Quasi-polynomials, partial symmetry, and metaplectic Whittaker functions

Vidya Venkateswaran (CCR-Princeton)

Based on https://arxiv.org/abs/2308.10844

Overview and outline

1. Whittaker functions
   - Metaplectic and non-metaplectic Whittaker functions
   - Brubaker-Buciumas-Bump-Gustafsson duality results and conjecture
   - Our result
   - Metaplectic Weyl group and Hecke algebra actions
2. Quasi-polynomial generalizations of Macdonald polynomials
   - Motivation and connections
   - (Anti-)symmetric quasi-polynomials, special elements
   - Partially (anti)-symmetric polynomials and duality results
3. Open questions

Metaplectic Whittaker functions: (minimal) setup

Focus of much work over past two decades by a number of people, including Brubaker, Buciumas, Bump, Chinta, Friedberg, Goldfeld, Gunnells, Gustafsson, Hoffstein, Licata, McNamara, Offen, Patnaik, Puskas ...

Metaplectic Whittaker functions: (minimal) setup

Focus of much work over past two decades by a number of people, including Brubaker, Buciumas, Bump, Chinta, Friedberg, Goldfeld, Gunnells, Gustafsson, Hoffstein, Licata, McNamara, Offen, Patnaik, Puskas ...

General context:

- F a local non-archimedean field with uniformizer ϖ
- parameter $v \in \mathbb{C}$
- $G = GL_r(F)$, $W = S_r$, rank $r-1$
- metaplectic parameter $n \in \mathbb{Z}_{\geq 1}$
- $\widetilde{G}$ is a particular n-fold metaplectic cover of G
- weight lattice $P = \mathbb{Z}^r \ni \lambda = (\lambda_1, \ldots, \lambda_r)$; $P^n = n \cdot P$
- dominant weights $P^+ = \{\lambda \in P : \lambda_i \geq \lambda_{i+1} \text{ for all } i\}$

We will focus on the GL_r-context, but objects in this talk have been defined and studied in arbitrary type.

Metaplectic Whittaker functions: (minimal) setup

There is a metaplectic spherical Whittaker function $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(x; g)$ that is of particular interest.

Some properties:

- p-parts of Weyl group multiple Dirichlet series
- for fixed g, $\widetilde{\mathcal{W}}(x; g) \in \mathbb{C}[P] = \mathbb{C}[x^\lambda : \lambda \in P]$
- determined by values at $g = \varpi^{\rho-\mu} = \mathrm{diag}(\varpi^{(\rho-\mu)_1}, \ldots, \varpi^{(\rho-\mu)_r})$
  - $\rho = (r-1, r-2, \ldots, 0)$, $\mu - \rho \in P^+$
- involves n^{th}-order Gauss sums $g_0, \ldots, g_{n-1}$
- Iwahori refinements $\widetilde{\phi}_w$ with $\widetilde{W} = \sum_{w \in W} \widetilde{\phi}_w$

Metaplectic Whittaker functions: (minimal) setup

There is a metaplectic spherical Whittaker function $\widetilde{\mathcal{W}} = \widetilde{\mathcal{W}}(x; g)$ that is of particular interest.

Some properties:

- p-parts of Weyl group multiple Dirichlet series
- for fixed g, $\widetilde{\mathcal{W}}(x; g) \in \mathbb{C}[P] = \mathbb{C}[x^\lambda : \lambda \in P]$
- determined by values at $g = \varpi^{\rho-\mu} = \mathrm{diag}(\varpi^{(\rho-\mu)_1}, \ldots, \varpi^{(\rho-\mu)_r})$
  - $\rho = (r-1, r-2, \ldots, 0)$, $\mu - \rho \in P^+$
- involves n^{th}-order Gauss sums $g_0, \ldots, g_{n-1}$
- Iwahori refinements $\widetilde{\phi}_w$ with $\widetilde{W} = \sum_{w \in W} \widetilde{\phi}_w$

Several different constructions / interpretations (number-theoretic, combinatorial, **algebraic**, lattice models) due to various subsets of previous list of authors.

This talk: new connections between metaplectic and non-metaplectic Whittaker functions.

Non-metaplectic Whittaker functions and MD pols

Special case $n = 1$:

- Spherical $\mathcal{W} = \sum_{w \in W} \phi_w \longleftrightarrow$ Anti-symmetric $E_\mu^-(\infty; v)$
- (Refinement) Iwahori $\phi_w \longleftrightarrow$ Nonsymmetric $E_{w\mu}(\infty; v)$

(Values at $\varpi^{\rho-\mu}$ in LHS)

Non-metaplectic Whittaker functions and MD pols

Special case $n = 1$:

- Spherical $\mathcal{W} = \sum_{w \in W} \phi_w \longleftrightarrow$ Anti-symmetric $E^-_\mu(\infty; v)$
- (Refinement) Iwahori $\phi_w \longleftrightarrow$ Nonsymmetric $E_{w\mu}(\infty; v)$

(Values at $\varpi^{\rho-\mu}$ in LHS)

BBBG recently introduced / studied *parahoric* Whittaker functions. Let $J \subseteq [1, r-1]$, $W_J = \langle s_j : j \in J \rangle$,
$w \in W^J =$ minimal length reps of W/W_J.

$$\text{Parahoric} \quad \psi^J_w = \sum_{u \in W_J} \phi_{wu},$$

$w = 1 \longleftrightarrow$ Macdonald polynomials with prescribed symmetry $E^J_\mu(\infty; v)$

Can be computed using Demazure-Whittaker operators

Metaplectic and non-metaplectic dualities

BBBG introduced / studied metaplectic component functions $\widetilde{\phi}^o_\theta$, where $\theta \in (\mathbb{Z}/n\mathbb{Z})^r = [0, n-1]^r$; these satisfy

$$\widetilde{\mathcal{W}} = \sum_{\theta \in (\mathbb{Z}/n\mathbb{Z})^r} \widetilde{\phi}^o_\theta, \qquad x^{\rho-\theta} \cdot \widetilde{\phi}^o_\theta(x; \varpi^{\rho-\mu}) \in \mathbb{C}[P^n] \subset \mathbb{C}[P]$$

Metaplectic and non-metaplectic dualities

BBBG introduced / studied metaplectic component functions $\widetilde{\phi}^o_\theta$, where $\theta \in (\mathbb{Z}/n\mathbb{Z})^r = [0, n-1]^r$; these satisfy

$$\widetilde{\mathcal{W}} = \sum_{\theta \in (\mathbb{Z}/n\mathbb{Z})^r} \widetilde{\phi}^o_\theta, \qquad x^{\rho-\theta} \cdot \widetilde{\phi}^o_\theta(x; \varpi^{\rho-\mu}) \in \mathbb{C}[P^n] \subset \mathbb{C}[P]$$

BBBG prove the following "duality" theorems:

- θ has all parts distinct: $\widetilde{\phi}^o_\theta(x; \varpi^{\rho-\mu}) \longleftrightarrow \phi_w(x^n; \varpi^{-\lambda} w')$
- θ has all parts equal: $\widetilde{\phi}^o_\theta(x; \varpi^{\rho-\mu}) \longleftrightarrow \mathcal{W}(x^n; \varpi^{-\lambda})$

if $\mu \mod n$ is a perm of θ; otherwise vanishes. Here $\lfloor \frac{\mu}{n} \rfloor = \lambda + \rho$; w' det'd from μ; w det'd from θ

Conjecture (BBBG '21)

For arbitrary θ, $\widetilde{\phi}^o_\theta(\mathbf{y}; \varpi^{\rho-\mu})$ can be expressed in terms of non-metaplectic parahoric Whittaker functions.

Parahoric-metaplectic duality

Theorem (V. '23)

Let $\theta \in (\mathbb{Z}/n\mathbb{Z})^r$ *and* $\lfloor \frac{\mu}{n} \rfloor = \lambda + \rho$. *Then*

$$w_0\Big(x^{\rho - \lfloor \theta \rfloor_n} \widetilde{\phi}^{o}_{\theta}(x; \varpi^{\rho-\mu})\Big) = \begin{cases} C \cdot x^{n\rho} \psi^J_u(x^n; \varpi^{-\lambda}\hat{w}), & \mu \mod n \text{ a perm of } \theta \\ 0, & else. \end{cases}$$

- $\hat{w}$ and the set J are determined from μ; u determined from θ
- w_0 is the longest element in W
- $C = C(\theta; w; u)$ is a ratio of statistics in terms of v and Gauss sums.
- θ distinct ($J = \emptyset$) and θ constant ($J = [1, r-1]$) recovers BBBG

Some comments

BBGG approach

- constructions of both metaplectic and non-metaplectic Whittaker functions as partition functions of solvable lattice models
- observed an exchangeability phenomenon in their models
- led to proofs using lattice models

Some comments

BBBG approach

- constructions of both metaplectic and non-metaplectic Whittaker functions as partition functions of solvable lattice models
- observed an exchangeability phenomenon in their models
- led to proofs using lattice models

Our approach

- purely representation-theoretic
  - use certain representations of the affine Hecke algebra
  - PBW-type decompositions in the Hecke algebra
  - properties of (anti-)symmetrizers
- continues the study of quasi-polynomials and associated representations, connects to (variants of) Macdonald polynomials

Rich interplay between lattice models $\longleftrightarrow$ Hecke algebras

Whittaker functions through metaplectic representations

- Chinta-Gunnells action σ^{CG} of W on $\mathbb{F}(P)$
  - key ingredient in their construction of WMDs
  - involves Gauss sums; reduces to usual action at $n = 1$
  - to show formulas actually define an action: explicit computations with rational functions and/ or computer checks, rank 2 reduction

Whittaker functions through metaplectic representations

- Chinta-Gunnells action σ^{CG} of W on $\mathbb{F}(P)$
  - key ingredient in their construction of WMDs
  - involves Gauss sums; reduces to usual action at $n = 1$
  - to show formulas actually define an action: explicit computations with rational functions and/ or computer checks, rank 2 reduction
- Chinta-Gunnells-Puskas used σ^{CG} to define an action of the Hecke algebra
  - metaplectic (Iwahori-) and spherical Whittaker functions can be understood through this action
  - similar methods to show formulas define an action

Whittaker functions through metaplectic representations

- Chinta-Gunnells action σ^{CG} of W on $\mathbb{F}(P)$
  - key ingredient in their construction of WMDs
  - involves Gauss sums; reduces to usual action at $n = 1$
  - to show formulas actually define an action: explicit computations with rational functions and/ or computer checks, rank 2 reduction
- Chinta-Gunnells-Puskas used σ^{CG} to define an action of the Hecke algebra
  - metaplectic (Iwahori-) and spherical Whittaker functions can be understood through this action
  - similar methods to show formulas define an action
- Earlier joint work with Sahi and Stokman: conceptual understanding of these actions

Affine Hecke algebras and Demazure-Lusztig operators

Definition

The affine Hecke algebra $\widetilde{\mathcal{H}}$ is the $\mathbb{F}$-algebra generated by $T_1, \ldots, T_{r-1}$ and $\mathbb{F}[P] = \mathbb{F}[x^\lambda : \lambda \in P]$ subject to quadratic and braid relations (among T_i) and commutation relations (between T_i and $\mathbb{F}[P]$). The finite Hecke algebra H is the subalgebra generated by T_i for $1 \leq i \leq r-1$.

Polynomial representation $\pi = \pi_t$ of $\widetilde{\mathcal{H}}$ on $\mathbb{F}[P]$, given by the Demazure-Lusztig operators:

$$\pi(T_j)x^\lambda = t \cdot x^{s_j\lambda} + (t - t^{-1})x^\lambda \cdot \frac{\left(\frac{x_j}{x_{j+1}}\right)^{-(\lambda_j - \lambda_{j+1})} - 1}{\frac{x_j}{x_{j+1}} - 1}$$

and $\pi(x^\mu)x^\lambda = x^{\mu+\lambda}$ for $\lambda, \mu \in P$ and $1 \leq j \leq r-1$.

Another parametrization: $\mathcal{T}_j = -tT_j^{-1}$, $v = t^2$.

Chinta-Gunnells-Puskas Hecke algebra action

CGP *metaplectic* action of the Hecke algebra H on $\mathbb{F}[P]$ is defined by

$$\mathcal{T}_j^{CGP}(f) = \left(1 - v\left(\frac{x_j}{x_{j+1}}\right)^n\right) \cdot \frac{f - \left(\frac{x_j}{x_{j+1}}\right)^n \cdot \sigma^{CG}(s_j)(f)}{1 - \left(\frac{x_j}{x_{j+1}}\right)^n} - f$$

for $1 \leq j \leq r-1$ and $f \in \mathbb{F}[P]$, and they showed that the metaplectic spherical Whittaker function is given by

$$\widetilde{\mathcal{W}}(x; \varpi^{\rho-\mu}) = \sum_{w \in W} \mathcal{T}_w^{CGP}(x^{w_0\mu}).$$

At $n = 1$: $\mathcal{T}_j^{CGP} = -v^{1/2} x^{\rho} \pi_{v^{1/2}}(T_j^{-1}) x^{-\rho}$.

Through quasi-polynomial representations

In earlier work with Siddhartha Sahi and Jasper Stokman, we:

- established a conceptual understanding of the Weyl group and Hecke algebras actions arising in the study of metaplectic Whittaker functions (e.g., CG and CGP actions)
- built a theory of DAHA representations and associated quasi-polynomials[1] paralleling / generalizing that for Macdonald polynomials; limiting case connects to metaplectic representation theory

In this talk: new results for (anti-)symmetric quasi-polynomials that imply the parahoric-metaplectic duality theorem.

[1] A *quasi-polynomial* is an element of the group algebra $\mathbb{F}[E] = \mathbb{F} - \text{span}\{x^y : y \in E\} \supset \mathbb{F}[P]$, where E is the ambient Euclidean space.

The representation π^{qp} of $\widetilde{\mathcal{H}}$

From now on, we let $c = (c_1, \ldots, c_r) \in \mathbb{R}^r$ with $0 \le c_i < 1$ and $c_i - c_{i+1} \ge 0$, and orbit $\widetilde{\mathcal{O}}_c = \{\mu + wc : \mu \in \mathbb{Z}^r, w \in S_r\}$.

Theorem (Sahi-Stokman-V. '22)

There is a representation π^{qp} of $\widetilde{\mathcal{H}}$ on $\mathbb{F}[\widetilde{\mathcal{O}}_c]$ given by the following explicit formulas: $\pi^{qp}(x^\lambda)x^y = x^{\lambda+y}$ for $y = (y_1, \ldots, y_r) \in \widetilde{\mathcal{O}}_c$, $\lambda \in \mathbb{Z}^r$ and

$$\pi^{qp}(T_j)x^y = \left\{\begin{matrix} 1, & y_j - y_{j+1} \in \mathbb{R} \setminus \mathbb{Z} \\ t, & y_j - y_{j+1} \in \mathbb{Z} \end{matrix}\right\} \cdot x^{s_j y}$$

$$+ (t - t^{-1})x^y \cdot \frac{\left(\frac{x_j}{x_{j+1}}\right)^{-\lfloor y_j - y_{j+1} \rfloor} - 1}{\frac{x_j}{x_{j+1}} - 1}$$

for $1 \le j \le r - 1$.

Special case $c = 0$: polynomial rep

Some key properties of π^{qp}

- Can be extended to a representation of DAHA $\mathbb{H}$ on $\mathbb{F}[\widetilde{\mathcal{O}}_c]$; T_0 involves a torus character $\tau : P \to \mathbb{F}$
  - Special case $c = (0, \ldots, 0)$: Cherednik's basic representation of $\mathbb{H}$, $\pi(T_j)$ are the Demazure-Lusztig operators
- Several families of interesting quasi-polynomials associated to π^{qp}, generalizing (nonsymmetric and symmetric) Macdonald polynomials
- Focusing on GL_r for this talk, but constructions work in arbitrary type
- Connects to metaplectic representation theory: $P = \mathbb{Z}^r$

$$P \subset \frac{1}{n} P \subset \mathbb{R}^r$$

$$x_i \to x_i^n : \quad P^n \subset P$$

Operators $\pi^{qp}(T_j)$ for $1 \leq j \leq r-1 \longleftrightarrow \mathcal{T}_j^{CGP}$ (introduce Gauss sums via a change of basis)

Quasi-polynomial generalizations of nonsymmetric Macdonald polynomials

Theorem (Sahi-Stokman-V. '22)

There is a family of quasi-polynomials $E_y = E_y(x; q, t, \tau) \in \mathbb{F}[\widetilde{\mathcal{O}}_c]$*, indexed by* $y \in \widetilde{\mathcal{O}}_c$*, with the following properties:*

- *simultaneous eigenfunctions of* $\pi^{qp}(Y^\lambda)$*,* $\lambda \in \mathbb{Z}^r$*,* $Y^\lambda \in \mathbb{H}$
- *admit a recursive construction via intertwiners in* $\mathbb{H}$
- $c = (0, \ldots, 0)$*:* $E_\lambda(x; q, t)$*,* $\lambda \in \mathbb{Z}^r$*, are nonsymmetric Macdonald polynomials*
- τ *satisfying a regularity condition:* $E_y(x; \infty, t, \tau)$ *can be identified with metaplectic Iwahori-Whittaker functions (up to an explicit reparametrization which introduces Gauss sums)*

(Anti-)symmetric variants

We also introduced and studied (anti-)symmetric variants through the (anti-)symmetrizer in the finite Hecke algebra H,

$$\mathbf{1}^{\pm} := \sum_{w \in S_r} (\pm t)^{\pm l(w)} T_w$$

$$E_y^{\pm} := \pi^{qp}(\mathbf{1}^{\pm}) E_y \in \mathbb{F}[\widetilde{\mathcal{O}}_c] \qquad (y \in \widetilde{\mathcal{O}}_c)$$

- $c = (0, \ldots, 0)$: $E_\lambda^{\pm}(x; q, t)$, $\lambda \in \mathbb{Z}^r$, are (anti-)symmetric Macdonald polynomials
- τ satisfying a regularity condition: $E_y^{-}(x; \infty, t, \tau)$ can be identified with metaplectic spherical Whittaker functions (up to an explicit reparametrization which introduces Gauss sums)
- symmetric w.r.t. a generalized action of the Weyl group (corresponds to CG action in metaplectic context)

The subspace of (anti)-symmetric quasi-polynomials

Let $\epsilon = 1$ (symmetric) or $\epsilon = -1$ (anti-symmetric).

Definition

The element $f \in \mathbb{F}[\widetilde{\mathcal{O}}_c]$ is an ϵ-symmetric quasi-polynomial iff $\pi^{qp}(T_j)f = \epsilon t^{\epsilon} f$ for all $1 \leq j \leq r-1$. Equivalently, $f = \pi^{qp}(\mathbf{1}^{\pm})g$ for some $g \in \mathbb{F}[\widetilde{\mathcal{O}}_c]$.

This uses the following key properties of the elements $\mathbf{1}^{\pm} \in H$:

$$T_j \mathbf{1}^{\pm} = \mathbf{1}^{\pm} T_j = \pm t^{\pm 1} \mathbf{1}^{\pm}$$

for $1 \leq j \leq r-1$.

Let $\mathbb{F}[\widetilde{\mathcal{O}}_c]^{\epsilon-\text{sym}} \subset \mathbb{F}[\widetilde{\mathcal{O}}_c]$ denote the subspace of ϵ-symmetric quasi-polynomials.

Special elements in $\mathbb{F}[\widetilde{\mathcal{O}}_c]^{\epsilon-\mathrm{sym}}$

We will give formulas for the (anti-)symmetric quasi-polynomials

$$p_y^{\pm} := \pi^{qp}(\mathbf{1}^{\pm})x^y \quad (y \in \widetilde{\mathcal{O}}_c)$$

in terms of *partial* symmetrizers and *classical* Demazure-Lusztig operators.

Special elements in $\mathbb{F}[\widetilde{\mathcal{O}}_c]^{\epsilon-\mathsf{sym}}$

We will give formulas for the (anti-)symmetric quasi-polynomials

$$p_y^{\pm} := \pi^{qp}(\mathbf{1}^{\pm})x^y \quad (y \in \widetilde{\mathcal{O}}_c)$$

in terms of *partial* symmetrizers and *classical* Demazure-Lusztig operators.

Some byproducts:

- By Sahi-Stokman-V '22, we have $E_y(x;\infty,t) = x^y$ for y antidominant $\Rightarrow$ formulas for $E_y^{\pm}(x;\infty,t)$
- Using relationship between $E_y^{-}(x;\infty,t)$ and metaplectic spherical Whittaker functions $\Rightarrow$ parahoric-duality theorem
- Since $p_y^{\pm}$ span $\mathbb{F}[\widetilde{\mathcal{O}}_c]^{\epsilon-\mathsf{sym}} \Rightarrow$ characterization of (anti-)symmetric quasi polynomials

Decomposing quasi-polynomials

Let $f \in \mathbb{F}[\widetilde{\mathcal{O}}_c]$. Let $J_c = \{1 \leq j \leq r-1 : c_j = c_{j+1}\}$, and set $W_c = W_{J_c}$ and $W^c = W^{J_c}$. We have

$$f = \sum_{w \in W^c} \gamma_w^{qp}(f) \cdot x^{wc},$$

where $\gamma_w^{qp} : \mathbb{F}[\widetilde{\mathcal{O}}_c] \to \mathbb{F}[P]$.

Decomposing quasi-polynomials

Let $f \in \mathbb{F}[\widetilde{\mathcal{O}}_c]$. Let $J_c = \{1 \leq j \leq r-1 : c_j = c_{j+1}\}$, and set $W_c = W_{J_c}$ and $W^c = W^{J_c}$. We have

$$f = \sum_{w \in W^c} \gamma_w^{qp}(f) \cdot x^{wc},$$

where $\gamma_w^{qp} : \mathbb{F}[\widetilde{\mathcal{O}}_c] \to \mathbb{F}[P]$. Two extreme cases to keep in mind:

1. c has distinct parts: $W_c = \{1\}$ and $W^c = W \Rightarrow$ full sum
2. c has identical parts: $W_c = W$ and $W^c = \{1\} \Rightarrow$ single term

Results

Theorem (V. '23)

Let $y \in \widetilde{\mathcal{O}}_c$ and decompose (uniquely) $y = \mu + \hat{w}c$, where $\mu \in P$ and $\hat{w} \in W^c$. Then for $w \in W^c$ we have

$$\gamma_w^{qp}(p_y^+) = c^+_{\hat{w},w}(t) \cdot w_0\pi\left(T^{-1}_{(w_0w)^{-1}}\mathbf{1}^+_{J_c}T_{\hat{w}^{-1}}\right)x^{\mu}$$
$$\gamma_w^{qp}(p_y^-) = c^-_{\hat{w},w}(t) \cdot \iota w_0\pi\left(T_{w_0w}\mathbf{1}^-_{J_c}T^{-1}_{\hat{w}}\right)\iota x^{\mu},$$

where w_0 is the longest word in W and $\iota : \mathbb{F}[P] \to \mathbb{F}[P]$ with $\iota(x^\mu) = x^{-\mu}$.

For a set $J \subset [1, r-1]$, $\mathbf{1}^\pm_J = \mathbf{1}^\pm_{W_J}$ is the **partial** (anti-)symmetrizer in $\mathcal{H}$:

$$\mathbf{1}^\pm_J := \sum_{w \in W_J} (\pm t)^{\pm l(w)} T_w.$$

Partially (anti-)symmetric polynomials

Let $\epsilon = 1$ (symmetric) or $\epsilon = -1$ (anti-symmetric) and $J \subset [1, r-1]$.

Definition

The polynomial $p \in \mathbb{F}[P]$ is a J-partially ϵ-symmetric polynomial iff $\pi(T_j)f = \epsilon t^{\epsilon} f$ for all $j \in J$. Equivalently, $f = \pi(\mathbf{1}_J^{\pm})g$ for some $g \in \mathbb{F}[P]$.

Let $\mathbb{F}[P]^{\epsilon, J-\text{sym}} \subset \mathbb{F}[P]$ denote the subspace of J-partially ϵ-symmetric polynomials.

Special elements: Macdonald polynomials with prescribed symmetry

$$p_\nu^{J,\pm} = \pi(\mathbf{1}_J^{\pm})E_\nu \qquad (\nu \in P).$$

Results, cont'd

Theorem (V. '23)

Decompose $w_0 = w_0^c w_{0c}$ *with* $w_0^c \in W^c, w_{0c} \in W_c$. *There is a bijection* $\phi_{c,\epsilon} : \mathbb{F}[\widetilde{\mathcal{O}}_c]^{\epsilon-\text{sym}} \to \mathbb{F}[P]^{\epsilon, J_c-\text{sym}}$ *defined by*

$$\phi_{c,\epsilon}(f) = t^{l(w_{0c})-l(w_0)} \begin{cases} w_0 \gamma_{w_0^c}^{qp}(f), & \epsilon = 1 \\ \iota w_0 \gamma_{w_0^c}^{qp}(f), & \epsilon = -1, \end{cases}$$

with explicit formulas for the inverse, for $f \in \mathbb{F}[\widetilde{\mathcal{O}}_c]^{\epsilon-\text{sym}}$.

Moreover, under this bijection, the quasi-polynomial generalizations of (anti-)symmetric Macdonald polynomials $E^{\pm}_{-\mu-\hat{w}c}(\infty)$ *correspond to Macdonald polynomials with prescribed symmetry* $p^{J_c,\pm}_{\hat{w}^{-1}\mu}$, *in the same limit.*

Proof ideas

Let $y \in \widetilde{\mathcal{O}}_c$, recall that we want to compute $\gamma_w^{qp}(p_y^{\pm}) = \gamma_w^{qp}(\pi^{qp}(\mathbf{1}^{\pm})x^y)$, for $w \in W^c$. Main steps:

(1) A useful property of the quasi-polynomial representation:

Theorem (Sahi-Stokman-V. '22)

Let $\mu \in P$ and $w \in W$ with $w = w^c w_c$ ($w^c \in W^c$ and $w_c \in W_c$). Then

$$\pi^{qp}(x^\mu T_w)x^c = t^{l(w_c)} \cdot x^{\mu + wc}.$$

In particular, if $y = \mu + \hat{w}c$ with $\hat{w} \in W^c$, we have

$$p_y^{\pm} = \pi^{qp}(\mathbf{1}^{\pm})x^y = \pi^{qp}(\mathbf{1}^{\pm}x^\mu T_{\hat{w}})x^c.$$

(2) PBW-type decompositions in $\widetilde{\mathcal{H}}$:

For $w \in W$, define coefficient functions $\gamma_w : \widetilde{\mathcal{H}} \to \mathbb{F}[P]$ by

$$h = \sum_{w \in W} \gamma_w(h) \cdot T_w$$

where $h \in \widetilde{\mathcal{H}}$. Crucially, for $w \in W^c$, can check

$$\gamma_w^{qp}(\pi^{qp}(h)x^c) = \sum_{u \in W_c} t^{l(u)} \gamma_{wu}(h).$$

Proof ideas, cont'd

(3) So, to compute $\gamma_w^{qp}(p_y^{\pm})$ for $w \in W^c$, it suffices to compute the coefficients

$$\gamma_{w'}(\mathbf{1}^{\pm} x^{\mu} T_{\hat{w}}) \quad \text{(for } w' \in W)$$

This is a computation that can be done entirely in the Hecke algebra!

Proof ideas, cont'd

(3) So, to compute $\gamma_w^{qp}(p_y^{\pm})$ for $w \in W^c$, it suffices to compute the coefficients

$$\gamma_{w'}(\mathbf{1}^{\pm} x^{\mu} T_{\hat{w}}) \quad (\text{for } w' \in W)$$

This is a computation that can be done entirely in the Hecke algebra!

Theorem (V. '23)

Let $w, \hat{w} \in W$ and $\mu \in P$. Then

$$\gamma_w(\mathbf{1}^{+} x^{\mu} T_{\hat{w}}) = t^{l(w_0)} w_0 \pi(T_{(w_0 w)^{-1}}^{-1}) \pi(T_{\hat{w}^{-1}}) x^{\mu}$$

(and an analogous statement holds for $\mathbf{1}^{-}$).

The sum over W_c (from previous slide) gives rise to the partial (anti-)symmetrizers $\mathbf{1}_J^{\pm}$.

Open questions

There are many! Here are some:

- q-level: by bijection,
$$\gamma^{qp}_{w_0^c}(E^{\pm}_y(x;q,t,\tau))$$
is (up to ι, w_0) a partially-symmetric polynomial. Obtain formulas, and possibly relate to Macdonald polynomials with prescribed symmetry.
- Interpret *symmetric* quasi-polynomials at $q \to \infty$, metaplectic context / Partially-*symmetric* polynomials in terms of p-adic special functions
- Other types: extend results from particular subspace of quasi-polynomials to all of $\mathbb{F}[E]$
- Lattice model interpretations
- Applications of dualities to Weyl group multiple Dirichlet series

Thank you!