A Kazdan-Warner type identity for the $\sigma_k$ curvature

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The Weyl-Schouten tensor $A_g$ of a metric $g$ is defined to be

$$A_g = \frac{1}{n-2} \{Ric_g - \frac{Scal_g}{2(n-1)} g\}.$$

The $\sigma_k$ curvature of $g$ is defined to be the $k$-th elementary symmetric function of the eigenvalues of the 1-1 tensor $g^{-1} \circ A_g$. $\sigma_1$ of $g$ is simply a dimensional constant multiple of the scalar curvature of $g$. Since the thesis of Viaclovsky [V1] and the fundamental work of A. Chang, M. Gursky, and P. Yang [CGY1], there has been very intensive research and progress on an extensive list of geometrical and PDE problems involving the $\sigma_k$ curvature of a metric for $k > 1$, mostly involving a conformal change of metric, see the bibliography for an incomplete list of recent work in this area. Since the Weyl-Schouten tensor transforms in the following way under a conformal change of metric $g = e^{2w(x)} g_0$,

$$A_g = A_{g_0} - \left[ \nabla^2 w - dw \otimes dw + \frac{1}{2} |\nabla w|^2 g_0 \right],$$

the $\sigma_k$ curvature of $g$, when $k \geq 2$, is then expressed as a fully nonlinear expression involving $w$ and its derivatives up to order 2. Almost all analytical work involving the $\sigma_k$ curvature restricts attention to the so called admissible metrics, for which the $\sigma_k$ curvature, regarded as a differential operator on $w$ is, elliptic. And for this reason,

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it is natural to consider $\sigma_k^{1/k}$, not $\sigma_k$, to be the analytical object of study, as $\sigma_k^{1/k}$, regarded as a differential operator on $w$, is concave on the second derivatives of $w$, and the concavity property is crucial for applying the Evans-Krylov regularity theory. For this reason in the PDE analysis of solvability results involving the $\sigma_k$ curvature one is often led to imposing conditions on $\sigma_k^{1/k}$. However, as this note indicates, global geometric obstruction conditions are naturally in terms of $\sigma_k$, not $\sigma_k^{1/k}$.

For $k = 1$, Kazdan and Warner first noticed a global geometric obstruction for a function $K(x)$ on the round sphere $\mathbb{S}^n$ to be the scalar curvature of a conformal metric, expressed as

$$\int_{\mathbb{S}^n} \langle \nabla x_j, \nabla K \rangle \, d\text{vol}_g = 0, \quad \text{for } j = 1, \ldots, n + 1,$$

where $x_j$ are the coordinate functions on $\mathbb{S}^n$ from the standard embedding. Later this obstruction was extended to a general manifold involving general conformal killing vector field by Bourguignon and Ezin—note that $\nabla x_j$ generates conformal killing vector fields on $\mathbb{S}^n$. Schoen also derived local versions and used them in obtaining a priori estimates for metrics of constant scalar curvature. Here we obtain a natural generalization of these obstructions for the $\sigma_k$ curvatures on a compact Riemannian manifold.

**Theorem 1.** Let $(M^n, g)$ be a compact Riemannian manifold of dimension $n$, $\sigma_k(g^{-1} \circ A_g)$ be the $\sigma_k$ curvatures of $g$, for $k = 1, \ldots, n$, and $X$ be a conformal Killing vectorfield on $(M^n, g)$. When $k > 2$, also assume that $(M^n, g)$ is locally conformally flat. Then

$$\int_M \langle X, \nabla \sigma_k(g^{-1} \circ A_g) \rangle \, d\text{vol}_g = 0. \quad (1)$$

When $(M^n, g) = (\mathbb{S}^n, g_{\text{can}})$, the obstruction can be obtained by a more elementary variational means, as was used in our work [CHY1]. In [B] Bourguinon uses the construction of a closed 1-form on the infinite dimensional manifold consisting of metrics conformal to $(M^n, g)$, which is also invariant under the action of conformal diffeomorphisms of $(M^n, g)$, to prove a generalized Kazdan-Warner type identity involving the differential of the scalar curvature of $g$ by a conformal killing vector field. He also sketches a way to obtain generalized integral identities involving the higher degree Pfaffian polynomials of the curvature of $g$. That method in fact can be adapted to prove (1). However, the most direct and elementary proof for (1) is to adapt the argument of Bourguinon-Ezin [BE], Schoen [S1], [S2], using the following elementary algebraic and analytic properties of the $\sigma_k$ curvature.
Proposition 1 ([Rei]). Defining the $k$-th Newton transform of a 1-1 tensor $\Lambda$ by

$$T_k(\Lambda) = \sum_{j=0}^{n} (-1)^j \sigma_{k-j}(\Lambda) \Lambda^j,$$

we have

(i) $(k+1)\sigma_{k+1}(\Lambda) = T_k(\Lambda)^a_a \Lambda^b_b$.

(ii) $(n-k)\sigma_k(\Lambda) = T_k(\Lambda)^a_a$.

(iii) $T_k(\Lambda)^a_a = \sigma_k(\Lambda) \delta^a_b - T_{k-1}(\Lambda)^a_a \Lambda^b_b$.

(iv) $T_k(\Lambda)^a_b = \frac{1}{k!} \sum \delta \left( \frac{a_1 \cdots a_k a}{b_1 \cdots b_k b} \right) \Lambda_{a_1}^{b_1} \cdots \Lambda_{a_k}^{b_k}$, where $\delta \left( \frac{a_1 \cdots a_k a}{b_1 \cdots b_k b} \right)$ is the Kronecker symbol, which has a value $+1$ (respectively, $-1$) if $a_1 \cdots a_k a$ are distinct and $b_1 \cdots b_k b$ is an even (respectively, odd) permutation of $a_1 \cdots a_k a$; and a value $0$ otherwise.

In the following, we will interchangeably use the $\nabla$ operator with indices or indices after a comma sign, whichever is more convenient, to denote covariant differentiation in $g$. $T_k(g^{-1} \circ A_g)$ as a 1-1 tensor field on $M$ enjoys the following analytic property.

Proposition 2 ([Rei], [V1]). If $g$ is locally conformally flat, or if $k = 1$, then $\nabla_a T_k(g^{-1} \circ A_g)^b_a = 0$.

Remark. Viaclovsky’s proof of Proposition 2 in [V1] is imbedded in the middle of his computations for his Proposition 6, using Cartan’s formalism. For ease of reference, we point out a straightforward tensor calculus argument adapting that of Reilly in [Rei]. The case of $k = 1$ is a direct consequence of Bianchi’s identity, as $T_1(g^{-1} \circ A_g)^b_a = \frac{1}{n-2} \left( \frac{n}{2} \delta^b_a - R^b_a \right)$. For the case $k \geq 2$, note that in a locally conformally flat metric $g$, we have the following classic, yet not widely used property:

$$A_{abc} = A_{ac,b}. \quad (2)$$

(2) means that $A_{abc}$ as a $(0,3)$ tensor is totally symmetric.

Using (2) we can follow Reilly to conclude

$$\nabla_a T_k^b = \frac{1}{k!} \sum_{j=1}^{k} \sum_{a=1}^{n} \sum \delta \left( \frac{a_1 \cdots a_k a}{b_1 \cdots b_k b} \right) A_{a_1}^{b_1} \cdots \nabla_a A_{a_j}^{b_j} \cdots A_{a_k}^{b_k}.$$
The summation turns out to be 0, because $\nabla_a A^b_{aj}$ is symmetric in $a_j$ and $a$ by (2), while the Kronecker symbol is antisymmetric in $a_j$ and $a$.

(2) can be deduced from the definition and property of the Cotton tensor. More directly, we start from the once contracted Bianchi identity,

$$R_{ab,c} - R_{ac,b} = -R^d_{abc,d}.$$  

Substituting the terms in the left hand side in terms of $A_{ab}$, and the right hand side in terms of the representation

$$R_{abcd} = W_{abcd} + A_{ac}g_{bd} - A_{ad}g_{bc},$$

we arrive at

$$(n - 3) [A_{ab,c} - A_{ac,b}] = -W^d_{abc,d},$$

which proves (2) in the case $n > 3$. The case $n = 3$ requires a separate argument using the conformal flatness in dimension 3.

**Proof of Theorem 1.** Let $\phi_t$ denote the local one-parameter family of conformal diffeomorphisms of $(M, g)$ generated by $X$. Thus for some function $w_t$ we have

$$\phi^*_t(g) = e^{2w_t}g =: g_t.$$  

We have the following properties:

$$\sigma_k(g^{-1} \circ A_g) \circ \phi_t = \sigma_k(g_t^{-1} \circ A_{g_t}),$$  

(3)

$$\dot{w} := \frac{d}{dt}\bigg|_{t=0} w_t = div X/n = \nabla a X^a/n,$$  

(4)

$$\frac{d}{dt}\bigg|_{t=0} (g_t^{-1} \circ A_{g_t})^a_b = -\nabla^a_b \dot{w} - 2\dot{w} A^a_b,$$  

(5)

We next point out the following useful

**Fact.**

$$\left(1 - \frac{2k}{n}\right) \langle X, \nabla \sigma_k \rangle = -\nabla_a \left[T^a_b \nabla^b \left(\frac{div X}{n}\right) + \frac{2k}{n} \sigma_k X^a\right],$$  

(6)

where we use $T^a_b$ to denote the components of $T_{k-1}$ and have dropped the dependence of $\sigma_k$ on $g$. We will do the same in the following, whenever there is no possibility of confusion.
Proof of (6). Using (i), (3), (4), (5) and Proposition 2, we have

\[ \langle X, \nabla \sigma_k \rangle = T^b_a [-\nabla^a_b \dot{w} - 2\dot{w}A^a_b] \]

\[ = -T^b_a \nabla^a_b \dot{w} - 2k\sigma_k \dot{w} \]

\[ = -T^b_a \nabla^a_b \dot{w} - \frac{2k}{n} \sigma_k \nabla_b X^b \]

\[ = -T^b_a \nabla^a_b \dot{w} + \frac{2k}{n} \langle X, \nabla \sigma_k \rangle - \frac{2k}{n} \nabla_b (\sigma_k X^b) \]

\[ = -\nabla_b \left[ T^b_a \nabla^a_b \dot{w} + \frac{2k}{n} \sigma_k X^b \right] + \frac{2k}{n} \langle X, \nabla \sigma_k \rangle \]

(7)

(6) now follows directly. \qed

We now continue our proof of Theorem 1. When $2k \neq n$, it follows directly from integrating (6). When $2k = n$, we first prove that

\[ \int_M \langle X, \nabla \sigma_k (g^{-1} \circ A_g) \rangle \, d\text{vol}_g = -\int_M \sigma_k(g^{-1} \circ A_g)\text{div}_g X \, d\text{vol}_g \]

is independent of $g$ within its conformal class. Let $g_t = e^{2t\eta}g$ denote any conformal variation of $g$. Then, noting that

\[ \frac{d}{dt} \bigg|_{t=0} \text{div}_{g_t} X = n\langle X, \nabla \eta \rangle, \]

\[ 5 \]
we have, using again (4), (5), (6), and Proposition 2,

\[ \frac{d}{dt} \bigg|_{t=0} \int_M \sigma_k(g^{-1} \circ A_{gr}) \text{div}_{gr} X \, dvol_{gr} \]

\[ = \int_M \left\{ -T^a_b \cdot (\nabla^b_a \eta) \, \text{div}_g X + n \sigma_k \langle X, \nabla \eta \rangle \right\} \, dvol_g \]

\[ = \int_M \left\{ -T^a_b \nabla^b_a (\text{div}_g X) - n \nabla_a (\sigma_k X^a) \right\} \, dvol_g \]

\[ = 0, \]

where, in the last line, we have used (6) for the case $2k = n$.

Now we can complete the proof of Theorem 1 in the case $2k = n$ by following the argument of Bourguignon and Ezin: Either the connected component of the identity of the conformal group $C_0(M, g)$ is compact, then there is a metric $\hat{g}$ conformal to $g$ admitting $C_0(M, g)$ as a group of isometries, from which it follows that $\text{div}_g X \equiv 0$ and (1) therefore holds; or, $C_0(M, g)$ is non-compact, then by a theorem of Obata-Ferrand, $(M, g)$ is conformal to the standard sphere, in which case we can pick the canonical metric to compute the integral on the left of (1) and conclude that it is 0. \qed

Another property in the case of $k = 1$ or $k < n$ and $(M, g)$ locally conformally flat is

\[ \langle X, \nabla \sigma_k \rangle = -\frac{n}{n-k} \nabla_b \left[ (T^a_k)_a X^a \right] + \nabla_b \left[ \sigma_k X^b \right], \]

where $(T^a_k)_a$ stand for the components of $T_k$. This can also be used to prove Theorem 1 in such cases.

This is proved in the case that $(M, g)$ is locally conformally flat by starting with

\[ \nabla_a \sigma_k = T^b_c \nabla_a A^c_b = T^b_c \nabla_b A^c_a = \nabla_b \left[ T^b_c A^c_a \right], \] using (2) and Proposition 2.

Set $H^b_a = T^b_c A^c_a$. Then $H^a_a = k \sigma_k$, and $H^a_a = \sigma_k \delta^b_a - (T^b_k)_a$ by (iii) of Proposition 1. Define

\[ \hat{H}^b_a = H^b_a - \frac{H^c_a \delta^b_a}{n} = H^b_a - \frac{k \sigma_k \delta^b_a}{n}. \] Then $\hat{H}^a_a = 0$, and

\[ \nabla_b \hat{H}^b_a = \nabla_b H^b_a - \frac{k}{n} \nabla_a \sigma_k = \frac{n-k}{n} \nabla_a \sigma_k. \]
This last property can also be checked using Bianchi identity in the case $k = 1$ without the locally conformally flat condition. Thus

$$\langle X, \nabla \sigma_k \rangle = \frac{n}{n-k} X^a \nabla_b H^b_a$$

$$= \frac{n}{n-k} \left\{ \nabla_b \left[ X^a H^b_a \right] - \nabla_b X^a H^b_a \right\}$$

$$= \frac{n}{n-k} \left\{ \nabla_b \left[ X^a H^b_a \right] - \frac{1}{2} \left( \nabla_b X^a + \nabla_a X^b \right) H^b_a \right\}$$

$$= \frac{n}{n-k} \nabla_b \left[ X^a H^b_a \right]$$

$$= \frac{n}{n-k} \nabla_b \left[ X^a \left( \frac{n-k}{n} \sigma_k b_a - (T_k)_a \right) \right]$$

$$= -\frac{n}{n-k} \nabla_b \left[ X^a (T_k)_a \right] + \nabla_b \left[ \sigma_k X^b \right].$$

References


