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Some Remarks on the Geometry of a Class of Locally Conformally Flat Metrics

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In honor of Gang Tian on the occasion of his sixtieth birthday

Abstract. We prove that conformal metrics on domains of the round sphere, with non-negative constant Q-curvature, and non-negative scalar curvature, has positive mean curvature on the boundary of embedded balls (in the round metric). As a result, such metrics have certain reflection symmetries if the complement of the domain is contained in a lower-dimensional round sphere. We also prove that the development map of a locally conformally flat metric with non-positive Schouten tensor is an embedding.

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1. Introduction

An important question in conformal geometry is to understand under what conditions the development map of a locally conformally flat manifold into the standard sphere is an embedding. Another related question is to understand conditions on a domain $\Omega \subset \mathbb{S}^n$ under which a complete conformal metric exists on Ω with constant scalar curvature; also relevant is the uniqueness of such a metric, or possibly cataloging such metrics when uniqueness fails and Ω is some canonical domain in $\mathbb{S}^n$ such as $\mathbb{S}^n \setminus \mathbb{S}^l$ for some $0 \leq l < n$.

In [SY88] Schoen and Yau found some sufficient conditions for the development map of a locally conformally flat manifold to be an embedding. In particular they proved that the answer is positive if the Yamabe constant of (M, g) is non-negative. No positive result is known, as far as we are aware, in the case when the Yamabe constant of (M, g) is negative. In general the development map may not be an embedding, as shown by the elementary examples $\mathbb{S}^1_r \times \mathbb{H}^{n-1}$ where r

is the radius of the circle; moreover, these examples also show that the holonomy representation of the fundamental group of M under the development map may not be discrete. However, Kulkarni and Pinkall showed in [KP86] that for a closed conformally flat n-manifold M with infinite fundamental group, its development map $d : M \mapsto \mathbb{S}^n$ is a covering map, *iff* d is not surjective.

In [SY88], Schoen–Yau also proved that if a complete metric $g = v^{-2}(x)|dx|^2$ exists on a domain $\Omega \subset \mathbb{R}^n$ with its scalar curvature having a positive lower bound, then the Hausdorff dimension of $\partial\Omega$ has to be $\leq \frac{n-2}{2}$. In another direction, Schoen constructed in [S88] complete conformal metrics with scalar curvature 1 on $\mathbb{S}^n \setminus \Lambda$ when Λ is a certain subset of $\mathbb{S}^n$, including the case when it is any finite set with at least two points. Later Mazzeo and Parcard [MP96] [MP99] proved that if $\Omega \subset \mathbb{S}^n$ is a domain such that $\mathbb{S}^n \setminus \Omega$ consists a finite number of smooth submanifold of dimension $< \frac{n-2}{2}$, then one can find a complete metric $g = v^{-2}(x)|dx|^2$ on Ω with its scalar curvature identical to $+1$.

For the negative scalar curvature case, the works of Löwner–Nirenberg [LN75], Aviles [A82], and Veron [V81] imply that if $\Omega \subset \mathbb{S}^n$ admits a complete, conformal metric with negative constant scalar curvature, then the Hausdorff dimension of $\partial\Omega > \frac{n-2}{2}$. Löwner–Nirenberg [LN75] also proved that if $\Omega \subset \mathbb{S}^n$ is a domain with smooth boundary $\partial\Omega$ of dimension $> \frac{n-2}{2}$, then there exists a complete metric $g = v^{-2}(x)|dx|^2$ on Ω with its scalar curvature $= -1$; such a metric is unique when $\partial\Omega$ consists of hypersurfaces. This result was later generalized by D. Finn [F95] to the case of $\partial\Omega$ consisting of smooth submanifolds of dimension $> \frac{n-2}{2}$ and with boundary.

It is natural to ask whether some kind of additional curvature condition in the negative Yamabe constant case would force the development map to be an embedding as well, and whether further curvature conditions would improve the estimate on the Hausdorff dimension of $\partial\Omega$?

The additional curvature conditions are often imposed in terms of the σ_k or Q-curvature of a representative metric g. The σ_k-curvature, denoted as $\sigma_k(A_g)$, refers to the kth elementary symmetric functions of the eigenvalues of the 1-1 tensor derived from the Weyl–Schouten tensor A_g of the conformal metric g,

$$A_g = \frac{1}{n-2}\left\{\mathrm{Ric} - \frac{R}{2(n-1)}g\right\}.$$

Note that the $\sigma_1(A_g)$ curvature is simply the scalar curvature of g, up to a dimensional constant.

The condition involving the σ_k-curvature often assumes that the Weyl–Schouten tensor A_g is in the Γ_k^+ class for some $k > 1$, *i.e.*, the eigenvalues, $\lambda_1 \leq \cdots \leq \lambda_n$, of A_g at each x satisfy $\sigma_j(\lambda_1, \ldots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$. It is also natural to consider metrics whose Weyl–Schouten tensor A_g is in Γ_k^- class, namely, $(-1)^j\sigma_j(\lambda_1, \ldots, \lambda_n) > 0$ for all $j, 1 \leq j \leq k$. It is known that the operator $w \mapsto \sigma_k(A_{e^{2w}g_0})$ is elliptic when the Weyl–Schouten tensor of $g = e^{2w}g_0$ is in either Γ_k^+ or Γ_k^- class.

The Q-curvature of a metric g is defined through

$$Q_g = c_n|Rc_g|^2 + d_n|R_g|^2 - \frac{\Delta_g R_g}{2(n-1)},$$

with c_n and d_n being some dimensional constants: $c_n = -\frac{2}{(n-2)^2}$ and $d_n = \frac{n^3-4n^3+16n-16}{8(n-1)^2(n-2)^2}$. Note that Q_g involves 4th-order derivatives of the metric. The Q-curvatures of two conformally related metrics g and $g_u = u^{\frac{n+4}{n-4}}g$ (for $n \neq 4$) have the following relation through a 4th-order differential operator P_g, called the Panietz-type operator:

$$P_g(u) = \frac{n-4}{2} Q_{g_u} u^{\frac{n+4}{n-4}} \quad \text{for } n \neq 4,$$

where $P_g(u) = (-\Delta_g)^2 u + \operatorname{div}[(a_n R_g g + b_n \operatorname{Ric}_g)du] + \frac{n-4}{2} Q_g u$ for some dimensional constants a_n and b_n. For $n = 4$, $P_g(u) = (-\Delta_g)^2 u + \operatorname{div}[(\frac{2}{3}R_g - 2\operatorname{Ric}_g)du]$, and the relation between the Q-curvatures takes the following form:

$$P_g(w) + 2Q_g = 2Q_{e^{2w}g} e^{4w}.$$

P_g enjoys certain conformal covariance properties much like those of the conformal Laplace operator; see [CY97] for more details.

In [CHgY04], Chang, Hang, and Yang proved that *if* $\Omega \subset \mathbb{S}^n$ $(n \geq 5)$ *admits a complete, conformal metric* g *with*

$$\sigma_1(A_g) \geq c_1 > 0, \quad \sigma_2(A_g) \geq 0, \quad \text{and}$$
$$|R_g| + |\nabla_g R|_g \leq c_0, \tag{1.1}$$

then $\dim(\mathbb{S}^n \setminus \Omega) < \frac{n-4}{2}$. This has been generalized by M. Gonzáles [G04] to the case of $2 < k < n/2$: *if* $\Omega \subset \mathbb{S}^n$ *admits a complete, conformal metric* g *with*

$$\sigma_1(A_g) \geq c_1 > 0, \quad \sigma_2(A_g), \ \ldots, \ \sigma_k(A_g) \geq 0, \quad \text{and (1.1)},$$

then $\dim(\mathbb{S}^n \setminus \Omega) < \frac{n-2k}{2}$. See also the work of Guan, Lin and Wang [GLW04].

[CHgY04] also contains a result involving conditions on the Q-curvature: *if* $\Omega \subset \mathbb{S}^n$ $(n \geq 3)$ *admits a complete, conformal metric* g *with* $R_g \geq c_1 > 0$ *and* $Q_g \geq c_2 > 0$*, then* $\dim(S^n \setminus \Omega) < \frac{n-4}{2}$*. In particular, this means* $\Omega = S^n$ *when* $n \leq 4$*. If we replace* $Q_g \geq c_2 > 0$ *by* $Q_g \geq 0$*, then when* $n \geq 5$*, we have* $\dim(S^n \setminus \Omega) \leq \frac{n-4}{2}$.

There are earlier results involving the Q-curvature that are relevant to the discussion here: they concern the radial symmetry and classification of solutions to constant Q-curvature equations on $\mathbb{R}^n$. When $g = u^{\frac{4}{n-4}}|dx|^2$ on a domain in $\mathbb{R}^n$, $u > 0$, $n \neq 4$, the Q-curvature Q_g of g is computed through

$$(-\Delta)^2 u = \frac{n-4}{2} Q_g u^{\frac{n+4}{n-4}};$$

while on a domain in $\mathbb{R}^4$, if $g = e^{2w}|dx|^2$, then

$$(-\Delta)^2 w = 2Q_g e^{4w}.$$

In [CY97] Chang and Yang proved that any entire smooth solution $u(x)$ to

$$(-\Delta)^{\frac{n}{2}} u(x) = (n-1)! e^{nu(x)} \quad \text{on} \quad \mathbb{R}^n \tag{1.2}$$

with the asymptotic behavior as $x \to \infty$:

$$\begin{gathered} u(x) = \log \frac{2}{1+|x|^2} + w\left(\frac{x}{1+|x|^2}\right) \text{ for some} \\ \text{smooth function } w \text{ defined near } 0, \end{gathered} \tag{1.3}$$

must be rotationally symmetric with respect to some point in $\mathbb{R}^n$, and of the form

$$\log \frac{2\lambda}{1+\lambda^2|x-x_0|^2} \text{ for some } x_0 \in \mathbb{R}^n \text{ and constant } \lambda > 0. \tag{1.4}$$

In [L98] C.S. Lin obtained related results. For (1.2) in the case $n=4$, Lin obtained the same result as in [CY97] under an integral assumption

$$\int_{\mathbb{R}^4} e^{4u(y)} dy = \frac{8\pi^2}{3}. \tag{1.5}$$

Lin's result actually implies that any solution u to (1.2) for the $n=4$ case with $\int_{\mathbb{R}^4} e^{4u(y)} dy < \infty$ must satisfy $\int_{\mathbb{R}^4} e^{4u(y)} dy \leq \frac{8\pi^2}{3}$, with equality iff u is of the form (1.4).

Lin also obtained a related result for the positive constant Q-curvature equation on $\mathbb{R}^n$, $n > 4$,

$$\begin{cases} \Delta^2 u(x) = u^{\frac{n+4}{n-4}}(x), & x \in \mathbb{R}^n; \\ \quad u(x) > 0, & x \in \mathbb{R}^n. \end{cases} \tag{1.6}$$

His result implies that any solution to (1.6) must be rotationally symmetric with respect to some point $x_0 \in \mathbb{R}^n$, and of the form

$$u(x) = c_n \left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{n-4}{2}}, \tag{1.7}$$

for some $\lambda > 0$ and a dimensional constant c_n.

The proofs in both [CY97] and [L98] involve the method of moving planes. In [X00] X. Xu provided a proof for the rotational symmetry of solutions to (1.6) using the method of moving spheres.

Considering also that the canonical locally conformally flat metric on $\mathbb{S}^n \setminus \mathbb{S}^{n-k} \cong \mathbb{S}^{k-1} \times \mathbb{H}^{n-k+1}$, given via stereographic coordinates of $\mathbb{S}^n \setminus \mathbb{S}^{n-k}$ as

$$\begin{aligned} \frac{dx_1^2 + \cdots + dx_n^2}{x_1^2 + \cdots + x_k^2} &= \frac{d\rho^2 + \rho^2 d\omega_{\mathbb{S}^{k-1}}^2 + dx_{k+1}^2 + \cdots + dx_n^2}{\rho^2} \\ &= d\omega_{\mathbb{S}^{k-1}}^2 + \frac{d\rho^2 + dx_{k+1}^2 + \cdots + dx_n^2}{\rho^2} \end{aligned}$$

with $\rho^2 = x_1^2 + \cdots + x_k^2$, has its scalar curvature equal to $(n-1)(2k-n-2)$ and its Q-curvature equal to $8(2k-n)(2k-n-4)/n$, the sign of the Q-curvature alone is a poor indicator of how the metric behaves. Our first result, stated below, stems from this observation that it is natural to impose some additional condition involving the scalar curvature when considering global properties of solutions to the Q-curvature equation.

Theorem 1.1. *Let g be a conformal metric on $\Omega \subset \mathbb{S}^n$ such that*

$$Q_g \equiv 1 \text{ or } 0 \quad \text{in } \Omega, \tag{1.8}$$

and

$$R_g \geq 0 \quad \text{in } \Omega. \tag{1.9}$$

(i) *If $\mathbb{S}^n \setminus \Omega$ contains more than one point and g is complete on Ω, then for any ball $B \subset\subset \Omega$ in the canonical metric $g_{\mathbb{S}^n}$, the mean curvature of its boundary ∂B in metric g with respect to its inner normal is positive;*

(ii) *If $\mathbb{S}^n \setminus \Omega$ is empty or consists of one point, then g is the round metric on $\mathbb{S}^n$ in the case $Q_g \equiv 1$; and the flat metric on $\mathbb{S}^n \setminus \{\infty\} \sim \mathbb{R}^n$ in the case $Q_g \equiv 0$ and $\mathbb{S}^n \setminus \Omega = \{\infty\}$.*

A corollary of Theorem 1.1 is the following

Corollary 1.2. *Suppose that $\Gamma \subset \mathbb{S}^l$ for $l \leq \frac{n-2}{2}$ and contains more than one point. Then any complete, conformal metric g on $\mathbb{S}^n \setminus \Gamma$ satisfying* (1.8) *and* (1.9) *has to be symmetric with respect to rotations of $\mathbb{S}^n$ which leave $\mathbb{S}^l$ invariant.*

A second corollary of Theorem 1.1 is the following

Corollary 1.3. *Suppose that $g = u(x)^{4/(n-4)} g_{\mathbb{S}^n}$ is a conformal metric on $\Omega \subsetneqq \mathbb{S}^n$ such that* (1.8) *and* (1.9) *hold, and that g is a complete metric on Ω or $u(x) \to \infty$ as $x \to \partial\Omega$, then there exists a constant $C > 0$ such that $u(x)^{2/(n-4)} \leq C\delta(x, \partial\Omega)^{-1}$, where $\delta(x, \partial\Omega)$ is the distance from x to $\partial\Omega$ in the metric $g_{\mathbb{S}^n}$.*

Remark 1.4. Corollary 1.2 can be considered as extending the consideration in [CY97, L98, X00] to cases where the solutions are not defined on $\mathbb{S}^n$ or $\mathbb{R}^n$, but on some more general $\Omega \subsetneqq \mathbb{S}^n$. Note that the corresponding (classification) results on $\mathbb{S}^n$ or $\mathbb{R}^n$ in [CY97, L98, X00] hold without assuming (1.9), but assuming (1.5) only when $n = 4$. We will see below – Remark 1.7 and the last 2 paragraphs for the $n = 4$ case in the proof of Theorem1.1 in the next section – that, in the $n = 4$ context, (1.5) is equivalent to (1.9).

In [Sc88] Schoen proved a version of Theorem 1.1 and Corollary 1.2 for a conformal, complete metric on $\Omega \subsetneqq \mathbb{S}^n$ with non-negative constant scalar curvature using a moving spheres argument, and proved a version of Corollary 1.3 in the same setting using a blow up argument. There have been many similar symmetry results on entire solutions or entire solutions with one point deleted to the constant σ_k curvature equation in the positive Γ_k class, which are generalizations of the Yamabe equation; a partial list of work in this direction includes those of Viaclovsky [V00a][V00b], Chang, Gursky and Yang [CGY02b][CGY03], Li and Li [LL03][LL05], Guan, Lin and Wang [GLW04], Li[L06].

Remark 1.5. In Theorem 1.1 we use the sign convention for the mean curvature as in [Sc88], namely, the mean curvature of the boundary of round Euclidean balls with respect to their inner normals is positive, as is the mean curvature of the boundary of round balls in $\mathbb{S}^n$ when they are confined to a hemisphere – note that as soon as a round ball in $\mathbb{S}^n$ contains a hemisphere, the mean curvature of its boundary with respect to its inner normal becomes negative in our convention.

Since umbilicity is invariant under a conformal change of metric, and round balls are umbilic in the canonical metric, our theorem implies that all principal curvatures of ∂B in metric g are positive.

Remark 1.6. The canonical locally conformally flat metric on $\mathbb{S}^n \setminus \mathbb{S}^{n-k} \cong \mathbb{S}^{k-1} \times \mathbb{H}^{n-k+1}$, for appropriate range of k, provides examples of metrics satisfying the assumptions in Theorem 1 with $\Gamma = \mathbb{S}^{n-k}$. The existence of conformal metrics satisfying the assumptions in Theorem 1 for more general Γ is an interesting question, but will not be addressed here.

In a local conformal representation for $g(x) = u(x)^{\frac{4}{n-4}}|dx|^2$ when $n \neq 4$, we have

$$(n-4)R_g(x) = -4(n-1)u(x)^{-\frac{n}{n-4}}\left(\Delta u(x) + \frac{2}{n-4}\frac{|\nabla u(x)|^2}{u(x)}\right). \tag{1.10}$$

We see that the condition (1.9) for $n > 4$ implies that

$$\Delta u(x) \leq 0. \tag{1.11}$$

The analog of (1.10) when $n = 4$ and $g = e^{2w}|dx|^2$ is

$$R_g(x) = -6e^{-2w(x)}\left(\Delta w(x) + |\nabla w(x)|^2\right). \tag{1.12}$$

We remark that in Y. Li's joint work [LL05] with A. Li on the study of entire solutions to a class of conformally invariant PDEs, and later on in his study of local behavior near isolated singularities of such solutions in [L06], condition (1.11) was used. One important ingredient in [LL05, L06] is that they work with the equations of u as well as of its classical Kelvin transforms with respect to the spheres $\partial B(x_0, R)$:

$$u_{x_0,R}(x) := \frac{R^{n-2}}{|x-x_0|^{n-2}}u\left(x_0 + \frac{R^2(x-x_0)}{|x-x_0|^2}\right),$$

and that $\Delta u(x) \leq 0$ for $|x - x_0| < (\geq)R$ is equivalent to $\Delta u_{x_0,R}(x) \leq 0$ for $|x - x_0| > (\leq)R$. This comes from computing the scalar curvature of a conformal metric g in the set up: $g(x) = u(x)^{\frac{4}{n-2}}|dx|^2$

$$R_g(x) = -4\frac{n-1}{n-2}u(x)^{-\frac{n+2}{n-2}}\Delta u(x). \tag{1.13}$$

This is different from (1.10). Note that under the inversion in $\partial B(x_0, R) : x \mapsto x_0 + \frac{R^2(x-x_0)}{|x-x_0|^2}$, the same metric g is represented as

$$\begin{aligned} g(x_0 + \frac{R^2(x-x_0)}{|x-x_0|^2}) &= u(x_0 + \frac{R^2(x-x_0)}{|x-x_0|^2})^{\frac{4}{n-2}}\left(\frac{R}{|x-x_0|}\right)^4|dx|^2 \\ &= u_{x_0,R}(x)^{\frac{4}{n-2}}|dx|^2, \end{aligned}$$

so

$$R_g\left(x_0+\frac{R^2(x-x_0)}{|x-x_0|^2}\right)=-4\frac{n-1}{n-2}u_{x_0,R}(x)^{-\frac{n+2}{n-2}}\Delta u_{x_0,R}(x).$$

In dealing with the Q-curvature equations, the metric g is often represented as $g(x)=u(x)^{\frac{4}{n-4}}|dx|^2$ (when $n\neq 4$), so the corresponding transformation under the inversion in $\partial B(x_0,R)$ is

$$u_{Q;x_0,R}(x)=\frac{R^{n-4}}{|x-x_0|^{n-4}}u\left(x_0+\frac{R^2(x-x_0)}{|x-x_0|^2}\right),$$

which gives

$$u(y)^{\frac{4}{n-4}}|dy|^2\Big|_{y=x_0+\frac{R^2(x-x_0)}{|x-x_0|^2}}=u_{Q;x_0,R}(x)^{\frac{4}{n-4}}|dx|^2.$$

In this set up, $u_{Q;x_0,R}(x)$ would satisfy (1.6) if $u(x)$ does, although $\Delta u(x_0+\frac{R^2(x-x_0)}{|x-x_0|^2})\leq 0$ is not equivalent to $\Delta u_{Q;x_0,R}(x)\leq 0$. However, condition (1.9) is a geometric condition and would imply (1.11) (when $n>4$) for any of its local representation in the form above, namely, (1.9) would imply $\Delta u(x_0+\frac{R^2(x-x_0)}{|x-x_0|^2})\leq 0$ as well as $\Delta u_{Q;x_0,R}(x)\leq 0$ – this is essential for our argument; (1.11) itself is not a geometrically invariant condition.

These discussions have their analogs in the $n=4$ case, where we write $g(x)=e^{2w(x)}|dx|^2$, and under the inversion in $\partial B(x_0,R)$, the metric g is represented as $g(x_0+\frac{R^2(x-x_0)}{|x-x_0|^2})=e^{2w_{Q;x_0,R}(x)}|dx|^2$, where

$$w_{Q;x_0,R}(x)=w\left(x_0+\frac{R^2(x-x_0)}{|x-x_0|^2}\right)+\ln\left(\frac{R^2}{|x-x_0|^2}\right).$$

Remark 1.7. Theorem 1 implies that $\mathbb{S}^n\setminus\Omega$ cannot be a single point for the case of $Q_g=1$ and g complete on Ω. For, if that were the case, we could place that single point at ∞ so as to obtain an entire solution on $\mathbb{R}^n$, and the conclusion in (i) of Theorem 1 would imply that, at any $x\in\mathbb{R}^n$, for any unit vector e in $\mathbb{R}^n$, and for any $r>0$, $\nabla_e u(x)+\frac{n-4}{2r}u(x)\geq 0$ for the $n>4$ case – see the set up in the next section, and this would imply $\nabla_e u(x)=0$, which would imply that u is a constant in $\mathbb{R}^n$, but constants are not solutions to (1.2) or (1.6).

In [CY97, L98, X00] cited above, one technical step is to establish that entire solutions u to (1.2) under their respective assumptions and to (1.6) are superharmonic on $\mathbb{R}^n$. In the $n>4$ case these authors proved that all entire solutions to (1.6) are of the standard form (1.7). But in the $n=4$ case the superharmonicity of u is not enough to lead to the same classification; condition (1.5) is needed. In fact, Theorem 1.2 in [L98] Lin gives some properties of entire solutions to (1.2) for $n=4$ satisfying $\int_{\mathbb{R}^4}e^{4u(y)}dy<\frac{8\pi^2}{3}$; and in [CC01] Chang and Chen constructed such solutions. These solutions are not of the form (1.7), and from the perspective of Theorem 1, the conclusions of Theorem 1 does not hold on all Euclidean spheres for these solutions. This means that condition (1.9) is related to, but different from, the superharmonicity of a particular representation of the metric, that it is needed for the conclusions of Theorem 1, and that these solutions do not satisfy (1.9). These properties can be verified directly on large Euclidean spheres using the expansion (1.10) and (1.11) of [L98].

Remark 1.8. Corollary 1.3 follows from Theorem 1.1 as Schoen did in [Sc88] in the case of constant positive scalar curvature equation. The outline of the argument goes as follows. If the upper bound does not hold, then a sequence of *rescaled solutions* centered along a sequence of points approaching $\partial\Omega$ would converge to an entire solution on $\mathbb{R}^n$ to the same equation. The solutions of the latter equation are completely classified; they correspond to the round metric on $\mathbb{S}^n$, therefore their mean curvatures (and principal curvatures) along large Euclidean spheres become negative. But in the closure of any such a large Euclidean ball, this metric is the uniform limit of a sequence of metrics whose principal curvatures along its boundary sphere is positive by the version of Theorem 1.1. This would cause a contradiction; therefore Corollary 1.3 must hold. We will not supply a detailed proof of Corollary 1.3 here.

Remark 1.9. Although Schoen's result in [Sc88] corresponding to Theorem 1.1 was stated and proved for a constant *positive* scalar curvature metric on a domain $\Omega \subsetneqq \mathbb{S}^n$, an examination of the proof indicates that, as long as the three main ingredients for the moving plane/sphere arguments are valid, the same conclusion can be drawn, namely, the same conclusion as given in Theorem 1 continues to hold if the following three steps are still valid: (i). the initiation of the inequality between a solution in a half-space/ball enclosing its singular set and its reflected solution; (ii). the above inequality is a strict point wise inequality unless it becomes a point wise equality in the entire comparison domain; and (iii). the strict inequality continues to hold if the half-space/ball is moved in a small open neighborhood.

Both (i) and (iii) involve proving that the solution in a neighborhood of its singular set stays above its reflected solution (which is a smooth solution to the equation near the singular set) by a positive amount – we did this here by using the maximum principle for superharmonic functions in a domain with a boundary component having zero Newtonian capacity, without imposing an explicit growth condition of the solution toward its singular set.

Both (ii) and (iii) involve using the strong maximum principle and the Hopf boundary lemma for the difference between the solution and its reflected solution, when this difference is assumed to be non-negative. But this part works for solutions to the constant scalar curvature equation, even if the constant is non-positive; in fact, it works even for the constant σ_k curvature equation, as long as the equation is elliptic. (i) and (iii) can be established if we assume that the conformal factor tends to ∞ uniformly upon approaching the boundary of its domain. We thus have

Theorem 1.10. *Let g be a conformal metric on $\Omega \subsetneqq \mathbb{S}^n$ such that* (a) $\sigma_k(A_g) =$ *a constant in Ω,* (b) $A_g \in \Gamma_k^+$ *(or Γ_k^- respectively) pointwise in Ω, and* (c) *if we write $g = e^{2w} g_{\mathbb{S}^n}$, then $w \to \infty$ uniformly upon approaching $\partial\Omega$. Then for any ball*

$B \subset\subset \Omega$ in the canonical metric $g_{\mathbb{S}^n}$, the mean curvature of its boundary ∂B in metric g with respect to its inner normal is positive.

Theorem 1.10 essentially appeared in earlier work, maybe not in such explicit formulation, see, for example, estimate (27) in [LL05]. Some computational sketches will be provided in the next section to illustrate the implementation of the argument outlined in the previous remark.

Remark 1.11. Our background discussion mentioned results which construct conformal metrics satisfying the assumptions in Theorem 1.10 for the $k = 1$ case. The construction of such conformal metrics for $k > 1$ and more general Ω (subject to the constraints on the dimension of $\mathbb{S}^n \setminus \Omega$) is an interesting question. In a recent work [GLN18] González, Li, and Nguyen construct viscosity solutions to a class of conformally invariant equations, which include the equations in (a) of Theorem 1.10, such that these solutions are in Γ_k^- when they solve the equations in (a), and these solutions satisfy (c) – under appropriate dimensional constraints on $\partial\Omega$. Maximum principle, the key tool in proving Theorems 1.1 and 1.10, is valid for viscosity solutions, see [LNW18]; so Theorems 1.10 applies to solutions in [GLN18].

Our next result provides a criterion for the development map of a locally conformally flat manifold to be an embedding in the negative Yamabe constant case.

Theorem 1.12. *Let (M, g) be a complete, locally conformally flat manifold, and $F : (M, g) \mapsto (\mathbb{S}^n, g_{\mathbb{S}^n})$ be a conformal immersion. If the Schouten tensor A_g of some metric in the conformal class of g is non-positive point wise on M, then F is an imbedding.*

Based on the following algebraic property that $\sigma_1(A_g) \leq 0$ and

$$\sigma_2(A_g) \geq \frac{(n-2)}{2(n-1)}(\sigma_1(A_g))^2 \tag{1.14}$$

imply $A_g \leq 0$, we have

Corollary 1.13. *If (M, g) is a complete, locally conformally flat manifold, and satisfies $\sigma_1(A_g) \leq 0$ and (1.14), and $F : (M, g) \mapsto (\mathbb{S}^n, g_{\mathbb{S}^n})$ is a conformal immersion, then F is an imbedding.*

Remark 1.14. Based on the following relation between the Schouten tensor A_g and the Einstein tensor E_g

$$A_g = \frac{E_g}{n-2} + \frac{R_g}{2n(n-1)}g = \frac{E_g}{n-2} + \frac{\sigma_1(A_g)}{n}g, \tag{1.15}$$

we have

$$2\sigma_2(A_g) = \frac{n-1}{n}(\sigma_1(A_g))^2 - \frac{||E_g||^2}{(n-2)^2}, \tag{1.16}$$

where $||E_g||$ is the metric norm of E with respect to g.

It then follows that condition (1.14) is a kind of pinching condition, as it is equivalent to

$$\frac{||E_g||^2}{(n-2)^2} \le \frac{(\sigma_1(A_g))^2}{(n-1)n}, \tag{1.17}$$

using (1.16). (1.14) is also equivalent to

$$(n-1)||A_g||^2 \le (\sigma_1(A_g))^2 \tag{1.18}$$

Remark 1.15. Theorem 1.12 and its corollary were obtained in the early 2000's, and were lectured by the second author in several seminar talks, including the fall 2003 CUNY Graduate Center Differential Geometry and Analysis Seminar.

When the condition $A_g \le 0$ is not satisfied, F may not be an embedding, as shown by the canonical locally conformally flat metric on $\mathbb{S}^1_r \times \mathbb{H}^{n-1}$, whose Schouten tensor is $\mathrm{diag}(\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2})$.

We will provide a proof of Theorem 1.1 and Corollary 1.2 in Section 2. The proof of Theorem 1.1 will follow the outlines as was done in [CY97], [L98], and [X00]. We will only sketch the main steps of the proof, in particular, indicate how to handle the behavior of the solution near $\partial\Omega$. Some computational sketches will also be provided for a proof of Theorem 1.10.

We will provide a proof of Theorem 1.12 in Section 3.

2. Proof of Theorems 1.1 and 1.10

Proof of Theorem 1.1. We first set up a stereographic coordinate for proving Theorem 1.1. Let $B \subset\subset \Omega$ be as given in Theorem 1.1. We can choose a stereographic coordinate such that B is mapped onto $\{x \in \mathbb{R}^n : x_1 < \lambda_0\}$ – this amounts to choosing coordinate such that the north pole lies on ∂B, and is equivalent to working with an appropriately transformed $u_{Q;x_0,R}$ in place of u. Define $\Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 > \lambda\}$, and $T_\lambda = \partial(\Sigma_\lambda) = \{x \in \mathbb{R}^n : x_1 = \lambda\}$. Let Γ be the image of $\mathbb{S}^n \setminus \Omega$ under this stereographic map. Then Γ is a compact subset in Σ_{λ_0}. Define $\Sigma'_\lambda = \Sigma_\lambda \setminus \Gamma$. In this stereographic coordinate we can write

$$g(x) = u(x)^{\frac{4}{n-4}}|dx|^2 \quad \text{for } x \in \mathbb{R}^n \setminus \Gamma. \tag{2.1}$$

Here, we first provide the details for *the* $n > 4$ *case*; the modifications needed for the $n = 3, 4$ cases will be sketched at the end.

The statement that the mean curvature of ∂B in metric g with respect to its inner normal is positive in the $n > 4$ case is equivalent to

$$u_{x_1}(x) > 0 \text{ for all } x \in T_{\lambda_0}. \tag{2.2}$$

Remark 2.1. If one represents B by a Euclidean ball $B(x_0, r)$ with x_0 as center and $r > 0$ as radius, then the statement that the mean curvature of ∂B at $x \in \partial B$ in metric g with respect to its inner normal is positive (when $n > 4$) is equivalent to

$$\nabla_\theta u(x) + \frac{n-4}{2r}u(x) := \frac{\partial u(x_0 + r\theta)}{\partial r} + \frac{n-4}{2r}u(x) > 0 \quad \text{for } x = x_0 + r\theta, \theta \in \mathbb{S}^{n-1}, \tag{2.3}$$

as the mean curvature in metric $g = u(x)^{4/(n-4)}|dx|^2$ at a point x on ∂B is given by

$$\frac{2u(x)^{\frac{2-n}{n-4}}}{n-4}\left[\nabla_\theta u(x) + \frac{n-4}{2r}u(x)\right].$$

It follows from this set up that (2.3) implies gradient estimate for u, for, if $\nabla u(x) = |\nabla u(x)|e$ for some $e \in \mathbb{S}^{n-1}$, then, with $x_0 = x + \frac{\delta(x,\Gamma)}{2}e$, $B_{\frac{\delta(x,\Gamma)}{2}}(x_0) \subset \Omega$, and $x \in \partial B_{\frac{\delta(x,\Gamma)}{2}}(x_0)$, thus (2.3) at x implies that $|\nabla u(x)| = -\nabla_{-e}u(x) < \frac{n-4}{\delta(x,\Gamma)}u(x)$. Y. Li and his collaborators also used estimates like (2.3) (see, for example [LL05] and [LN14]), or rather an inequality of the form $u(y) \geq u_{x_0,r}(y)$ (or $u(y) \geq u_{Q;x_0,r}(y)$ in our setting, which is used in deriving (2.3)), to obtain gradient estimates.

Remark 2.2. It follows from Theorem 2.7 in [SY88] that, in the situation of our Theorem 1.1, the Newtonian capacity $\mathrm{cap}(\mathbb{S}^n \setminus \Omega) = 0$, which implies that $\mathrm{cap}(\Gamma) = 0$. We will use this to deal with the behavior of $u(x)$ and that of $\Delta u(x)$ near Γ.

Let $v(x) = -\Delta u(x)$. Then based on our set up, we have

$$\begin{cases} \Delta v(x) = -u^{\frac{n+4}{n-4}}(x) \leq 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \\ \quad v(x) \geq 0 & \text{in } \mathbb{R}^n \setminus \Gamma, \end{cases} \tag{2.4}$$

(the $Q \equiv 0$ case can be handled by a straightforward modification) and that $u(x)$ has an expansion at $x = \infty$:

$$\begin{cases} u(x) = c_1|x|^{4-n} + \displaystyle\sum_{j=1}^n \frac{b_j x_j}{|x|^{n-2}} + O\left(\frac{1}{|x|^{n-2}}\right) \\ u_{x_i} = -(n-2)c_1 x_i |x|^{2-n} + O\left(\frac{1}{|x|^{n-2}}\right) \\ u_{x_i x_j}(x) = O\left(\frac{1}{|x|^{n-2}}\right) \end{cases} \tag{2.5}$$

for some constants $c_1 > 0$ and b_j's. It follows from this expansion that $v(x) = -\Delta u(x)$ has the following expansion at $x = \infty$:

$$\begin{cases} v(x) = c_0|x|^{2-n} + \displaystyle\sum_{j=1}^n \frac{a_j x_j}{|x|^{n}} + O\left(\frac{1}{|x|^{n}}\right) \\ v_{x_i} = -(n-2)c_0 x_i |x|^{-n} + O\left(\frac{1}{|x|^{n}}\right) \\ v_{x_i x_j}(x) = O\left(\frac{1}{|x|^{n}}\right) \end{cases} \tag{2.6}$$

for some constants $c_0 > 0$ and a_j's.

Set $x^\lambda = (2\lambda - x_1, x_2, \ldots, x_n)$, which is the reflection of x with respect to T_λ, and

$$w_\lambda(x) = u(x) - u(x^\lambda) \quad \text{for } x \in \Sigma_\lambda^{'}.$$

We will prove using the moving planes method that, when g cannot be extended as a smooth metric across Γ,

$$u(x) - u(x^\lambda) > 0 \text{ and } v(x) - v(x^\lambda) > 0, \text{ for all } x \in \Sigma_\lambda' \text{ and } \lambda \le \lambda_0. \tag{2.7}$$

It would then follow from (2.7) that

$$u_{x_1}(x) \ge 0 \text{ and } \partial_{x_1}(\Delta u(x)) \le 0, \text{ for any } x \text{ with } x_1 \le \lambda_0, \tag{2.8}$$

which, together with the strong maximum principle applied to $u(x) - u(x^\lambda)$ and $v(x) - v(x^\lambda)$, would conclude our proof.

In our setting it is impossible for $v(x) \equiv 0$ on $\mathbb{R}^n \setminus \Gamma$ due to (2.6). Then it follows from (2.4) and the strong maximum principle that $v(x) > 0$ in $\mathbb{R}^n \setminus \Gamma$.

We may suppose that $\Gamma \subset B(0, R_0)$ for some $R_0 > 0$. Now for any $R \ge R_0$, since $v > 0$ in $\overline{B(0,R)} \setminus \Gamma$, there exists $\delta > 0$ depending on R such that $v(x) \ge \delta$ for all $x \in \partial B(0, R)$. It now follows, using $\operatorname{cap}(\Gamma) = 0$ and (2.4), that

$$v(x) \ge \delta \quad \text{ for all } x \in B(0,R) \setminus \Gamma. \tag{2.9}$$

A reference for this kind of extended maximum principle is [L72, Chap. III, Thm. 3.4]. A formulation of this kind extended maximum principle in our setting is

Lemma 2.3. *Suppose that* (i) *Ω is a bounded domain in $\mathbb{R}^n$ and that $\Gamma \subset \Omega$ has capacity* 0*,* (ii) *v is superharmonic in $\Omega \setminus \Gamma$, and* (iii) *v is bounded below in $\Omega \setminus \Gamma$, and there exists M such that for any $z \in \partial\Omega$, $\liminf_{x \in \Omega, x \to z} v(x) \ge M$. Then $v(x) \ge M$ in $\Omega \setminus \Gamma$.*

The expansion (2.6) of $v(x)$ at ∞ and Lemma 2.3 in [CGS89] implies that

$$\begin{cases} \text{there exists } \lambda_1 \le \lambda_0 \text{ and } R_1 \ge R_0 \text{ such that} \\ v(x) > v(x^\lambda) \quad \text{for all } x \in \Sigma_\lambda' \text{ with } |x| \ge R_1, \text{ and } \lambda \le \lambda_1. \end{cases} \tag{2.10}$$

Then using (2.9) and the expansion (2.6) of $v(x)$ at ∞, we conclude that there exists $\lambda_2 \le \lambda_1$ such that

$$v(x) > v(x^\lambda) \quad \text{for all } x \in \Sigma_\lambda', \ \lambda \le \lambda_2. \tag{2.11}$$

Next, $w_\lambda(x)$ satisfies

$$\Delta w_\lambda(x) = v(x^\lambda) - v(x) \le 0 \quad \text{for all } x \in \Sigma_\lambda', \tag{2.12}$$

and $\lambda \le \lambda_2$. The expansion (2.5) of $u(x)$ at ∞ implies that

$$w_\lambda(x) \to 0 \quad \text{ as } x \to \infty. \tag{2.13}$$

Using (2.12), (2.13), $w_\lambda(x) = 0$ for all $x \in T_\lambda$, and the observation that $w_\lambda(x) = u(x) - u(x^\lambda) \ge -u(x^\lambda)$ is bounded below in a neighborhood of Γ and the information that $\operatorname{cap}(\Gamma) = 0$, we conclude that $w_\lambda(x) \ge 0$ for all $x \in \Sigma_\lambda'$, $\lambda \le \lambda_2$.

In the situation of (i), the completeness assumption on g and $\Omega \ne \mathbb{S}^n$ imply that $w_\lambda(x)$ cannot be $\equiv 0$, so with the strong maximum principle, we conclude that

$$w_\lambda(x) > 0 \quad \text{for all } x \in \Sigma_\lambda', \tag{2.14}$$

and $\lambda \le \lambda_2$.

We now define

$$\lambda_* = \sup\{\lambda \leq \lambda_0 : v(x^\mu) < v(x) \quad \text{for all } x \in \Sigma_\mu^{'}, \text{ and all } \mu \leq \lambda,\}$$

and proceed to prove that $\lambda_* = \lambda_0$.

By continuity (together with strong maximum principle and completeness of g), (2.12) and (2.14) continue to hold for λ_* replacing λ. We now have, using (2.14) for λ_* replacing λ, that

$$\Delta\left[v(x^{\lambda_*}) - v(x)\right] = u^{\frac{n+4}{n-4}}(x) - u^{\frac{n+4}{n-4}}(x^{\lambda_*}) \geq 0 \quad \text{for all } x \in \Sigma_{\lambda_*}^{'}. \tag{2.15}$$

$v(x^{\lambda_*}) - v(x) \leq 0$ for all $x \in \Sigma_{\lambda_*}^{'}$. Now strong maximum principle, (2.14) and (2.15) imply that $v(x^{\lambda_*}) - v(x) < 0$ for all $x \in \Sigma_{\lambda_*}^{'}$ – the $Q \equiv 0$ case would need a modified argument to rule out $v(x^{\lambda_*}) - v(x) \equiv 0$ using the Liouville theorem on the harmonic function $v(x)$ and (2.6). Furthermore, using $\text{cap}(\Gamma) = 0$, there exists some $\delta_* > 0$ such that

$$v(x^{\lambda_*}) - v(x) \leq -\delta_* \quad \text{for } x \text{ in a neighborhood of } \Gamma.$$

This, together with (2.15) and Lemma 2.4 in [CGS89], implies that $\lambda_* = \lambda_0$, and concludes the case for (i).

In the $\Omega = \mathbb{S}^n$ subclass of (ii), the set up in the proof of (i) is used to prove, in a more standard fashion as in [CY97], that $u(x)$ is rotationally symmetric about some point; then in the $Q \equiv 1$ case the argument in [L98] proves that $u(x)$ is of the standard form; while in the $Q \equiv 0$ case standard properties on entire positive harmonic functions implies that $u(x)$ must be a positive constant, but the associated metric would not be a smooth metric over $\Omega = \mathbb{S}^n$, so this latter case cannot occur.

In the remaining case of (ii): $\Omega = \mathbb{S}^n \setminus \{\text{a point}\}$, the set up in the proof of (i) works identically, and proves that the solution is rotationally symmetric about the image point of ∞ under the inversion used in the set up. But the sphere with respect to which the inversion is done can be chosen arbitrarily, so the solution is shown to be rotationally symmetric about any point, therefore is a positive constant. This cannot happen in the $Q \equiv 1$ case, and in the $Q \equiv 0$ case leads to the conclusion that the metric is the flat one on $\Omega = \mathbb{S}^n \setminus \{\text{a point}\}$.

We now indicate the modifications needed for the $n = 3$ case. (2.2) turns into

$$u_{x_1} < 0 \quad \text{for all } x \in T_{\lambda_0}; \tag{2.16}$$

(2.3) turns into

$$\nabla_\theta u(x) - \frac{u(x)}{2r} < 0; \tag{2.17}$$

The condition $R_g \geq 0$ turns into

$$\Delta u(x) - \frac{2|\nabla u(x)|^2}{u(x)} \geq 0; \tag{2.18}$$

and the three-dimensional version of (1.6) for $Q = 2$ is

$$(-\Delta)^2 u = -u^{-7}, \quad x \in \Omega \subset \mathbb{R}^3. \tag{2.19}$$

Setting $\tilde{v}(x) = \Delta u(x)$, we find that under (2.18), $\tilde{v}(x) \geq 0$; and $\eta(x) := \tilde{v}(x) - \tilde{v}(x^\lambda)$ satisfies $\eta(x) \geq -\Delta u(x^\lambda)$, as well as $\Delta \eta(x) \leq 0$ whenever $u(x) \leq u(x^\lambda)$. The version of (2.7) that we need to establish in 3 dimension is

$$u(x) - u(x^\lambda) < 0 \text{ and } \eta(x) = \tilde{v}(x) - \tilde{v}(x^\lambda) > 0 \quad \text{for all } x \in \Sigma_\lambda^{'} \text{ and } \lambda \leq \lambda_0. \quad (2.20)$$

Given (2.19) and the information on η above, (2.20) is established in almost identical way as in the $n > 4$ case.

For the $n = 4$ case, we use $g(x) = e^{2w(x)}|dx|^2$; (2.18) is replaced by $\Delta w(x) + |\nabla w(x)|^2 \leq 0$; (2.3) is replaced by $\frac{\partial w}{\partial r} + \frac{1}{r} \geq 0$; in place of (2.5), we have a similar expansion for $e^{w(x)}$ at ∞ (in an appropriately chosen stereographic coordinate) whose leading order term is $2|x|^{-2}$, or equivalently, an expansion for $w(x)$ whose leading order term is $-2 \ln |x|$ – the expansions for $w_{x_j}(x)$ and $\Delta w(x)$ come as consequences of the expansion for $w(x)$. Our objective in this set up is still to establish (2.7) with $w(x)$ replacing $u(x)$ there. (2.9) is established in the same way for $v(x) = -\Delta w(x)$, as we still have $\Delta w(x) \leq 0$ (< 0 in fact) based on $R_g \geq 0$, and $\Delta v(x) \leq 0$ in $\mathbb{R}^4 \setminus \Gamma$. The analog of (2.14) we need is $w(x) - w(x^\lambda) > 0$ for $x \in \Sigma'_\lambda$ and all $\lambda \leq \lambda_0$, and one key ingredient in proving this is a lower bound for $w(x)$ in a neighborhood of Γ. This is done using

$$\Delta e^{w(x)} = e^{w(x)} \left[\Delta w(x) + |\nabla w(x)|^2 \right] \leq 0 \quad \text{for } x \in \mathbb{R}^4 \setminus \Gamma,$$

from which it follows from the extended maximum principle applied to $e^{w(x)}$ over $B_R \setminus \Gamma$ for a fixed, sufficiently large $R > 0$ that $e^{w(x)}$ has a positive lower bound in $B_R \setminus \Gamma$, which then implies a lower bound for $w(x)$ in $B_R \setminus \Gamma$. These modifications suffice to complete a proof for the $n = 4$ case.

For the $n = 4$ and $\Gamma = \{\text{one point}\}$ case, we can arrange coordinates such that $\Gamma = \{0\}$. The argument in the above paragraph applies, except that it is possible that $w(x) - w(x^\lambda) \equiv 0$ for some λ – in fact, this will always happen. Then it's easy to see that g must be the round metric, and as a consequence, (1.5) holds. If (1.5) is assumed in place of (1.9), then it follows from [L98] that g must be the round metric, and as a consequence, (1.9) holds. Thus in this context, (1.9) and (1.5) are equivalent. □

Remark 2.4. An examination of the proof shows three crucial ingredients for completing the proof for Theorem 1:

(i) (2.9) for initiating of the relation that $v(x) - v(x^\lambda) = -\Delta u(x) + \Delta u(x^\lambda) \geq 0$ for $x \in \Sigma'_\lambda$ and $\lambda \leq \lambda_1$ for $|\lambda_1|$ large (for the $n > 4$ cases; the $n = 3, 4$ cases can be formulated appropriately);

(ii) $u(x) - u(x^\lambda) \equiv 0$ cannot happen in Σ'_λ; and

(iii) once $u(x) - u(x^\lambda) \geq 0$ in Σ'_λ is established, there exists $\delta > 0$ such that

$$u(x) - u(x^\lambda) > \delta \text{ and } -\Delta u(x) + \Delta u(x^\lambda) \geq \delta \text{ in a neighborhood of } \Gamma.$$

(2.9) is proved using the equation for $v(x) = -\Delta u(x)$, the property $v(x) \geq 0$, which follows from $R_g \geq 0$, and $cap(\Gamma) = 0$; (iii) also relies on $R_g \geq 0$ and $cap(\Gamma) = 0$ crucially; while (ii) relies on the assumption that g is a complete metric on $\Omega = \mathbb{S}^n \setminus \Gamma$ – this assumption, together with $R_g \geq 0$ (and locally conformal flatness of g), implies $cap(\Gamma) = 0$, based on [SY88]. Based on this examination, the assumption in Theorem 1 that g is a complete metric on $\mathbb{S}^n \setminus \Gamma$ can be replaced by the assumption that g cannot be extended as a smooth metric over Γ and that $cap(\Gamma) = 0$.

Remark 2.5. Here is another illustration why the assumption that $R_g \geq 0$ cannot be dropped: in the case of $Q \equiv 0$ on $\mathbb{S}^n \setminus \{\text{a point}\}$, which we identify as $\mathbb{R}^n$, we would be studying positive solutions $u(x)$ on $\mathbb{R}^n$ to $\Delta^2 u(x) = 0$. A simple argument using the Green's formula to $\Delta u(x)$:

$$\begin{aligned}\Delta u(x) &= \frac{1}{|B_r(x)|}\int_{B_r(x)} \Delta u(y)dy \\ &= \frac{1}{|B_r(x)|}\int_{\partial B_r(x)} \frac{\partial u(y)}{\partial \nu(y)} d\sigma(y) = \frac{n}{r}\partial_r\left(\int_{\mathbb{S}^{n-1}} u(x+r\omega)d\omega\right),\end{aligned}$$

and $u > 0$ on $\mathbb{R}^n$ shows that $\Delta u(x) \geq 0$. This then makes $\Delta u(x)$ a non-negative entire harmonic function, so $\Delta u(x) = c$ for some non-negative constant c. $u(x) = u_0 + \sum_{j=1}^n a_j x_j^2$, for $u_0 > 0$ and appropriately chosen $a_j \geq 0$, are positive solutions. These solutions have reflection symmetries, but do not have rotational symmetry unless a_j's are all equal; and, in any case, do not satisfy the conclusions of Theorem 1.1 unless a_j's are all 0.

Unless $c = 0$, these solutions do not correspond to metrics with $R_g \geq 0$. If we were to follow the set up in the proof of (i) of Theorem 1.1, we would work with

$$u_{Q;0,1}(x) = |x|^{4-n} u(\frac{x}{|x|^2}) = u_0|x|^{4-n} + |x|^{-n}\sum_{j=1}^n a_j x_j^2.$$

But $\Delta u_{Q;0,1}(x)$ may become unbounded near $x = 0$ when $a_j \neq a_k$ for some $j \neq k$. This would prevent an estimate like (2.9) for $v(x) := -\Delta u_{Q;0,1}(x)$, which is needed for the initiation of step (i) alluded to in the previous remark.

Proof of Corollary 1.2. It suffices to prove that, when $\mathbb{S}^l \setminus \{\infty\}$ is represented via a stereographic projection as $\mathbb{R}^l = \{x \in \mathbb{R}^n : x_{l+1} = \cdots = x_n = 0\}$, and for any $x \in \mathbb{R}^n \setminus \mathbb{R}^l$, and for any (unit) vector $e = (0, \ldots, 0, e_{l+1}, \ldots, e_n) \perp x$, we have $\nabla_e u(x) = 0$ – this set up would require $n - l \geq 2$, which we have from $l \leq (n-2)/2$. This would imply that, in this set up, $u = u(x_1, \ldots, x_n)$ depends on $x_{l+1}, \ldots, x_n$ only through $\sqrt{x_{l+1}^2 + \cdots + x_n^2}$.

For any $r > 0$, we see that $B(x - re, r) \subset\subset \mathbb{R}^n \setminus \mathbb{R}^l$, so the conclusion of Theorem 1 is valid on $\partial B(x - re, r)$. In particular, for the $n > 4$ case and at $x \in \partial B(x - re, r)$, we have, by (2.3)

$$\nabla_e u(x) + \frac{n-4}{2r} u(x) > 0. \tag{2.21}$$

Since we can take $r > 0$ arbitrarily large, we conclude that $\nabla_e u(x) \geq 0$. Repeating this argument with $-e$ replacing e, we obtain $\nabla_{-e} u(x) \geq 0$, and therefore conclude that $\nabla_e u(x) = 0$. The $n = 3, 4$ cases need only minor modifications. $\square$

Sketch of proof of Theorem 1.10. Here we will express the metric g in the form of $e^{2w(x)}|dx|^2$, and express the equation in the form of $f(\lambda(A[w])) = 1$, where $A[w] = -\nabla^2 w + \nabla w \otimes \nabla w - \frac{|\nabla w|^2}{2} I$ denotes the matrix representing the Schouten tensor, and $\lambda(A[w])$ refers to the eigenvalues of $A[w]$. Again we have set up coordinates such that $\Gamma \subset \mathbb{R}^n$, and that $w(x)$ has an expansion at ∞ in the spirit of (2.5), but with $-2\ln|x|$ as the leading order term. $w(x^\lambda)$ satisfies the same equation.

To initiate the moving plane method, we need a positive lower bound for $e^{w(x)}$ near Γ. This is provided for by assumption (c). Then traditional method is used to establish $w(x) - w(x^\lambda) \geq 0$ for $x \in \Sigma'_\lambda$ for $\lambda \leq \lambda_1$ for some large $|\lambda_1|$. To carry through the moving plane method, namely, to establish the above inequality for all expected range of λ, we use the equations for $w(x)$ and $w(x^\lambda)$ to obtain a linear, second-order, elliptic equation for $w(x) - w(x^\lambda)$: $L[w(x) - w(x^\lambda)] = 0$ in Σ'_λ, thanks to assumption (b). Using assumption (c), $w(x) - w(x^\lambda) \geq 0$, and the strong maximum principle, we obtain $w(x) - w(x^\lambda) > 0$ – the version used here is for non-negative solutions, which can be derived from Lemma 3.4 in [GT], and does not require a condition on the sign of the coefficient of the zeroth-order term in L; an explicitly formulated version for such a setting appears, e.g., as Lemma 3.5 in [CY97]; it is for this reason that the argument for Theorem 1.10 does not distinguish between the solutions in Γ_k^+ class from those in the Γ_k^- class. Assumption (c) further implies that there exists $\delta > 0$ such that $w(x) - w(x^\lambda) > \delta$ in a neighborhood of Γ. This, the Hopf Lemma and Lemma 2.4 in [CGS89], imply that $w(x) - w(x^\lambda) > 0$ holds for all expected range of λ. $\square$

3. Proof of Theorem 1.12

Proof of Theorem 1.12 . We may assume that g itself satisfies that its Schouten tensor A_g is non-positive point wise on M. For any point $z_0 \in F(M) \subset \mathbb{S}^n$, using stereographic coordinates, there is a smooth function u on $\mathbb{S}^n$ such that $u > 0$ on $\mathbb{S}^n \setminus \{z_0\}$, $u(z_0) = 0$, and $u^{-2} g_0$ is flat. Writing $F^*(u^{-2} g_0) = v^{-2} g \stackrel{\text{def}}{=} \widehat{g}$ on $M \setminus F^{-1}(z_0)$, then $\widehat{g}$ is flat. Hence, on $M \setminus F^{-1}(z_0)$, $\widehat{E} = 0$, $\widehat{R} = 0$.

Under a (pointwise) conformal change of the metric g, $\widehat{g} = v^{-2} g$, the Einstein tensor and scalar curvature transform as follows.

$$\widehat{E} = E + \frac{n-2}{v}\left\{\nabla^2 v - \frac{\Delta v}{n} g\right\}, \tag{3.1}$$

$$\widehat{R} = v^2\left\{R + 2(n-1)\frac{\Delta v}{v} - n(n-1)\frac{|\nabla v|^2}{v^2}\right\}. \tag{3.2}$$

Thus in the situation here, we have, by (3.1) and (3.2),

$$E = -\frac{n-2}{v}\left\{\nabla^2 v - \frac{\Delta v}{n} g\right\}, \tag{3.3}$$

$$R = -(n-1)\left\{2\frac{\Delta v}{v} - n\frac{|\nabla v|^2}{v^2}\right\}. \tag{3.4}$$

It now follows that

$$A = -\frac{\nabla^2 v}{v} + \frac{|\nabla v|^2}{2v^2} g. \tag{3.5}$$

Under our assumption that $A \leq 0$, we therefore have

$$\nabla^2 v \geq \frac{|\nabla v|^2}{2v} g, \tag{3.6}$$

on $M \setminus F^{-1}(z_0)$. By a limiting argument, $v(\gamma(s))$ is a non-negative convex function along any geodesic (in metric g) $\gamma(s)$ on M.

If $P_0 \neq P_1 \in M$ are such that $F(P_0) = F(P_1)$, we set $z_0 = F(P_0) = F(P_1)$ and carry out the computations in the paragraph above. Since (M, g) is assumed to be complete, we may joint P_0 and P_1 by a geodesic (in metric g) $\gamma(s)$ parametrized over $s \in [0,1]$ with $\gamma(0) = P_0$ and $\gamma(1) = P_1$, then $v(\gamma(0)) = v(\gamma(1)) = 0$. Since $v > 0$ on $M \setminus F^{-1}(z_0)$, this would imply that $\gamma(s) \in F^{-1}(z_0)$ for all $s \in [0,1]$, using the convexity of v. But this is not possible, and this contradiction implies that F must be an imbedding. □

Proof of Corollary 1.13. We just need to establish the algebraic property that $\sigma_1(A_g) \leq 0$ and (1.14) imply $A_g \leq 0$. Since E is trace free, we have the sharp inequality

$$-\sqrt{\frac{n-1}{n}}||E_g||g \leq E \leq \sqrt{\frac{n-1}{n}}||E_g||g. \tag{3.7}$$

Thus, when (1.14) holds, we have (1.17), as remarked earlier, which then implies that

$$\frac{||E_g||}{(n-2)} \leq \frac{|\sigma_1(A_g)|}{\sqrt{n(n-1)}}. \tag{3.8}$$

It now follows from (3.8), (1.15) and (3.7) that $A_g \leq 0$. □

Note added to galley proof

After our submission was accepted in 2018 for publication in this volume, we became aware of the paper "Asymptotic symmetry and local behavior of solutions of higher order conformally invariant equations with isolated singularities" by Tianling Jin and Jingang Xiong (https://arxiv.org/abs/1901.01678), which, among other results, proves sharp blow up rates of solutions of higher order conformally invariant equations in a bounded domain with an isolated singularity, and the asymptotic radial symmetry of the solutions near the singularity. It also contains some recent relevant references.

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