Modular invariance of (logarithmic) intertwining operators

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The modular invariance conjecture of Moore and Seiberg

- Let V be a vertex operator algebra and W_1, W_2 and W_3 lower-bounded generalized V-modules.
- A (non-logarithmic) intertwining operator is a linear map

$$\begin{aligned} \mathcal{Y} : W_1 \otimes W_2 &\to W_3\{x\} \\ w_1 \otimes w_2 &\mapsto \mathcal{Y}(w_1, x)w_2 \end{aligned}$$

  satisfying the lower-truncation property, the Jacobi identity and the $L(-1)$-derivative property.
- In 1988, Moore and Seiberg conjectured that for rational conformal field theories, the space spanned by q-traces (shifted by $-\frac{c}{24}$) of products of chiral vertex operators (intertwining operators) is invariant under modular transformations.
- Based on these two conjectures, they derived the Verlinde formula and discovered modular tensor category structures.

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Zhu's theorem: A special case of the conjecture of Moore and Seiberg

- In 1990, Zhu formulated precisely and proved a special case of the modular invariance conjecture of Moore and Seiberg.
- Let V be a vertex operator algebra V satisfying the following conditions:
  1. V has no nonzero elements of negative weights,
  2. V is C_2-cofinite (that is, $\dim V/C_2(V) < \infty$, where $C_2(V) = \langle u_{-2}v \mid u, v \in V \rangle$),
  3. Every lower-bounded generalized V-module is completely reducible.
- Zhu proved that the space spanned by q-traces (shifted by $-\frac{c}{24}$) of products of vertex operators for V-modules is invariant under modular transformations.
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Miyamoto's pseudo-q-traces and nonsemisimple generalization of Zhu's theorem

- In 2002, Miyamoto proved a nonsemisimple generalization of Zhu's theorem. In the nonsemisimple case, q-traces are not enough. Miyamoto introduced pseudo-q-traces of operators on grading-restricted generalized V-modules.
- Let V be a vertex operator algebra V satisfying the conditions:
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- Miyamoto proved that the space spanned by pseudo-q-traces (shifted by $-\frac{c}{24}$) of products of vertex operators for grading-restricted generalized V-modules is invariant under modular transformations.
- Arike and Nagatomo first noticed that Condition 3 is needed in Miyamoto's paper. There are examples of vertex operators satisfying Conditions 1 and 2 but not Condition 3.

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The precise formulation and proof of the modular invariance conjecture of Moore-Seiberg

- In 2003, I formulated precisely and proved the modular invariance conjecture of Moore and Seiberg.
- Let V be a vertex operator algebra satisfying the three conditions needed in Zhu's theorem. Then the modular invariance conjecture of Moore and Seiberg is true. That is, the space spanned by q-traces (shifted by $-\frac{c}{24}$) of products of intertwining operators is invariant under modular transformations.
- The method used by Zhu (which is also used by Miyamoto) cannot be used to prove this conjecture of Moore and Seiberg for the q-traces shifted by $\frac{c}{24}$ of products of more than one intertwining operators.
- A completely different method is developed to prove this conjecture. This is the reason why it took 13 years after Zhu's theorem to prove this conjecture.

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Conjecture on the modular invariance of (logarithmic) intertwining operators

- In the nonsemisimple case, we have to consider more general intertwining operators: A (logarithmic) intertwining operator is a linear map

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  satisfying the same conditions above for intertwining operators. We will sometimes call a (logarithmic) intertwining operator simply an intertwining operator.

- **Conjecture (2003)**: For a C_2-cofinite vertex operator algebra V without nonzero elements of negative weights, the space of pseudo-q-traces (shifted by $-\frac{c}{24}$) of products of (logarithmic) intertwining operators is invariant under modular transformations.

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Pseudo-traces

- Let P be a finite-dimensional associative algebra and M a finitely generated right projective P-module.
- Then M has a pair of sets $\{m_i\}_{i=1}^n \subset M$, $\{\alpha_i\}_{i=1}^n \subset \mathrm{Hom}_P(M, P)$ (called a projective basis) such that for all $m \in M$,

$$m = \sum_{i=1}^{n} m_i(\alpha_i(m)).$$

- Let $\phi : P \to \mathbb{C}$ be a symmetric linear function on P, that is, a linear map satisfying $\phi(p_1 p_2) = \phi(p_2 p_1)$ for all $p_1, p_2 \in P$.
- The pseudo-trace of $A \in \mathrm{End}_P M$ associated to ϕ is defined to be

$$\mathrm{Tr}_M^{\phi} A = \phi\left(\sum_{i=1}^{n} \alpha_i(A(m_i))\right).$$

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Pseudo-q-traces

- Let V be a vertex operator algebra, P a finite-dimensional associative algebra and ϕ a symmetric linear function on P.
- Let W be a grading-restricted generalized V-P-bimodule (a grading-restricted generalized V-modue which is also a right P-module such that the vertex operators and the action of P commutes), projective as a right P-module and $L_W(0)_N^{K+1}W = 0$ for some $K \in \mathbb{Z}_+$. In particular, the homogeneous subspaces of W are finitely generated projective P-modules.
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$$\mathrm{Tr}_W^{\phi} A q^{L_W(0)-\frac{c}{24}} = \sum_{k=0}^{K} \sum_{n\in\mathbb{C}} \mathrm{Tr}_{W_{[n]}}^{\phi} \pi_n A \bigg|_{W_{[n]}} L_W(0)_N^k (\log q)^k q^{n-\frac{c}{24}},$$

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Products of intertwining operators

- Let $W_1, \ldots, W_n$, $\widetilde{W}_1, \ldots, \widetilde{W}_{n-1}$ be grading-restricted generalized V-modules and let W be a grading-restricted generalized V-P-bimodule which is projective as a right P-module.
- Let $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ be intertwining operators of types $\binom{W}{W_1\, \widetilde{W}_1}, \ldots, \binom{\widetilde{W}_{n-1}}{W_n\, W}$, respectively.
- In the case that V is C_2-cofinite, I had proved that for $w_1 \in W_1, \ldots, w_n \in W_n$,

$$\mathcal{Y}_1(w_1, z_1) \cdots \mathcal{Y}_n(w_n, z_n) \in \mathrm{Hom}(W, \overline{W})$$

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Pseudo-q_τ-traces of products of intertwining operators

- In 2015, Fiordalisi proved in his Ph.D. thesis that

$$\mathrm{Tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}_{W_1}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}_{W_n}(q_{z_n})w_n, q_{z_n})q_\tau^{L(0)-\frac{c}{24}}$$

(where $q_{z_j} = e^{2\pi i z_j}$ for $j = 1, \ldots, n$ and $q_\tau = e^{2\pi i\tau}$) is absolutely convergent in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ and can be analytically extended to a multivalued analytic function

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1, \ldots, w_n; z_1, \ldots, z_n; \tau)$$

in the region $\Im(\tau) > 0$, $z_i \neq z_j + k\tau + l$ for $i \neq j$, $k, l \in \mathbb{Z}$.

- For $w_1 \in W_1, \ldots, w_n \in W_n$, let $\mathcal{F}_{w_1,\ldots,w_n}$ be the space spanned by all such multivalued analytic functions.
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Modular transformations

- We define the modular transformation of

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The modular invariance theorem of (logarithmic) intertwining operators

The conjecture on the modular invariance of (logarithmic) intertwining operators has been proved. Here is the precise formulation of the theorem:

Theorem (H., 2023)

Let V be a C_2-cofinite vertex operator algebra without nonzero elements of negative weights. Then the modular transformation of

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- We have the genus-one associativity

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Associative algebras and modular invariance

- In the proof of the modular invariance conjecture of Moore and Seiberg, one needs to use the modular transformation of the pseudo-q_τ-trace of an intertwining operator to obtain a symmetric linear function on a bimodule for Zhu's algebra $A(V)$.
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- Later, I found that in order to prove the modular invariance of (logaritmic) intertwining operators, one needs to introduce much larger associative algebras. These are the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ introduced in 2020.

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- Later, I found that in order to prove the modular invariance of (logaritmic) intertwining operators, one needs to introduce much larger associative algebras. These are the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ introduced in 2020.

Lower-bounded generalized V-modules and infinite matrices with entries in V

- Given a lower-bounded generalized V-module W, there is a canonical $\mathbb{N}$-grading $W = \coprod_{n\in\mathbb{N}} W_{[\![n]\!]}$ of W such that for homogeneous $v \in V$ and $k, l \in \mathbb{N}$, the coefficient

$$\mathrm{Res}_x x^{l-k-1} Y_W(x^{L(0)}v, x) = v_{\mathrm{wt}\ v+l-k-1}$$

  of the vertex operator $Y_W(v, x)$ maps $W_{[\![l]\!]}$ to $W_{[\![k]\!]}$.
- Let $U^\infty(V)$ be the space of column-finite infinite matrices with entries in V and indexed by $\mathbb{N}$. For $v \in V$ and $k, l \in \mathbb{N}$, let $[v]_{kl}$ be the matrix in $U^\infty(V)$ with the (k, l)-entry being v and all the other entries being 0. Then we have a linear map $\vartheta_W : U^\infty(V) \to \mathrm{End}\ W$ given by

$$\vartheta_W([v]_{kl}) = \mathrm{Res}_x x^{l-k-1} Y_W(x^{L(0)}v, x).$$

- Let $Q^\infty(V)$ be the intersection of $\ker \vartheta_W$ for all lower-bounded generalized V-modules.

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Associative algebras $A^{\infty}(V)$ and $A^{N}(V)$

- There exists a product $\diamond$ in $U^{\infty}(V)$, unique modulo $Q^{\infty}(V)$, such that $\vartheta_W([u]_{kl} \diamond [v]_{mn}) = \vartheta_W([u]_{kl})\vartheta_W([v]_{mn})$ for any lower-bounded generalized V-module W.
- Let $A^{\infty}(V) = U^{\infty}(V)/Q^{\infty}(V)$. Then $A^{\infty}(V)$ is an associative algebra and ϑ_W induces an $A^{\infty}(V)$-module structure on W.
- For $N \in \mathbb{N}$, let $U^{N}(V)$ be the subspace of $U^{\infty}(V)$ consisting of matrices with (k, l)-entries being 0 for $k, l \geq N + 1$. Let $A^{N}(V)$ be the subset of $A^{\infty}(V)$ consisting of the cosets containing elements of $U^{N}(V)$. Then $A^{N}(V)$ is a subalgebra of $A^{\infty}(V)$.
- Let

$$\Omega_N^0(W) = \coprod_{n=0}^{N} W_{[\![n]\!]}.$$

Then the restriction of ϑ_W to $A^{N}(V)$ gives an $A^{N}(V)$-module structure on $\Omega_N^0(W)$.

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$A^\infty(V)$-bimodules, $A^N(V)$-bimodules and intertwining operators

- For each lower-bounded generalized V-module W, we also construct an $A^\infty(V)$-bimodule $A^\infty(W)$ using the space of column-finite infinite matrices with entries in W. The space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ is isomorphic to $\mathrm{Hom}_{A^\infty(V)}(A^\infty(W_1) \otimes_{A^\infty(V)} W_2, W_3)$.
- For $N \in \mathbb{N}$, we have a subspace $A^N(W)$ of $A^\infty(W)$ obtained from $(N+1) \times (N+1)$ matrices with entries in W. The actions of $A^\infty(V)$ on $A^\infty(W)$ induce actions of $A^N(V)$ on $A^N(W)$ such that $A^N(W)$ becomes an $A^N(V)$-bimodule.
- Given a graded $A^N(V)$-module M, there is a lower-bounded generalized V-module $S^N(M)$ satisfying a universal property. In the case that N is sufficiently large and W_2 and W_3' are equivalent to $S^N(\Omega_N^0(W_2))$ and $S^N(\Omega_N^0(W_3'))$, respectively, the space of intertwining operators of type $\binom{W_3}{W_1 W_2}$ is isomorphic to $\mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} W_2, W_3)$.

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1-point genus-one correlation functions and symmetric linear functions

- The analytic extension $\overline{F}^{\phi}_{\mathcal{Y}}(w_1; z; \tau)$ of the pseudo-q_τ-trace of an intertwining operator $\mathcal{Y}$ of type $\binom{W_2}{W_1\, W_2}$ is in fact a 1-point genus-one correlation function. Then a modular transformation of $\overline{F}^{\phi}_{\mathcal{Y}}(w_1; z; \tau)$ is also a 1-point genus-one correlation function.
- Using the properties of 1-point genus-one correlation functions, we prove that for $N \in \mathbb{N}$, a modular transformation of $\overline{F}^{\phi}_{\mathcal{Y}}(w_1; z; \tau)$ gives a symmetric linear function on $A^N(W_1)$. This step is the most difficult part of the proof of this conjecture.
- By a theorem of Fiordalisi proved in his thesis using a theorem of Miyamoto and Arike on pseudo-traces and square-zero extension of associative algebras, a symmetric linear function on $A^N(W_1)$ is a sum of pseudo-traces of elements of $\mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} M, M)$ for some finite-dimensional $A^N(V)$-module M.

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The last step of the proof

- For N sufficiently large, the C_2-cofiniteness of V implies that W_2 and W_2' are equivalent to $S^N(\Omega_N^0(W_2))$ and $S^N(\Omega_N^0(W_2'))$, respectively.
- Then $\mathrm{Hom}_{A^N(V)}(A^N(W_1) \otimes_{A^N(V)} W_2, W_2)$ is isomorphic to the space of intertwining operators of type $\binom{W_2}{W_1\, W_2}$. In particular, we obtain intertwining operators of types $\binom{W_2}{W_1\, W_2}$.
- We take the difference of the modular transformation of $\overline{F}^{\phi}_{\mathcal{Y}}(w_1; z; \tau)$ and the sum of the analytic extensions of pseudo-q_τ-traces of these intertwining operators. This is still a genus-one correlation function.
- Repeat the step above. Finally we can find finitely many intertwining operators such that the sum of the analytic extensions of pseudo-q_τ-traces of these intertwining operators is equal to the modular transformation of $\overline{F}^{\phi}_{\mathcal{Y}}(w_1; z; \tau)$, proving the modular invariance theorem.

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A main possible application

- The proof of the modular invariance conjecture of Moore and Seiberg led to the proof of the Verlinde formula and the proof of the rigidity and modularity of the braided tensor category of V-modules when V satisfies in addition a condition on the weight-one subspace of V, the existence of a nondegenerate bilinear invariant form on V and a complete reducibility condition.
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- In the case that complete reducibility condition is not satisfied, I also conjectured many years ago that the braided tensor category of grading-restricted generalized V-modules is rigidity. Later, it was also conjectured bu other people that this category should be modular in a suitable sense.
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Thank you!