $||x|| \leq C||(L - \lambda I_Y)x||$ for all x), $(L - \lambda I_Y)^{-1}$ is called the resolvent operator of L, and λ is said to be in the resolvent of L. Scalars not in the resolvent of L are said to be in the spectrum of L. L is said to have compact resolvent if $(L-\lambda I_Y)^{-1} : Y \mapsto X$ is a compact operator, namely, it maps any bounded sequence in Y into a sequence having a convergent subsequence in Y. This notion does not depend on the particular choice of λ in the resolvent of L: if both λ_1 and λ_2 are in the resolvent of L, then

$$\begin{aligned}(L - \lambda_1 I_Y)^{-1} - (L - \lambda_2 I_Y)^{-1} &= (\lambda_1 - \lambda_2)(L - \lambda_1 I_Y)^{-1} \cdot (L - \lambda_2 I_Y)^{-1} \\ &= (\lambda_1 - \lambda_2)(L - \lambda_2 I_Y)^{-1} \cdot (L - \lambda_1 I_Y)^{-1},\end{aligned}$$

so $(L - \lambda_2 I_Y)^{-1} y_m$ converges iff $(L - \lambda_1 I_Y)^{-1} y_m$ converges.

We will sketch below an argument for why the operators in a regular Sturm-Liouville problem (3.14) or in a boundary value problems for second order elliptic operators on a bounded domain discussed in this chapter have compact resolvent. In such a situation, the strategy for understanding the spectrum of L goes as follows: first suppose that there exists some $\lambda_* \in \mathbb{R}$ in the resolvent of L; then, using the relation

$$(L - \lambda I_Y) = (L - \lambda_* I_Y) - (\lambda - \lambda_*) I_Y = (L - \lambda_* I_Y) \left[I_Y - (\lambda - \lambda_*)(L - \lambda_* I_Y)^{-1} \right],$$

we see that $(L-\lambda I_Y)$ has a bounded inverse iff $I_Y - (\lambda-\lambda_*)(L-\lambda_* I_Y)^{-1}$ has a bounded inverse; equivalently, λ is in the spectrum of L iff $(\lambda - \lambda_*)^{-1}$ is in the spectrum of $(L - \lambda_* I_Y)^{-1}$.

Denoting $(L - \lambda_* I_Y)^{-1}$ by K. We note that, if $(Lx_1, x_2) = (x_1, Lx_2)$ for all $x_1, x_2 \in X$, then K also has the property $(Ky_1, y_2) = (y_1, Ky_2)$ for all $y_1, y_2 \in Y$. This is seen as follows. Let $x_1 = Ky_1, x_2 = Ky_2$, then $Lx_1 - \lambda_* x_1 = y_1$, and

$Lx_2 - \lambda_* x_2 = y_2$, so

$$\begin{aligned}(Ky_1, y_2) &= (x_1, Lx_2 - \lambda_* x_2) \\ &= (Lx_1, x_2) - (x_1, \lambda_* x_2) \\ &= (Lx_1 - \lambda_* x_1, x_2) \\ &= (y_1, Ky_2).\end{aligned}$$

Thus we have reduced the study of the spectrum of L to that of K, which is a bounded symmetric operator on Y, and we can appeal to a relatively simple spectrum theory for compact symmetric operators.

Historically, the spectrum theory for linear differential operators arose first from the theory of eigenfunction expansions in solving boundary value problems of differential equations, as in Fourier series expansion, and in the (regular) Sturm-Liouville problems. Subsequent development involves integral equations extensively. Although Fourier transforms can be interpreted as providing (generalized) "eigenfunction expansion" of the differential operator $\frac{d^2}{dx^2}$ on $L^2(\mathbb{R}^2)$ ($e^{i\xi x}$ are the bounded generalized eigenfunctions of $\frac{d^2}{dx^2}$), it was Weyl's 1910 work that made the first systematic study of spectral properties of singular Sturm-Liouville problems, which included the case of (3.14) on a compact interval with $p(x) \to 0$ at one or both ends or $q(x)$ or $w(x)$ singular somewhere in the interval, and also included the case of (3.14) on an infinite interval. The main new features are the possible presence of continuous spectrum. In the late 1920's von Neumann, and Stone, independently, developed a spectral theory of abstract unbounded self-adjoint operators, which can be applied to the study of spectral properties of boundary value problems of differential equations; but the implementing of the abstract theory still requires a detailed analysis of the solutions to the differential equations with appropriate boundary conditions.

We will not have space here to discuss the theory of singular Sturm-Liouville problems or of abstract unbounded self-adjoint operators; we will limit our discussion to setting up the resolvent $K = (L - \lambda_*)^{-1}$ in the context of our boundary value problems—using integral representation via Green's function for Sturm-Liouville problems for one dimensional problems and variational methods for Dirichlet or Neumann boundary value problems in multi-dimensions, and providing the necessary development of the spectral properties of compact symmetric operators. The construction for the resolvent operator of the regular Sturm-Liouville problems can be largely subsumed by the latter approach. But it pays to see how things work out in the one dimensional case; in addition, this explicit approach handles the more general linear homogeneous boundary conditions with ease.

The construction of $K = (L - \lambda)^{-1}$ in the case of a regular Sturm-Liouville problem can be carried out in a straight forward fashion, in an almost explicit way; and the construction will show the needed conditions on λ. The presence of $w(x)$ in (3.14) would need us to work in the weighted L^2 space $L^2_w[a,b]$—mainly to get the L^2 symmetry, to be discussed below. We would define a linear operator L as $Lu = w(x)^{-1}\left[-(p(x)u'(x))' + q(x)u(x)\right]$ (use the spherical Laplace operator as a guide, where $w(x) = sin(x)$ for $x \in [a,b] \subset (0,\pi)$). To identify the domain $X = \mathcal{D}(L)$ of L, we would need $u'(x)$ to be well defined, and $p(x)u'(x)$ to be absolutely continuous over $[a,b]$, and its derivative to be L^2 integrable over $[a,b]$ with appropriate weight.

Since we are assuming $0 < m \le w(x) \le M$ for all $x \in [a,b]$, for some $0 < m \le M < \infty$, $L^2_w[a,b]$ and $L^2[a,b]$ are actually equivalent as normed spaces, though not isometrically. For simplicity, we will assume that $p \in C^1[a,b]$, although the set up can be extended as long as $p^{-1}(x) \in L[a,b]$. We now define

$$X = \mathcal{D}(L) = \left\{ u \in AC[a,b] : \begin{array}{l} p(x)u'(x) \in AC[a,b], [p(x)u'(x)]' \in L^2[a,b], \\ u \text{ satisfies the BC's in (3.14)} \end{array} \right\}.$$

With this set up, we have, for any $u, v \in X$, $Lu, Lv \in L^2_w[a,b]$, and

$$\begin{aligned}
(Lu, v) &= \int_a^b w(x)^{-1}\left[-(p(x)u'(x))' + q(x)u(x)\right] v(x)w(x)dx \\
&= \int_a^b \left[-(p(x)u'(x))' + q(x)u(x)\right] v(x)dx \\
&= \int_a^b \left[p(x)u'(x)v'(x) + q(x)u(x)v(x)\right] dx - \left[p(x)u'(x)v(x)\right]\Big|_a^b \\
&= \int_a^b \left[p(x)u'(x)v'(x) + q(x)u(x)v(x)\right] dx \\
&\quad - \left[p(b)u'(b)v(b) - p(a)u(a)v'(a)\right] \\
&= \int_a^b \left[-(p(x)v'(x))' + q(x)v(x)\right] u(x)dx \\
&\quad - p(b)\left[u'(b)v(b) - u(b)v'(b)\right] + p(a)\left[u(a)v'(a) - u'(a)v(a)\right] \\
&= (u, Lv) \quad \text{ if } u, v \in \mathcal{D}(L)
\end{aligned}$$

The boundary conditions in (3.14) are crucial for establishing this symmetry; one key component in understanding the spectrum property of singular Sturm-Liouville problems is to identify appropriate boundary conditions to guarantee such symmetry and the unique solvability u of $(L - \lambda)u = f$ for $f \in L^2_w[a,b]$ *satisfying the prescribed* *boundary conditions* for appropriate λ—these boundary conditions are no longer in the form of point wise conditions as in (3.14), but may be integral conditions or asymptotic conditions towards the ends of the interval.

What's needed next is,

(I) For a given $f \in L^2_w[a,b]$, to construct a unique $u \in X$ (which encodes the boundary conditions) solving $Lu - \lambda u = f$ and to find a constant $C > 0$ independent of f such that

$$||u||_{L^2_w[a,b]} \leq C||f||_{L^2_w[a,b]}. \tag{10.29}$$

(II) Prove that $f \mapsto u = (L-\lambda)^{-1}f$ is a compact operator in $L^2_w[a,b]$, namely, for a sequence f_n bounded in $L^2_w[a,b]$, a subsequence of $(L-\lambda)^{-1}f_n$ converges in $L^2_w[a,b]$.

Remark 10.8. For the purpose of establishing $(Lu, v) = (u, Lv)$, one can use the simpler space of $C^2_c(a,b)$; the choice of our X is that the BCs for functions in X help to determine the unique solvability in the resolution of (I), while there may not be any solution in $C^2_c(a,b)$ which solves $Lu - \lambda u = f$. It turns out that X also arises in extending L, first defined on $C^2_c(a,b)$, to a *closed* operator in the abstract theory of unbounded operators; but such an extension depends on the specific BCs for functions in X.

(I) amounts to solving $-(p(x)u'(x))' + q(x)u(x) - \lambda w(x)u(x) = w(x)f(x)$ subject to the BCs in (3.14). Such an u can be constructed using the variation-of-parameters method. Let $u_a(x;\lambda)$ denote a solution of $-(p(x)u'(x))' + q(x)u(x) - \lambda w(x)u(x) = 0$ over $[a,b]$ subject to the condition at $x = a$: $\cos\alpha\, u_a(a;\lambda) + \sin\alpha\, u'_a(a;\lambda) = 0$. This can be considered an IVP for the linear ODE in (3.14), so $u_a(x;\lambda)$ exists, and is unique up to a scalar multiple due to the linear, homogeneous relation between $u_a(x;\lambda)$ and $u'_a(x;\lambda)$. Similarly, we define $u_b(x;\lambda)$ to be a solution of the same ODE subject to the condition at $x = b$: $\cos\beta\, u_b(b;\lambda) + \sin\beta\, u'_b(b;\lambda) = 0$. Then we construct $u(x)$ in the form of $C_1(x)u_a(x;\lambda) + C_2(x)u_b(x;\lambda)$ using the variation-of-parameters method. $C_1(x)$ and $C_2(x)$ are chosen so that

$$\begin{cases} C'_1(x)u_a(x;\lambda) + C'_2(x)u_b(x;\lambda) = 0 \\ p(x)u'_a(x;\lambda)C'_1(x) + p(x)u'_b(x;\lambda)C'_2(x) = -f(x)w(x). \end{cases}$$

We can determined $C'_1(x)$ and $C'_2(x)$ uniquely, provided the **Wronskian** between $u_a(x;\lambda)$ and $u_b(x;\lambda)$,

$$W[u_a, u_b; \lambda](x) \stackrel{\text{def}}{=} p(x)[u'_a(x;\lambda)u_b(x;\lambda) - u'_b(x;\lambda)u_a(x;\lambda)] \neq 0.$$

Note that

$$\frac{dW[u_a, u_b; \lambda](x)}{dx} = [p(x)u_a'(x;\lambda)]'u_b(x;\lambda) - [p(x)u_b'(x;\lambda)]'u_a(x;\lambda) = 0,$$

so $W[u_a, u_b; \lambda](x)$ is a constant in $[a, b]$.

We claim that $(L - \lambda)$ has a well defined inverse with $(L - \lambda)^{-1} f \in X$ iff $W[u_a, u_b; \lambda] \neq 0$. This condition turns out to characterize the eigenvalues of L, for, if $W[u_a, u_b; \lambda] = 0$, it would imply that $(u_a(x;\lambda), u_a'(x;\lambda))$ is a scalar multiple of $(u_b(x;\lambda), u_b'(x;\lambda))$, and since $u_a(x;\lambda)$ satisfies the BC at $x = a$ in (3.14), this would imply that $u_b(x;\lambda)$ also satisfies the same BC; since $u_b(x;\lambda)$ satisfies the BC at $x = b$, this would imply that it is an eigenfunction of L. To see the other direction of the claim, suppose that $W[u_a, u_b; \lambda] \neq 0$, then we are able to carry through the variation-of-parameter argument to construct $u(x) = C_1(x)u_a(x;\lambda) + C_2(x)u_b(x;\lambda)$ solving $(L-\lambda)u = f$. $C_1'(x)$ and $C_2'(x)$ are uniquely determined, but $C_1(x)$ and $C_2(x)$ each still has one free parameter, which we take to be $C_1(b)$ and $C_2(a)$. In order for $u(x)$ to satisfy the BC at $x = a$, we find that we need

$$\cos\alpha\, u(a;\lambda) + \sin\alpha\, u'(a;\lambda) = C_2(a)\left[\cos\alpha\, u_b(a;\lambda) + \sin\alpha\, u_b'(a;\lambda)\right] = 0.$$

This forces $C_2(a) = 0$, since $\cos\alpha\, u_b(a;\lambda) + \sin\alpha\, u_b'(a;\lambda) \neq 0$, for otherwise, it would make $W[u_a, u_b; \lambda] = 0$. Similarly, $C_1(b) = 0$. This determines uniquely $u(x) \in X$ solving $(L - \lambda)u = f$.

Remark 10.9. Note that the above analysis shows that, under the condition $W[u_a, u_b; \lambda] \neq 0$, we can actually uniquely solve $(L - \lambda)u = f$ subject to a non-homogeneous boundary condition at each end of the type $\cos\alpha\, u(a;\lambda) + \sin\alpha\, u'(a;\lambda) = \Gamma_a$ and $\cos\beta\, u(b;\lambda) + \sin\beta\, u'(b;\lambda) = \Gamma_b$ for some prescribed Γ_a and Γ_b.

Next, note that $W[u_a, u_b; \lambda]$ is an entire function in λ*, thus $W[u_a, u_b; \lambda] \neq 0$ except possibly on a discrete set of λ values. Assuming $W[u_a, u_b; \lambda] \neq 0$ for a particular λ, and incorporating the BC's in (3.14), we obtain the following integral representation for $u(x)$:

$$u(x) = W[u_a, u_b; \lambda]^{-1}\left[u_a(x;\lambda)\int_x^b u_b(y;\lambda)f(y)w(y)\,dy + u_b(x;\lambda)\int_a^x u_a(y;\lambda)f(y)w(y)\,dy\right].$$

Defining

$$G(x, y; \lambda) = \begin{cases} W[u_a, u_b; \lambda]^{-1}u_a(y;\lambda)u_b(x;\lambda) & \text{if } y \le x, \\ W[u_a, u_b; \lambda]^{-1}u_a(x;\lambda)u_b(y;\lambda) & \text{if } y > x, \end{cases}$$

*This is because $u_a(x;\lambda)$ and $u_b(x;\lambda)$ can be constructed as the uniform limit of a sequence of approximating sequence using Picard's iteration, and each function in the iteration is a holomorphic function of λ on $\mathbb{C}$

the above integral representation can be expressed as

$$u(x) = K[f] \stackrel{\text{def}}{=} \int_a^b G(x, y; \lambda) f(y) w(y) \, dy.$$

This $G(x, y; \lambda)$, as a function of $x \in [a, b]$ for each fixed $y \in [a, b]$, satisfies $L[G(x, y; \lambda)] = 0$ at $x \neq y$, $p(y)[G_x(y + 0, y; \lambda) - G_x(y - 0, y; \lambda)] = -1$ at each $y \in (a, b)$, and the two BC's in (3.14). It is called the Green's function of (3.14).

What remains is to prove that $K[f] \in L^2_w[a, b]$ for each $f \in L^2_w[a, b]$, and that $K : L^2_w[a, b] \mapsto L^2_w[a, b]$ is compact. The first statement follows simply by noting that $G(x, y; \lambda) \in C([a, b] \times [a, b])$, while the second statement relies on the compactness criterion of Ascoli-Arzela: we just need to verify that if $\mathcal{F} \subset L^2_w[a, b]$ is a family with bounded $L^2_w[a, b]$ norms, then $\{K[f] : f \in \mathcal{F}\}$ is equicontinuous in $L^2_w[a, b]$. Let $B > 0$ be such that $|u_a(x; \lambda)|, |u_b(x; \lambda)| \leq B$ for all $x \in [a, b]$, then it is easy to see that for any $x_1, x_2 \in [a, b]$,

$$|G(x_1, y; \lambda) - G(x_2, y; \lambda)| \leq B|W[u_a, u_b; \lambda]^{-1}| \int_{x_1}^{x_2} [|u_a'(y; \lambda)| + |u_b'(y; \lambda)|] \, dy.$$

It then follows that, for $a \leq x_1 < x_2 \leq b$ and $f \in \mathcal{F}$,

$$\begin{aligned} &|K[f](x_1) - K[f](x_2)]| \\ \leq &\int_a^b |G(x_1, y; \lambda) - G(x_2, y; \lambda)||f(y)|w(y)dy \\ &\leq B|W[u_a, u_b; \lambda]^{-1}| \left(\int_{x_1}^{x_2} [|u_a'(y; \lambda)| + |u_b'(y; \lambda)|] \, dy \right) \int_a^b |f(y)|w(y)dy, \end{aligned}$$

from which the equicontinuity of $\{K[f] : f \in \mathcal{F}\}$ in $L^2_w[a, b]$ follows easily. In fact, the equicontinuity of $\{K[f] : f \in \mathcal{F}\}$ would follow from the property that $\int_a^b \int_a^b |G(x + h, y; \lambda) - G(x, y; \lambda)|^2 w(y) dy dx \to 0$ as $h \to 0$, which can be proved under more general settings and conditions.

Remark 10.10. We will develop concepts later on which can be used to prove that K is compact without using an explicit representation for $K[f]$. We could prove here directly that the K (as well as L) has a set of complete orthonormal eigenfunctions in the sense that the set of linear combination of such eigenfunctions is dense in $L^2_w[a, b]$, but it is more economical to leave it to the next section, where this property is treated in a more general context.

In the abstract theory of functional analysis, many eigenvalue problems are set up as an eigenvalue problem for an **unbounded** operator defined on a dense subspace of $L^2(\Omega)$. From the discussion above, one can see that a so called unbounded operator, such as L, may arise naturally as the inverse of a bounded operator K on $L^2(\Omega)$, and since $K(L^2(\Omega))$ can be regarded as a dense subspace of $L^2(\Omega)$, but not the entire $L^2(\Omega)$, the definition of L would necessarily involve a dense subspace of $L^2(\Omega)$. In the one dimensional case, K was worked out as an integral operator before the notion of abstract operators became popular, and these eigenvalue and eigenfunction problems were often solved via the study of integral equations.

It is possible to treat L as a bounded linear operator, such as from $C_c^2(\Omega)$ to $L^2(\Omega)$, but this would not produce a closed image space, and it would not easy to apply the tools of functional analysis in such a setting; in addition, one prefers to treat the domain and the image to be in the same space for eigenvalue problems: imagine how to make sense of $L[v] = \lambda v$ when $L : X \mapsto Y$ and X and Y are unrelated. More importantly, one usually wouldn't be able to solve $L[u] = f$ for $u \in C_c^2(\Omega)$; and even if one works with an appropriate completion or extension, one may not be able to determine a solution u uniquely, unless the extension encodes appropriate BCs.

A large group of differential operators such as our L have a formal symmetry property $(L[u], v)_{L^2(\Omega)} = (u, L[v])_{L^2(\Omega)}$ when u and v are in appropriate subspaces of $L^2(\Omega)$ (defined in terms of boundary conditions), and this analogy with real symmetric and Hermitian operators in finite dimensional vector spaces is the reason why the $L^2(\Omega)$ inner product and the function space $L^2(\Omega)$ is most appropriate in studying the spectrum of L. So, even though we can treat $L : C_c^2(\Omega) \mapsto L^2(\Omega)$ as a bounded linear operator, for the purpose of understanding the spectrum of L, we would treat L as an unbounded operator from $X \mapsto L^2(\Omega)$, where X consists of functions in $L^2(\Omega)$ with appropriate differentiability and boundary conditions; often $C_c^2(\Omega) \subset X \subset L^2(\Omega)$.

Exercises

Exercise 10.4.1. Work out the eigenvalues, associated eigenfunctions, and Green's function for the boundary value problem

$$\begin{cases} -u''(x) = \lambda u(x) \quad x \in [0, 2\pi] \\ u(0) = 0 \\ u'(2\pi) = \beta u(2\pi) \end{cases}$$

where $\beta \in \mathbb{R}$ is a fixed parameter.

Exercise 10.4.2. Assume $G(x,y) \in L^2([a,b] \times [a,b])$. Prove that $f \in L^2[a,b] \mapsto \int_a^b G(x,y) f(y) dy$ defines a bounded compact operator.

Exercise 10.4.3. Model the discussion of this section and use the solutions $e^{\pm\sqrt{|\lambda|}x}$ to $(L-\lambda)u \stackrel{\text{def}}{=} -u''(x) - \lambda u(x) = 0$ over $\mathbb{R}$ for real $\lambda < 0$ to show that

$$f \in L^2(\mathbb{R}) \mapsto (2\sqrt{|\lambda|})^{-1} \int_{\mathbb{R}} f(y) e^{-\sqrt{|\lambda|}|x-y|} dy \in L^2(\mathbb{R})$$

is the resolvent for $(-\frac{d^2}{dx^2} - \lambda)$ for $\lambda < 0$.

Next, show that (10.29) can't hold for $(-\frac{d^2}{dx^2} - \lambda)$ for real $\lambda \geq 0$, which proves that $\mathbb{R}_{\geq 0}$ are in the spectrum of $-\frac{d^2}{dx^2}$. (Hint: work with smooth cut-off of the functions $e^{\pm i\sqrt{\lambda}x}$, which satisfy $(-\frac{d^2}{dx^2} - \lambda)e^{\pm i\sqrt{\lambda}x} = 0$.)

Exercise 10.4.4. The operator $L \stackrel{\text{def}}{=} y^2 \frac{d^2}{dy^2}$ is related to the Laplace operator $y^2(\partial_x^2 + \partial_y^2)$ on the hyperbolic plane $\mathbb{H}^2$. Note that for such an L, the L^2 space would need to use $w(y) = y^{-2}$ as a weight. For any $\lambda > -1/4$, the equation $s(s-1) = \lambda$ in s has two distinct real roots, one of which is $> 1/2$. Let s denote that root. Use the information that $u_1(y) = y^s$ and $u_2(y) = y^{1-s}$ are solutions to $(L-\lambda)u = 0$ to show that

$$f \in L^2_{y^{-2}}(\mathbb{R}) \mapsto (2s-1)^{-1} \left[u_1(y) \int_y^\infty u_2(\zeta) f(\zeta) \zeta^{-2} d\zeta + u_2(y) \int_0^y u_1(\zeta) f(\zeta) \zeta^{-2} d\zeta \right] \in L^2_{y^{-2}}(\mathbb{R})$$

is the resolvent for $L - \lambda$ on $L^2_{y^{-2}}(\mathbb{R})$ for real $\lambda > -1/4$.

For real $\lambda \leq -1/4$, both roots to $s(s-1) = \lambda$ satisfy $\mathfrak{Re}(s) = 1/2$. Use smooth cut-off of the functions y^s and y^{1-s} near $y = 0$ and $y = \infty$ to show that (10.29) can't hold for $(L - \lambda)$ and such λ, which proves that $\mathbb{R}_{\leq -1/4}$ are in the spectrum of L.

Exercise 10.4.5. Verify that the operator $T_0 = -i\frac{d}{dx}$ with domain $X_0 = \{u \in AC[0,1] : u(0) = u(1) = 0, u' \in L^2[0,1]\}$ defines a symmetric operator in $L^2[0,1]$, and that for any scalar λ, $T_0 - \lambda I : X_0 \mapsto L^2[0,1]$ is injective, yet $T_0 - \lambda I$ does not have a well defined inverse $L^2[0,1] \mapsto X_0$. Note that $L^2[0,1]$ refers to complex valued L^2 functions, and the inner product between two functions $u, v \in L^2[0,1]$ is defined as $(u,v) = \int_0^1 u(x)\overline{v(x)}dx$. This example demonstrates that an unbounded operator may have empty resolvent, and that the spectrum theory of a finite dimensional symmetric operator may not directly extend to an unbounded symmetric operator.

Exercise 10.4.6. Define the operator $T_\zeta = -i\frac{d}{dx}$ with domain $X_\zeta = \{u \in AC[0,1] : u(1) = e^{i\beta}u(0), u' \in L^2[0,1]\}$, where $\zeta = e^{i\beta}$ for some $\beta \in \mathbb{R}$. Note that $X_0 \subset X_\zeta$, so T_ζ may be considered an extension of T_0. Verify that T_ζ defines a symmetric operator in $L^2[0,1]$, and that a scalar λ is in the spectrum of T_ζ iff $\lambda - \beta \in 2\pi\mathbb{Z}$.

10.5 Variational Characterization of Eigenvalues and Eigenfunctions

The implementation of the strategy to relate the eigenfunction expansion for equations such as (4.29) to the spectrum of a compact symmetric operator would require some additional adaptation, which we will describe below. We will also discuss the variational approach, which allows us to construct the resolvent K and prove its properties without having an explicit integral representation as in the previous section.

10.5.1 Set up of a Symmetric Second Order Elliptic Operator and its Resolvent by Variational Method

One main technical issue in extending the approach to (4.29) is that it's not easy to provide an easily identifiable description for functions to be in the domain for the natural operator L associated to (4.29):

$$L[v] = -\sum_{i,j=1}^{n} \left(a_{ij}(x)v_{x_i}(x)\right)_{x_j} + c(x)v(x).$$

For the Dirichelt problem with homogeneous boundary condition, a natural candidate space would be $H_0^1(\Omega)$; but it is harder to characterize those $v \in H_0^1(\Omega)$ which make $L[v] \in L^2(\Omega)$. We handle this issue in one of two closely related ways.

The first approach is to define L on $H_0^1(\Omega)$, allowing $L[v]$ to lie in an extended space of $L^2(\Omega)$. When $v \in H_0^1(\Omega)$ and the $a_{ij}(x)$'s satisfy (4.27), the terms $a_{ij}(x)v_{x_i}(x)$ are in $L^2(\Omega)$, but we can't necessarily make sense of $\left(a_{ij}(x)v_{x_i}(x)\right)_{x_j}$ as an L^2 function without knowing more regularity about $a_{ij}(x)v_{x_i}(x)$, but $a_{ij}(x)v_{x_i}(x)$ define continuous linear functionals on $H_0^1(\Omega)$ through

$$w \in H_0^1(\Omega) \mapsto \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}(x)w_{x_j}(x)\,dx,$$

as there exists $C > 0$ depending on $a_{ij}(x)$ and $v_{x_i}(x)$ such that

$$\left| \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}(x)w_{x_j}(x)\,dx \right| \le C||w||_{H_0^1(\Omega)}.$$

The space of continuous linear functionals on $H_0^1(\Omega)$ comes into play when discussing the solvability of (4.29), and is denoted as $H^{-1}(\Omega)$*. Thus for any $v \in H_0^1(\Omega)$, $L[v] \in H^{-1}(\Omega)$, so $L : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ is a natural set up for L. Then the proper domain of L for the purpose of discussing spectrum properties would be the subspace of $H_0^1(\Omega)$ whose members v would make $L[v] \in L^2(\Omega)$; more precisely, v needs to have the property that the $L^2(\Omega)$ vector field

$$x \in \Omega \mapsto (\sum_{i=1}^{n} a_{i1}(x)v_{x_i}(x), \cdots, \sum_{i=1}^{n} a_{in}(x)v_{x_i}(x))$$

has an $L^2(\Omega)$ divergence in the sense that there exists some $h \in L^2(\Omega)$ such that for any $w \in H_0^1(\Omega)$,

$$\int_\Omega (\sum_{i=1}^{n} a_{i1}(x)v_{x_i}(x), \cdots, \sum_{i=1}^{n} a_{in}(x)v_{x_i}(x)) \cdot \nabla w(x)\, dx = -\int_\Omega h(x)w(x)\, dx.$$

Note that this description is a lot less explicit about v.

The second approach is to work directly with a bilinear form $B[v,w]$ on $H_0^1(\Omega) \times H_0^1(\Omega)$

$$B[v,w] = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x)v_{x_i}(x)w_{x_j}(x) + c(x)v(x)w(x) \right) dx,$$

and produce a compact symmetric operator $K : L^2(\Omega) \mapsto H_0^1(\Omega) \subset L^2(\Omega)$, to be described below, by applying the variational method to the quadratic form $B[v,v]$ associated with $B[v,w]$, without necessarily defining the domain of L for $L[v] \in L^2(\Omega)$ in full detail — K would serve as a resolvent for L, if L is properly defined. In general, for any fixed $v \in H_0^1(\Omega)$, $w \mapsto B[v,w]$ defines a bounded linear functional on $H_0^1(\Omega)$, thus giving rise to an element in $H^{-1}(\Omega)$, which we can label as $L[v]$; and $B[\cdot,\cdot]$ are related by the relation $\langle L[v], w\rangle = B[v,w]$ for $w \in H_0^1(\Omega)$.

When $0 \le c(x) \le M$ for all $x \in \Omega$, the variational approach in the earlier section sets up a well defined map

$$f \in L^2(\Omega) \mapsto v \in H_0^1(\Omega) \quad \text{such that (4.28) holds,}$$

namely,

$$B[v,w] = (f,w)_{L^2(\Omega)} \quad \text{for all } w \in H_0^1(\Omega).$$

*Riesz's Theorem in Functional Analysis implies that any continuous linear functional on $H_0^1(\Omega)$ can be represented as the inner product with an element in $H_0^1(\Omega)$; but continuous linear functionals often arise in other forms than as the inner product with an element. Thus there is a need to to study $H^{-1}(\Omega)$ separately.

Define this map to be $K[f] = v$, and regard

$$K : L^2(\Omega) \mapsto H_0^1(\Omega) \subset L^2(\Omega).$$

Then K is a **symmetric, bounded, injective** linear operator on $L^2(\Omega)$ in the sense that

$$K[f] = 0 \Longrightarrow f = 0 \tag{10.30}$$

$$\exists C > 0 \text{ depending only on } L \text{ through } M,\, m \text{ and } \Omega \text{ s.t. } ||K[f]||_{L^2(\Omega)} \le C||f||_{L^2(\Omega)}, \tag{10.31}$$

$$(K[f], g)_{L^2(\Omega)} = (f, K[g])_{L^2(\Omega)} \text{ for all } f, g \in L^2(\Omega). \tag{10.32}$$

(10.30) follows directly from the integral formulation for (4.28). (10.31) follows from using $w = v$ in the integral formulation for (4.28):

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) v_{x_i}(x) v_{x_j}(x) + (c(x)v(x) - f(x))\, v(x) \right] dx = 0,$$

and using (4.27) to imply

$$m \int_\Omega |\nabla v(x)|^2 \, dx \le \int_\Omega f(x) v(x)\, dx \le \left(\int_\Omega f^2(x)\, dx \right)^{1/2} \left(\int_\Omega v^2(x)\, dx \right)^{1/2},$$

which, together with (4.22), concludes a proof for (10.31).

In fact, we get the following stronger inequality if we use the estimate $\left(\int_\Omega v^2(x)\, dx\right)^{1/2} \le \sqrt{C} \left(\int_\Omega |\nabla v|^2(x)\, dx\right)^{1/2}$ in the above:

$$||K[f]||_{H_0^1(\Omega)} \le ||\nabla v||_{L^2(\Omega)} \le \frac{\sqrt{C}}{m} ||f||_{L^2(\Omega)}. \tag{10.33}$$

(10.32) is proved as follows. Set $K[f] = v$ and $K[g] = w$. By their defining property, we have

$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) v_{x_i}(x) w_{x_j}(x) + (c(x)v(x) - f(x))\, w(x) \right] dx = 0,$$
$$\int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) w_{x_i}(x) v_{x_j}(x) + (c(x)w(x) - g(x))\, v(x) \right] dx = 0,$$

from which it follows that

$$\int_\Omega f(x) w(x)\, dx = \int_\Omega g(x) v(x)\, dx,$$

namely, (10.32).

Based on (10.31) and (10.32), the eigenvalue and eigenfunction expansion problem for L reduces to that for the bounded and symmetric linear operator K on $L^2(\Omega)$: $v \in H_0^1(\Omega)$ is an eigenfunction for L, $L[v] = \lambda v$ for some $\lambda \neq 0$, iff $v = K[\lambda v] = \lambda K[v]$, iff $K[v] = \lambda^{-1} v$ and $v \in L^2(\Omega)$. It follows from (10.31) that $\lambda = 0$ is not an eigenvalue of L under our assumptions, thus we have reduced the problem to whether K, as a bounded and symmetric linear operator on $L^2(\Omega)$, has a complete set of eigenfunctions which spans $L^2(\Omega)$, with the additional orthogonality properties as described for the eigenfunctions of a Sturm-Liouville problem.

In situations where we don't have the condition $c(x) \geq 0$ in Ω, but have a bound of the form $|c(x)| \leq M$ in Ω, the operator $L + M$ would satisfy the set up above, and can be used to set up the map $K : L^2(\Omega) \mapsto H_0^1(\Omega) \subset L^2(\Omega)$ such that $K[f] = v$ satisfies

$$B[v, \eta] + M(v, \eta)_{L^2(\Omega)} = (f, \eta)_{L^2(\Omega)} \text{ for all } \eta \in H_0^1(\Omega),$$

and $L[v] = \lambda v$, iff $(L + M)[v] = (\lambda + M)v$, iff $K[v] = (\lambda + M)^{-1} v$.

Remark 10.11. For the one-dimensional Sturm-Liouville problem, we were able to identity more precisely a space X incorporating the BCs on which to consider $L : X \mapsto L^2(a, b)$. Here we used $X = H_0^1(\Omega)$, or rather a not explicitly identified subspace of it, in this discussion for homogeneous Dirichlet boundary condition; but other subspaces reflecting other boundary conditions may also be used.

Remark 10.12. In applications involving curvilinear coordinates or manifolds, the integrals in the integral formulation would involve a density function, like the weight function in the Sturm-Liouville problems; in fact, the integration may not be carried out in one coordinate patch, and there is a coordinate independent formulation of the volume element as well as the bilinear expressions in the integrand in the integrals.

The analysis carried out above extends to such situations without too much extra work; one can reformulate certain arguments in a slightly more abstract way, and when it comes time to work with an explicit form of the equation in a specific coordinate, one can choose the test function to have support in that coordinate patch, and carry out the computations largely as above.

For instance, on a Riemannian manifold (M, g) with a given Riemannian metric g, a C^1 function u has a coordinate-free definition of gradient ∇u and its length $||\nabla u||$ at any point, such that, if a coordinate is chosen in a neighborhood of a point x, in which the length square of a tangent vector v at x with coordinate $(v^1, \cdots, v^n)$ is given by $|v|^2 = \sum_{i,j=1}^n g_{ij}(x) v^i v^j$, then the length square of ∇u at x is given by

$|\nabla u(x)|^2 = \sum_{i,j=1}^n g^{ij}(x) u_{x_i}(x) u_{x_j}(x)$, and the volume element in this coordinate is expressed as $\sqrt{\det(g_{ij}(x))}dx_1 \cdots dx_n$, where $(g^{ij}(x))$ is the inverse matrix of $(g_{ij}(x))$. There is a coordinate-free definition of a bilinear form on the gradients of a pair of functions u and v, which is expressed as

$$B[u,v] = \int \left(\sum_{i,j=1}^n g^{ij}(x) u_{x_i}(x) v_{x_j}(x) \right) \sqrt{\det(g_{ij}(x))}\ dx_1 \cdots dx_n$$

when one of them has compact support in this coordinate patch. The linear differential operator associated with $B[u,v]$, the Laplace operator Δ_g of the metric g, then has a local expression in this coordinate

$$\Delta_g u(x) = \sum_{i,j=1}^n \frac{1}{\sqrt{\det(g_{ij}(x))}} \left(\sqrt{\det(g_{ij}(x))} g^{ij}(x) u_{x_i}(x) \right)_{x_j},$$

as

$$\int \left(\sum_{i,j=1}^n g^{ij}(x) u_{x_i}(x) v_{x_j}(x) \right) \sqrt{\det(g_{ij}(x))}\ dx_1 \cdots dx_n$$
$$= -\int \sum_{i,j=1}^n \frac{1}{\sqrt{\det(g_{ij}(x))}} \left(\sqrt{\det(g_{ij}(x))} g^{ij}(x) u_{x_i}(x) \right)_{x_j} v(x) \sqrt{\det(g_{ij}(x))}\ dx_1 \cdots dx_n.$$

Note that, in comparison with the expression for the linear operator L used earlier in the section, this Δ_g has the weight factor $\frac{1}{\sqrt{\det(g_{ij}(x))}}$ in front of the "divergence expression".

Again the analysis of the spherical Laplace operator may serve as a concrete guide, where, in spherical polar coordinate (θ, ϕ) on $\mathbb{S}^2$, $|\nabla u|^2 = u_\theta^2 + \sin^{-2}\theta u_\phi^2$, and the area element is $\sin\theta\, d\theta d\phi$; while, if we use a graph representation for $\mathbb{S}^2$ or stereographic coordinate for $\mathbb{S}^2$, the expressions for $|\nabla u|^2$, the area element, and the spherical Laplace operator would take on a different form.

10.5.2 Eigenvalues and Eigenfunctions of Symmetric Second Order Elliptic Operators

Some standard properties of eigenvalues and eigenfunctions of L follow from (10.32).

Theorem 10.22. *Suppose that the $a_{ij}(x)$'s and $c(x)$ are real-valued, satisfy (4.27) and $|c(x)| \le M$ for all $x \in \Omega$. Then all eigenvalues of L are real-valued, and eigenfunctions of L associated to distinct eigenvalues are orthogonal in $L^2(\Omega)$, and can be taken to be real-valued.*

Furthermore, we have

Theorem 10.23. *Suppose that* Ω *is a bounded domain in* $\mathbb{R}^n$*, that the* $a_{ij}(x)$*'s and* $c(x)$ *are real-valued, satisfy* (4.27) *and* $|c(x)| \le M$ *for all* $x \in \Omega$*. Then the eigenvalues of* L *can not have a finite accumulation point, and the eigenspace of* L *associated to any single eigenvalue is finite dimensional.*

Theorem 10.23 is based on the general spectrum property of a linear compact operator.

Theorem 10.24. *Let* Y *be a Banach space,* $K : Y \mapsto Y$ *be a linear compact operator. Then the spectrum of* K *can not have a non-zero accumulation point, and the eigenspace of* K *associated to any single non-zero eigenvalue is finite dimensional.*

See Theorem 6.8 in Brezis's text [B] for a simple proof of this theorem. To apply Theorem 10.24 to our setting, recall that we may assume 0 is in the resolvent of L by adding a multiple of the identity map if necessary; assuming for the moment that K has been verified to be compact, then $\lambda \neq 0$ is in the spectrum of L iff λ^{-1} is in the spectrum of K, then the conclusions of Theorem 10.23 follow readily. The verification of the compactness of K associated with Theorem 10.23 is based on the following compactness theorem.

Theorem 10.25. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$*. Then for any sequence* $\{w_j\}$ *in* $H_0^1(\Omega)$ *with bounded* $H_0^1(\Omega)$ *norms, there is a* $w \in H_0^1(\Omega)$ *and a subsequence* $\{w_{j_k}\}$ *of* $\{w_j\}$ *such that* $w_{j_k} \to w$ *in* $L^2(\Omega)$*, and for each* $1 \le a \le n$*,* $\{\partial_{x_a} w_{j_k}\}$ *converges weakly in* $L^2(\Omega)$*. As a consequence,* $\int_\Omega |\nabla w(x)|^2\, dx \le \liminf_{k\to\infty} \int_\Omega |\nabla w_{j_k}(x)|^2\, dx$*.*

Recall that a sequence $\{f_j\}$ in $L^2(\Omega)$ converges **weakly** to f^* in $L^2(\Omega)$ if

$$\int_\Omega f_j(x)\eta(x)\, dx \to \int_\Omega f^*(x)\eta(x)\, dx \quad \forall \eta \in L^2(\Omega).$$

Taking $\eta = f^*$, we see that

$$\int_\Omega |f^*(x)|^2\, dx = \lim_{j\to\infty} \int_\Omega f_j(x) f^*(x)\, dx \le \liminf_{j\to\infty} \left(\int_\Omega |f_j(x)|^2\, dx\right)^{1/2} \left(\int_\Omega |f^*(x)|^2\, dx\right)^{1/2},$$

from which it follows that

$$\int_\Omega |f^*(x)|^2\, dx \le \liminf_{j\to\infty} \int_\Omega |f_j(x)|^2\, dx. \tag{10.34}$$

The weak $L^2(\Omega)$ limits of $\{\partial_{x_a} w_{j_k}\}$ are the $L^2(\Omega)$ derivatives of the $w \in H_0^1(\Omega)$: w_{x_a}.

Under (4.27), we also have

$$\int_\Omega \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) w_{x_\beta}(x) \right) dx \le \liminf_{k\to\infty} \int_\Omega \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) (w_{j_k}(x))_{x_\alpha} (w_{j_k}(x))_{x_\beta} \right) dx \tag{10.35}$$

This follows from a similar argument. Since $\sum_{\alpha=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) \in L^2(\Omega)$ for each β, we have

$$\begin{aligned}
&\int_\Omega \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) w_{x_\beta}(x) \right) dx \\
&= \lim_{k\to\infty} \int_\Omega \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) (w_{j_k}(x))_{x_\beta} \right) dx \\
&\le \liminf_{k\to\infty} \left(\int_\Omega \sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) w_{x_\beta}(x)\, dx \right)^{1/2} \left(\int_\Omega \sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) (w_{j_k}(x))_{x_\alpha} (w_{j_k}(x))_{x_\beta}\, dx \right)^{1/2},
\end{aligned}$$

from which (10.35) follows. In the last two lines above, we have used the algebraic inequality

$$\begin{aligned}
&\left| \sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) (w_{j_k}(x))_{x_\beta} \right| \\
&\le \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) w_{x_\alpha}(x) w_{x_\beta}(x) \right)^{1/2} \left(\sum_{\alpha,\beta=1}^n a_{\alpha\beta}(x) (w_{j_k}(x))_{x_\alpha} (w_{j_k}(x))_{x_\beta} \right)^{1/2},
\end{aligned}$$

and the Cauchy-Schwarz inequality applied to the integration of the above estimate.

Remark 10.13. Since any eigenfunction $v_j(x)$: $Lv_j = \lambda_j v_j$, satisfies

$$B[v_j, \eta] = \lambda_j (v_j, \eta)_{L^2(\Omega)} \quad \text{for any } \eta \in H_0^1(\Omega),$$

the $L^2(\Omega)$-orthogonality relation in **Theorem** 10.22 also gives rise to an orthogonality relation in terms of the bilinear form B:

$$B[v_j, v_k] = 0 \quad \text{when } \lambda_j \neq \lambda_k.$$

But the underlying symmetry for the spectrum property is the $L^2(\Omega)$ symmetry property of K.

10.5.3 Variational Characterization of Eigenvalues and Eigenfunctions

It turns out that the eigenvalues and eigenfunctions of L (or of K) have a variational characterization. Let $Q[w]$ be the quadratic form associated with the bilinear form $B[v,w]$

$$Q[w] = B[w,w] = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x) w_{x_i}(x) w_{x_j}(x) + c(x) w^2(x) \right) dx.$$

Theorem 10.26. *Suppose that Ω, $a_{ij}(x)$'s, and $c(x)$ satisfy the same assumptions as in Theorem 10.23. Then*

$$\inf \left\{ Q[w] : w \in H_0^1(\Omega), \int_\Omega w^2(x)\, dx = 1 \right\}$$

is attained by some $v \in H_0^1(\Omega)$. Let this minimum be λ_1. Then $L[v] = \lambda_1 v$. Furthermore, any eigenvalue λ of L satisfies $\lambda \geq \lambda_1$. Thus this λ_1 is the smallest eigenvalue of L.

Theorem 10.27. *Suppose that Ω, $a_{ij}(x)$'s, and $c(x)$ satisfy the same assumptions as in Theorem 10.23. Suppose further that $\{v_1(x), \cdots, v_j(x), \cdots\}$ is a collection of eigenfunctions of L, which can be taken to be orthonormal in $L^2(\Omega)$ by Theorems 10.22 and 10.23. Then if*

$$\{w \in H_0^1(\Omega) : \int_\Omega w^2(x)\, dx = 1, \int_\Omega w(x) v_j(x)\, dx = 0 \text{ for all } v_j\text{'s}\}$$

is non-empty,

$$\inf \left\{ Q[w] : w \in H_0^1(\Omega), \int_\Omega w^2(x)\, dx = 1, \int_\Omega w(x) v_j(x)\, dx = 0 \text{ for all } v_j\text{'s} \right\}$$

is attained by some $v \in H_0^1(\Omega)$, and $L[v] = \lambda v$ for some λ.

It turns out that one can provide a proof for both of the above theorems based on the compactness Theorem 10.25.

Remark 10.14. A symmetric, compact, bounded linear operator has similar spectrum properties, but after stating a general theorem for the spectrum decomposition of a symmetric, compact, bounded linear operator on a Hilbert space, we still present a brute force proof for Theorem 10.27 working directly with L and its associated bilinear form B.

Theorem 10.28. *Let H be a separable Hilbert space and let K be a compact symmetric operator on H. Then there exists a Hilbert basis composed of eigenvectors of K. More specifically, let (λ_n) be the sequence of all (distinct) nonzero eigenvalues of K, $E_n = \{x \in H : (K - \lambda_n I)x = 0\}$, $E_0 = \{x \in H : Kx = 0\}$, then*

(i) each E_n for $n \neq 0$ is finite dimensional;

(ii) the E_n' are mutually orthogonal; and

(iii) $\bigoplus_{n=0}^{\infty} E_n$ is dense in H.

See Theorem 6.11 in Brezis's text [B]. Although K may have only a finite number of non-zero eigenvalues in the general context of Theorem 10.28, in applying Theorem 10.28 to our settings, K is constructed to be injective, so it follows that we will get an infinite sequence $(\lambda_n) \to \infty$ as $n \to \infty$.

Theorem 10.27 provides a characterization for the n-th eigenvalue λ_n of L as

$$\lambda_n = \inf\{Q[w]/(w,w) : w \neq 0, (w, v_1) = \cdots = (w, v_{n-1}) = 0\},$$

where $v_1, \cdots, v_{n-1}$ are (orthonormal) eigenfunctions associated with the first $(n-1)$ eigenvalues $\lambda_1, \cdots, \lambda_{n-1}$ (counting with multiplicity). However, this characterization depends on knowing the first $(n-1)$ eigenvalues and their associated eigenfunctions. The following variational characterization for the n-th eigenvalue λ_n of L, by a minimax procedure, is due to Rayleigh and Ritz.

Theorem 10.29. *The n-th eigenvalue λ_n of L can be characterized as*

$$\lambda_n = \inf_{W_n : n dimensional\ subspace\ of\ H_0^1(\Omega)} \max\{Q[w]/(w,w,) : w \neq 0, w \in W_n\}.$$

Since there is a short proof for Theorem 10.29, assuming Theorem 10.27, we first supply a proof of Theorem 10.29.

Proof of Theorem 10.29. Let W_n be any n-dimensional subspace of $H_0^1(\Omega)$, spanned by $\{w_1, w_2, \cdots, w_n\}$, and let $\{v_1, v_2, \cdots, v_n\}$ denote a set of orthonormal eigenfunctions of L associated with the first n eigenvalues (counting with multiplicity). Then there are constants $c_1, \cdots, c_n$, not all zero, such that $w = \sum_{i=1}^n c_i w_i$ is orthogonal to $v_1, \cdots, v_{n-1}$ in $L^2(\Omega)$. This is because that the orthogonality conditions $(\sum_{i=1}^n c_i w_i, v_j) = 0$, $j = 1, \cdots, n-1$, are $n-1$ linear, homogeneous equations on the n unknowns $c_1, \cdots, c_n$. Thus by Theorem 10.27,

$$\max\{Q[w]/(w,w) : w \neq 0, w \in W_n\} \geq Q[\sum_{i=1}^n c_i w_i]/||\sum_{i=1}^n c_i w_i||^2 \geq \lambda_n.$$

Next, if we set $V_n = \text{span}\{v_1, v_2, \cdots, v_n\}$, then $\max\{Q[w]/(w, w,) : w \neq 0, w \in V_n\} = \lambda_n$. This completes a proof for Theorem 10.29. □

Proof of Theorem 10.26. First, we need to prove that the inf in the Theorem is finite. This is easy to prove here: $|\int_\Omega c(x)w^2(x)\,dx| \leq M\int_\Omega w^2(x)\,dx = M$, when $\int_\Omega w^2(x)\,dx = 1$, so $Q[w] \geq m\int_\Omega |\nabla w(x)|^2\,dx - M$ when $w \in H_0^1(\Omega)$ and $\int_\Omega w^2(x)\,dx = 1$.

Next, let $\{w_j(x)\} \subset H_0^1(\Omega)$ be a minimizing sequence. Then the lower bound for $Q[w]$ in the line above shows that $\int_\Omega |\nabla w_j(x)|^2\,dx$ is bounded. We now apply Theorem 10.25 to $\{w_j(x)\}$ to find a subsequence, still denoted as $\{w_j(x)\}$, a $v \in H_0^1(\Omega)$ such that $w_j \to v$ in $L^2(\Omega)$, and $\int_\Omega |\nabla v(x)|^2\,dx \leq \liminf_{j\to\infty}\int_\Omega |\nabla w_j(x)|^2\,dx$. Using $w_j \to v$ in $L^2(\Omega)$, we see that $\int_\Omega v^2(x)\,dx = 1$, and $\int_\Omega c(x)w_j^2(x)\,dx \to \int_\Omega c(x)v^2(x)\,dx$. Together with (10.35), it follows that

$$Q[v] \leq \liminf_{j\to\infty} Q[w_j].$$

This shows that v attains the infimum.

Finally, take any $\eta \in H_0^1(\Omega)$ and define $\mu(t) > 0$ for small $t \in \mathbb{R}$ through $\mu(t)^2 = \int_\Omega (v(x) + t\eta(x))^2\,dx$. Then $\mu(0) = 1$, and $\mu(t)$ is C^1 in t with $\mu'(0) = \int_\Omega v(x)\eta(x)\,dx$. Note that $\mu(t)^{-1}(v(x) + t\eta(x)) \in H_0^1(\Omega)$ satisfies the constraint

$$\int_\Omega \left|\mu(t)^{-1}(v(x) + t\eta(x))\right|^2\,dx = 1,$$

so

$$Q[\mu(t)^{-1}(v(x) + t\eta(x))] \text{ has a mimimum at } t = 0.$$

By taking derivative in t and setting $t = 0$, using $\mu(0) = 1$, and $\mu'(0) = \int_\Omega v(x)\eta(x)\,dx$, it follows that

$$2\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)v_{x_i}(x)\eta_{x_j}(x) + c(x)v(x)\eta(x) - Q[v]v(x)\eta(x)\right)\,dx = 0.$$

This shows that $L[v] = \lambda_1 v$ with $\lambda_1 = Q[v]$, which is the infimum value of $Q[w]$ under the constraint.

Let $w \in H_0^1(\Omega)$ be any eigenfunction of L: $L[w] = \lambda w$. We may normalize w such that $\int_\Omega w^2(x)\,dx = 1$. Then it follows from the integral form for $L[w] = \lambda w$ that

$$Q[w] = \int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)w_{x_i}(x)w_{x_j}(x) + c(x)w^2(x)\right)\,dx = \lambda\int_\Omega w^2(x)\,dx = \lambda,$$

which shows that $\lambda = Q[w] \geq \lambda_1$. □

Proof for Theorem 10.27. The above proof works verbatim to produce a $w \in H_0^1(\Omega)$ such that

$$\int_\Omega w^2(x)\,dx = 1, \int_\Omega w(x)v_j(x)\,dx = 0 \text{ for all } v_j\text{'s, and}$$

$$\frac{d}{dt}\Big|_{t=0} Q[\mu(t)^{-1}(w(x)+t\eta(x))] = 0,$$

for any $\eta \in H_0^1(\Omega)$ satisfying $\int_\Omega \eta(x)v_j(x)\,dx = 0$ for all v_j's, and $\mu(t) > 0$ is defined through $\mu(t)^2 = \int_\Omega (w(x)+t\eta(x))^2\,dx$. This again leads to

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij}(x)w_{x_i}(x)\eta_{x_j}(x) + c(x)w(x)\eta(x) - Q[w]w(x)\eta(x) \right) dx = 0, \tag{10.36}$$

for such η.

Given any $\eta \in H_0^1(\Omega)$, we are going to define a projection $P\eta \in H_0^1(\Omega)$ such that $\eta - P\eta$ satisfies all the constraints required.

Define $\eta_j = \int_\Omega \eta(x)v_j(x)\,dx$, and $P_N\eta = \sum_{j=1}^N \eta_j v_j \in H_0^1(\Omega)$. In fact, $P_N\eta$ is well defined for any $\eta \in L^2(\Omega)$. We will prove

> **Claim**. For any $\eta \in L^2(\Omega)$, $\{P_N\eta\}$ is Cauchy in $L^2(\Omega)$. If $\eta \in H_0^1(\Omega)$, then $\{P_N\eta\}$ is also Cauchy in $H_0^1(\Omega)$.

When $\eta \in H_0^1(\Omega)$, we define $P\eta$ to be the $H_0^1(\Omega)$ limit of $\{P_N\eta\}$. We will provide a proof for this Claim at the end of this proof. In addition, we will use the following

> **Property**. For any finite dimensional span of $\{v_1, v_2, \cdots\}$, where $\{v_1, v_2, \cdots\}$ is a collection of eigenfunctions of L,
>
> $$\begin{aligned} &w \in H_0^1(\Omega) \text{ and } (w,v)_{L^2(\Omega)} = 0 \text{ for } v \in \operatorname{span}\{v_{j_1}, \cdots, v_{j_n}\} \\ \Longrightarrow\ & B[w,v] = 0 \text{ for } v \in \operatorname{span}\{v_{j_1}, \cdots, v_{j_n}\}. \end{aligned} \tag{10.37}$$
>
> This follows from
>
> $$B[w,v] = \int_\Omega w(x)L[v](x)\,dx = \sum_{k=1}^n c_k \int_\Omega w(x)v_{j_k}(x)\,dx = 0,$$

when $v = \sum_{k=1}^n c_k v_{j_k}$. Note that the condition $w \in H_0^1(\Omega)$ is used in an inconspicuous way, but the zero boundary value is important; The above property may not hold if $w \in H^1(\Omega)$.

We now prove that (10.36) holds for any $\eta \in H_0^1(\Omega)$ without any constraints, from which it follows that $L[w] = Q[w]w$. For any $\eta \in H_0^1(\Omega)$, note that, for any j,

$$(\eta - P\eta, v_j)_{L^2(\Omega)} = \lim_{N\to\infty} (\eta - P_N\eta, v_j)_{L^2(\Omega)} = 0.$$

So $\eta - P\eta$ satisfies the constraints for applying (10.36), which now leads to

$$\begin{aligned} 0 &= B[w, \eta - P\eta] - Q[w]\,(w, \eta - P\eta)_{L^2(\Omega)} \\ &= B[w, \eta] - Q[w]\,(w, \eta)_{L^2(\Omega)} - B[w, P\eta] + Q[w]\,(w, P\eta)_{L^2(\Omega)}\,. \end{aligned}$$

Finally, since $(w, P_N\eta)_{L^2(\Omega)} = 0$ for all N, it follows from the **Property** stated above that $B[w, P_N\eta] = 0$ for all N. Then, using $P_N\eta \to P\eta$ in $H_0^1(\Omega)$,

$$B[w, P\eta] - Q[w]\,(w, P\eta)_{L^2(\Omega)} = \lim_{N\to\infty} \left\{ B[w, P_N\eta] - Q[w]\,(w, P_N\eta)_{L^2(\Omega)} \right\} = 0.$$

Thus we have proved that

$$B[w, \eta] - Q[w]\,(w, \eta)_{L^2(\Omega)} = 0$$

for all $\eta \in H_0^1(\Omega)$, from which it follows that w is an eigenfunction of L: $L[w] = Q[w]w$.

We now supply a proof for the **Claim** stated above. Note that $(\eta - P_N\eta, v_j)_{L^2(\Omega)} = 0$ for each $1 \le j \le N$, so $(\eta - P_N\eta, P_N\eta)_{L^2(\Omega)} = 0$, from which it follows that

$$||\eta||^2_{L^2(\Omega)} = ||\eta - P_N\eta||^2_{L^2(\Omega)} + ||P_N\eta||^2_{L^2(\Omega)}.$$

Since $||P_N\eta||^2_{L^2(\Omega)} = \sum_{j=1}^N \eta_j^2$, this produces the Bessel's inequality for this setting:

$$\sum_{j=1}^{\infty} \eta_j^2 \le ||\eta||^2_{L^2(\Omega)}.$$

Furthermore, we have $||P_{N'}\eta - P_N\eta||^2_{L^2(\Omega)} = \sum_{j=N+1}^{N'} \eta_j^2$. By the Bessel's inequality, $\{P_N\eta\}$ is Cauchy in $L^2(\Omega)$.

We next show that, when $\eta \in H_0^1(\Omega)$, $\{P_N\eta\}$ is Cauchy in $H_0^1(\Omega)$.

Since $(\eta - P_N\eta, P_N\eta)_{L^2(\Omega)} = 0$, it follows that

$$B[\eta - P_N\eta, P_N\eta] = 0,$$

and

$$B[\eta, \eta] = B[\eta - P_N\eta, \eta - P_N\eta] + B[P_N\eta, P_N\eta].$$

Note that

$$\begin{aligned} B[P_N\eta, P_N\eta] &= \int_\Omega P_N\eta L[P_N\eta]\,dx \\ &= \int_\Omega P_N\eta \left(\sum_{j=1}^N \lambda_j \eta_j v_j(x) \right)\,dx \\ &= \sum_{j=1}^N \lambda_j \eta_j^2. \end{aligned}$$

Using (**??**), we have

$$\begin{aligned}
&B[P_N\eta, P_N\eta]\\
=&B[\eta,\eta] - B[\eta - P_N\eta, \eta - P_N\eta]\\
\le&M\left(||\nabla\eta||^2_{L^2(\Omega)} + ||\eta||^2_{L^2(\Omega)}\right) - m||\nabla(\eta - P_N\eta)||^2_{L^2(\Omega)} + M||\eta - P_N\eta||^2_{L^2(\Omega)}\\
\le&M\left(||\nabla\eta||^2_{L^2(\Omega)} + 2||\eta||^2_{L^2(\Omega)}\right).
\end{aligned}$$

Thus $\sum_{j=1}^{\infty}\lambda_j\eta_j^2$ is convergent. Next, using $L[P_{N'}\eta - P_N\eta] = \sum_{j=N+1}^{N'}\lambda_j\eta_j v_j(x)$, we see that, for $N' > N$,

$$\begin{aligned}
B[P_{N'}\eta - P_N\eta, P_{N'}\eta - P_N\eta] &= \int_\Omega (P_{N'}\eta - P_N\eta)\, L[P_{N'}\eta - P_N\eta]\, dx\\
&= \int_\Omega (P_{N'}\eta - P_N\eta)\left(\sum_{j=N+1}^{N'}\lambda_j\eta_j v_j(x)\right) dx\\
&= \sum_{j=N+1}^{N'}\lambda_j\eta_j^2.
\end{aligned}$$

This implies that

$$m||\nabla(P_{N'}\eta - P_N\eta)||^2_{L^2(\Omega)} \le B[P_{N'}\eta - P_N\eta, P_{N'}\eta - P_N\eta] + M||P_{N'}\eta - P_N\eta||^2_{L^2(\Omega)} \to 0,$$

as $N', N \to \infty$, which proves that $\{P_N\eta\}$ is Cauchy in $H_0^1(\Omega)$. □

Based on Theorems 10.22, 10.23, and 10.27, we can enumerate all the eigenvalues of L as $\lambda_1 \le \lambda_2 \le \cdots$ (allowing finite multiplicity); and it is easy to see that

$$\{w \in H_0^1(\Omega) : \int_\Omega w^2(x)\, dx = 1, \int_\Omega w(x)v_j(x)\, dx = 0 \text{ for all } v_j\text{'s}\}$$

is non-empty for any finite collection of v_j's, so the procedure in Theorem 10.27 can not stop after a finite number of steps, and $\lambda_n \to \infty$ as $n \to \infty$, producing associated eigenfunctions $v_1, v_2, \cdots$: $L[v_n] = \lambda_n v_n$, such that $\{v_n\}$ is orthonormal in $L^2(\Omega)$. Then, according to Theorems 10.22,

$$\{w \in H_0^1(\Omega) : \int_\Omega w(x)v_n(x)\, dx = 0 \text{ for all } n\} \text{ must consist only of } w = 0.$$

We now claim that

$$\{f \in L^2(\Omega) : \int_\Omega f(x)v_n(x)\, dx = 0 \text{ for all } n\} \text{ must also consist only of } f = 0.$$

For, if $f \in L^2(\Omega)$ and $\int_\Omega f(x)v_n(x)\,dx = 0$ for all n. Let $w = K[f] \in H_0^1(\Omega)$. Then

$$B[w, v_n] = (f, v_n)_{L^2(\Omega)} = 0 \text{ for all } n,$$

but

$$B[w, v_n] = \int_\Omega w(x)L[v_n](x)\,dx = \int_\Omega \lambda_n w(x)v_n(x)\,dx,$$

which implies that $\int_\Omega w(x)v_n(x)\,dx = 0$ for all n, from which we conclude that $w = 0$ and therefore $f = 0$. This shows that the $L^2(\Omega)$ closure of $\{v_1, v_2, \cdots\}$ is $L^2(\Omega)$. We have thus proved

Theorem 10.30. *Under the same assumptions as Theorem 10.22, the $L^2(\Omega)$ closure of a complete set of eigenfunctions of L (and of K) $\{v_1, v_2, \cdots\}$ is $L^2(\Omega)$. We can take $\{v_1, v_2, \cdots\}$ to be orthonormal in $L^2(\Omega)$. Then any function $f \in L^2(\Omega)$ has a Fourier expansion in $\{v_1, v_2, \cdots\}$: $f = \sum_{n=1}^\infty f_n v_n$, where the convergence is in $L^2(\Omega)$. Furthermore, if $f \in H_0^1(\Omega)$, then the same Fourier series also converges in $H_0^1(\Omega)$.*

The compactness Theorem 10.25 is based on the following compactness criterion for subsets of $L^p(\Omega)$ (see Theorem 4.26 of Brezis's text [B]).

Theorem 10.31. *Let $\mathcal{F}$ be a bounded set in $L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. Assume that*

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } ||f(\cdot + h) - f(\cdot)||_{L^p(\mathbb{R}^n)} < \epsilon \;\forall f \in \mathcal{F}, \forall h \in \mathbb{R}^n \text{ with } |h| < \delta.$$

Then the closure of $\mathcal{F}\big|_\Omega$ in $L^p(\Omega)$ is compact for any measurable $\Omega \subset \mathbb{R}^n$ with finite measure.

We can apply this theorem in the setting of Theorem 10.25 after we establish

Lemma 10.32. *For any $v \in C_c^1(\mathbb{R}^n)$, we have*

$$||v(\cdot + h) - v(\cdot)||_{L^p(\mathbb{R}^n)} \le |h|||\nabla v||_{L^p(\mathbb{R}^n)}. \tag{10.38}$$

Proof. The Fundamental Theorem of Calculus gives

$$|v(x+h)-v(x)| = \Big|\int_0^1 h\cdot\nabla v(x+th)dt\Big| \le \int_0^1 |h||\nabla v(x+th)|dt \le |h|\left(\int_0^1 |\nabla v(x+th)|^p dt\right)^{1/p}.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^n} |v(x+h) - v(x)|^p\,dx \le& |h|^p \int_{\mathbb{R}^n}\int_0^1 |\nabla v(x+th)|^p dt\,dx \\ \le& |h|^p \int_0^1\int_{\mathbb{R}^n} |\nabla v(x+th)|^p\,dxdt \\ \le& |h|^p \int_0^1 ||\nabla v||^p_{L^p(\mathbb{R}^n)} dt \\ =& |h|^p ||\nabla v||^p_{L^p(\mathbb{R}^n)},\end{aligned}$$

where we have used $||\nabla v(\cdot + th)||_{L^p(\mathbb{R}^n)} = ||\nabla v||_{L^p(\mathbb{R}^n)}$ is independent of t, this concludes the proof for (10.38). □

Let Ω be a domain in $\mathbb{R}^n$, $1 \le p < \infty$, define $W_0^{1,p}(\Omega)$ be the completion of $C_c^1(\Omega)$ under the norm $||u||_{W_0^{1,p}(\Omega)} = ||\nabla u||_{L^p(\Omega)} + ||u||_{L^p(\Omega)}$. Then for any $u \in W_0^{1,p}(\Omega)$, there exists a sequence $\{v_j\} \subset C_c^1(\Omega)$, such that $v_j \to u$, and $\partial_{x_a} v_j \to \partial_{x_a} u$ in $L^p(\Omega)$ as $j \to \infty$ for each $a = 1, \cdots, n$. Applying (10.38) to v_j and passing to $j \to \infty$ shows that (10.38) continues to hold for functions in $W_0^{1,p}(\Omega)$; in particular, when $p = 2$, it holds for $v \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$.

Proof for Theorem 10.25. Let $\{w_j\}$ be a sequence in $H_0^1(\Omega)$ with bounded $H_0^1(\Omega)$ norms. (10.38) and Theorem 10.31 applied to $\{w_j\}$ shows that it has a convergent subsequence in $L^2(\Omega)$. Let's assume that $w_{j_k} \to w^*$ in $L^2(\Omega)$ as $k \to \infty$. For each $1 \le a \le n$, $\{\partial_{x_a} w_j\}$ is a bounded sequence in $L^2(\Omega)$. We now appeal to a property that *any bounded sequence in $L^2(\Omega)$ has a subsequence which converges weakly in $L^2(\Omega)$*. In our setting, we can get a subsequence of w_{j_k}, still denoted as w_{j_k}, and n functions $v_{[1]}, v_{[2]}, \cdots, v_{[n]}$ in $L^2(\Omega)$ such that

$$\partial_{x_a} w_{j_k} \text{converges weakly to } v_{[a]} \text{ as } k \to \infty.$$

If we take $\eta \in C_c^1(\Omega)$, we have

$$\int_\Omega \partial_{x_a} w_{j_k}(x)\eta(x)\,dx = -\int_\Omega w_{j_k}(x)\partial_{x_a}\eta(x)\,dx;$$

passing to $k \to \infty$, we obtain

$$\int_\Omega v_{[a]}(x)\eta(x)\,dx = -\int_\Omega w^*(x)\partial_{x_a}\eta(x)\,dx.$$

This shows that w^* has weak L^2 derivatives in Ω, and $\partial_{x_a} w^*(x) = v_{[a]}(x)$. Later we will prove that a function having weak L^2 derivatives in the sense here also has L^2 derivatives by $L^2(\Omega)$ norm approximation as defined earlier.

Applying (10.34) to each $v_{[a]}(x)$, we see that

$$||\partial_{x_a} w^*(x)||_{L^2(\Omega)} \le \liminf_{k\to\infty} ||\partial_{x_a} w_{j_k}(x)||_{L^2(\Omega)},$$

which shows that

$$||\nabla w^*(x)||_{L^2(\Omega)} \le \liminf_{k\to\infty} ||\nabla w_{j_k}(x)||_{L^2(\Omega)}.$$

□

Remark 10.15. The notion of weak convergence and weak compactness has wide applications. An infinite dimensional normed space no longer has the Bolzano-Weierstrass property that *any bounded sequence has a convergent (in norm) subsequence*; but the notion of weak compactness provides a substitute for this property.

A concrete case to understand the notion of weak compactness is the space l^p, which is defined as $\{x = (x_1, \cdots, x_n, \cdots) : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ for $1 < p < \infty$. Let $\{x[m]\}$ be a bounded sequence in l^p for some $1 < p < \infty$, namely, each $x[m] = (x[m]_1, \cdots, x[m]_n, \cdots) \in l^p$, and there exists $M > 0$ such that $\sum_{n=1}^{\infty} |x[m]_n|^p < M^p$ for all m. This implies that, for each fixed n, the sequence $\{x[m]_n\}_{m=1}^{\infty}$ is a bounded sequence, so it has a convergent subsequence. Through a diagonal process, we can find a subsequence $\{x'[m]\}$ of $\{x[m]\}$, and x_n such that $x'[m]_n \to x_n$ as $m \to \infty$, namely, $x'[m]_n \to x_n$ componentwise. Furthermore, for any N,

$$\sum_{n=1}^{N} |x_n|^p = \lim_{m\to\infty} \sum_{n=1}^{N} |x'[m]_n|^p \le M^p,$$

so $\sum_{n=1}^{\infty} |x_n|^p \le M^p$, namely, $(x_n)_n \in l^p$.

While we can't imply that $\{x'[m]\}$ converges to $x = (x_n)$ in l^p norm, we can see that $\{x'[m]\}$ converges weakly to (x_n) in the following sense: Given any $g = (g_n) \in l^{p'}$, where p' is the conjugate exponent of p determined by $p^{-1} + p'^{-1} = 1$, in examining the relation between

$$\sum_{n=1}^{\infty} x'[m]_n g_n \text{ and } \sum_{n=1}^{\infty} x_n g_n \text{ as } m \to \infty,$$

we can use the convergence of $\sum_{n=1}^{\infty} |g_n|^{p'}$ to control the tail part of both sums as follows. For any given $\epsilon > 0$, we can find N such that $\sum_{n=N}^{\infty} |g_n|^{p'} < \epsilon^{p'}$. Then

$$\begin{aligned}
&\Big|\sum_{n=1}^{\infty} x'[m]_n g_n - \sum_{n=1}^{\infty} x_n g_n\Big| \\
&\le \sum_{n=1}^{N-1} \Big|x'[m]_n - x_n\Big|\Big|g_n\Big| + \sum_{n=N}^{\infty} \left(\Big|x'[m]_n g_n\Big| + \Big|x_n g_n\Big|\right) \\
&\le \sum_{n=1}^{N-1} \Big|x'[m]_n - x_n\Big|\Big|g_n\Big| + \left[\left(\sum_{n=N}^{\infty} |x'[m]_n|^p\right)^{1/p} + \left(\sum_{n=N}^{\infty} |x_n|^p\right)^{1/p}\right]\left(\sum_{n=N}^{\infty} |g_n|^{p'}\right)^{1/p'} \\
&\le \sum_{n=1}^{N-1} \Big|x'[m]_n - x_n\Big|\Big|g_n\Big| + 2M\epsilon.
\end{aligned}$$

We can now use $x'[m]_n \to x_n$ as $m \to \infty$ to control the finite sum $\sum_{n=1}^{N-1} \Big|x'[m]_n -$

$x_n\Big||g_n\big|$: to find m' such that $\sum_{n=1}^{N-1}\Big|x'[m]_n - x_n\Big||g_n\big| < \epsilon$ when $m > m'$. To summarize, weak l^p convergence is equivalent to componentwise convergence, plus a uniform bound on the l^p norms.

The key role played by the element g in the **dual space** $l^{p'}$ is to control the tail part of the error. This feature is not visible in the more abstract set up for weak convergence and weak compactness, but it does provide some helpful guidance on the notion of weak convergence and weak compactness.

Exercises

Exercise 10.5.1. In the last statement of Theorem 10.30, *if $f \in H_0^1(\Omega)$, then the same Fourier series also converges in* $H_0^1(\Omega)$, is the statement still valid if $H_0^1(\Omega)$ is replaced by $H^1(\Omega)$?

Exercise 10.5.2. In the set up and proof of Theorem 10.26, if we replace the constraint $\int_\Omega w^2(x) = 1$ by the constraint $\int_\Omega |\nabla w|^2(x) = 1$, would the same proof for the existence of a minimizer go through under this constaint? Would the conclusion that a minimizer exsits hold using this constraint?

Exercise 10.5.3. Prove that $\sup\{(K[f], f)_{L^2(\Omega)} : f \in L^2(\Omega), ||f||_{L^2(\Omega)} = 1\}$ is attained by some f_1, and that f_1 is an eigenfunction of K. Recall that, under our formulation, $(K[f], f)_{L^2(\Omega)} = Q(K[f]) \geq c||K[f]||^2_{L^2(\Omega)}$ for some $c > 0$ depending only on K and Ω, and it's then easy to see that $\sup\{(K[f], f)_{L^2(\Omega)} : f \in L^2(\Omega), ||f||_{L^2(\Omega)} = 1\} > 0$. Does the conclusion of this exercise still hold in the abstract if one only relies on (10.32) and (10.31)?

Exercise 10.5.4. Let $\{f_1, \cdots, f_k, \cdots\}$ be a collection of eigenfunctions of K. Prove that

$$\sup\{(K[f], f)_{L^2(\Omega)} : f \in L^2(\Omega), ||f||_{L^2(\Omega)} = 1, (f, f_j)_{L^2(\Omega)} = 0 \text{ for all } f_j\text{'s}\}$$

is attained by some f^*, and that f^* is an eigenfunction of K.

Exercise 10.5.5. Prove that if $\{u_j\}$ converges in $L^p(\Omega)$, and has uniformly bounded $L^q(\Omega)$ norms for some $q > p$, then for any $p < r < q$, $\{u_j\}$ converges in $L^r(\Omega)$.

Exercise 10.5.6. (i). Prove that, if Ω is bounded and $1 < p < \infty$, then for any sequence $\{u_j\}$ in $W_0^{1,p}(\Omega)$ with bounded $W_0^{1,p}(\Omega)$ norms, there exist $u \in W_0^{1,p}(\Omega)$ and a subsequence $\{u_{j_k}\}$ such that $u_{j_k} \to u$ in $L^q(\Omega)$ for any $1 \leq q \leq p$.

(ii). Exhibit a sequence $\{u_j\}$ in $W_0^{1,1}(-1,1)$ with $W_0^{1,1}(-1,1)$ norms $\to 1$ as $j \to \infty$, such that $u_j'(x) \to 0$ for $x \neq 0$ in $(-1,1)$, $\{u_j\}$ converges in $L^1(-1,1)$, yet $\{u_j'\}$ won't converge weakly in $L^1(-1,1)$.

(iii). Exhibit a sequence $\{u_j\}$ in $W_0^{1,p}(\mathbb{R}^n)$ with bounded $W_0^{1,p}(\mathbb{R}^n)$ norms, but with no convergent subsequence in $L^p(\mathbb{R}^n)$.

Exercise 10.5.7. Consider the functional $I[u] = \int_\Omega \left\{ \frac{1}{2}|\nabla u(x)|^2 - \frac{|u(x)|^p}{p} \right\} dx$ on $H_0^1(\Omega)$, where $1 < p < 2$, and Ω is a bounded domain in $\mathbb{R}^n$. Prove that $\inf\{I[u] : u \in H_0^1(\Omega)\}$ is attained by some $v \in H_0^1(\Omega)$, and that v satisfies

$$\int_\Omega \left\{ \nabla v(x) \cdot \nabla \eta(x) - |v(x)|^{p-2} v(x) \eta(x) \right\} dx = 0$$

for all $\eta \in H_0^1(\Omega)$. Here we interpret $|v(x)|^{p-2}v(x)$ to be equal to 0 when $v(x) = 0$.

10.6 Additional Problems

Problem 10.6.1. Consider the heat equation $u_t = \Delta u$ in a bounded domain $\Omega \subset \mathbb{R}^n$, with initial condition $u = u_0(x)$ at $t = 0$ and boundary condition $\frac{\partial u}{\partial \nu} = -k(u - U)$ at $\partial\Omega$, where ν is the outward unit normal and $k \geq 0$. Use the energy method to show that there can be at most one solution. Does a similar assertion hold also for $k < 0$? (Try to play with 1-dimensional interval case.)

Problem 10.6.2. Suppose $a(x,t) = (a_{ij}(x,t))$ takes values in the class of symmetric, positive definite $n \times n$ matrices, with the bounds $m|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq M|\xi|^2$ for all $\xi \in \mathbb{R}^n$ and (x,t) in $\Omega \times [0,T]$ and for some $0 < m \leq M$, and $c(x,t)$ is bounded. Consider the PDE

$$u_t(x,t) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x,t) \frac{\partial u(x,t)}{\partial x_j} \right) + c(x,t)u(x,t)$$

in a bounded domain Ω, with initial condition $u = u_0(x)$ at $t = 0$ and a Dirichlet boundary condition $u = g$ at $\partial\Omega$. Use the energy method to show that there can be at most one solution.

Problem 10.6.3. This problem illustrates that the energy method can be adapted to deal with certain variable coefficient wave equations. Prove that if u is a C^2 solution of

$$u_{tt} - c^2 u_{xx} + \alpha(x,t)u_t + \beta(x,t)u_x + \gamma(x,t)u = 0,$$

with $u(x,0) = u_t(x,0) = 0$ for $|x - x_0| \leq R$, and $\alpha(x,t), \beta(x,t), \gamma(x,t)$ are assumed to be in x^∞, then $u(x,t) = 0$ for $|x - x_0| \leq R - ct$ for $0 < t < R/c$. (Hint: formulate and prove a version of the energy estimates.)

Let $u(x,t)$ be a C^2 solution of the n-dimensional wave equation $u_{tt}(x,t) - c^2 \Delta u(x,t) = 0$ in $B(X,R)$. Prove that for any $0 < t \leq R/c$,

$$\int_{|x-X|\leq R-ct} \left[u_t^2(x,t) + c^2|\nabla_x u(x,t)|^2\right] dx \leq \int_{|x-X|\leq R} \left[u_t^2(x,0) + c^2|\nabla_x u(x,0)|^2\right] dx.$$

You may take n to be 2 or 3.

Problem 10.6.4. Suppose that $u \in C^1(\overline{\mathbb{D}}) \cap C^2(\mathbb{D})$ is harmonic in $\mathbb{D}$, where $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$ centered at 0. Let $g(e^{i\theta}) = u(e^{i\theta})$ and $h(e^{i\theta}) = \frac{\partial u(re^{i\theta})}{\partial r}|_{r=1}$ for $e^{i\theta} \in \partial\mathbb{D}$. Let g_n and h_n be the Fourier coefficient of g and h respectively: $g_n = (2\pi)^{-1} \int_{\partial\mathbb{D}} g(e^{i\theta}) e^{-ni\theta} d\theta$, $h_n = (2\pi)^{-1} \int_{\partial\mathbb{D}} h(e^{i\theta}) e^{-ni\theta} d\theta$. Prove that $h_n = ng_n$. (Hint: Apply Green's formula to $u(re^{i\theta})$ and $v = r^n e^{-ni\theta}$ on $\mathbb{D}$.)

Suppose that $U, V \subset \mathbb{C}$ are open domains in $\mathbb{C}$, and $\phi : U \mapsto V$ is holomorphic. Let $u \in C^2(V)$. Prove that $\Delta[u(\phi(z))] = (\Delta u)(\phi(z))|\phi'(z)|^2$.

Chapter 11

SOLVABILITY OF IVP TO PARABOLIC EQUATIONS

In Chapter 1 we used the fundamental solution to the heat equation

$$K(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & \text{when } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ 0 & \text{when } (x,t) \in \mathbb{R}^n \times (-\infty, 0] \end{cases}$$

to generate a solution to the homogeneous heat equation with initial data $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = 0 & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n \end{cases} \tag{11.1}$$

in the form

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t) g(y) dy.$$

The same representation also provides a solution when $g \in L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, and can be used with Duhamel principle to generate a solution to the nonhomogeneous heat equation:

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) f(y,s) dy ds + \int_{\mathbb{R}^n} K(x-y,t) g(y) dy. \tag{11.2}$$

We will provide a justification in section 1 below that, under appropriate regularity assumption on f — mere continuity of f in $\mathbb{R}^n \times (0,T]$ will not be enough, (11.2) provides a classical $C^{2,1}(\mathbb{R}^n \times (0,T])$ solution (namely u and up to 2-derivatives in x and 1-derivative in t: $u_{x_i}, u_t, u_{x_i x_j} \in C(\mathbb{R}^n \times (0,T])$—when there is a need to distinguish the order of differentiation in x and t, we use the notation $C^{2,1}_{x,t}(\mathbb{R}^n\times(0,T]))$ to

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t) & \text{for } (x,t) \in \mathbb{R}^n \times (0,T] \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n. \end{cases} \tag{11.3}$$

We will also prove, using a device based on Green's identity concerning solutions to (11.3) and involving the adjoint operator to the heat equation, that any $C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solution $u(x,t)$ of (11.3) with some growth control of u and f at ∞ is represented as in (11.2).

The condition $f \in C^{2,1}(\mathbb{R}^n \times (0,T])$ is too strong for getting a $C^{2,1}(\mathbb{R}^n \times (0,T])$ solution to (11.3); we will find in section 2 that a natural condition is to impose some kind of **Hölder** continuity condition on f.

In section 2 we also study the IVP for perturbations of the standard heat equation—perturbations only on the lower order terms at this stage, and learn how to use the available estimates for the standard heat equation and an iteration procedure to construct solutions to the perturbed equation.

In section 3 we study the estimates of more general second order variable coefficient heat equations, and the solvability of corresponding IVPs.

11.1 Solvability of IVP (11.3)

We first prove

Theorem 11.1. *Let $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, and $f \in C^1_x(\mathbb{R}^n \times [0,T])$ with compact support in $\mathbb{R}^n \times [0,T]$. Then (11.2) provides a solution to (11.3) in $C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$.*

Proof. We will rely on a basic fact used in the proof of Theorem in Chapter 1: for any $g \in L^p(\mathbb{R}^n)$, for some $1 \le p \le \infty$,

$$\int_{\mathbb{R}^n} K(x-y,t)g(y)dy.$$

provides a $C^{2,1}(\mathbb{R}^n \times (0,\infty))$ solution to (11.1).

Set

$$F(x,t;s) = \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dy.$$

Then

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)g(y)dy + \int_0^t F(x,t;s)ds.$$

$F(x,t;s)$ is a smooth function of (x,t) in the domain $t>s$, and

$$\begin{aligned}\partial_{x_i}F(x,t;s) &= \int_{\mathbb{R}^n}\partial_{x_i}K(x-y,t-s)f(y,s)dy,\\ \partial_{x_ix_j}F(x,t;s) &= \int_{\mathbb{R}^n}\partial_{x_ix_j}K(x-y,t-s)f(y,s)dy,\\ \partial_tF(x,t;s) &= \int_{\mathbb{R}^n}\partial_tK(x-y,t-s)f(y,s)dy.\end{aligned}$$

We already know that, under our condition on g, $\int_{\mathbb{R}^n}K(x-y,t)g(y)dy$ provides a $C^{2,1}(\mathbb{R}^n\times(0,\infty))$ solution to (11.1). It remains to prove that the second integral, $\int_0^t F(x,t;s)ds$, is a solution to (11.3), and $\int_0^t F(\tilde{x},t;s)ds\to 0$ as $(\tilde{x},t)\to(x,0)$.

To apply (v) of Lemma A.1 to the second integral, $\int_0^t F(x,t;s)ds$, we need to establish that, for any $(x,t)\in\mathbb{R}^n\times(0,T]$, there exist a $\delta>0$ and an integrable bound $G(s;x,t)$ over $s\in[0,t]$ such that

$$|\partial_{x_i}F(\tilde{x},t;s)|,\left|\partial_{x_ix_j}F(\tilde{x},t;s)\right|,\left|\partial_tF(x,\tilde{t};s)\right|\le G(s;x,t),\quad \text{for } |\tilde{x}-x|<\delta \text{ and } |\tilde{t}-t|<\delta.$$

Note first that

$$\begin{aligned}|\partial_{x_i}F(\tilde{x},t;s)| &\le ||f||_{L^\infty(\mathbb{R}^n\times[0,T])}\int_{\mathbb{R}^n}|\nabla_xK(\tilde{x}-y,t-s)|dy\\ &= ||f||_{L^\infty(\mathbb{R}^n\times[0,T])}\int_{\mathbb{R}^n}\frac{|\tilde{x}-y|}{(4\pi(t-s))^{n/2}2(t-s)}e^{-\frac{|\tilde{x}-y|^2}{4(t-s)}}dy \qquad (11.4)\\ &\le \frac{||f||_{L^\infty(\mathbb{R}^n\times[0,T])}}{2(4\pi)^{n/2}\sqrt{t-s}}\int_{\mathbb{R}^n}|z|e^{-|z|^2/4}dz,\end{aligned}$$

which is (Lebesgue) integrable in $s\in[0,t]$. So

$$\partial_{x_i}\left(\int_0^t F(x,t;s)ds\right)=\int_0^t\partial_{x_i}F(x,t;s)ds=\int_0^t\int_{\mathbb{R}^n}\partial_{x_i}K(x-y,t-s)f(y,s)dyds.$$

Under the sole assumption $||f||_{L^\infty(\mathbb{R}^n\times[0,T])}<\infty$ (even if $f\in C(\mathbb{R}^n\times[0,T])$), the best direct upper bound one can get for $|\partial^2_{x_ix_j}F(x,t;s)|$ is

$$\begin{aligned}&\left|\partial^2_{x_ix_j}F(x,t;s)\right|\\ \le& ||f||_{L^\infty(\mathbb{R}^n\times[0,T])}\int_{\mathbb{R}^n}\left|\partial^2_{x_ix_j}K(x-y,t-s)\right|dy\\ \le& C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}/(t-s),\end{aligned}$$

which is not (Lebesgue) integrable in $s\in[0,t]$, so one can not conclude that one can pass $\partial^2_{x_ix_j}$ under the integral $\int_0^t F(x,t;s)ds$, under the assumption of $f\in C(\mathbb{R}^n\times[0,T])\cap L^\infty(\mathbb{R}^n\times[0,T])$, by appealing to (v) of Lemma A.1.

However, under the assumption that $f \in C_x^1(\mathbb{R}^n \times [0,T])$ with compact support in $\mathbb{R}^n \times [0,T])$, we see that

$$\begin{aligned}
&\left|\partial^2_{x_ix_j} F(\tilde{x},t;s)\right| \\
=&\left|\int_{\mathbb{R}^n} \partial_{x_ix_j} K(\tilde{x}-y,t-s) f(y,s)dy\right| \\
=&\left|-\int_{\mathbb{R}^n} \partial_{x_iy_j} K(\tilde{x}-y,t-s) f(y,s)dy\right| \\
=&\left|\int_{\mathbb{R}^n} \partial_{x_i} K(\tilde{x}-y,t-s) \partial_{y_j} f(y,s)dy\right| \\
\leq&||\nabla f||_{L^\infty(\mathbb{R}^n\times[0,T])} \int_{\mathbb{R}^n} |\partial_{x_i} K(\tilde{x}-y,t-s)|\, dy \\
\leq&C(n)||\nabla f||_{L^\infty(\mathbb{R}^n\times[0,T])}/\sqrt{t-s},
\end{aligned}$$

which is (Lebesgue) integrable in $s \in [0,t]$. So under this assumption on f, we have

$$\begin{aligned}
\partial^2_{x_ix_j}\left(\int_0^t F(x,t;s)ds\right) &= \int_0^t \left(\partial^2_{x_ix_j} F(x,t;s)\right) ds \\
&= \int_0^t \int_{\mathbb{R}^n} \partial_{x_ix_j} K(\tilde{x}-y,t-s) f(y,s)dyds.
\end{aligned}$$

In particular, $\Delta_x \left(\int_0^t F(x,t;s)ds\right) = \int_0^t \int_{\mathbb{R}^n} \Delta_x K(\tilde{x}-y,t-s)f(y,s)dyds.$

Next, note that

$$\begin{aligned}
\partial_t F(x,t;s) &= \int_{\mathbb{R}^n} \partial_t K(x-y,t-s) f(y,s)dy \\
&= \int_{\mathbb{R}^n} \Delta_x K(x-y,t-s) f(y,s)dy \\
&= \int_{\mathbb{R}^n} \Delta_y K(x-y,t-s) f(y,s)dy \\
&= -\int_{\mathbb{R}^n} \nabla_y K(x-y,t-s)\cdot \nabla_y f(y,s)dy.
\end{aligned}$$

Thus

$$|\partial_t F(x,t;s)| \leq C(n)||\nabla f||_{L^\infty(\mathbb{R}^n\times[0,T])}/\sqrt{t-s}.$$

To determine $\partial_t \left(\int_0^t F(x,t;s)ds\right)$, we first examine, for $h > 0$

$$\begin{aligned}
&h^{-1}\left(\int_0^{t+h} F(x,t+h;s)ds - \int_0^t F(x,t;s)ds\right) \\
=&h^{-1}\left(\int_0^t \left[F(x,t+h;s) - F(x,t;s)\right] ds + \int_t^{t+h} F(x,t+h;s)ds\right) \\
=&h^{-1}\left(\int_0^t \left[F(x,t+h;s) - F(x,t;s)\right] ds + \int_t^{t+h} \left[F(x,t+h;s) - f(x,t)\right] ds\right) + f(x,t).
\end{aligned}$$

We can now apply (v) of Lemma A.1 to the first integral to conclude that

$$\begin{aligned}\lim_{h\searrow 0} h^{-1}\left(\int_0^t [F(x,t+h;s)-F(x,t;s)]\,ds\right) &= \int_0^t \partial_t F(x,t;s)ds\\ &= \int_0^t\int_{\mathbb{R}^n} \partial_t K(x-y,t-s)f(y,s)dyds.\end{aligned}$$

Next

$$\begin{aligned}F(x,t+h;s)-f(x,t) &= \int_{\mathbb{R}^n} K(x-y,t+h-s)f(y,s)dy - f(x,t)\\ &= \int_{\mathbb{R}^n} K(x-y,t+h-s)\left[f(y,s)-f(x,t)\right]dy.\end{aligned}$$

Using the continuity of $f(y,s)$ at (x,t), for a given $\epsilon>0$, we find $\delta>0$ such that $|f(y,s)-f(x,t)|<\epsilon$ when $|y-x|<\delta$, and $|s-t|<\delta$. When $0<h<\delta$, and $t\le s\le t+h$, we have

$$\begin{aligned}&|F(x,t+h;s)-f(x,t)|\\ \le&\left(\int_{|y-x|<\delta}+\int_{|y-x|\ge\delta}\right)K(x-y,t+h-s)\left|f(y,s)-f(x,t)\right|dy\\ \le&\epsilon\int_{|y-x|<\delta}K(x-y,t+h-s)dy+2||f||_{L^\infty(\mathbb{R}^n\times[0,T])}\int_{|y-x|\ge\delta}K(x-y,t+h-s)dy.\end{aligned}$$

Noting that

$$\left|\int_{|y-x|<\delta}K(x-y,t+h-s)dy\right|<1,$$

and

$$\int_{|y-x|\ge\delta}K(x-y,t+h-s)dy=\int_{|z|\ge\frac{\delta}{\sqrt{t+h-s}}}\frac{e^{-\frac{|z|^2}{4}}}{(4\pi)^{\frac{n}{2}}}dz,$$

since there exists $M>1$ such that

$$2||f||_{L^\infty(\mathbb{R}^n\times[0,T])}\left|\int_{|z|\ge M}\frac{e^{-\frac{|z|^2}{4}}}{(4\pi)^{\frac{n}{2}}}dz\right|<\epsilon,$$

so if we set $\sigma=(\delta/M)^2$, then when $0<h<\min\{\delta,\sigma\}$, and $t\le s\le t+h$, we will have $0\le t+h-s\le h<\sigma$, thus $\frac{\delta}{\sqrt{t+h-s}}\ge M$, we can then conclude that

$$h^{-1}\left|\int_t^{t+h}[F(x,t+h;s)-f(x,t)]\,ds\right|\le h^{-1}\int_t^{t+h}2\epsilon ds\le 2\epsilon,$$

therefore proving

$$\lim_{h\searrow 0}h^{-1}\left(\int_0^{t+h}F(x,t+h;s)ds-\int_0^t F(x,t;s)ds\right)=\int_0^t\int_{\mathbb{R}^n}\partial_t K(x-y,t-s)f(y,s)dyds+f(x,t).$$

The case $\lim_{h\nearrow 0}$ is handled in a similar way. We have thus proved that

$$\partial_t \left(\int_0^t F(x,t;s)ds \right) = \int_0^t \int_{\mathbb{R}^n} \partial_t K(x-y,t-s) f(y,s) dyds + f(x,t).$$

Finally, since

$$|F(x,t;s)| \le \int_{\mathbb{R}^n} ||f||_{L^\infty(\mathbb{R}^n\times[0,T])} K(x-y,t-s)dy \le ||f||_{L^\infty(\mathbb{R}^n\times[0,T])},$$

we conclude that

$$\left| \int_0^t F(x,t;s)ds \right| \le ||f||_{L^\infty(\mathbb{R}^n\times[0,T])} t \to 0 \quad \text{as } t \searrow 0,$$

thus proving that (11.2) takes on the initial value $g(x)$ in the classical sense. □

Theorem 11.2. *Let $u(x,t)$ be any $C^{2,1}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solution of (11.3) satisfying, for some constants $A, B > 0$,*

$$|u(x,t)| \le Ae^{B|x|^2},\ |\nabla u(x,t)| \le Ae^{B|x|^2},\ \textit{and } |f(x,t)| \le Ae^{B|x|^2} \textit{ for } (x,t) \in \mathbb{R}^n \times (0,T]. \tag{11.5}$$

Then, for $0 < T < (4B)^{-1}$, (11.2) holds; in fact, (11.2) holds without the assumption on $|\nabla u(x,t)|$, or under the weaker integral growth assumption

$$\iint_{\mathbb{R}^n\times(0,T]} [|u(x,t)| + |f(x,t)|]\, e^{-B|x|^2} dxdt < \infty. \tag{11.6}$$

Remark 11.1. Under the assumption (11.5) or (11.6), and the restriction on T, the integrals in (11.2) are well defined. The function $u(x,t) = \frac{e^{\frac{|x|^2}{4(T-t)}}}{(T-t)^{\frac{n}{2}}}$ satisfies (11.3) with $f \equiv 0$ for $t < T$, $u(x,0) = \frac{e^{\frac{|x|^2}{4T}}}{T^{\frac{n}{2}}}$, but $u(x,t) \nearrow \infty$ as $t \nearrow T$. This example illustrates that the restriction on T is needed; furthermore, that, a solution to (11.3) may exist for only a finite time interval when the initial data has sufficiently fast growth as $x \to \infty$.

This Theorem will be proved by exploiting the relation

$$\begin{aligned} &v(y,s)\,[\partial_s - \Delta_y]\, u(y,s) + u(y,s)\,[\partial_s + \Delta_y]\, v(y,s) \\ &= [u(y,s)v(y,s)]_s + \nabla_y\,[u(y,s)\nabla_y v(y,s) - v(y,s)\nabla_y u(y,s)]\,. \end{aligned}$$

$\partial_s + \Delta_y$ is called the **adjoint operator** of $\partial_s - \Delta_y$. If we do integration by parts of the above over $B_R \times [0,t]$, we obtain

$$\begin{aligned}
&\iint_{B_R\times[0,t]} v(y,s)\left[\partial_s - \Delta_y\right] u(y,s)dyds + \iint_{B_R\times[0,t]} u(y,s)\left[\partial_s + \Delta_y\right] v(y,s)dyds \\
&= \iint_{\partial B_R\times[0,t]} \left[u(y,s)\nabla_{n(y)}v(y,s) - v(y,s)\nabla_{n(y)}u(y,s)\right] d\sigma(y)ds \\
&\quad + \int_{B_R} u(y,t)v(y,t)dy - \int_{B_R} u(y,0)v(y,0)dy.
\end{aligned} \tag{11.7}$$

(11.7) can be exploited in various ways, mainly by working with pairs u and v such that some of the terms will either vanish or be controlled. For example, if v vanishes near the boundary of $B_R \times [0,t]$, then all the integrands on the right hand side vanish, giving us

$$\iint_{B_R\times[0,t]} v(y,s)\left[\partial_s - \Delta_y\right] u(y,s)dyds + \iint_{B_R\times[0,t]} u(y,s)\left[\partial_s + \Delta_y\right] v(y,s)dyds = 0.$$

This is similar to the Green's identity for the Laplace operator. We will first illustrate an application of using such integral relations by proving

Theorem 11.3. *Let $u(x,t)$ be a $C^{2,1}(B_R \times (0,T])$ solution to $u_t(x,t) - \Delta u(x,t) = 0$ for $(x,t) \in B_R \times (0,T]$, then $u \in C^\infty(B_R \times (0,T])$.*

Proof. For any $(x,t) \in B_R \times (0,T]$, we may assume $u \in C(\overline{B_R} \times [0,T])$ by working with a slightly smaller R and resetting initial time to some $0 < \epsilon < t$. If we choose $v(y,s)$ such that $\left[\partial_s + \Delta_y\right] v(y,s) = 0$ for $y \in B_R$ and $0 < s < t$, we then obtain

$$\begin{aligned}
\int_{B_R} u(y,t)v(y,t)dy = \int_{B_R} u(y,0)v(y,0)dy + \iint_{B_R\times[0,t]} f(y,s)v(y,s)dyds \\
- \iint_{\partial B_R\times[0,t]} \left[u(y,s)\nabla_{n(y)}v(y,s) - v(y,s)\nabla_{n(y)}u(y,s)\right] d\sigma(y)ds.
\end{aligned}$$

Now if we set $v_\epsilon(y,s) = K(x-y, t+\epsilon-s)$, where $\epsilon > 0$ is a (small) positive parameter, then $\left[\partial_s + \Delta_y\right] v_\epsilon(y,s) = 0$ for $y \in B_R$ and $0 < s < t$, but as $\epsilon \searrow 0$,

$$\int_{B_R} u(y,t)v_\epsilon(y,t)dy \to u(x,t) \quad \int_{B_R} u(y,0)v_\epsilon(y,0)dy \to \int_{B_R} u(y,0)K(x-y,t)dy,$$

and

$$\iint_{B_R\times[0,t]} f(y,s)v(y,s)dyds \to \iint_{B_R\times[0,t]} f(y,s)K(x-y,t-s)dyds. \tag{11.8}$$

In the last integral above, a somewhat technical argument is needed to justify the limiting process due to the singularity of $K(x-y,t-s)$ at $(y,s)=(x,t)$—see below; or we may assume $f \equiv 0$ in the context of this Theorem here. So we obtain

$$\begin{aligned} u(x,t) = &\int_{B_R} u(y,0)K(x-y,t)dy + \iint_{B_R\times[0,t]} f(y,s)K(x-y,t-s)dyds \\ &- \iint_{\partial B_R\times[0,t]} \left[u(y,s)\nabla_{n(y)}K(x-y,t-s) - K(x-y,t-s)\nabla_{n(y)}u(y,s)\right] d\sigma(y)ds. \end{aligned} \tag{11.9}$$

This representation formula is similar to the Poisson representation formula obtained through the Green's Theorem for the Laplace operator. Note that $K(x-y,t-s)$ in C^∞ when $(x,t) \in B_R \times (0,T]$ and $(y,s) \in \partial B_R \times [0,t]$ or $B_R \times \{0\}$; and the only singular point in the integrals above is at $y=x$ and $s=t$ in $\iint_{B_R\times[0,t]} f(y,s)K(x-y,t-s)dyds$. If $f \equiv 0$, then this integral is absent and we can appeal to Lemma A.1 through the above integral representation to conclude that $u(x,t)$ is C^∞ in $(x,t) \in B_R \times (0,T]$.

Here is a justification for (11.8), under the assumption that $f \in L^\infty(B_R \times [0,t])$. Setting $F_R(x,t;s) = \int_{B_R} f(y,s)K(x-y,t-s)dy$, we know that, for any x and $t > s$, $F_R(x,t+\epsilon;s) \to F_R(x,t;s)$ as $\epsilon \searrow 0$, and

$$|F_R(x,t+\epsilon;s)| \le \int_{B_R} |f(y,s)|\, K(x-y,t+\epsilon-s)dy \le ||f||_{L^\infty(B_R\times[0,t])},$$

for any $0 \le s \le t$ and $\epsilon > 0$, thus we can apply (v) of Lemma A.1 to conclude that

$$\begin{aligned} &\iint_{B_R\times[0,t]} f(y,s)K(x-y,t+\epsilon-s)dyds = \int_0^t F_R(x,t+\epsilon;s)ds \\ \to &\int_0^t F_R(x,t;s)ds = \iint_{B_R\times[0,t]} f(y,s)K(x-y,t-s)dyds \end{aligned}$$

as $\epsilon \searrow 0$. □

Next, let u be a solution to (11.3), and we proceed to derive (11.2) under appropriate growth assumptions on u.

Proof of Theorem 4.2. Let's first handle the simpler case under (11.5). For the given x pick $R > 0$ such that $|x| < R/2$. We will apply (11.9) on increasingly large R and prove that the boundary integrals converge to 0, and the integral over $B_R \times (0,t]$ converges to the integral over $\mathbb{R}^n \times (0,t]$.

More specifically we will let $R \to \infty$ and appeal to Lemma A.1 to prove

$$\int_{B_R^c} |u(y,0)|K(x-y,t)dy \to 0,$$
$$\iint_{B_R^c \times [0,t]} K(x-y,t-s)|f(y,s)|dyds \to 0,$$
$$\iint_{\partial B_R \times [0,t]} \left[u(y,s)\nabla_{n(y)}K(x-y,t-s) - K(x-y,t-s)\nabla_{n(y)}u(y,s)\right] d\sigma(y)ds \to 0, \tag{11.10}$$

and as a result,

$$\int_{B_R} u(y,0)K(x-y,t)dy \to \int_{\mathbb{R}^n} u(y,0)K(x-y,t)dy$$
$$\iint_{B_R\times[0,t]} K(x-y,t-s)f(y,s)dyds \to \iint_{\mathbb{R}^n\times[0,t]} K(x-y,t-s)f(y,s)dyds,$$

thus establishing (11.2).

Under our assumptions, there exists $\theta > 0$ such that $B < \frac{1-\theta}{4t}$. Then we also have $B < \frac{1-\theta}{4(t-s)}$ for all $0 \le s \le t$. Thus, under either (11.5) or (11.6), we see that (11.10) hold; for example,

$$\begin{aligned} K(x-y,t-s)|f(y,s)| &\le \left|f(y,s)e^{-B|y|^2}\right| \frac{e^{\frac{(1-\theta)|y|^2-|x-y|^2}{4(t-s)}}}{(4\pi(t-s))^{\frac{n}{2}}} \\ &\le \left|f(y,s)e^{-B|y|^2}\right| \frac{e^{\frac{-\theta|y|^2+2x\cdot y-|x|^2}{4(t-s)}}}{(4\pi(t-s))^{\frac{n}{2}}}, \end{aligned}$$

which is integrable over $\mathbb{R}^n \times [0,t]$, thus the second limit in (11.10) holds; and as to the last limit in (11.10), under (11.5), there exists some $C > 0$ such that $|u(x,t)|, |\nabla u(x,t)| \le Ce^{B|x|^2}$. We then use

$$|\nabla_y K(x-y,t-s)| \le \frac{|x-y|}{2(t-s)}K(x-y,t-s)$$

and

$$\begin{aligned} B|y|^2 = B\left[|y-x|^2 + 2(y-x)\cdot x + |x|^2\right] &\le B(1+\theta)|y-x|^2 + B(\theta^{-1}+1)|x|^2 \\ &\le \frac{(1-\theta^2)|y-x|^2}{4(t-s)} + B(\theta^{-1}+1)|x|^2 \end{aligned}$$

to estimate

$$\begin{aligned} &\left|u(y,s)\nabla_{n(y)}K(x-y,t-s) - K(x-y,t-s)\nabla_{n(y)}u(y,s)\right| \\ \le& C\left(1+\frac{|x-y|}{2(t-s)}\right)e^{B|y|^2}K(x-y,t-s) \\ \le& C\left(1+\frac{|x-y|}{2(t-s)}\right)\frac{e^{-\frac{\theta^2|y-x|^2}{4(t-s)}+B(\theta^{-1}+1)|x|^2}}{(4\pi(t-s))^{\frac{n}{2}}}. \end{aligned}$$

When $|x| < R/2$, and $|y| = R$, we have $R/2 \le |x-y| \le 2R$, so we have, for some constant $C' > 0$, that

$$\left| \iint_{\partial B_R \times [0,t]} \left[u(y,s)\nabla_{n(y)}K(x-y,t-s) - K(x-y,t-s)\nabla_{n(y)}u(y,s) \right] d\sigma(y)ds \right|$$
$$\le C' e^{B(\theta^{-1}+1)|x|^2} \int_0^t R^{n-1}(1+\frac{R}{t-s})\frac{e^{-\frac{\theta^2 R^2}{16(t-s)}}}{(t-s)^{\frac{n}{2}}}ds.$$

Making the change of variable $\tau = \frac{R^2}{t-s}$, we find $d\tau = \frac{R^2}{(t-s)^2}ds$, and

$$\int_0^t R^{n-1}\left(1+\frac{R}{t-s}\right)\frac{e^{-\frac{\theta^2 R^2}{16(t-s)}}}{(t-s)^{\frac{n}{2}}}\ ds$$
$$= \int_{\frac{R^2}{t}}^{\infty} (R+\tau)\tau^{\frac{n}{2}-2}e^{-\frac{\theta^2\tau}{16}}d\tau,$$

which $\to 0$ as $R \to \infty$. Thus by sending $R \to \infty$, we obtain (11.2) in this case.

The above argument made a point wise growth assumption on ∇u. If we would like to avoid such kind of assumptions, we need to treat the boundary integral $\iint_{\partial B_R\times[0,t]}$ differently: if we choose $v_\epsilon(y,s) = \eta(y)K(x-y,t+\epsilon-s)$, where $\eta(y)$ is a smooth cut-off function, equal to 1 on $B_{R/2}$ and supported in B_R, $\epsilon > 0$ is a small parameter. Then the integral term $\iint_{\partial B_R\times[0,t]}$ vanishes*, but

$$\left[\partial_s + \Delta_y\right] v_\epsilon(y,s) = \left(\Delta_y \eta(y)\right) K(x-y,t+\epsilon-s) + 2\nabla_y\eta(y)\cdot\nabla_y K(x-y,t+\epsilon-s),$$

so (11.7) would give us

$$\int_{B_R} u(y,t)v_\epsilon(y,t)dy$$
$$= \int_{B_R} u(y,0)K(x-y,t+\epsilon)\eta(y)dy + \iint_{B_R\times[0,t]} \eta(y)K(x-y,t+\epsilon-s)f(y,s)dyds$$
$$- \iint_{B_R\times[0,t]} u(y,s)\left[\left(\Delta_y\eta(y)\right)K(x-y,t+\epsilon-s) + 2\nabla_y\eta(y)\cdot\nabla_y K(x-y,t+\epsilon-s)\right]dyds.$$

We would like to take $\epsilon \searrow 0$ and $R \to \infty$ in the above relation to obtain a relation that does not involve ϵ or R. We first send $\epsilon \searrow 0$, as done in the earlier part of the proof, to obtain

$$u(x,t) = \int_{B_R} u(y,0)K(x-y,t)\eta(y)dy + \iint_{B_R\times[0,t]} \eta(y)K(x-y,t-s)f(y,s)dyds$$
$$- \iint_{B_R\times[0,t]} u(y,s)\left[\left(\Delta_y\eta(y)\right)K(x-y,t-s) + 2\nabla_y\eta(y)\cdot\nabla_y K(x-y,t-s)\right]dyds.$$

*the argument below is on the technical side, and may be omitted on a first reading.

As done earlier, we have

$$\int_{B_R} u(y,0)K(x-y,t)\eta(y)dy \to \int_{\mathbb{R}^n} u(y,0)K(x-y,t)dy.$$

So it remains to prove

$$\iint_{B_R\times[0,t]} \eta(y)K(x-y,t-s)f(y,s)dyds \to \iint_{\mathbb{R}^n\times[0,t]} K(x-y,t-s)f(y,s)dyds, \tag{11.11}$$

$$\iint_{B_R\times[0,t]} u(y,s)\left[(\Delta_y\eta(y))\,K(x-y,t-s)+2\nabla_y\eta(y)\cdot\nabla_y K(x-y,t-s)\right]dyds \to 0. \tag{11.12}$$

The factor $(\Delta_y\eta(y))\,K(x-y,t-s)+2\nabla_y\eta(y)\cdot\nabla_y K(x-y,t-s)$ in the last integral vanishes for $y\in B_{R/2}$, as $\Delta_y\eta(y)=0$ and $\nabla_y\eta(y)=0$ for $y\in B_{R/2}$, so the integral is carried out in $\left(B_R\setminus B_{R/2}\right)\times[0,t]$. Using the bounds

$$|\Delta_y\eta(y)|\le \frac{C}{R^2},\qquad |\nabla_y\eta(y)|\le\frac{C}{R},$$

for some constant $C>0$ independent of $R>0$, and estimate for $e^{B|y|^2}$ as done earlier, we have a bound for

$$\begin{aligned}
&|u(y,s)\left[(\Delta_y\eta(y))\,K(x-y,t-s)+2\nabla_y\eta(y)\cdot\nabla_y K(x-y,t-s)\right]|\\
\le& C\left|u(y,s)e^{-B|y|^2}\right|\left(\frac{1}{R^2}+\frac{|x-y|}{R(t-s)}\right)\frac{e^{B|y|^2-\frac{|x-y|^2}{4(t-s)}}}{(4\pi(t-s))^{\frac{n}{2}}}\\
\le& C'\left|u(y,s)e^{-B|y|^2}\right|\left(\frac{1}{R^2}+\frac{|x-y|}{R(t-s)}\right)\frac{e^{-\frac{\theta^2|x-y|^2}{4(t-s)}+B(\theta^{-1}+1)|x|^2}}{(t-s)^{\frac{n}{2}}}\\
\le& C'\left|u(y,s)e^{-B|y|^2}\right|\left(\frac{1}{R^2}+\frac{1}{t-s}\right)\frac{e^{-\frac{\theta^2R^2}{16(t-s)}+B(\theta^{-1}+1)|x|^2}}{(t-s)^{\frac{n}{2}}},
\end{aligned}$$

when $|x|<R/2\le|y|\le R$, and $0\le s<t$. We will take $R^2>t$, so

$$\left(\frac{1}{R^2}+\frac{1}{t-s}\right)\frac{e^{-\frac{\theta^2R^2}{16(t-s)}+B(\theta^{-1}+1)|x|^2}}{(t-s)^{\frac{n}{2}}}\le\frac{2e^{B(\theta^{-1}+1)|x|^2}e^{-\frac{\theta^2R^2}{16(t-s)}}}{(t-s)^{\frac{n}{2}+1}}.$$

But elementary one variable calculus shows that

$$\frac{e^{-\frac{\theta^2R^2}{16(t-s)}}}{(t-s)^{\frac{n}{2}+1}}\le\frac{C(n)}{R^{n+2}},$$

which $\to 0$ as $R \to \infty$. In addition, we also have

$$\iint_{\{R/2 \le |y| \le R\} \times [0,t]} \left| u(y,s) e^{-B|y|^2} \right| dyds \to 0$$

as $R \to \infty$, under the assumption (11.6). This concludes the proof for (11.12). (11.11) is proved in a similar fashion. $\square$

11.2 Hölder estimates and improved solvability of IVP (11.3)

As we saw in last section, some smoothness of f is imposed to justify differentiation under the integral sign, as $\partial_x^2 K(x,t), \partial_t K(x,t)$ are not (Lebesgue) integrable in $\mathbb{R}^n \times (0,T]$, so one can not simply carry out ∂_x^2 or ∂_t under the integral sign in (11.2) assuming only $f \in L^\infty(\mathbb{R}^n \times (0,T])$ or $C(\mathbb{R}^n \times (0,T])$.

It is desirable to weaken the smoothness assumption on f to still have u as given by (11.2) in $C^{2,1}(\mathbb{R}^n \times (0,T])$. We will study this issue now and also carry out a brief discussion on how to study the solvability of IVP to perturbations of the heat equation. For instance, we would like to solve

$$\begin{cases} u_t(x,t) - \Delta u(x,t) + \sum_{i=1}^n b_i(x,t) u_{x_i}(x,t) + c(x,t)u(x,t) = f(x,t) & \text{for } (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{11.13}$$

where $b_i(x,t), c(x,t), f(x,t)$ are given functions on $\mathbb{R}^n \times [0,T]$ with reasonable regularity and growth condition, and $g(x)$ is a given function on $\mathbb{R}^n$ with reasonable regularity and growth condition.

One natural approach is to establish the existence of a solution to (11.13) by iteration, treating the lower order terms $\sum_{i=1}^n b_i(x,t)u_{x_i}(x,t) + c(x,t)u(x,t)$ as part of the source term on the right hand side, namely, we define a map

$$\begin{aligned} S[u](x,t) = &\int_0^t \int_{\mathbb{R}^n} K(x-y,t-s) \left[f(y,s) - \sum_{i=1}^n b_i(y,s) u_{x_i}(y,s) - c(y,s)u(y,s) \right] dyds \\ &+ \int_{\mathbb{R}^n} K(x-y,t) g(y) dy, \end{aligned} \tag{11.14}$$

and hope to establish the existence of a fixed point of the map S in an appropriate function space.

The following Theorem helps to provide a natural function space on which to study S.

Theorem 11.4. *Suppose that $f \in L^\infty(\mathbb{R}^n \times [0,T])$. Then*

$$v(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dyds \in C(\mathbb{R}^n \times [0,T]).$$

Furthermore, $\nabla_x v \in C(\mathbb{R}^n \times [0,T])$, and, for any $0 < \delta < 1$, there exists $C = C(n,\delta) > 0$ such that

$$|v|_{0;\mathbb{R}^n\times[0,T]} + \sqrt{T}|\nabla_x v|_{0;\mathbb{R}^n\times[0,T]} + \sqrt{T}^{1+\delta}[\nabla_x v]_{\delta;\mathbb{R}^n\times[0,T]} \leq CT||f||_{L^\infty(\mathbb{R}^n\times[0,T])}, \quad (11.15)$$

where

$$|v|_{0;\mathbb{R}^n\times[0,T]} := \sup\{|v(x,t)| : (x,t) \in \mathbb{R}^n \times [0,T]\},$$

and

$$\begin{aligned}[\nabla_x v]_{\delta;\mathbb{R}^n\times[0,T]} :=& \sup\{\frac{|\nabla_x v(x_1,t) - \nabla_x v(x_2,t)|}{|x_1-x_2|^\delta} : x_1,x_2 \in \mathbb{R}^n, x_1 \neq x_2, 0 \leq t \leq T\} \\ &+ \sup\{\frac{|\nabla_x v(x,t_1) - \nabla_x v(x,t_2)|}{|t_1-t_2|^{\delta/2}} : x \in \mathbb{R}^n, 0 \leq t_1,t_2 \leq T, t_1 \neq t_2\}.\end{aligned}$$

Let

$$\begin{aligned}C^{\delta,\delta/2}_{x,t}(\mathbb{R}^n \times [0,T]) =& C^{\delta,\delta/2}(\mathbb{R}^n \times [0,T]) \\ =& \{u \in C(\mathbb{R}^n \times [0,T]) : |v|_{0;\mathbb{R}^n\times[0,T]} + [u]_{\delta;\mathbb{R}^n\times[0,T]} < \infty\}.\end{aligned}$$

(Some authors would write $[u]_{\delta,\delta/2;\mathbb{R}^n\times[0,T]}$ for $[u]_{\delta;\mathbb{R}^n\times[0,T]}$.) If we make the following assumptions on $f(x,t), b_i(x,t), c(x,t)$ and $g(x)$:

$$f(x,t), b_i(x,t), c(x,t) \in L^\infty(\mathbb{R}^n \times [0,T]), \quad g \in C^1(\mathbb{R}^n) \text{ with } g, \partial_x g \in L^\infty(\mathbb{R}^n) \quad (11.16)$$

then, using only the estimate for the first two terms on the left hand side of (11.15) in Theorem 11.4, (11.14) gives a well defined map $S : X \mapsto X$, where

$$X = \{u(x,t) : u, \partial_x u(x,t) \in C(\mathbb{R}^n \times [0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T])\}$$

equipped with $||u||_X := |u|_{0;\mathbb{R}^n\times[0,T]} + \sqrt{T}|\partial_x u|_{0;\mathbb{R}^n\times[0,T]}$. Furthermore, using (11.15), we have

$$||S[u_1] - S[u_2]||_X \leq C\sqrt{T}\left(\sup_i ||b_i||_{L^\infty(\mathbb{R}^n\times[0,T])} + \sqrt{T}||c||_{L^\infty(\mathbb{R}^n\times[0,T])}\right)||u_1 - u_2||_X.$$

So, for $T > 0$ sufficiently small, $S : X \mapsto X$ is a contraction and therefore has a unique fixed point.

Remark 11.2. The form of the estimate (11.15) is dictated by scaling.

This method also works to prove short-time existence of generalized solutions in function spaces such as X when the lower order terms are nonlinear in ∇u and u.

A solution to (11.13) is a fixed point u of S; but a fixed point u of S can only be regarded as a generalized solution of (11.13) at this point, as we don't know whether it is $C^{2,1}_{x,t}$; although Theorem 11.4 actually shows that the fixed point $u \in X$ has additional differentiability in x, namely, $\partial_x u \in C^{\delta,\delta/2}(\mathbb{R}^n \times [\tau, T])$ for any $0 < \delta < 1$ and any $0 < \tau < T$. If we make the following further assumptions on $f(x,t), b_i(x,t)$ and $c(x,t)$:

$$[f, b_i, c]_{\delta;\mathbb{R}^n \times [0,T]} < \infty, \tag{11.17}$$

for some $0 < \delta < 1$, then with the next Theorem, we can conclude that the fixed point $u \in X$ actually has the better regularity that $u_t(x,t), \partial^2_{x_i x_j} u(x,t)$ exist in $\mathbb{R}^n \times (0,T]$ and continuous there, therefore solves (11.13) in the classical sense.

Theorem 11.5. *Assume that* $|f|_{0;\mathbb{R}^n \times [0,T]} + [f]_{\delta;\mathbb{R}^n \times [0,T]} < \infty$ *for some* $0 < \delta < 1$*, then* $v(x,t)$ *as defined in Theorem 11.4 has the property that* $v_t(x,t), \partial^2_{x_i x_j} v(x,t)$ *exist in* $\mathbb{R}^n \times [0,T]$ *and continuous there; furthermore, there exists* $C = C(n,\delta) > 0$,

$$[v_t, \partial^2_{x_i x_j} v]_{\delta;\mathbb{R}^n \times [0,T]} \le C[f]_{\delta;\mathbb{R}^n \times [0,T]}. \tag{11.18}$$

We summarize the solvability for (11.13) as

Theorem 11.6. *Suppose* (11.16)*, then* S *as defined by* (11.14) *has a unique fixed point* $u \in X$*, which serves as a generalized solution to* (11.13)*. If we further assume* (11.17)*, then* (11.13) *has a unique solution in* $C(\mathbb{R}^n \times [0,T]) \cap C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times (0,T])$*.*

Here and from now on, to simplify notations, when Q is a closed set, $C(Q)$ will denote the space of continuous functions on Q with finite $C(Q)$ norm; $C^\alpha(Q)$, etc, will be used in a similar way.

Proof of Theorem 11.6. We only need to clarify two issues: (a) to remove the smallness assumption on T in the earlier argument, and (b) to make sure that we can apply Theorem 11.5.

For (a), the smallness of $T > 0$ needed to make $S : X \mapsto X$ a contraction depends only on the size of the L^∞ norms of the coefficients, so after establishing the existence of a fixed point $u = S[u]$ on $0 \le t \le T$ for some $T > 0$, one can use the value of $u(x,T)$ as an initial value to construct a fixed point $u = S[u]$ on $T \le t \le 2T$; and one can carry out this procedure iteratively to obtain a solution on any given interval. Technically one can use the property

$$\int_{\mathbb{R}^n} K(x-y,t-\tau)K(y-z,\tau-s)dy = K(x-z,t-s) \quad \text{for } t>\tau>s,$$

to rewrite

$$\int_0^\tau \int_{\mathbb{R}^n} K(x-y,t-s)\left[f(y,s)-\sum_{i=1}^n b_i(y,s)u_{x_i}(y,s)-c(y,s)u(y,s)\right]dyds$$
$$=\int_0^\tau \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K(x-z,t-\tau)K(z-y,\tau-s)\left[f(y,s)-\sum_{i=1}^n b_i(y,s)u_{x_i}(y,s)-c(y,s)u(y,s)\right]dzdyds,$$

which, after integrating out $dyds$ first, gives us

$$\int_{\mathbb{R}^n} K(x-z,t-\tau)\left(\int_0^\tau \int_{\mathbb{R}^n} K(z-y,\tau-s)\left[f(y,s)-\sum_{i=1}^n b_i(y,s)u_{x_i}(y,s)-c(y,s)u(y,s)\right]dyds\right)dz.$$

Likewise

$$\begin{aligned}&\int_{\mathbb{R}^n} K(x-y,t)g(y)dy\\ &=\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K(x-z,t-\tau)K(z-y,\tau)dz\right)g(y)dy\\ &=\int_{\mathbb{R}^n} K(x-z,t-\tau)\left(\int_{\mathbb{R}^n} K(z-y,\tau)g(y)dy\right)dz.\end{aligned}$$

Putting these together, we see that

$$\begin{aligned}&\int_0^\tau \int_{\mathbb{R}^n} K(x-y,t-s)\left[f(y,s)-\sum_{i=1}^n b_i(y,s)u_{x_i}(y,s)-c(y,s)u(y,s)\right]dyds\\ &\quad+\int_{\mathbb{R}^n} K(x-y,t)g(y)dy\\ &=\int_{\mathbb{R}^n} K(x-z,t-\tau)S[u](z,\tau)dz\end{aligned}$$

using the definition of $S[u](z,\tau)$. Then one can rewrite $S[u](x,t)$, for $t>\tau$, as

$$\begin{aligned}&\left\{\int_0^\tau+\int_\tau^t\right\}\int_{\mathbb{R}^n} K(x-y,t-s)\left[f(y,s)-\sum_{i=1}^n b_i(y,s)u_{x_i}(y,s)-c(y,s)u(y,s)\right]dyds\\ &\quad+\int_{\mathbb{R}^n} K(x-y,t)g(y)dy\\ &=\int_\tau^t \int_{\mathbb{R}^n} K(x-y,t-s)\left[f(y,s)-\sum_{i=1}^n b_i(y,s)u_{x_i}(y,s)-c(y,s)u(y,s)\right]dyds\\ &\quad+\int_{\mathbb{R}^n} K(x-y,t-\tau)S[u](y,\tau)dy,\end{aligned}$$

which gives the iteration formula to be used to extend the solution from $[0, T]$ to $[T, 2T]$.

For (b), we use a similar technique. Under assumption (11.16) and (11.17), for $u \in X$, Theorem 11.5 proves that the fixed point $u = S[u]$ has the property that $u, \partial_x u \in C^{\delta,\delta/2}(\mathbb{R}^n \times (0, T])$, and provides an upper bound for the $[\cdot]_{\delta;\mathbb{R}^n\times[0,T]}$ norm of the first integral in (11.14) and its derivative in x, but provides no upper bound in the same norm over $\mathbb{R}^n \times [0, T]$ for the second integral $\int_{\mathbb{R}^n} K(x-y, t)g(y)dy$ without assuming $g \in C^{1+\delta}(\mathbb{R}^n)$, although it is C^∞ in $\mathbb{R}^n \times (0, T]$, so one cannot apply Theorem 11.5 directly. For any $0 < \tau < t$, one simply rewrites $S[u](x, t)$ as above, and applies Theorem 11.5 to the first integral $\int_\tau^t \int_{\mathbb{R}^n} \dots dyds$, while the remaining two integrals are C^∞ in $\mathbb{R}^n \times (\tau, T]$. Finally, We can now differentiate under the integral sign to verify that the fixed point u satisfies (11.13). □

Proof of Theorems 11.4 11.5. We will use the observation that

$$\partial_x^l \partial_t^m K(x, t) = \frac{1}{t^{\frac{l}{2}+m}} p_{l+2m}(\frac{x}{\sqrt{t}})K(x, t),$$

where p_{l+2m} is a polynomial of degree $l + 2m$, and the consequence

$$\int_{\mathbb{R}^n} \left|\partial_x^l \partial_t^m K(x, t)\right| dx = \frac{C_{l,m;n}}{t^{\frac{l}{2}+m}}.$$

First, $v \in C(\mathbb{R}^n \times [0, T])$ by Lebesgue's Dominated Convergence Theorem, and

$$|v|_{0;\mathbb{R}^n\times[0,T]} \le ||f||_{L^\infty(\mathbb{R}^n\times[0,T])} \int_0^T \int_{\mathbb{R}^n} K(x-y, t-s)dyds = T||f||_{L^\infty(\mathbb{R}^n\times[0,T])}.$$

Next set

$$F(x, t; s) = \int_{\mathbb{R}^n} K(x-y, t-s)f(y, s)dy.$$

Then

$$v(x, t) = \int_0^t F(x, t; s)ds,$$

$F(x, t; s)$ is a smooth function of (x, t) in the domain $t > s$, and

$$\partial_{x_i} F(x, t; s) = \int_{\mathbb{R}^n} \partial_{x_i} K(x-y, t-s)f(y, s)dy,$$

from which we obtain

$$\begin{aligned} |\partial_{x_i} F(x, t; s)| &\le ||f||_{L^\infty(\mathbb{R}^n\times[0,T])} \int_{\mathbb{R}^n} |\nabla_x K(x-y, t-s)|dy \\ &= ||f||_{L^\infty(\mathbb{R}^n\times[0,T])} \int_{\mathbb{R}^n} \frac{|x-y|}{(4\pi(t-s))^{n/2} 2(t-s)} e^{-\frac{|x-y|^2}{4(t-s)}} dy \qquad (11.19) \\ &\le \frac{C_{1,0;n}||f||_{L^\infty(\mathbb{R}^n\times[0,T])}}{\sqrt{t-s}}, \end{aligned}$$

which is Lebesgue integrable in $s \in [0,t]$. So

$$\partial_{x_i} v(x,t) = \int_0^t \partial_{x_i} F(x,t;s)ds = \int_0^t \int_{\mathbb{R}^n} \partial_{x_i} K(x-y,t-s) f(y,s) dy ds,$$

from which one obtains

$$\begin{aligned} &|\nabla_x v|_{0;\mathbb{R}^n \times [0,T]} \\ \leq &C_{1,0;n} ||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \int_0^t \frac{ds}{\sqrt{t-s}} \\ \leq &C(n)\sqrt{T} ||f||_{L^\infty(\mathbb{R}^n \times [0,T])}. \end{aligned}$$

We will prove the following slightly stronger estimates than (11.15).

$$|\partial_{x_i} v(x_1,t) - \partial_{x_i} v(x_2,t)| \leq C(n) ||f||_{L^\infty(\mathbb{R}^n \times [0,T])} |x_1 - x_2| \left(\log_+ \frac{t}{|x_1-x_2|^2} + 1 \right), \tag{11.20}$$

and for any $0 < t_1 < t_2$,

$$|\partial_{x_i} v(x,t_2) - \partial_{x_i} v(x,t_1)| \leq C(n) ||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t_2 - t_1}. \tag{11.21}$$

In the last section we already discussed the reason why one does not have

$$\partial^2_{x_i x_j} v(x,t) = \int_0^t \int_{\mathbb{R}^n} \partial^2_{x_i x_j} K(x-y,t-s) f(y,s) dy ds$$

under the sole assumption $||f||_{L^\infty(\mathbb{R}^n \times [0,T])} < \infty$ (even if $f \in C^(\mathbb{R}^n \times [0,T])$). However, with the assumption that $[f]_{\delta;\mathbb{R}^n \times [0,T])} < \infty$ and the property

$$\int_{\mathbb{R}^n} \partial^2_{x_i x_j} K(x-y,t-s) dy = 0 \quad \text{for all } t > s,$$

we have

$$\begin{aligned} &\left| \partial^2_{x_i x_j} F(x,t;s) \right| \\ \leq &\left| \int_{\mathbb{R}^n} \partial^2_{x_i x_j} K(x-y,t-s) \left[f(y,s) - f(x,s) \right] dy \right| \\ \leq &C(n)[f]_\delta \int_{\mathbb{R}^n} |x-y|^\delta \left(\frac{1}{(t-s)^{n/2+1}} + \frac{|x-y|^2}{(t-s)^{n/2+2}} \right) e^{-\frac{|x-y|^2}{4(t-s)}} dy \\ = &\frac{C(n)[f]_\delta}{(t-s)^{1-\delta/2}} \int_{\mathbb{R}^n} |z|^\delta \left(1 + |z|^2\right) e^{-|z|^2/4} dz \\ = &\frac{C'(n)[f]_\delta}{(t-s)^{1-\delta/2}}, \end{aligned}$$

which is Lebesgue integrable in $s \in [0, t]$. Thus we have

$$\begin{aligned}
&\partial^2_{x_i x_j} v(x,t) \\
=&\int_0^t \int_{\mathbb{R}^n} \partial^2_{x_i x_j} K(x-y, t-s) \left[f(y,s) - f(x,s) \right] dyds,
\end{aligned}$$

and

$$\begin{aligned}
&\left| \partial^2_{x_i x_j} v(x,t) \right| \\
\leq & C'(n)[f]_\delta \int_0^t \frac{ds}{(t-s)^{1-\delta/2}} \\
=& 2\delta^{-1} C'(n)[f]_\delta t^{\delta/2}.
\end{aligned}$$

Next, to prove (11.20), we estimate

$$\begin{aligned}
&|\partial_{x_i} F(x_1, t; s) - \partial_{x_i} F(x_2, t; s)| \\
=&\left| \int_{\mathbb{R}^n} \left[\partial_{x_i} K(x_1 - y, t-s) - \partial_{x_i} K(x_2 - y, t-s) \right] f(y,s) dy \right| \\
=&\left| \int_{\mathbb{R}^n} \int_0^1 \frac{\partial}{\partial \theta} \left[\partial_{x_i} K(\theta x_1 + (1-\theta)x_2 - y, t-s) \right] f(y,s) d\theta dy \right| \\
=&\left| \sum_{j=1}^n \int_{\mathbb{R}^n} \int_0^1 (x_1 - x_2)_j \frac{\partial^2}{\partial x_i \partial x_j} K(\theta x_1 + (1-\theta)x_2 - y, t-s) f(y,s) d\theta dy \right| \\
\leq & |x_1 - x_2| \int_0^1 \int_{\mathbb{R}^n} \left| \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} K(\theta x_1 + (1-\theta)x_2 - y, t-s) \right| |f(y,s)| dy d\theta \\
\leq & \frac{C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} |x_1 - x_2|}{t-s}.
\end{aligned} \tag{11.22}$$

We will estimate

$$|\partial_{x_i} v(x_1, t) - \partial_{x_i} v(x_2, t)| = | \int_0^t \left[\partial_{x_i} F(x_1, t; s) - \partial_{x_i} F(x_2, t; s) \right] ds|$$

depending on the relation between $|x_1 - x_2|^2$ and t. If $t \leq |x_1 - x_2|^2$, then

$$\begin{aligned}
&|\partial_{x_i} v(x_1, t) - \partial_{x_i} v(x_2, t)| \\
\leq & \int_0^t \left[|\partial_{x_i} F(x_1, t; s)| + |\partial_{x_i} F(x_2, t; s)| \right] ds \\
\leq & \int_0^t \frac{C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])}}{\sqrt{t-s}} ds \\
\leq & 2C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t} \\
\leq & 2C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} |x_1 - x_2|,
\end{aligned}$$

using (11.19); while if $t > |x_1 - x_2|^2$, then

$$\begin{aligned}&|\partial_{x_i} v(x_1,t) - \partial_{x_i} v(x_2,t)|\\ \le&\left(\int_0^{t-|x_1-x_2|^2} + \int_{t-|x_1-x_2|^2}^{t}\right)|\partial_{x_i}F(x_1,t;s) - \partial_{x_i}F(x_2,t;s)|\,ds,\end{aligned}$$

and use (11.22) to estimate the first integral

$$\begin{aligned}&\int_0^{t-|x_1-x_2|^2}|\partial_{x_i}F(x_1,t;s) - \partial_{x_i}F(x_2,t;s)|\,ds\\ \le&\int_0^{t-|x_1-x_2|^2}\frac{C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}|x_1-x_2|}{t-s}ds\\ \le&C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}|x_1-x_2|\log\frac{t}{|x_1-x_2|^2},\end{aligned}$$

and use (11.19) to estimate the second integral

$$\begin{aligned}&\int_{t-|x_1-x_2|^2}^{t}|\partial_{x_i}F(x_1,t;s) - \partial_{x_i}F(x_2,t;s)|\,ds\\ \le&\int_{t-|x_1-x_2|^2}^{t}\frac{C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}}{\sqrt{t-s}}ds\\ \le&2C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}|x_1-x_2|.\end{aligned}$$

The above estimates concludes (11.20). We can estimate $|\partial_{x_i}v(x,t_2) - \partial_{x_i}v(x,t_1)|$ according to the relation $t_2 - t_1 < t_1$ or otherwise. Set $0 \le t_0 < t_1$. Then

$$\begin{aligned}&|\partial_{x_i}v(x,t_2) - \partial_{x_i}v(x,t_1)|\\ =&\left|\int_0^{t_0}(\partial_{x_i}F(x,t_2;s) - \partial_{x_i}F(x,t_1;s))\,ds + \int_{t_0}^{t_2}\partial_{x_i}F(x,t_2;s)ds - \int_{t_0}^{t_1}\partial_{x_i}F(x,t_1;s)ds\right|\\ \le&\int_0^{t_0}|\partial_{x_i}F(x,t_2;s) - \partial_{x_i}F(x,t_1;s)|\,ds + \int_{t_0}^{t_2}|\partial_{x_i}F(x,t_2;s)|\,ds + \int_{t_0}^{t_1}|\partial_{x_i}F(x,t_1;s)|\,ds\end{aligned}$$

The first integral can be estimated as

$$\begin{aligned}&\int_0^{t_0}|\partial_{x_i}F(x,t_2;s) - \partial_{x_i}F(x,t_1;s)|\,ds\\ \le&\int_{t_1}^{t_2}\int_0^{t_0}|\partial_t\partial_{x_i}F(x,\tau;s)|\,dsd\tau\\ \le&C(n)\int_{t_1}^{t_2}\int_0^{t_0}\frac{||f||_{L^\infty(\mathbb{R}^n\times[0,T])}}{(\tau-s)^{3/2}}dsd\tau\\ \le&C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}\left(\sqrt{t_2-t_0} - \sqrt{t_1-t_0}\right),\end{aligned} \tag{11.23}$$

the second integral can be estimated as

$$\begin{aligned}&\int_{t_0}^{t_2} |\partial_{x_i} F(x, t_2; s)| \, ds \\ \leq & C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \int_{t_0}^{t_2} \frac{ds}{(t_2 - s)^{1/2}} \\ \leq & 2C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t_2 - t_0},\end{aligned} \tag{11.24}$$

and the third integral can be estimated as

$$\begin{aligned}&\int_{t_0}^{t_1} |\partial_{x_i} F(x, t_1; s)| \, ds \\ \leq & C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \int_{t_0}^{t_1} \frac{ds}{(t_1 - s)^{1/2}} \\ \leq & 2C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t_1 - t_0}.\end{aligned} \tag{11.25}$$

If $t_2 - t_1 < t_1$, we can set $t_0 = t_1 - (t_2 - t_1)$, so $t_1 - t_0 = t_2 - t_1$ and $t_2 - t_0 = 2(t_2 - t_1)$, then (11.23)(11.24)(11.25) imply that

$$|\partial_{x_i} v(x, t_2) - \partial_{x_i} v(x, t_1)| \leq C'(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t_2 - t_1};$$

while if $t_2 - t_1 \geq t_1$, then we can set $t_0 = 0$, and using $t_2 \leq 2(t_2 - t_1)$ in this case, we can still get

$$\begin{aligned}&|\partial_{x_i} v(x, t_2) - \partial_{x_i} v(x, t_1)| \\ \leq & C'(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t_2} \\ \leq & C''(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} \sqrt{t_2 - t_1}.\end{aligned}$$

(11.15) follws from (11.20) and (11.21) as follows. If $|x_1 - x_2| \leq \sqrt{t}$, then (11.20) implies that

$$\begin{aligned}&\frac{|\partial_{x_i} v(x_1, t) - \partial_{x_i} v(x_2, t)|}{|x_1 - x_2|^\delta} \\ \leq & C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} |x_1 - x_2|^{1-\delta} \left(\log_+ \frac{t}{|x_1 - x_2|^2} + 1 \right) \\ \leq & C'(n, \delta) t^{\frac{1-\delta}{2}};\end{aligned}$$

while if $|x_1 - x_2| > \sqrt{t}$, then

$$\begin{aligned}&\frac{|\partial_{x_i} v(x_1, t) - \partial_{x_i} v(x_2, t)|}{|x_1 - x_2|^\delta} \\ \leq & \frac{|\partial_{x_i} v(x_1, t) - \partial_{x_i} v(x_2, t)|}{t^{\delta/2}} \\ \leq & \frac{C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} t^{1/2}}{t^{\delta/2}} \\ = & C(n)||f||_{L^\infty(\mathbb{R}^n \times [0,T])} t^{\frac{1-\delta}{2}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\frac{|\partial_{x_i}v(x,t_2)-\partial_{x_i}v(x,t_1)|}{|t_2-t_1|^{\delta/2}}\\ \leq &C^{''}||f||_{L^\infty(\mathbb{R}^n\times[0,T])}|t_2-t_1|^{\frac{1-\delta}{2}}\\ \leq &C^{''}||f||_{L^\infty(\mathbb{R}^n\times[0,T])}T^{\frac{1-\delta}{2}},\end{aligned}$$

from which we conclude that

$$[\partial_x v]_{\delta;\mathbb{R}^n\times[0,T]}\leq C(n)||f||_{L^\infty(\mathbb{R}^n\times[0,T])}T^{\frac{1-\delta}{2}}.$$

Finally, (11.18) is proved using similar techniques. We will just pick one term to illustrate the method. First, we can establish in a similar way that

$$v_t(x,t)=\int_0^t\int_{\mathbb{R}^n}K_t(x-y,t-s)\,[f(y,s)-f(x,s)]+f(x,t)=\int_0^t F_t(x,t;s)ds+f(x,t).$$

For $x_1,x_2\in\mathbb{R}^n$, we estimate

$$\begin{aligned}&\left|\int_0^t F_t(x_1,t;s)ds-\int_0^t F_t(x_2,t;s)ds\right|\\ \leq&\int_0^{t-|x_1-x_2|^2}|F_t(x_1,t;s)-F_t(x_2,t;s)|\,ds+\int_{t-|x_1-x_2|^2}^t\left(|F_t(x_1,t;s)-F_t(x_2,t;s)|\right)ds.\end{aligned}$$

The second integral is bounded above by

$$C(n)[f]_\delta\int_{t-|x_1-x_2|^2}^t(t-s)^{-1+\delta/2}ds\leq\delta^{-1}C(n)[f]_\delta|x_1-x_2|^\delta.$$

We will estimate the first integral using

$$|\partial_t\partial_x F(x,t;s)|\leq C(n)[f]_\delta(t-s)^{-\frac{3}{2}+\frac{\delta}{2}}.$$

So using $|F_t(x_1,t;s)-F_t(x_2,t;s)|=|x_1-x_2|\,|\partial_t\partial_x F(\xi,t;s)|$ for some ξ between x_1 and x_2, and the bound above, the first integral is bounded above by

$$C(n)[f]_\delta|x_1-x_2|\int_0^{t-|x_1-x_2|^2}(t-s)^{-\frac{3}{2}+\frac{\delta}{2}}ds\leq\frac{C(n)[f]_\delta}{1-\delta}|x_1-x_2|^\delta.$$

□

Remark 11.3. Note that in order for $v(x,t)$ as given in Theorem 11.4 to be in $C^{2,1}(\mathbb{R}^n\times[0,T])$, it suffices to assume $|f(x_1,t)-f(x_2,t)|\leq M|x_1-x_2|^\delta$ for some $0<M<\infty$ and $0<\delta\leq 1$. The $\delta<1$ assumption enters in proving the Hölder continuity of the integral $\int_0^t\partial_x^2F(x,t;s)ds$ in $\partial_x^2v(x,t)$ (respectively, $\int_0^t\partial_tF(x,t;s)ds$ in $v_t(x,t)$), and the Hölder continuity of f in t is used only in proving the Hölder continuity of v_t in t through

$$\partial_t v(x,t)=\int_0^t\partial_tF(x,t;s)ds+f(x,t).$$

11.3 Local Hölder estimates and solvability of IVPs of more general second order parabolic equations*

We next localize the Hölder estimates in Theorems 11.4 and 11.5 to prepare for a priori Schauder estimates for solutions to second order parabolic equations with variable, Hölder continuous coefficients, which we will use to solve initial value problems for such equations and prove Hölder regularity of u_t and $u_{x_ix_j}$ for $C^{2,1}$ solutions of such equations.

Let $Q_R = \{(x,t) : |x| < R, -R^2 < t \le 0\}$.

Theorem 11.7. *Suppose that $u \in C^{2,1}(Q_{2R})$ is a solution to $u_t - \Delta u = f(x,t)$ in Q_{2R}, and that $f \in C^{\delta,\delta/2}(Q_{2R})$ for some $0 < \delta < 1$. Then $u \in C^{2+\delta,1+\delta/2}(Q_R)$, and there exists $C = C(n,\delta) > 0$ such that*

$$\begin{aligned} &|u|'_{2+\delta,1+\delta/2;Q_R} \\ :=&|u|_{0;Q_R} + R|\partial_x u|_{0;Q_R} + R^2\left(|\partial_x^2 u|_{0;Q_R} + |u_t|_{0;Q_R}\right) + R^{2+\delta}\left([u_t]_{\delta,\delta/2;Q_R} + [\partial_x^2 u]_{\delta,\delta/2;Q_R}\right) \\ \le& C\left\{|u|_{0;Q_{2R}} + R^2|f|'_{\delta,\delta/2;Q_{2R}}\right\} \end{aligned} \tag{11.26}$$

where $|f|'_{\delta,\delta/2;Q_{2R}} = |f|_{0;Q_{2R}} + (2R)^\delta[f]_{\delta,\delta/2;Q_{2R}}$.

The following is the interior a priori Schauder estimate for solutions to second order parabolic equations with variable, Hölder continuous coefficients.

Theorem 11.8. *Suppose that $a_{ij}(x,t), b_i(x,t), c(x,t) \in C^{\delta,\delta/2}(Q_{2R})$ for some $0 < \delta < 1$, and that for some $0 < \lambda \le \Lambda$,*

$$|a_{ij}|_{0;Q_{2R}} + R^\delta[a_{ij}]_{\delta,\delta/2;Q_{2R}} + R|b_i|_{0;Q_{2R}} + R^{1+\delta}[b_i]_{\delta,\delta/2;Q_{2R}} + R^2|c|_{0;Q_{2R}} + R^{2+\delta}[c]_{\delta,\delta/2;Q_{2R}} \le \Lambda, \tag{11.27}$$

and

$$\lambda|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \quad \forall (x,t) \in Q_{2R}, \xi \in \mathbb{R}^n. \tag{11.28}$$

Suppose $u \in C^{2+\delta,1+\delta/2}(Q_{2R})$ and let

$$u_t(x,t) - \sum_{i,j=1}^n a_{ij}(x,t)u_{x_ix_j}(x,t) + \sum_{i=1}^n b_i(x,t)u_{x_i}(x,t) + c(x,t)u(x,t) = f(x,t). \tag{11.29}$$

*May be skipped in a first course.

Then there exists some $C = C(n, \delta, \lambda, \Lambda) > 0$ such that

$$|u|'_{2+\delta,1+\delta/2;Q_R} \le C \left\{ |u|_{0;Q_{2R}} + R^2 |f|'_{\delta,\delta/2;Q_{2R}} \right\} \tag{11.30}$$

Remark 11.4. Theorem 11.8 is different from Theorem 11.7 in that it assumes that $u \in C^{2+\delta,1+\delta/2}(Q_{2R})$ a priori. Later we will use Theorem 11.8 to prove that any $C^{2,1}(Q_{2R})$ solution u to (11.29) with $C^{\delta,\delta/2}(Q_{2R})$ Hölder continuous coefficients and right hand side is actually $C^{2+\delta,1+\delta/2}(Q_{2R})$ Hölder continuous.

The hypotheses and conclusion are formulated in a scaling invariant way so as to easily exhibit how the size of the coefficients and domain impact the constant in the estimate; hypothesis (11.27) does impose conditions on the size of the coefficients in relation to the size of the domain in order to have control on the constant C in (11.30). For example, if one would like to use (11.30) on a solution to (11.29) with $f \equiv 0$ on $\mathbb{R}^n \times (-\infty, T]$ on $Q_R(x,t)$ for arbitrary $(x,t) \in \mathbb{R}^n \times (-\infty, T]$ and arbitrarily large $R > 0$, then (11.27) demands that $a_{ij}(x,t)$ must be constants, and $b_i(x,t) = c(x,t) = 0$ in $\mathbb{R}^n \times (-\infty, T]$. But in such cases (11.30) implies that any bounded solution u on $\mathbb{R}^n \times (-\infty, T]$ (so called ancient solution) satisfies $|u|'_{2+\delta,1+\delta/2;Q_R} \le C|u|_{0;Q_{2R}} \le C|u|_{0;\mathbb{R}^n \times (-\infty,T]}$ for all $R > 0$, which implies that u must be a constant.

The following global a priori Schauder estimate for solutions to second order parabolic equations with variable, Hölder continuous coefficients will be useful for proving existence of solutions to the initial value problem for such equations.

Theorem 11.9. *Suppose that $a_{ij}(x,t), b_i(x,t), c(x,t) \in C^{\delta,\delta/2}(\mathbb{R}^n \times [0,T])$ for some $0 < \delta < 1$ and $T > 0$, and that, for some $0 < \lambda \le \Lambda$, (11.28) holds in $\mathbb{R}^n \times [0,T]$, and*

$$\begin{aligned} &|a_{ij}|_{0;\mathbb{R}^n\times[0,T]} + T^{\delta/2}[a_{ij}]_{\delta,\delta/2;\mathbb{R}^n\times[0,T]} + \sqrt{T}|b_i|_{0;\mathbb{R}^n\times[0,T]} + T^{\frac{1+\delta}{2}}[b_i]_{\delta,\delta/2;\mathbb{R}^n\times[0,T]} \\ &+T|c|_{0;\mathbb{R}^n\times[0,T]} + T^{1+\delta/2}[c]_{\delta,\delta/2;\mathbb{R}^n\times[0,T]} \le \Lambda. \end{aligned} \tag{11.31}$$

Suppose that $u \in C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times [0,T])$ and satisfies (11.29) in $\mathbb{R}^n \times (0,T]$, and that $u(x,0) = g(x)$ for $x \in \mathbb{R}^n$. Then there exists some $C = C(n,\delta,\lambda,\Lambda) > 0$ such that

$$|u|'_{2+\delta,1+\delta/2;\mathbb{R}^n\times[0,T]} \le C \left\{ |g|'_{2+\delta;\mathbb{R}^n} + T|f|'_{\delta,\delta/2;\mathbb{R}^n\times[0,T]} \right\}, \tag{11.32}$$

where $|u|'_{2+\delta,1+\delta/2;\mathbb{R}^n\times[0,T]}$, $|g|'_{2+\delta;\mathbb{R}^n}$ and $|f|'_{\delta,\delta/2;\mathbb{R}^n\times[0,T]}$ are defined in a similar way as in Theorem 11.7, replacing R there by $\sqrt{T}$.

We will defer outlining some of the steps for proving Theorems 11.7, 11.8 and 11.9 until after we describe their applications.

Theorem 11.10. *Suppose that* $a_{ij}(x,t), b_i(x,t), c(x,t)$ *satisfy the same hypotheses as in Theorem 11.9, that* $f \in C^{\delta,\delta/2}(\mathbb{R}^n \times [0,T])$*, and that* $g \in C^{2+\delta}(\mathbb{R}^n)$*. Then there exists a unique solution* $u \in C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times [0,T])$ *to the Cauchy problem*

$$\begin{cases} u_t(x,t) - \sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_ix_j}(x,t) + \sum_{i=1}^{n} b_i(x,t)u_{x_i}(x,t) + c(x,t)u(x,t) = f(x,t) & \text{in } \mathbb{R}^n \times (0,T], \\ u(x,0) = g(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{11.33}$$

Proof of Theorem 11.10. By working with the rescaling $(x,t) \mapsto (x/\sqrt{T}, t/T)$, we may assume $T = 1$. The existence and uniqueness of u follows from the method of continuity and the estimate (11.32). More specifically, we consider the family of maps

$$M_\theta : X = C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times [0,T]) \mapsto Y := C^{\delta,\delta/2}(\mathbb{R}^n \times [0,T]) \times C^{2+\delta}(\mathbb{R}^n)$$

defined by

$$X \ni u \mapsto ([\partial_t + L_\theta]u, u(\cdot,0)) \in Y.$$

where $L_\theta = \theta L - (1-\theta)\Delta$ for $\theta \in [0,1]$. Then it follows from (11.32) that for some $C = C(n,\delta,\lambda,\Lambda)$ independent of $\theta \in [0,1]$,

$$||u||_X \le C||M_\theta u||_Y \quad \text{for all } u \in X \text{ and } \theta \in [0,1] \tag{11.34}$$

where the norms in X and Y are scaled by T according to our definition. Since when $\theta = 0$, $M_0 : X \mapsto Y$ is an isomorphism from X onto Y, it follows that $M_1 : X \mapsto Y$ is also an isomorphism from X onto Y, proving the existence of a unique solution $u \in C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times [0,T])$ to the Cauchy problem (11.33). □

We can use the interior estimates (11.30) to weaken the hypotheses on g in Theorem 11.10.

Theorem 11.11. *Suppose that* $a_{ij}(x,t), b_i(x,t), c(x,t)$ *satisfy the same hypotheses as in Theorem 11.9, that* $f \in C^{\delta,\delta/2}(\mathbb{R}^n \times [0,T])$*, and that* $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Then there exists a unique solution* $u \in C(\mathbb{R}^n \times [0,T]) \cap C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times (0,T])$ *to the Cauchy problem (11.33).*

Proof of Theorem 11.11. . Take a sequence $g_j(x) \in C^\infty(\mathbb{R}^n)$ with finite $C^{2+\delta,1+\delta/2}(\mathbb{R}^n \times [0,T])$ norms such that, for any compact subset K of $\mathbb{R}^n$, $|g_j - g|_{0;K} \to 0$, and $|g_j|_{0;\mathbb{R}^n} \le |g|_{0;\mathbb{R}^n}$. Let u_j be the unique solution to (11.33) with g_j replacing g.

If we know $|g_j - g|_{0;\mathbb{R}^n} \to 0$, then by the maximum principle, there exists $C = C(n, \lambda, \Lambda, T) > 0$ such that

$$|u_j - u_k|_{0;\mathbb{R}^n\times[0,T]} \le C|g_j - g_k|_{0;\mathbb{R}^n} \to 0, \text{ and } \lim_{j\to\infty} u_j(x,t) := u(x,t) \in C(\mathbb{R}^n \times [0,T]) \text{ exists.}$$

Furthermore $u(x, 0) = g(x)$ for all $x \in \mathbb{R}^n$. Next, for any $Q_R(x_0, t_0) \subset \mathbb{R}^n \times [0, T]$, we can apply the interior estimates (11.30) to $u_j - u_k$ over $Q_{R/2}(x_0, t_0)$ to conclude that $\{u_j\}$ is Cauchy in $C^{2+\delta,1+\delta/2}(Q_{R/2}(x_0, t_0))$, therefore $u \in C^{2+\delta,1+\delta/2}(Q_{R/2}(x_0, t_0))$ as well and satisfies the first equation in (11.33) there.

In the absence of having $|g_j - g|_{0;\mathbb{R}^n} \to 0$, we still have, by the maximum principle, that

$$|u_j|_{0;\mathbb{R}^n\times[0,T]} \le C(\Lambda, T)\left(|f|_{0;\mathbb{R}^n\times[0,T]} + |g_j|_{0;\mathbb{R}^n}\right) \le C(\Lambda, T)\left(|f|_{0;\mathbb{R}^n\times[0,T]} + |g|_{0;\mathbb{R}^n}\right) =: M. \tag{11.35}$$

Thus, applying the interior estimates (11.30) to u_j over $Q_R(x_0, t_0)$ would imply that $\{u_j\}$ is bounded in $C^{2+\delta,1+\delta/2}(Q_{R/2}(x_0, t_0))$, so a subsequence of $\{u_j\}$ would converge to some u in $C^{2+\delta',1+\delta'/2}(Q_{R/2}(x_0, t_0))$ for any $0 < \delta' < \delta$, proving that $u \in C^{2+\delta',1+\delta'/2}(Q_{R/2}(x_0, t_0))$ for any $0 < \delta' < \delta$ and satisfies the first equation in (11.33) there. In addition,

$$[u]_{2+\delta,1+\delta/2;Q_{R/2}(x_0,t_0)} \le \liminf_{j\to\infty} [u_j]_{2+\delta,1+\delta/2;Q_{R/2}(x_0,t_0)},$$

proving that $u \in C^{2+\delta,1+\delta/2}(Q_{R/2}(x_0, t_0))$.

Finally, to prove that $u \in C(\mathbb{R}^n \times [0, T])$ and satisfies $u(x, 0) = g(x)$ for all $x \in \mathbb{R}^n$, we use the barrier argument. If all lower order terms are absent, then we can choose $\beta > 0$ depending on Λ and $|f|_{0;\mathbb{R}^n\times[0,T]}$ such that for any $\alpha, \epsilon > 0$, $w(x,t) = \alpha(|x - x_0|^2 + \beta t) + g(x_0) + \epsilon$ is a super solution. We can also choose $\alpha > 0$ depending on ϵ and g such that for all large j, $g_j(x) \le w(x, 0)$ for all $x \in \mathbb{R}^n$. Then by the maximum principle, $u_j(x,t) \le w(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. Sending $j \to \infty$, we obtain $u(x,t) \le w(x,t)$ for all $(x,t) \in \mathbb{R}^n \times [0,T]$. Thus we can choose $\sigma > 0$ such that when $|x - x_0|^2 + t \le \sigma^2$, $u(x,t) - g(x_0) \le 2\epsilon$. Similarly, we can prove that $-u(x,t) + g(x_0) \le 2\epsilon$ when $|x - x_0|^2 + t \le \sigma^2$. Thus $u(x,t)$ is continuous at each $(x_0, 0)$ and $u(x, 0) = g(x)$.

In the general case, for any x_0 and $\epsilon > 0$, a local barrier in $Q = B_\sigma(x_0) \times [0, \sigma^2]$ can be constructed in the form of $w(x,t) = [\alpha(|x - x_0|^2 + \beta t) + g(x_0) + \epsilon]\, e^{\gamma t}$ for appropriate choice of α, β, γ and σ. First we choose σ such that $|g(x) - g(x_0)| \le \epsilon/2$ for $x \in B_\sigma(x_0)$. Denoting the operator on the left hand side of the first equation in (11.33) by P, then a direct computation shows that

$$Pw \ge \alpha\,[\beta - C\Lambda(1 + |x - x_0|)]\, e^{\gamma t} + [c(x,t) + \gamma]\, w(x,t).$$

We now choose $\gamma \geq \Lambda \geq |c|_{0;Q}$, which leads to $[c(x,t)+\gamma]\, w(x,t) \geq [c(x,t)+\gamma]\, g(x_0)e^{\gamma t}$, so

$$Pw \geq \{\alpha\, [\beta - C\Lambda(1+|x-x_0|)] + [c(x,t)+\gamma]\, g(x_0)\}\, e^{\gamma t}.$$

We then choose $\beta > 0$ so that $\beta - C\Lambda(1+|x-x_0|) \geq 1$ in $B_\sigma(x_0)$, and choose $\alpha > |c(x,t)+\gamma|_{0;Q}|g(x_0)| + |f|_{0;\mathbb{R}^n\times[0,T]}$ so that $Pw \geq |f|_{0;\mathbb{R}^n\times[0,T]}$ in Q. Finally, recalling the C^0 bound (11.35) for u_j, we further require that $w(x,t) \geq M \geq |u_j(x,t)|$ for $(x,t) \in \partial B_\sigma(x_0) \times [0,\sigma^2]$, which can be achieved if $\alpha\sigma^2 \geq |g(x_0)| + M$.

Now the maximum principle applied to u_j and w on Q implies that, for all large j, $u_j(x,t) \leq w(x,t)$ for $(x,t) \in Q$. Since $\lim_{(x,t)\to(x_0,0)} w(x,t) = g(x_0)+\epsilon$, we can find $0 < \rho \leq \sigma$, such that when $|x-x_0|^2 + t \leq \rho^2$, $u_j(x,t) \leq w(x,t) \leq g(x_0) + 2\epsilon$. The barrier construction can also be applied to $-u_j$, which finally leads to the continuity of $u(x,t)$ at $(x_0,0)$ and $u(x_0,0) = g(x_0)$. □

We can also use interior estimates (11.30) to prove $C^{2+\delta,1+\delta/2}$ regularity of any $C^{2,1}$ solution to (11.29) in a region.

Theorem 11.12. *Suppose that $a_{ij}(x,t), b_i(x,t), c(x,t) \in C^{\delta,\delta/2}(Q_{2R})$ for some $0 < \delta < 1$, and that $u \in C^{2,1}(Q_{2R})$ satisfies $[\partial_t + L]\, u \in C^{\delta,\delta/2}(Q_{2R})$. Then $u \in C^{2+\delta,1+\delta/2}(Q_{2R})$.*

Proof. It suffices to prove $u \in C^{2+\delta,1+\delta/2}(Q_{R'})$ for any $R' < 2R$. We can extend $a_{ij}(x,t), b_i(x,t)$, and $c(x,t)$ to $\mathbb{R}^n \times [-4R^2, 0]$ so that their extensions belong to $C^{\delta,\delta/2}(\mathbb{R}^n \times [-4R^2,0])$. Let $\zeta(x,t)$ be a smooth cut-off function such that it is identically 1 in $Q_{R'}$ and supported in Q_{2R}. Then $v(x,t) = u(x,t)\zeta(x,t) \in C^{2,1}(\mathbb{R}^n \times [-4R^2,0])$, and also satisfies $[\partial_t + L]\, v \in C^{\delta,\delta/2}(\mathbb{R}^n \times [-4R^2,0])$, and $v(x,-4R^2) = 0$ for all $x \in \mathbb{R}^n$. According to Theorem 11.10, there exists a unique $w \in X$ (shifting the initial time to $t = -4R^2$) satisfying $[\partial_t + L]\, w = [\partial_t + L]\, v$ in $\mathbb{R}^n \times [-4R^2,0]$. But by the maximum principle, $w(x,t) = v(x,t)$ in $\mathbb{R}^n \times [-4R^2,0]$. Thus in $Q_{R'}$, $u = v = w \in C^{2+\delta,1+\delta/2}(Q_{R'})$. □

As a byproduct of our Schauder theory for variable coefficients second order parabolic equations, we obtain the corresponding results for variable coefficients second order elliptic equations.

Corollary 11.13. *Suppose that $a_{ij}(x), b_i(x), c(x) \in C^\delta(B_{2R})$ for some $0 < \delta < 1$, and that for some $0 < \lambda \leq \Lambda$,*

$$|a_{ij}|_{0;R_{2R}} + R^\delta[a_{ij}]_{\delta;B_{2R}} + R|b_i|_{0;B_{2R}} + R^{1+\delta}[b_i]_{\delta;B_{2R}} + R^2|c|_{0;B_{2R}} + R^{2+\delta}[c]_{\delta;B_{2R}} \leq \Lambda, \tag{11.36}$$

and

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \quad \forall x \in B_{2R}, \xi \in \mathbb{R}^n. \tag{11.37}$$

Suppose $u \in C^{2+\delta}(B_{2R})$ *and let*

$$-\sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j}(x) + \sum_{i=1}^{n} b_i(x)u_{x_i}(x) + c(x)u(x) = f(x). \tag{11.38}$$

Then there exists some $C = C(n, \delta, \lambda, \Lambda) > 0$ *such that*

$$|u|'_{2+\delta;B_R} \leq C\left\{|u|_{0;B_{2R}} + R^2|f|'_{\delta;B_{2R}}\right\} \tag{11.39}$$

Corollary 11.14. *Under the same hypotheses for* $a_{ij}(x), b_i(x), c(x)$ *as in Corollary 11.13, suppose that* $u \in C^2(B_{2R})$ *satisfies* $-\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}(x) + \sum_{i=1}^n b_i(x)u_{x_i}(x) + c(x)u(x) \in C^\delta(B_{2R})$*. Then* $u \in C^{2+\delta}(B_{2R})$*.*

We now outline some steps in the proof for Theorems 11.7, 11.8 and 11.9.

Proof for Theorem 11.7. Again we may assume $R = 1$. Let $\zeta(x,t)$ be a smooth cut-off function supported in Q_2 and identically equal to 1 in $Q_{3/2}$. Let

$$v(x,t) = \int_{-4}^{t}\int_{B_2} K(x-y, t-s)f(y,s)\zeta(y,s)dyds.$$

Then $v \in C^{2+\delta,1+\delta/2}(Q_2)$ and satisfies

$$\begin{aligned} v_t(x,t) - \Delta v(x,t) &= f(x,t)\zeta(x,t) \qquad \text{in } Q_2, \\ &= f(x,t) \qquad \text{in } Q_{3/2}. \end{aligned}$$

Set $w(x,t) = u(x,t) - v(x,t)$. Then $w_t(x,t) - \Delta w(x,t) = 0$ in $Q_{3/2}$. Note that $|v|_{0;Q_2} \leq 4|f|_{0;Q_2}$, so $|w|_{0;Q_2} \leq |u|_{0;Q_2} + 4|f|_{0;Q_2}$. By the gradient estimates for solutions to the homogeneous heat equation, $w \in C^\infty(Q_{3/2})$, and $|\partial_t^k\partial_x^l w|_{0;Q_1} \leq C(n,k,l)|w|_{0;Q_2}$ for any $k, l \in \mathbb{N}$. In particular $|w|_{2+\delta,1+\delta/2;Q_1} \leq C(n,\delta)|w|_{0;Q_2}$. We now apply Theorem 2 to v to imply that

$$\begin{aligned} [u]_{2+\delta,1+\delta/2;Q_1} &\leq [v]_{2+\delta,1+\delta/2;Q_1} + [w]_{2+\delta,1+\delta/2;Q_1} \\ &\leq C[f\zeta]_{\delta,\delta/2;Q_2} + C\left(|u|_{0;Q_2} + 4|f|_{0;Q_2}\right) \\ &\leq C(n,\delta)\left(|u|_{0;Q_2} + |f|_{\delta,\delta/2;Q_2}\right) \end{aligned}$$

Finally we apply the interpolation inequality which bounds any intermediate semi-norms of u in terms of $|u|_{0;Q_1}$ and $[u]_{2+\delta,1+\delta/2;Q_1}$ to conclude the proof for Theorem 2. □

Proof for Theorem 11.8. Here we will introduce three new techniques: (i) how to handle interior estimates; (ii) how to use the method of "freezing coefficients"; and (iii) how to use interpolation inequalities.

We will denote (x,t) by X, (y,s) by Y to simplify notation. The basic idea is to study (11.29) near each $Y_0 = (y_0, s_0)$ as $u_t(X) - \sum_{i,j=1}^n a_{ij}(Y_0)\partial_{ij}^2 u(X) = F(X)$, where

$$F(X) = \sum_{i,j=1}^n \left[a_{ij}(X) - a_{ij}(Y_0)\right] \partial_{ij}^2 u(X) - \sum_{i=1}^n b_i(X) u_{x_i}(X) - c(X)u(X) + f(X),$$

Thus we are treating equation (11.29) near Y_0 as if we are freezing the coefficents of the principal term at Y_0. We can then apply an extension of (11.26) in Theorem 11.7 to the constant coefficients operator $\partial_t - \sum_{i,j=1}^n a_{ij}(Y_0)\partial_{ij}^2$ on $Q_{2r}(Y_0)$ for $r > 0$ small, to estimate $[u]_{2+\delta,1+\delta/2;Q_r(Y_0)}$ in terms of the $[\cdot]_{\delta,\delta/2;Q_{2r}(Y_0)}$ norm of F and $|u|_{0;Q_{2r}(Y_0)}$. Using that, when $r > 0$ is small, the $[\cdot]_{\delta,\delta/2;Q_{2r}(Y_0)}$ norm of the term $\sum_{i,j=1}^n \left[a_{ij}(X) - a_{ij}(Y_0)\right] \partial_{ij}^2 u(X)$ in F is a small multiple of $[u]_{2+\delta,1+\delta/2;Q_{2r}(Y_0)}$ plus a multiple of $|\partial_x^2 u|_{0;Q_{2r}(Y_0)}$, we hope to absorb the term $[u]_{2+\delta,1+\delta/2;Q_{2r}(Y_0)}$ to the left hand side, except that this semi-norm is evaluated on $Q_{2r}(Y_0)$ instead of $Q_r(Y_0)$. The technique below overcomes this difficulty. The lower order terms will be estimated, with the help of interpolation inequalities, in terms of a small multiple of $[u]_{2+\delta,1+\delta/2;Q_{2r}(Y_0)}$ plus a multiple of $|u|_{0;Q_{2r}(Y_0)}$.

Again it suffices to prove the theorem for $R = 1$. It's convenient to use $d(X,Y) = |X-Y| + \sqrt{t-s}$ as the distance between X and Y. Let $d_{X,Y} = \min\{d(X, \partial' Q_2), d(Y, \partial' Q_2)\}$. Note $d_{X,Y} = 0$ when either X or $Y \in \partial' Q_2$, and $d_{X,Y} \geq 1$ when $X, Y \in Q_1$.

Due to the interpolation inequalities, it suffices to estimate

$$M := \sup\{d_{X,Y}^{2+\delta} \frac{|\partial_x^2 u(X) - \partial_x^2 u(Y)|}{d(X,Y)^\delta} : X \neq Y \in Q_2\}.$$

in terms of $|u|_{0;Q_2}$ and $|f|_{\delta,\delta/2;Q_2}$. The scale $d_{X,Y}^{2+\delta}$, in addition to making M scaling invariant, will help us locate points X_0, Y_0 in the *interior* of Q_2 through which we will estimate M.

Suppose $M > 0$ and pick $X_0 \neq Y_0 \in Q_2$ such that

$$\frac{M}{2} \leq d_{X_0,Y_0}^{2+\delta} \frac{|\partial_x^2 u(X_0) - \partial_x^2 u(Y_0)|}{d(X_0,Y_0)^\delta}.$$

Let $d_0 := d_{X_0,Y_0}$. Then $d_0 > 0$ and $Q_{d_0}(X), Q_{d_0}(Y) \subset Q_2$. We will estimate M depending on whether $d(X_0,Y_0) < \frac{\theta d_0}{2}$ or $d(X_0,Y_0) \geq \frac{\theta d_0}{2}$, where $0 < \theta < 1$ is to be chosen later.

In the case $d(X_0, Y_0) < \frac{\theta d_0}{2}$, then either $X_0 \in Q_{\frac{\theta d_0}{2}}(Y_0)$ or $Y_0 \in Q_{\frac{\theta d_0}{2}}(X_0)$. We may assume the former, and have

$$\frac{M}{2} \leq d_0^{2+\delta} \frac{|\partial_x^2 u(X_0) - \partial_x^2 u(Y_0)|}{d(X_0, Y_0)^\delta} \leq d_0^{2+\delta} [\partial_x^2 u]_{\delta,\delta/2;Q_{\frac{\theta d_0}{2}}(Y_0)};$$

while in the case $d(X_0, Y_0) \geq \frac{\theta d_0}{2}$, we have

$$\frac{M}{2} \leq \frac{2^{1+\delta} d_0^2}{\theta^\delta} |\partial_x^2 u|_{0;Q_{2-d_0}},$$

so we have

$$\begin{aligned}
\frac{M}{2} &\leq d_0^{2+\delta} [\partial_x^2 u]_{\delta,\delta/2;Q_{\frac{\theta d_0}{2}}(Y_0)} + \frac{2^{1+\delta} d_0^2}{\theta^\delta} |\partial_x^2 u|_{0;Q_{2-d_0}} \\
&\leq C d_0^{2+\delta} \left\{ \frac{|u|_{0;Q_{\theta d_0}(Y_0)}}{(\theta d_0)^{2+\delta}} + \frac{|F|_{0;Q_{\theta d_0}(Y_0)}}{(\theta d_0)^\delta} + [F]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} \right\} + \frac{2^{1+\delta} d_0^2}{\theta^\delta} |\partial_x^2 u|_{0;Q_{2-d_0}} \\
&\leq C \left\{ \frac{|u|_{0;Q_{\theta d_0}(Y_0)}}{\theta^{2+\delta}} + \frac{d_0^2 |F|_{0;Q_{\theta d_0}(Y_0)}}{\theta^\delta} + d_0^{2+\delta} [F]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} \right\} + \frac{2^{1+\delta} d_0^2}{\theta^\delta} |\partial_x^2 u|_{0;Q_{2-d_0}}
\end{aligned}$$

where $F(X)$ is defined as above. In the above we have used an extension of (11.26) in Theorem 11.7 to the constant coefficient operator $\partial_t - \sum_{i,j=1}^n a_{ij}(Y_0) \partial_{ij}^2$.

In the following we will assume that $b_i, c \equiv 0$ and put our focus on the term

$$\sum_{i,j=1}^n [a_{ij}(X) - a_{ij}(Y_0)] \, \partial_{ij}^2 u(X),$$

—when the $b_i u_{x_i}(X)$ terms are present, they will be handled by interpolation inequalities. Note that

$$[F]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} \leq \Lambda \left[(\theta d_0)^\delta [\partial_x^2 u]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} + |\partial_x^2 u|_{0;Q_{\theta d_0}(Y_0)} \right] + [f]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)},$$

$$|F|_{0;Q_{\theta d_0}(Y_0)} \leq \Lambda (\theta d_0)^\delta |\partial_x^2 u|_{0;Q_{\theta d_0}(Y_0)} + |f|_{0;Q_{\theta d_0}(Y_0)}.$$

By the definition of M, $[(1-\theta) d_0]^{2+\delta} [\partial_x^2 u]_{\delta;Q_{\theta d_0}(Y_0)} \leq M$, so

$$\begin{aligned}
\frac{M}{2} \leq& C \left\{ \frac{|u|_{0;Q_{\theta d_0}(Y_0)}}{\theta^{2+\delta}} + \frac{d_0^2 [\Lambda (\theta d_0)^\delta |\partial_x^2 u|_{0;Q_{\theta d_0}(Y_0)} + |f|_{0;Q_{\theta d_0}(Y_0)}]}{\theta^\delta} \right. \\
& \left. + d_0^{2+\delta} \left\{ \Lambda \left[(\theta d_0)^\delta [\partial_x^2 u]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} + |\partial_x^2 u|_{0;Q_{\theta d_0}(Y_0)} \right] + [f]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} \right\} \right\} + \frac{2^{1+\delta} d_0^2}{\theta^\delta} |\partial_x^2 u|_{0;Q_{2-d_0}} \\
\leq& C \left\{ \frac{\Lambda (\theta d_0)^\delta}{(1-\theta)^{2+\delta}} M + d_0^2 \left(\Lambda d_0^\delta + \frac{2^{1+\delta}}{\theta^\delta} \right) |\partial_x^2 u|_{0;Q_{2-d_0}} \right\} \\
& + C \left[d_0^{2+\delta} [f]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)} + \frac{d_0^2}{\theta^\delta} |f|_{0;Q_{\theta d_0}(Y_0)} \right] + \frac{|u|_{0;Q_{\theta d_0}(Y_0)}}{\theta^{2+\delta}}.
\end{aligned}$$

We now fix θ so that $\frac{C\Lambda\theta^\delta}{(1-\theta)^{2+\delta}} \le \frac{1}{8}$ ($d_0 \le 2$ here) to obtain

$$\begin{aligned}\frac{M}{4} \le & C\left\{d_0^2\left[\left(\Lambda d_0^\delta + \frac{2^{1+\delta}}{\theta^\delta}\right)|\partial_x^2 u|_{0;Q_{2-d_0}} + [f]_{\delta,\delta/2;Q_{\theta d_0}(Y_0)}\right] + \frac{d_0^2}{\theta^\delta}|f|_{0;Q_{\theta d_0}(Y_0)} + \frac{|u|_{0;Q_{\theta d_0}(Y_0)}}{\theta^{2+\delta}}\right\}\\ \le & C d_0^2 |\partial_x^2 u|_{0;Q_{2-d_0}} + C'[|f|'_{\delta,\delta/2;Q_2} + |u|_{0;Q_2}]\end{aligned} \tag{11.40}$$

We finally use interpolation inequalities to estimate $Cd_0^{2+\delta}|\partial_x^2 u|_{0;Q_{2-d_0}}$ in terms of a small multiple of M plus a multiple of $|u|_{0;Q_2}$, which leads to our desired bound (11.30). □

Exercise 11.3.1. Let $K(x,t)$ denote the standard heat kernel, and $u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)g(y)dy$ for $g \in L^p(\mathbb{R}^n)$. Prove that for any non-negative integers l and m,

- $||\partial_t^l \partial_x^m K(\cdot,t)||_{L^q(\mathbb{R}^n)} = C_{q,l,m,n} t^{-\frac{n}{2}(1-\frac{1}{q})-l-\frac{m}{2}}$ for $t > 0$.
- with r given by $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$,

$$||u(\cdot,t)||_{L^r(\mathbb{R}^n)} \le C_{p,r,l,m,n} t^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{r})-l-\frac{m}{2}} ||f||_{L^p(\mathbb{R}^n)}.$$

Exercise 11.3.2. Prove that $\int_{\mathbb{R}^n} K(x-y,t-\tau)K(y-z,\tau-s)dy = K(x-z,t-s)$ for $t > \tau > s$.

Exercise 11.3.3. Let $v(x,t) = \int_0^t F(x,t;s)ds$ as defined in Theorem 1. Complete the details in the proof for $[v_t]_{\delta;\mathbb{R}^n\times[0,T]} \le C[f]_{\delta;\mathbb{R}^n\times[0,T]}$ by proving the bound of the Hölder semi-norm of v_t in the t-direction.

Exercise 11.3.4. Let $u \in C^{2,1}_{x,t}(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ be a solution to (11.3), satisfying for some $A > 0$, $a > 0$,

$$|u(x,t)| \le Ae^{a|x|^2} \quad \text{for all } (x,t) \in \mathbb{R}^n \times [0,T].$$

Prove that for $t < \frac{1}{4a}$, (11.2) holds, namely,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-s)f(y,s)dyds + \int_{\mathbb{R}^n} K(x-y,t)g(y)dy.$$

Exercise 11.3.5. Suppose $u \in C^{2+\delta,1+\delta/2}(Q_R)$ satisfies

$$u_t(x,t) - \sum_{i,j=1}^n a_{ij}(x,t)u_{x_ix_j}(x,t) + \sum_{i=1}^n b_i(x,t)u_{x_i}(x,t) + c(x,t)u(x,t) = f(x,t)$$

in Q_R. Define $v(y,s) = u(Ry, R^2 s)$ for $(y,s) \in Q_1$. (a). Express $|v|_{2+\delta,1+\delta/2;Q_1}$ in terms of corresponding norms of u on Q_R. (b). Transform the equation above for u into an equation for v, and express the $|\cdot|_{\delta,\delta/2;Q_1}$norms of the coefficients in the equation for v in terms of the corresponding norms of the coefficients in the equation for u over Q_R.

Exercise 11.3.6. Suppose that $\{u_j\}$ is bounded in $C^{2+\delta,1+\delta/2}(Q_R(x_0,t_0))$. Prove that there exists a subsequence $\{u_{j_k}\}$ of $\{u_j\}$ and $u \in C^{2+\delta,1+\delta/2}(Q_R(x_0,t_0))$ such that $\{u_{j_k}\}$ converges to u in $C^{2+\delta',1+\delta'/2}(Q_{R/2}(x_0,t_0))$ for any $0 < \delta' < \delta$.

Exercise 11.3.7. Interpolation inequalities are used in deriving the Schauder estimates. There are many different forms of interpolation inequalities, all tracing their origin to the 1-dimensional version. Let I denote a closed interval on $\mathbb{R}$.

(1). There exists $C > 0$ such that for any $u \in C^2(I)$, and any $|I| > \epsilon > 0$, $|u'|_{0;I} \le \epsilon |u''|_{0;I} + C\epsilon^{-1}|u|_{0;I}$. (Hint: For any $x \in I$, and for h such that $x + h \in I$, use

$$u(x+h) - u(x) = \int_x^{x+h} u'(t)dt = u'(x)h + \int_x^{x+h} (x+h-t)u''(t)dt$$

to express $|u'|_{0;I}$ in terms of $|u|_{0;I}$ and $|u''|_{0;I}$.)

(ii). There exists $C > 0$ such that for any $u \in C^{1+\delta}(I)$ for some $0 < \delta \le 1$, and any $|I| > \epsilon > 0$, $|u'|_{0;I} \le \epsilon^\delta [u']_{\delta;I} + C\epsilon^{-1}|u|_{0;I}$. (Hint: Modify the above relation to $u(x+h) - u(x) = u'(x)h + \int_x^{x+h}[u'(t) - u'(x)]dt$ and use $|u'(t) - u'(x)| \le [u']_{\delta;I}|t-x|^\delta$.)

(iii). There exists $C > 0$ such that for any $u \in C^{2+\delta}(I)$ for some $0 < \delta \le 1$, and any $|I| > \epsilon > 0$, $|u''|_{0;I} \le \epsilon^\delta [u'']_{\delta;I} + C\epsilon^{-2}|u|_{0;I}$.

Appendix A

Interchange of Order of Differentiation and Integral or Sum

We often need to interchange the order of differentiation and an infinite sum or integral. We list below some commonly used criteria to justify such an interchange. The most convenient tool is Lebesgue's Dominated Convergence Theorem, formulated as (iv) and (v) below; but we have also included more elementary versions for those readers who are not familiar with or comfortable with Lebesgue's integral.

Lemma A.1. *(i). Suppose that $s_N(x) \to s(x)$ as $N \to \infty$ for some $x \in [a, b]$, that $s'_N(x)$ exists and is continuous for $x \in [a, b]$, and that $s'_N(x)$ converges to $t(x)$ uniformly for $x \in [a, b]$ as $N \to \infty$, then $s(x)$ is continuously differentiable for $x \in [a, b]$ and $s'(x) = t(x)$ for $x \in [a, b]$.*

(ii). Suppose that each $a_n(x)$ is $C^1[a, b]$, that the series $\sum_{n=1}^{\infty} a'_n(x)$ converges uniformly over $[a, b]$, and that $\sum_{n=1}^{\infty} a_n(x)$ converges at some $x \in [a, b]$, then $\sum_{n=1}^{\infty} a_n(x)$ defines a $C^1[a, b]$ function and

$$\left(\sum_{n=1}^{\infty} a_n(x)\right)' = \sum_{n=1}^{\infty} a'_n(x) \quad \text{for } x \in [a, b].$$

(iii). Suppose that U is a **bounded closed** *set in $\mathbb{R}^n$, that $s(x; \xi)$ and $s_x(x; \xi)$ are continuous for $(x; \xi) \in (x_0 - \delta, x_0 + \delta) \times U$, then $\int_U s(x; \xi) d\xi$ is continuously*

differentiable in $x \in (x_0 - \delta, x_0 + \delta)$*, and*

$$\frac{d}{dx}\left(\int_U s(x;\xi)d\xi\right) = \int_U s_x(x;\xi)d\xi \quad \textit{for } x \in (x_0 - \delta, x_0 + \delta).$$

(iv). Suppose that $U \subset \mathbb{R}^n$*, that* $s_j(\xi)$ *and* $s(\xi)$ *are integrable with* $s_j(\xi) \to s(\xi)$ *as* $j \to \infty$ *for any* $\xi \in U$*, and that there exists an integrable function* $M(\xi)$ *over* U *such that* $|s_j(\xi)| \le M(\xi)$ *for all* $\xi \in U$*, then* $\lim_{j\to\infty} \int_U s_j(\xi)d\xi = \int_U s(\xi)d\xi$*.*

(v). Suppose that $U \subset \mathbb{R}^n$*, that* $s(x;\xi)$ *and* $s_x(x;\xi)$ *are continuous for* $(x;\xi) \in (x_0 - \delta, x_0 + \delta) \times U$*, and that there exists an integrable function* $M(\xi)$ *over* U *such that*

$$|s_x(x;\xi)| \le M(\xi) \quad \textit{for } (x;\xi) \in (x_0 - \delta, x_0 + \delta) \times U,$$

then $\int_U s(x;\xi)d\xi$ *is continuously differentiable in* $x \in (x_0 - \delta, x_0 + \delta)$*, and*

$$\frac{d}{dx}\left(\int_U s(x;\xi)d\xi\right) = \int_U s_x(x;\xi)d\xi \quad \textit{for } x \in (x_0 - \delta, x_0 + \delta).$$

Remark.

- In (iv)–(v) the integrability can be either Riemann or Lebesgue integrability, and the continuity in the ξ in (v) can be replaced by the integrability in ξ.

- The formulation in (iii)-(v) above may seem like that U is an n-dimensional region in $\mathbb{R}^n$, and $d\xi$ is the volume integral in $\mathbb{R}^n$; but they are equally valid when U is a lower dimensional surface in $\mathbb{R}^n$, and $d\xi$ is the corresponding surface integral.

- (iii) has two key ingredients: the uniform continuity of $s_x(x;\xi)$ in $[x_0 - \delta_1, x_0 + \delta_1] \times U$ for any $0 < \delta_1 < \delta$, and the finiteness of the measure of U. Then, for any $x \in (x_0 - \delta, x_0 + \delta)$, we can find $0 < \delta_1 < \delta$, such that $x \in (x_0 - \delta_1, x_0 + \delta_1)$, and $\delta_2 = \delta_1 - |x - x_0| > 0$; and for any $0 < |h| < \delta_2$, $s(x + h;\xi) - s(x;\xi) = s_x(x + \theta h;\xi)h$ for some $0 < \theta < 1$ depending on h, x and ξ. But due to the uniform continuity of $s_x(x;\xi)$ in $[x_0 - \delta_1, x_0 + \delta_1] \times U$, for any given $\epsilon > 0$, we can find $0 < h_0 \le \delta_2$, such that

$$|h^{-1}(s(x + h;\xi) - s(x;\xi)) - s_x(x,\xi)| < \epsilon \qquad \text{(uniform)}$$

for any $0 < |h| < h_0$, $(x, \xi) \in [x_0 - \delta_1, x_0 + \delta_1] \times U$, and

$$\left|h^{-1}\left[\int_U (s(x + h;\xi) - s(x;\xi))\, d\xi\right] - \int_U s_x(x,\xi)\, d\xi\right| < \epsilon|U|,$$

which provides a proof for (iii). Even if we can establish *(uniform)* when U is not closed or bounded, but $|U|$ is ∞, (iii) may not be valid without a condition such as in (v). Here is a simple example when $U = \mathbb{R}$ and $x = x_0 = 0$: $s(x,\xi) = \frac{x^2}{1+x^2\xi^2}$. At issue is that the integrals in the "tail part", i.e., when $|\xi|$ is large, is not small uniformly in h when $|h| \to 0$.

- In most applications of (v), the dominating function $M(\xi)$ is needed only in the tail part as alluded to above, or near isolated points where the integral may become an improper integral. With the assumption in (v), we can use a "divide-and-conquer" strategy to tackle the problem. For many problems, for any given $\epsilon > 0$, we can find a *bounded and closed* set V in U such that (a) $\int_{U\setminus V} |M(\xi)|\, d\xi < \epsilon$; and (b) $s_x(x;\tau)$ becomes uniformly continuous in $(x_0 - \delta/2, x_0+\delta/2) \times V$. Then we estimate, when $x, x+h \in (x_0-\delta/2, x_0+\delta/2)$,

$$\begin{aligned}
&\left| h^{-1}\left[\int_U (s(x+h;\xi) - s(x;\xi))\, d\xi\right] - \int_U s_x(x;\xi)\, d\xi \right| \\
\le &\int_{U\setminus V} \left|s_x(x+\theta h;\xi) - s_x(x;\xi)\right| d\xi + \int_V \left|s_x(x+\theta h;\xi) - s_x(x;\xi)\right| d\xi \\
\le &\int_{U\setminus V} 2M(\xi)\, d\xi + \int_V \left|s_x(x+\theta h;\xi) - s_x(x;\xi)\right| d\xi \\
\le &2\epsilon + \int_V \left|s_x(x+\theta h;\xi) - s_x(x;\xi)\right| d\xi.
\end{aligned}$$

  Finally we use the uniform continuity of $s_x(x;\xi)$ over $(x_0-\delta/2, x_0+\delta/2)\times V$ to make $\int_V \left|s_x(x+\theta h;\xi) - s_x(x;\xi)\right| d\xi < \epsilon$ when $|h|$ is sufficiently small. We will provide two examples below to illustrate this.

Example A.1. We provide here a justifcation for (2.17). Set $s(x,t;\xi) = c(\xi)e^{ix\xi - \xi^2 t}$ for $(x,t) \in \mathbb{R}\times\mathbb{R}^+$, and $\xi \in \mathbb{R}$. $s(x,t;\xi)$ is a smooth function of $(x,t;\xi)$ in this domain. For any $(x,t)\in\mathbb{R}\times\mathbb{R}^+$, we can take $0 < t_0 < t < t_1$, and apply Lemma A.1 on $\mathbb{R}\times(t_0,t_1)\times\mathbb{R}$. If the integral were $\int_{-L}^{L}$ for some finite $L>0$, then we can directly apply (iii) in Lemma A.1. For this integral over $\mathbb{R}$, our attention in constructing an $M(\xi)$ as in (v) should be focused on the large $|\xi|$ region.

$$|s_t(x,t;\xi)| = |-\xi^2 c(\xi) e^{ix\xi-\xi^2 t}| = |\xi|^2 |c(\xi)| e^{-\xi^2 t}| \le |\xi|^2 |c(\xi)| e^{-\xi^2 t_0},$$

if $t\in(t_0,t_1)$. Under fairly flexible assumptions on $|c(\xi)|$, $|\xi||c(\xi)|e^{-\xi^2 t_0}$ is integrable in ξ over $\mathbb{R}$. The key here is that $t_0>0$ can be fixed in advance depending on the given $t>0$, and the bound above applies uniformly in (t_0,t_1).

Example A.2. Here we provide a partial justification for **Exercise** 3.2.2: if f is bounded, then

$$u(x,t) = \int_0^t \int_{\mathbb{R}} K(x-y, t-\tau) f(y,\tau)\, dy d\tau \in C(\mathbb{R} \times \overline{\mathbb{R}^+})$$

and is continuously differentiable in x for $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. For any $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, we first prove the differentiability of $u(x,t)$ in x, and

$$u_x(x,t) = \int_0^t \int_{\mathbb{R}} K_x(x-y, t-\tau) f(y,\tau)\, dy d\tau,$$

and then use this integral representation to prove that $u_x(x,t)$ is continuous in $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. In proving the differentiability in x, we may regard $t > 0$ as a fixed parameter and let $s(x,t;\tau) = \int_{\mathbb{R}} K(x-y,t-\tau)f(y,\tau)\,dy$. Then we can apply (v) of Lemma A.1 to the integral $\int_{\mathbb{R}} K(x-y,t-\tau)f(y,\tau)\,dy$, as in the previous example, to prove that, for any $0 \le \tau < t$,

$$s_x(x,t;\tau) = \int_{\mathbb{R}} K_x(x-y,t-\tau) f(y,\tau)\, dy.$$

Now $u(x,t) = \int_0^t s(x,t;\tau)\,d\tau$. Since

$$K_x(x-y,t-\tau) = -\frac{x-y}{2\sqrt{4\pi}(t-\tau)^{3/2}} e^{-\frac{|x-y|^2}{4(t-\tau)}},$$

so if $|f(y,\tau)| \le M$ for all $(y,\tau) \in \mathbb{R} \times \mathbb{R}^+$, then

$$|s_x(x,t;\tau)| \le M \int_{\mathbb{R}} \frac{|x-y|}{2\sqrt{4\pi}(t-\tau)^{3/2}} e^{-\frac{|x-y|^2}{4(t-\tau)}}\, dy.$$

Making the change of variables $z = \frac{x-y}{2\sqrt{t-\tau}}$ in the above integral, we find

$$\int_{\mathbb{R}} \frac{|x-y|}{2\sqrt{4\pi}(t-\tau)^{3/2}} e^{-\frac{|x-y|^2}{4(t-\tau)}}\, dy = \int_{\mathbb{R}} \frac{|z|e^{-z^2}}{\sqrt{\pi(t-\tau)}}\, dz,$$

and

$$|s_x(x,t;\tau)| \le \frac{CM}{\sqrt{t-\tau}},$$

with $C = \int_{\mathbb{R}} \frac{|z|e^{-z^2}}{\sqrt{\pi}}\, dz < \infty$. Since $\int_0^t \frac{CM}{\sqrt{t-\tau}} d\tau < \infty$, we can apply (v) of Lemma A.1 again to conclude that $u_x(x,t) = \int_0^t s_x(x,t;\tau)\,d\tau$.

Next we show how to prove the continuity of $u(x,t)$ in $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. Since the domain of integration depends on t, and the integral at $\tau = t$ is an improper one, it's not easy to directly apply (iv). We will provide a direct proof, which illustrates how we handle such analysis.

Fix any $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, we will select $0 < \delta < t$, and for $t - \delta/2 < t' < t + \delta/2$, we estimate

$$\begin{aligned} \left|u(x,t) - u(x',t')\right| =& \left|\int_0^{t-\delta} \left[s(x,t;\tau) - s(x',t';\tau)\right]\, d\tau + \int_{t-\delta}^{t} s(x,t;\tau) - \int_{t-\delta}^{t'} s(x',t';\tau)\, d\tau\right| \\ \leq& \int_0^{t-\delta} \left|s(x,t;\tau) - s(x',t';\tau)\right| d\tau + \int_{t-\delta}^{t} \left|s(x,t;\tau)\right| d\tau + \int_{t-\delta}^{t'} \left|s(x',t';\tau)\right| d\tau. \end{aligned}$$

We can easily estimate that $|s(x,t;\tau)| \leq M$, so $\int_{t-\delta}^{t} |s(x,t;\tau)| d\tau \leq M\delta$, and

$$\int_{t-\delta}^{t'} |s(x',t';\tau)|\, d\tau \leq M(t' - t + \delta) \leq 2M\delta.$$

For any given $\epsilon > 0$, we first fix $0 < \delta < t$ such that $2M\delta < \epsilon$.

For $\int_0^{t-\delta} \left|s(x,t;\tau) - s(x',t';\tau)\right| d\tau$, we can apply (iv) of Lemma A.1 as follows. Take any $(x'_j, t'_j) \to (x,t)$, since $|s(x',t';\tau)| \leq M$ for any $t' > \tau$, to apply (iv) of Lemma A.1 it suffices to prove the continuity of $s(x',t';\tau)$ at (x,t) when $\tau \leq t - \delta$, as then it would imply $\lim_{j\to\infty} \left|s(x,t;\tau) - s(x'_j, t'_j;\tau)\right| \to 0$ verifying the remaining condition to apply (iv) of Lemma A.1.

Since $s(x',t';\tau) = \int_{\mathbb{R}} K(x'-y, t'-\tau) f(y,\tau)\, dy$,and

$$K(x'_j - y, t'_j - \tau) f(y,\tau) \to K(x-y, t-\tau) f(y,\tau)$$

as $j \to \infty$ for every $y \in \mathbb{R}$, it suffices to find a dominating function for $K(x'_j - y, t'_j - \tau) f(y,\tau)$. But when $|x - x'_j| \leq \delta/2$, $\tau \leq t - \delta$ and $t'_j \geq t - \delta/2$, we will have $t'_j - \tau \geq \delta/2$, so

$$\left|K(x'_j - y, t'_j - \tau) f(y,\tau)\right| \leq M \frac{e^{-\frac{|x'_j - y|^2}{2\delta}}}{\sqrt{2\pi\delta}} \leq M \frac{e^{-\frac{|x-y|^2 - \delta^2/2}{4\delta}}}{\sqrt{2\pi\delta}}$$

using $|x-y| \leq |x - x'_j| + |x'_j - y| \leq \delta/2 + |x'_j - y|$, so $|x-y|^2 \leq \delta^2/2 + 2|x'_j - y|^2$. Since the upper bound is an integrable function of y over $\mathbb{R}$, we are now in a position to apply (iv) of Lemma A.1 to conclude that $s(x',t';\tau)$ is continuous at (x,t) when $\tau \leq t - \delta$. A similar proof works for $u_x(x,t)$.

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