Representation theory of vertex operator algebras and orbifold conformal field theory

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Outline

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1. The main problem in orbifold conformal field theory
2. Vertex operator algebras, twisted modules and twisted intertwining operators
3. Conjectures and problems
4. Conjectural properties of twisted intertwining operators
5. Some further thoughts and possible applications

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Conformal field theories

- A conformal field theory is a projective algebra over the PROP of the moduli space of Riemann surfaces with parametrized boundaries (Kontsevich, Segal).
- The first main problem in conformal field theory is to give a construction.
- A program to construct conformal field theories using the representation theory of vertex operator algebras has been very successful in the past thirty years.
- To construct a conformal field theory, one first construct a vertex operator algebra, then use modules for this vertex operator algebra to construct an intertwining operator algebra and prove the modular invariance property.
- Finally one has to prove properties of higher-genus correlation functions.

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- The first example of orbifold conformal field theories, the moonshine module, was constructed in mathematics by Frenkel, Lepowsky and Meurman.
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The main problem

- **Main problem**: Given a vertex operator algebra V and a group G of automorphisms of V, construct and classify all the conformal field theories whose vertex operator algebras contain the fixed point vertex operator subalgebra V^G of V as a subalgebra.
- It is difficult to study directly V^G-modules and intertwining operators among V^G-modules.
- But we expect that V^G-modules can all be obtained from twisted V-modules. In the case that G is the cyclic group generated by an automorphism g of V, this has been proved (H., 2019).
- Thus our conjectures and problems will be on twisted intertwining operators among suitable twisted modules.

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Vertex operator algebras

- Vertex operator algebras can be viewed as analogues of commutative associative algebras with their multiplications controlled by Riemann spheres with three punctures (and local coordinates).
- More precisely, a **vertex operator algebra** is a $\mathbb{Z}$-graded vector space $V = \coprod_{n\in\mathbb{Z}} V_{(n)}$ equipped with multiplications

$$\begin{aligned} Y_V : V \otimes V &\to V[[z, z^{-1}]], \\ u \otimes v &\mapsto Y(u, z)v. \end{aligned}$$

parametrized by a nonzero complex number z, called **vertex operator map**, a **vacuum** $\mathbf{1} \in V_{(0)}$ (analogous to the identity in an associative algebra) and a **conformal vector** ω (which generates the conformal symmetry) satisfying axioms either similar to those for associative algebras or about the conformal symmetry.

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Vertex operator algebras

- The main axiom is the **associativity** (**operator product expansion**). It says that when $|z_1 > |z_2| > |z_1 - z_2| > 0$,

$$Y(u_1, z_1)Y(u_2, z_2)u_3 = Y(Y(u_1, z_1 - z_2)u_2, z_2)u_3$$

  for $u_1, u_2, u_3 \in V$. Moreover, we also require that these expressions can be analytically extended to a rational function of z_1 and z_2 in a suitable sense.
- Another main axiom is the **commutativity**, which is the analogue of the commutativity of a commutative associative algebra in a more subtle sense. Roughly speaking, it says that for $u_1, u_2 \in V$, we require that $Y(u_1, z_1)Y(u_2, z_2)$ and $Y(u_2, z_2)Y(u_1, z_1)$ are analytic extensions of each other.

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Some conditions on vertex operator algebras

- Our main conjectures are based on some of the following conditions:
  1. **Positive energy (CFT type)**: $V_{(0)} = \mathbb{C}\mathbf{1}$, $V_{(n)} = 0$ for $n < 0$.
  2. **Self-contragredient**: The contragredient V', as a V-module, is equivalent to V.
  3. **C_n-cofinite**: $\dim V/C_n(V) < \infty$, where $C_n(V)$ is the subspace of V spanned by the elements of the form $\mathrm{Res}_x x^{-n} Y(u, x)v$ for $u, v \in V$ and $Y : V \otimes V \to V[[x, x^{-1}]]$ is the vertex operator map for V.
  4. **Reductive (rational)**: Every lower-bounded generalized V-module is completely reducible.
- A vertex operator algebra is sometimes called **(strongly) rational** if it is of positive energy, self-contragredient, C_2-cofinite and reductive. The conformal field theory constructed from intertwining operators for such a vertex operator algebra is **rational**.

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Twisted modules

- Twisted modules associated to automorphisms of finite orders appeared first in the works of Frenkel-Lepowsky-Meurman (1984) and Lepowsky (1985).
- Twisted modules associated to general automorphisms, including automorphisms of infinite orders, were introduced by H. (2009).
- In the case that the automorphism is of finite order, the twisted vertex operators involve powers of the variable z, but not the logarithm of the variable.
- In general, the logarithm $\log z$ of the variable z might appear in twisted vertex operators when the order of the automorphism is of infinite order.

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Definition of generalized twisted module (H., 2009)

- A **generalized g-twisted V-module** is a $\mathbb{C}$-graded vector space $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ (graded by *weights*) equipped with a linear map (*twisted vertex operator map*)

$$\begin{aligned} Y_W^g : V \otimes W &\to W\{x\}[\log x], \\ v \otimes w &\mapsto Y_W^g(v, x)w \end{aligned}$$

  satisfying natural axioms.
- The first main axiom is the **equivariance property**. It states that for $z \in \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{Z}$, the $p+1$-value $(Y_W^g)^{p+1}(gv, z)$ of the twisted vertex operator $Y_W^g(gv, x)$ evaluated at z is equal to the p-th value $(Y_W^g)^p(v, z)$ of the twisted vertex operator $Y_W^g(v, x)$ evaluated at z.

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  satisfying natural axioms.
- The first main axiom is the **equivariance property**. It states that for $z \in \mathbb{C} \setminus \{0\}$ and $p \in \mathbb{Z}$, the $p+1$-value $(Y_W^g)^{p+1}(gv, z)$ of the twisted vertex operator $Y_W^g(gv, x)$ evaluated at z is equal to the p-th value $(Y_W^g)^p(v, z)$ of the twisted vertex operator $Y_W^g(v, x)$ evaluated at z.

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Lower-bounded generalized, grading-restricted generalized and (ordinary) g-twisted modules

- A generalized g-twisted V-module $W = \coprod_{n\in\mathbb{C}} W_{[n]}$ is said to be **lower-bounded** if $W_{[n]} = 0$ for $\Re(n)$ sufficiently negative.
- A lower-bounded generalized g-twisted V-module is said to be **grading restricted** if $\dim W_{[n]} < \infty$ for $n \in \mathbb{C}$.
- A grading-restricted generalized g-twisted V-module is said to be an **(ordinary) g-twisted V-module** if $L_W(0) = \mathrm{Res}_x x^{-1} Y_W^g(\omega, x)$ acts on W semisimply.
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Construction and existence of twisted modules

- The construction and existence of twisted modules were open problems for many years and they have been solved last year. (H. 2019)
- A construction theorem was obtained and using this construction theorem, universal lower-bounded generalized twisted modules are constructed.
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Twisted intertwining operators

- In conformal field theory, the main objects to study are intertwining operators. In orbifold conformal field theory, the main objects to study are intertwining operators among grading-restricted generalized twisted modules. These intertwining operators are called twisted intertwining operators.
- It is straightforward to generalize intertwining operators to twisted intertwining operators using the Jacobi identity when the automorphisms involved are of finite orders and commute with each other (Xu, 1995).
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- For twenty two years, no such general definition was given in the literature.
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$$(Y_{W_3}^{g_3})^{p_1}(u, z_1)\mathcal{Y}^{p_2}(w_1, z_2)w_2 \\ = \mathcal{Y}^{p_2}((Y_{W_1}^{g_1})^{p_{12}}(u, z_1 - z_2)w_1, z_2)w_2$$

  for $u \in V$, $w_1 \in W_1$ and $w_2 \in W_2$. Moreover, the functions can be analytically extended to a multivalued analytic function with regular singular points at $z_1 = 0$, $z_2 = 0$ and $z_1 - z_2 = 0$.
- The commutativity states that $(Y_{W_3}^{g_3})^{p_1}(u, z_1)\mathcal{Y}^{p_2}(w_1, z_2)w_2$ and $\mathcal{Y}^{p_2}(w_1, z_2)(Y_{W_2}^{g_2})^{p_1}(u, z_1)w_2$ are analytic extensions of each other along suitable paths.
- Some basic properties of twisted intertwining operators, including the skew-symmetry and contragredient isomorphisms between spaces of twisted intertwining operators, were proved (H. 2017).

Outline

Orbifold theory conjectures for finite groups

Conjecture

For a C_1-cofinite vertex operator algebra V and a finite group G of automorphisms of V satisfying suitable additional conditions (too techinical to state here), twisted intertwining operators among grading-restricted generalized g-twisted V-modules of finite lengths for $g \in G$ satisfy convergence, associativity and commutativity properties.

Conjecture

The category of grading-restricted generalized g-twisted V-modules for all $g \in G$ has a natural structure of G-crossed tensor category.

- In the case of $G = \{1_V\}$, these conjectures are theorems (H. 2007).

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Orbifold theory conjectures for (strongly) rational vertex operator algebras and finite groups

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For a (strongly) rational vertex operator algebra V and a finite group G of automorphisms of V, twisted intertwining operators among g-twisted V-modules for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties.

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- In the case that $G = \{1_V\}$ or that V is holomorphic and G is finite cyclic, these conjectures have been proved (H., 2002, 2003, 2005, Carnahan -Miyamoto 2016, van Ekeren-Möller-Scheithauer 2015, Möller 2016).

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Problems in the general case

Problem

Let V be a vertex operator algebra and let G be a group of automorphisms of V. Under what conditions do the twisted intertwining operators among grading-restricted generalized g-twisted V-modules of finite lengths for all $g \in G$ satisfy the convergence, associativity, commutativity and modular invariance properties?

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Under what conditions does the category of grading-restricted generalized g-twisted V-modules of finite lengths for all $g \in G$ has a structure of a G-crossed tensor category?

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Convergence and extension property of products of n twisted intertwining operators

- Let $\mathcal{Y}_1, \dots, \mathcal{Y}_i, \dots, \mathcal{Y}_n$ be twisted intertwining operators such that they can be multiplied. Then the series

$$\langle w_0', \mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) w_{n+1} \rangle$$

is absolutely convergent in the region $|z_1| > \cdots > |z_n| > 0$ and its sum can be analytically extended to a multivalued analytic function

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on the region

$$\{(z_1, \dots, z_n) \mid z_i \neq 0, z_i - z_j \neq 0 \text{ for } i \neq j\} \subset \mathbb{C}^n$$

and the only possible singular points $z_i = 0, \infty$ and $z_i = z_j$ are regular singular points.

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Associativity and commutativity of twisted intertwining operators

- For twisted intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$, there exist intertwining operators $\mathcal{Y}_3$ and $\mathcal{Y}_4$ such that

$$F(\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle) = F(\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle).$$

- For grading-restricted generalized g_i-twisted V-module W_i for $i = 1, 2, 3, 4, 5$ and twisted intertwining operators $\mathcal{Y}_1$ and $\mathcal{Y}_2$ of types $\binom{W_4}{W_1\, W_5}$ and $\binom{W_5}{W_2\, W_3}$, respectively, there exist intertwining operators $\mathcal{Y}_5$ and $\mathcal{Y}_6$ of types $\binom{W_4}{W_2\, W_6}$ and $\binom{W_6}{\phi_{g_2^{-1}}(W_1)\, W_3}$, respectively, such that

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Pseudo-traces

- Let P be an finite-dimensional associative algebra P over $\mathbb{C}$. For a finitely generated projective right P-module M, there exists a projective basis, that is, a pair of sets $\{w_i\}_{i=1}^n \subseteq M$, $\{w_i'\}_{i=1}^n \subseteq \mathrm{Hom}_P(M, P)$ such that for all $w \in M$, $w = \sum_{i=1}^n w_i(w_i'(w))$.
- A linear function $\phi : P \to \mathbb{C}$ is said to be symmetric if $\phi(pq) = \phi(qp)$ for all $p, q \in P$. For a symmetric linear function ϕ, the pseudo-trace $\mathrm{Tr}_M^\phi \alpha$ for $\alpha \in \mathrm{End}_P(M)$ associated to ϕ is the function Tr_M^ϕ defined by

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Convergence and extension property of pseudo-q-traces of products of n geometrically-modified twisted intertwining operators

- Let $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be twisted intertwining operators such that they can be multiplied and the grading-restricted generalized twisted module W_0 coming out from $\mathcal{Y}_1(w_1, x)$ is equal to the grading-restricted generalized twisted module W_{n+1} acted on by $\mathcal{Y}_n(w_n, x)$.
- Let P be a finite-dimensional associative algebra and ϕ a symmetric linear function on P.
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- Assume in addition that the product $\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)$ for $w_1 \in W_1, \ldots, w_n \in W_n$ commutes with the action of P.
- For a grading-restricted generalized twisted V-modules W, let $\mathcal{U}(q_z) = (2\pi i q_z)^{L_W(0)} e^{-\sum_{j \in \mathbb{N}} A_j L_W(j)}$, where $q^z = e^{2\pi i z}$ and $A_j \in \mathbb{C}$ for $j \in \mathbb{Z}_+$ are defined by

$$\frac{1}{2\pi i} \log(1 + 2\pi i y) = \left(\exp \left(\sum_{j \in \mathbb{Z}_+} A_j y^{j+1} \frac{\partial}{\partial y} \right) \right) y.$$

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- Then

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is absolutely convergent in the region $1 > |q_{z_1}| > \ldots > |q_{z_n}| > |q_\tau| > 0$ and can be extended to a multivalued analytic function

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in the region $\Im(\tau) > 0$, $z_i \neq z_j + l + m\tau$ for $i \neq j$, $l, m \in \mathbb{Z}$.

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Modular invariance of twisted intertwining operators

- Let $\mathcal{F}_{w_1,\dots,w_n}$ be the vector space spanned by functions of the form

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for all finite-dimensional associative algebras P, all symmetric linear functions ϕ, all projective right P-module structures on $W_0 = W_{n+1}$ commuting with its twisted vertex operators, all twisted intertwining operators $\mathcal{Y}_i$ such that their product commutes with the right action of P on $W_0 = W_{n+1}$.

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Orbifold theory conjectures and (strong) rationality of fixed point subalgebras

- Canahan and Miyamoto (2016) proved that the fixed point subalgebra of a (strongly) rational vertex operator algebra under a finite solvable group is still (strongly) rational.
- Their proof is for one automorphism of finite order and already uses all the results on intertwining operators. For non-solvable groups, their method does not seem to work.
- If the orbifold theory conjectures can be proved proved without using the (strong) rationality of the fixed vertex operator subalgebra, we would obtain orbifold conformal field theories associated to non-solvable groups without this (strong) rationality.
- Instead of using the (strong) rationality of the fixed point vertex operator subalgebra, we might be able to prove this (strong) rationality using these conjectures.

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- The uniqueness of the moonshine module conjectured by Frenkel, Lepowsky and Meurman says that a vertex operator algebra of central charge 24, without nonzero weight one vectors and whose only irreducible module is itself must be isomorphic to the moonshine module.
- This uniqueness conjecture is obtained by the analogy between the Leech lattice and the moonshine module.
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Mirror conformal field theories

- The construction of mirror manifolds by Greene and Plesser is in fact a construction in conformal field theory. Orbifold theories play an important role.
- To understand mirror symmetry completely, certainly it is important to construct $N = 2$ superconformal field theory associated to Calabi-Yau manifolds. But it is also important to develop a general orbifold theory construction.
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THANK YOU VERY MUCH!