Modular invariance of (logarithmic) intertwining operators

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Abstract

Let V be a C_2-cofinite vertex operator algebra without nonzero elements of negative weights. We prove the conjecture that the spaces spanned by analytic extensions of pseudo-q-traces ($q = e^{2\pi i\tau}$) shifted by $-\frac{c}{24}$ of products of geometrically-modified (logarithmic) intertwining operators among grading-restricted generalized V-modules are invariant under modular transformations. The convergence and analytic extension result needed to formulate this conjecture and some consequences on such shifted pseudo-q-traces were proved by Fiordalisi in [F1] and [F2] using the method developed in [H2]. The method that we use to prove this conjecture is based on the theory of the associative algebras $A^N(V)$ for $N \in \mathbb{N}$, their graded modules and their bimodules introduced and studied by the author in [H8] and [H9]. This modular invariance result gives a construction of C_2-cofinite genus-one logarithmic conformal field theories from the corresponding genus-zero logarithmic conformal field theories.

1 Introduction

The modular invariance of (logarithmic) intertwining operators was conjectured by the author almost twenty years ago. It is a conjecture on logarithmic conformal field theories.

In this paper, we prove this conjecture. In the language of conformal field theory, this modular invariance result says that in the C_2-cofinite case, genus-one logarithmic conformal field theories can be constructed by sewing the corresponding genus-zero logarithmic conformal field theories. We expect that this modular invariance result will play an important role in the study of problems and conjectures on C_2-cofinite logarithmic conformal field theories.

Modular invariance plays a crucial role in the construction and study of conformal field theories. In 1988, Moore and Seiberg [MS1] [MS2] conjectured that for rational conformal field theories, the space spanned by q-traces ($q = e^{2\pi i\tau}$) shifted by $-\frac{c}{24}$ of products of n chiral vertex operators (intertwining operators) is invariant under modular transformations for each $n \in \mathbb{N}$. (For simplicity, we shall use "shifted (pseudo-)q-traces" below to mean "(pseudo-)q-traces ($q = e^{2\pi i\tau}$) shifted by $-\frac{c}{24}$.") Based on this conjecture in the most important case of $n = 2$ and the conjecture that intertwining operators have operator product expansion, they derived the Verlinde formula [V] and discovered a mathematical structure now called modular tensor category (see [T] for a precise definition and its connection with three-dimensional topological quantum field theories).

In 1990, Zhu [Z] proved a special case of the modular invariance conjecture of Moore and Seiberg under suitable conditions formulated precisely also in [Z]. Let V be a vertex operator algebra satisfying the conditions that (i) V has no nonzero elements of negative weights, (ii) V is C_2-cofinite (that is, $\dim V/C_2(V) < \infty$, where $C_2(V) = \langle \mathrm{Res}_x x^{-2} Y_V(u, x)v \mid u, v \in V\rangle$), and (iii) every lower-bounded generalized V-module is completely reducible. Zhu proved in [Z] that for each $n \in \mathbb{N}$ and each subset $\{v_1, \ldots, v_n\}$ of V, the space spanned by the analytic extensions of shifted q-traces of products of suitably modified vertex operators associated to $v_1, \ldots, v_n$ for V-modules is invariant under modular transformations. In particular, the space spanned by the vacuum characters of irreducible V-modules is invariant under modular transformations. In 2000, using the the method developed by Zhu in [Z], Miyamoto in [Mi1] generalized this modular invariance result of Zhu to the space spanned by the analytic extensions of shifted q-traces of products of suitably modified vertex operators associated to $v_1, \ldots v_{n-1}$ for V-modules and one suitably modified intertwining operator of a special type associated to an element $w \in W$ for each $n \in \mathbb{Z}_+$, each subset $\{v_1, \ldots, v_{n-1}\}$ of V, each V-module W and each $w \in W$. Unfortunately, the method used in [Z] and [Mi1] cannot be used to prove the modular invariance conjecture of Moore and Seiberg in the most important case $n = 2$ and the important cases $n > 2$,

In 2002, Miyamoto [Mi2] proved a nonsemisimple generalization of Zhu's theorem. It was observed in [Mi2] that in the nonsemisimple case, the space of shifted q-traces of suitably modified vertex operators for grading-restricted generalized V-modules is in general not modular invariant and one needs shifted pseudo-q-traces of suitable operators on grading-restricted generalized V-modules introduced and studied in [Mi2]. Let V be a vertex operator algebra satisfying Conditions (i) and (ii) above and also satisfying an additional condition that (iv) there are no finite-dimensional irreducible V-modules. Miyamoto proved that for each $n \in \mathbb{N}$ and each subset $\{v_1, \ldots, v_n\}$ of V, the space spanned by the analytic extensions of shifted pseudo-q-traces of products of suitably modified vertex operators associated to $v_1, \ldots, v_n$ for grading-restricted generalized V-modules is invariant under modular transformations. The fact that Condition (iv) above is needed in [Mi2] was pointed out explicitly in [ArN]. There are examples of vertex operators satisfying Conditions (i) and (ii) above but not Condition (iv) (see also [ArN]).

In 2003, the author proved in [H2] the modular invariance conjecture of Moore and Seiberg under the same conditions as in [Z]. The precise statement of the modular invariance theorem in [H2] is that for a vertex operator algebra V satisfying the same conditions (i)[1], (ii) and (iii) above as in [Z] and for each $n \in \mathbb{N}$, each set of n grading-restricted generalized V-modules $W_1, \ldots, W_n$ and each set $\{w_1, \ldots, w_n\}$ for $w_1 \in W_1, \ldots, w_n \in W_n$, the space

[1]Note that the statement of the modular invariance theorem in [H2] also has an additional condition $V_{(0)} = \mathbb{C}\mathbf{1}$. This condition is added in [H2] because Theorem 7.2 in [H2] needs some results of [ABD], which are in turn proved using a result of Buhl [B] giving a spanning set of a weak V-module. It is this result in [B] that needs the condition $V_{(0)} = \mathbb{C}\mathbf{1}$. But in Lemma 2.4 in [Mi2], Miyamoto obtained such a spanning set of a weak module without using the condition $V_{(0)} = \mathbb{C}\mathbf{1}$. So the modular invariance theorem in [H2] in fact does not need this condition.

spanned by the analytic extensions of shifted q-traces of products of geometrically-modified intertwining operators associated to $w_1, \dots, w_n$ is invariant under modular transformations. As is mentioned above, the method used by Zhu and Miyamoto cannot be used to prove this conjecture of Moore and Seiberg in the main important cases $n \geq 2$. A completely different method was developed in [H2] to prove this conjecture in these cases, including the most important case $n = 2$. Using this modular invariance and the associativity of intertwining operators proved in [H1], the author proved the Verlinde conjecture and Verlinde formula in [H3] and the rigidity and modularity of the braided tensor category of V-modules in [H4].

Around 2003, after the work of Miyamoto [Mi2] and the proof in [H2] of the modular invariance conjecture of Moore and Seiberg, the following modular invariance was conjectured by the author: For a C_2-cofinite vertex operator algebra V without nonzero elements of negative weights and for each $n \in \mathbb{N}$, each set of n grading-restricted generalized V-modules $W_1, \dots, W_n$ and each set $\{w_1, \dots, w_n\}$ for $w_1 \in W_1, \dots, w_n \in W_n$, the space of analytic extensions of shifted pseudo-q-traces of products of geometrically-modified (logarithmic) intertwining operators associated to $w_1, \dots, w_n$ is invariant under modular transformations. We put the word "logarithmic" in parenthesis before "intertwining operators" since both intertwining operators without logarithm and logarithmic intertwining operators are needed. In the main body of this paper, we shall omit "(logarithm)" so that intertwining operators in general might have the logarithms of the variables involved.

In fact, in the original version of this conjecture, the convergence and analytic extension of shifted pseudo-q-traces of products of geometrically-modified (logarithmic) intertwining operators is also part of the conjecture. In 2015, using the method developed in [H2], Fiordalisi [F1] [F2] proved that such shifted pseudo-q-traces are convergent absolutely in a suitable region and can be analytically extended to multivalued analytic functions on a maximal region. He also proved that these multivalued analytic functions satisfy the genus-one associativity, genus-one commutativity and other properties using the associativity and commutativity of (logarithmic) intertwining operators and other properties for a C_2-cofinite vertex operator algebra proved in [H5]. As in the proof in [H2] of the modular invariance conjecture of Moore and Seiberg, by the genus-one associativity proved in [F1] and [F2], the modular invariance conjecture in the case $n \geq 2$ can be reduced to the modular invariance conjecture in the case $n = 1$. Here the author would like to emphasize the importance of the convergence and analytic extension results proved in [H2], [F1] and [F2]. Without the convergence and analytic extension results in [H2], [F1] and [F2], we could not even formulate the modular invariance conjecture of Moore and Seiberg and the modular invariance conjecture for (logarithmic) intertwining operators above. See [H10] for a survey on the convergence and analytic extension results and conjectures in the approach to conformal field theory using the representation theory of vertex operator algebras.

To prove the modular invariance conjecture for (logarithmic) intertwining operators above, we need to show that the modular transformations of the multivalued analytic functions obtained from shifted pseudo-q-traces of geometrically-modified (logarithmic) intertwining operators are sums of the multivalued analytic functions obtained from such shifted pseudo-q-traces. In [Z], [Mi1] and [H2], Zhu algebra $A(V) = A_0(V)$ associated to V, its modules and its bimodules are needed to prove the modular invariance. In [Mi2], the generalizations $A_n(V)$ for $n \in \mathbb{N}$ of Zhu algebra by Dong, Li and Mason [DLM] and their modules are needed to prove the modular invariance. On the other hand, the additional condition (Condition (iv) above) in [Mi2] that there are no finite-dimensional irreducible V-modules is needed exactly because the associative algebras $A_n(V)$ for $n \in \mathbb{N}$ cannot be used to handle the case that there exist finite-dimensional irreducible V-modules. More importantly, to study general (logarithmic) intertwining operators using the theory of associative algebras, the associative algebras $A_n(V)$ for $n \in \mathbb{N}$ are not enough.

In [H8], the author introduced new associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ associated to a vertex operator algebra V. These associative algebras contain $A_n(V)$ for $n \in \mathbb{N}$ as (very small) subalgebras. In fact, $A_n(V)$ for $n \in \mathbb{N}$ are all algebras of zero modes but acting on different homogeneous subspaces of lower-bounded generalized V-modules. On the other hand, the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$ introduced by the author in [H8] are algebras of all modes, including all nonzero modes. In [H9], the author introduced bimodules $A^\infty(W)$ and $A^N(W)$ for $N \in \mathbb{N}$ for these new associative algebras associated to a lower-bounded generalized V-module W and proved that the spaces of (logarithmic) intertwining operators are linearly isomorphic to the corresponding spaces of module maps between suitable modules for these associative algebras. In Section 3 of the present paper, we also introduced associative algebras $\widetilde{A}^\infty(V)$ and $\widetilde{A}^N(V)$ for $N \in \mathbb{N}$ isomorphic to $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$, respectively, and the corresponding bimodules $\widetilde{A}^\infty(W)$ and $\widetilde{A}^N(W)$ for $N \in \mathbb{N}$ associated to a lower-bounded generalized V-module W. We then transport the results on the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$, their graded modules and bimodules obtained in [H8] and [H9] to the corresponding results on $A^\infty(V)$ and $A^N(V)$ for $N \in \mathbb{N}$, their graded modules and bimodules.

In this paper, using the results of Fiordalisi in [F1] and [F2] and the results on these new associative algebras, their modules and bimodules in [H8], [H9] and Section 3 of the present paper, we prove the modular invariance conjecture for (logarithmic) intertwining operators discussed above. For the precise statement, see Theorem 5.5. Note that all the modular invariance theorems mentioned above are special cases of Theorem 5.5. In particular, in the special case studied in [Mi2] that the intertwining operators involved are vertex operators for grading-restricted generalized V-modules, we obtain a proof of the modular invariance result in [Mi2] without Condition (iv) above requiring that there are no finite-dimensional irreducible V-modules. Also, as special cases of the proofs of (4.38) and (4.61), we recover the proof by McRae [Mc] of Propositions 4.4 in [Mi2] and obtain a proof of Proposition 4.5 in [Mi2] (see Remarks 4.9 and 4.11).

Here we give a sketch of the proof of this modular invariance conjecture: As is mentioned above, we need only prove the modular invariance conjecture in the case $n = 1$. To prove the modular invariance conjecture in this case, we introduce a notion of genus-one 1-point conformal block labeled by a grading-restricted generalized V-module W. Then the conjecture follows from the following two results: (1) For grading-restricted generalized V-modules W and $\widetilde{W}$, the modular transformation of the analytic extension of the shifted pseudo-q-trace of a geometrically-modified intertwining operator of type $\binom{\widetilde{W}}{W\widetilde{W}}$ is a genus-one 1-point conformal block labeled by W (see Proposition 5.2). (2) Every genus-one 1-point conformal block labeled by W is the sum of the analytic extensions of the shifted pseudo-q-traces of geometrically-modified intertwining operators of type $\binom{\widetilde{W}_i}{W\widetilde{W}_i}$ for finitely many grading-restricted generalized V-modules $\widetilde{W}_i$ (see Theorem 5.4). The proof of (1) can be obtained easily from the properties of the shifted pseudo-q-traces of (logarithmic) intertwining operators in [F1] and [F2]. To prove (2), we first prove that a genus-one 1-point conformal block labeled by W gives a symmetric linear function on $\widetilde{A}^N(W)$ satisfying some additional properties for each $N \in \mathbb{N}$. This proof is technically the most difficult part of this paper. Then using the results on symmetric linear functions and pseudo-traces proved by Miyamoto [Mi2], Arike [Ar] and Fiordalisi [F1] [F2], we prove that for N sufficiently large, these symmetric linear functions on $A^N(W)$ are in fact finite sums of pseudo-traces of suitable linear operators on $A^N(V)$-modules. Next we use a main result in [H9] to show that for N sufficiently large, pseudo-traces of such linear operators on $A^N(V)$-modules are in fact obtained from shifted pseudo-q-traces of geometrically-modified (logarithmic) intertwining operators. Finally we show that when N is sufficiently large, the genus-one 1-point conformal block labeled by W that we start with is equal to the sum of the analytic extensions of the shifted pseudo-q-traces of these geometrically-modified (logarithmic) intertwining operators.

The present paper is organized as follows: In Section 2, we recall some basic definitions and results needed in this paper. In Subsection 2.1, we recall the definition of pseudo-traces and the results about pseudo-traces and symmetric linear functions obtained by Miyaomoto [Mi2], Arike [Ar] and Fiordalisi [F1]. In Subsection 2.2, we recall the results of Fiordalisi in [F1] and [F2] on the convergence and analytic extensions of shifted pseudo-q-traces of products of intertwining operators and their properties. In Section 3, for a vertex operator algebra V, we study further the $A^\infty(V)$-bimodule $A^\infty(W)$ and the $A^N(V)$-bimodules $A^N(W)$ for $N \in \mathbb{N}$ constructed from a lower-bounded generalized V-module W in [H9]. Then we introduce and study new associative algebras $\widetilde{A}^\infty(V)$ and $\widetilde{A}^N(V)$ $N \in \mathbb{N}$, the $\widetilde{A}^\infty(V)$-bimodule $\widetilde{A}^\infty(W)$ and $\widetilde{A}^N(V)$-bimodules $\widetilde{A}^N(W)$ for $N \in \mathbb{N}$ in this section. In Section 4, we give two constructions of symmetric linear functions on $\widetilde{A}^N(V)$-bimodules $\widetilde{A}^N(W)$ for a grading-restricted generalized V-module W. The first construction is given by shifted pseudo-q-traces of intertwining operators. The second construction is given by suitable maps satisfying properties involving the Weierstrass $\wp$- and ζ-functions. In Section 5, we prove our modular invariance theorem (Theorem 5.5) by proving Proposition 5.2 and Theorem 5.4 (the two results (1) and (2) discussed above).

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2 Pseudo-traces, symmetric linear functions and pseudo-q-traces

We recall some basic definitions and results in this section. In Subsection 2.1, we recall the basic definitions and results on pseudo-traces and symmetric linear functions introduced and obtained by Miyamoto [Mi2], Arike [Ar] and Fiordalisi [F1]. In Subsection 2.2, we recall the results obtained by Fiordalisi [F1] [F2] on genus-one correlation functions constructed using shifted pseudo-q-traces of products of (logarithmic) intertwining operators.

2.1 Pseudo-traces and symmetric linear functions

In this subsection, we first recall the definition of pseudo-traces for a finitely generated right projective module M for a finite-dimensional associative algebra A introduced by Miyamoto [Mi2] and further studied by Arike [Ar]. Then we recall a result on symmetric linear functions on A obtained by Miyamoto [Mi2] and Arike [Ar] and a result on symmetric linear functions on finite-dimensional A-bimodules obtained by Fiordalisi [F1]. See [Ar] and Fiordalisi [F1] for details.

Let A be a finite-dimensional associative algebra. Recall that a right A-module M is said to be projective if every short exact sequence

$$0 \to K \to P \to M \to 0$$

for right A-modules K and P splits.

Let M be a finitely generated right A-module. A projective basis for M is a pair of sets $\{m_i\}_{i=1}^n \subset M$, $\{\alpha_i\}_{i=1}^n \subset \mathrm{Hom}_A(M, A)$ such that for all $m \in M$,

$$m = \sum_{i=1}^{n} m_i \alpha_i(m).$$

A finitely generated right A-module M has a projective basis if and only if it is projective.

Let M be a right A-module. Then $\mathrm{Hom}_A(M, A)$ has a left A-module structure given by

$$(a\alpha)(m) = \alpha(ma)$$

for $a \in A, \alpha \in \mathrm{Hom}_A(M, A)$ and $m \in M$. We also have a contraction map

$$\begin{aligned} \pi_M : M \otimes_A \mathrm{Hom}_A(M, A) &\to A \\ m \otimes_A \alpha &\mapsto \alpha(m) \end{aligned}$$

For right A-modules M_1 and M_2, let

$$\tau_{M_1,M_2} : M_2 \otimes_A \mathrm{Hom}_A(M_1, A) \to \mathrm{Hom}_A(M_1, M_2)$$
$$m_2 \otimes_A \alpha \mapsto \tau_{M_1,M_2}(m_2 \otimes_A \alpha)$$

be the natural linear map defined by $(\tau_{M_1,M_2}(m_2 \otimes \alpha))(m_1) = m_2\alpha(m_1)$ for $m_1 \in M_1$, $m_2 \in M_2$ and $\alpha \in \mathrm{Hom}_A(M_1, M_2)$. In the case that $M_1 = M_2 = M$, let $\tau_M = \tau_{M,M}$.

For a finitely generated right A-module M, the map $\tau_M : M \otimes_A \mathrm{Hom}_A(M, A) \to \mathrm{End}_A M$ is an isomorphism so that $\tau_M^{-1} : \mathrm{End}_A M \to M \otimes_A \mathrm{Hom}_A(M, A)$ exists. The Hattori-Stallings trace of an endomorphism $\alpha \in \mathrm{End}_A M$ is the element

$$\mathrm{Tr}_M \alpha = \pi_M(\tau_M^{-1}(\alpha)) + [A, A]$$

of $A/[A, A]$. For finitely generated projective right A-modules M_1, M_2, $f \in \mathrm{Hom}_A(M_1, M_2)$ and $g \in \mathrm{Hom}_A(M_2, M_1)$, we have

$$\mathrm{Tr}_{M_2} f \circ g = \mathrm{Tr}_{M_1} g \circ f.$$

A linear function $\phi : A \to \mathbb{C}$ is said to be symmetric if $\phi(ab) = \phi(ba)$ for all $a, b \in A$. We denote the the space of symmetric linear functions on A by $SLF(A)$. Then $SLF(A)$ is linearly isomorphic to $(A/[A, A])^*$. Any symmetric linear function on A defines a symmetric bilinear form

$$\langle\ ,\ \rangle : A \times A \to \mathbb{C}$$

by $\langle a, b\rangle = \phi(ab)$. A symmetric linear function ϕ on A is said to be nondegenerate if the corresponding bilinear form is nondegenerate. The radical of a symmetric linear function on A is defined to be the two sided ideal

$$\mathrm{rad}\ \phi = \left\{a \in A \mid \langle a, b\rangle = 0 \quad \text{for all } b \in A\right\}$$

of A. A symmetric function ϕ is nondegenerate if and only if $\mathrm{rad}\ \phi = \{0\}$.

Let M be a finitely generated projective right A-module and $\{m_i\}_{i=1}^n, \{\alpha_i\}_{i=1}^n$ a projective basis. The pseudo-trace function ϕ_M on $\mathrm{End}_A M$ associated to a linear symmetric function ϕ on A is the map $\phi_M = \phi \circ \mathrm{Tr}_M : \mathrm{End}_A M \to \mathbb{C}$. We can express the pseudo-trace of $\alpha \in \mathrm{End}_A M$ in terms of the projective basis as

$$\phi_M(\alpha) = \phi\left(\sum_{i=1}^n \alpha_i(\alpha(m_i))\right).$$

For right projective A-modules M_1 and M_2, $\alpha \in \mathrm{Hom}_A(M_1, M_2)$ and $\beta \in \mathrm{Hom}_A(M_2, M_1)$, we have

$$\phi_{M_1}(\beta \circ \alpha) = \phi_{M_2}(\alpha \circ \beta).$$

A symmetric algebra (or Frobenius algebra) is an associative algebra equipped with a nondegenerate symmetric linear function. A basic algebra is an associative algebra A such that $A/J(A)$ is isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$, where $J(A)$ is the Jacobson radical of A.

Theorem 2.1 (Miyamoto [Mi2], Arike [Ar]) *Let A be a finite-dimensional associative algebra and $\phi \in SLF(A)$. Let $1_A = e_1 + \cdots + e_n$ with the largest n, where 1_A is the identity of A and $e_1, \ldots, e_n$ are orthognal central idempodents. Then $A = A_1 \oplus \cdots \oplus A_n$ is a decomposition of A as a direct sum of indecomposable A-bimodules, where $A_i = Ae_i$ for $i = 1, \ldots, n$. Let $\phi_i = \phi|_{A_i}$ for $i = 1, \ldots, n$. For each i, let $P_i = \bar{e}_i(A/\mathrm{Rad}(\phi_i))\bar{e}_i$ and $M_i = (A/\mathrm{Rad}(\phi_i))\bar{e}_i$, where $\bar{e}_i = e_i + \mathrm{Rad}(\phi_i) \in A/\mathrm{Rad}(\phi_i)$. Then for $i = 1, \ldots, n$, P_i are basic symmetric algebras with symmetric linear functions given by ϕ_i (still denoted by ϕ_i), and M_i are A-P_i-bimodules , finitely generated and projective as right P_i-modules, such that*

$$\phi(a) = \sum_{i=1}^{n} (\phi_i)_{M_i}(a)$$

where in each term in the right-hand side, $a \in A$ is viewed an element of $\mathrm{End}_{P_i} M_i$ *given by the left action a on M_i. Furthermore, if ν is an element of* $\mathrm{rad}\ \phi$*, that is,*

$$\phi(\nu a) = 0$$

for all $a \in A$, then $\nu M_i = 0$.

Let A be an associative algebra and M an A-bimodule. A linear function $\phi : M \to \mathbb{C}$ is said to be symmetric if for all $m \in M$ and $a \in A$,

$$\phi(am) = \phi(ma).$$

Let P be another associative algebra and U an A-P-bimodule. Then the endomorphism ring $\mathrm{End}_P U$ is an A-A-bimodule with the actions given by $(a\tau)(u) = a(\tau(u))$ and $(\tau a)(u) = \tau(au)$ for $a \in A$, $\tau \in \mathrm{End}_P U$ and $u \in U$.

Now assume that P is finite dimensional and U is finitely generated and projective as a right P-module. Then for $\phi \in SLF(P)$, the pseudo-trace ϕ_U on $\mathrm{End}_P U$ is a symmetric linear function on the A-bimodule $\mathrm{End}_P U$.

Let M be an A-bimodule and let $\mathrm{Hom}_{A,P}(M \otimes_A U, U)$ be the set of all A-P-bimodule maps from $M \otimes_A U$ to U. Let $f \in \mathrm{Hom}_{A,P}(M \otimes_A U, U)$. Then for any $m \in M$, the map

$$\begin{aligned} U &\to U \\ u &\mapsto f(m \otimes u) \end{aligned}$$

is an element of $\mathrm{End}_P U$. Define $T_f : M \to \mathrm{End}_P U$ by $T_f(m) = (u \mapsto f(m \otimes u))$. Then the map T_f is an A-bimodule homomorphism. Let $\phi_U^f : M \to \mathbb{C}$ be the linear function on M defined by $\phi_U^f(m) = \phi_U(T_f(m))$. Then the linear function ϕ_U^f is symmetric.

We have obtained a map $SLF(P) \otimes \mathrm{Hom}_{A,P}(M \otimes_A U, U) \to SLF(M)$. We now want to give an "inverse" map in a suitable sense.

We consider the trivial square-zero extension $\bar{A} = A \oplus M$ of A by M with the product given by

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2)$$

for $(a_1, m_1), (a_2, m_2) \in \bar{A}$. Let $\phi \in SLF(M)$. We extend ϕ to a linear function $\bar{\phi}$ on $\bar{A}$ by $\bar{\phi}(a, m) = \phi(m)$. Then

$$\begin{aligned}\bar{\phi}((a_1, m_1)(a_2, m_2)) &= \bar{\phi}(a_1a_2, a_1m_2 + m_1a_2)\\ &= \phi(a_1m_2 + m_1a_2)\\ &= \phi(m_2a_1 + a_2m_1)\\ &= \bar{\phi}(a_2a_1, m_2a_1 + a_2m_1)\\ &= \bar{\phi}((a_2, m_2)(a_1, m_1)),\end{aligned}$$

which says that $\bar{\phi}$ is in fact symmetric.

Let $1_A = e_1 + \cdots + e_n$ with the largest n, where 1_A is the identity of A and $e_1, \dots, e_n$ are orthogonal central idempodents. Then $A = A_1 \oplus \cdots \oplus A_n$ is a decomposition of A as a direct sum of indecomposable A-bimodules, where $A_i = Ae_i$ for $i = 1, \dots, n$. Then $A_1, \dots, A_n$ are in fact two-sided ideal of A and as an associative algebra, A is canonically isomorphic to the direct product of $A_1, \dots, A_n$. Let $M = M_1 \oplus \cdots \oplus M_n$ be a decomposition of M as a direct sum of A_1-, …, A_n-bimodules. Since $M = M_1 \oplus \cdots \oplus M_n$ and for each i, M_i is an A_i-bimodule, we have $\bar{A} = \bar{A}_1 \oplus \cdots \oplus \bar{A}_n$, where for each i, $\bar{A}_i = A_i + M_i$ is the trivial square-zero extension of A_i by M_i. Let $\phi_i = \bar{\phi}|_{\bar{A}_i}$ for $i = 1, \dots, n$.

Theorem 2.2 (Fiordalisi [F1]) *Let A be a finite-dimensional associative algebra and M a finite-dimensional A-bimodule, and let $\phi \in SLF(M)$. Let $1_A = e_1 + \cdots + e_n$ with the largest n, where 1_A is the identity of A and $e_1, \dots, e_n$ are orthogonal central idempodents such that $A = A_1 \oplus \cdots \oplus A_n$ is a decomposition of A as a direct sum of A-bimodules, where $A_i = Ae_i$ for $i = 1, \dots, n$, and let $M = M_1 \oplus \cdots \oplus M_n$ be a decomposition of M as a direct sum of A_1-, …, A_n-modules. For each i, let $P_i = \bar{e}_i(\bar{A}/\mathrm{Rad}(\phi_i))\bar{e}_i$ and $U_i = (\bar{A}/\mathrm{Rad}(\phi_i))\bar{e}_i$, where $\bar{e}_i = e_i + \mathrm{Rad}(\phi_i)$. Then for $i = 1, \dots, n$, P_i are basic symmetric algebras with symmetric linear functions given by ϕ_i, U_i and A-P_i-bimodules, finitely generated and projective as right P_i-modules. For each i, let $f_i \in \mathrm{Hom}(M \otimes U_i, U_i)$ be defined by $f_i(m \otimes u_i) = (0, m)u_i$ for $m \in M$ and $u_i \in U_i$. Then $f_i \in \mathrm{Hom}_{A,P}(M \otimes_A U_i, U_i)$ and for any $m \in M$,*

$$\phi(m) = \sum_{i=1}^{n} (\phi_i)_{U_i}^{f_i}(m).$$

Moreover, if ν is an element in A such that $\phi(\nu m) = 0$ for all $m \in M$, then the modules U_i can be chosen in such a way that $\nu U_i = 0$ for $i = 1, \dots, n$.

Proof. The proof of this result is essentially in [F1]. But since we need to give P_i, ϕ_i, U_i and f_i explicitly, we give a complete proof here.

We apply Theorem 2.1 to the algebra $\bar{A}$. Then $\bar{P}_i$ for $i = 1, \dots, n$ are basic symmetric linear algebras with symmetric linear functions given by ϕ_i (still denoted by ϕ_i), and U_i for $i = 1, \dots, n$ are $\bar{A}$-$\bar{P}_i$-bimodules, finitely generated and projective as right $\bar{P}_i$-modules, such that

$$\bar{\phi}(\bar{a}) = \sum_{n=1}^{n} (\phi_i)_{U_i}(\bar{a})$$

for $\bar{a} \in \bar{A}$, where in each term in the right-hand side, $\bar{a} \in \bar{A}$ is viewed as an element of $\mathrm{End}_{\bar{P}_i} U_i$ given by the left action of $\bar{a}$ on U_i. By definition, we have $\bar{\phi}(a, m) = \phi(m)$ for $a \in A$ and $m \in M$. Then we have

$$\phi(m) = \bar{\phi}(0, m) = \sum_{n=1}^{n} (\phi_i)_{U_i}(0, m)$$

for $m \in M$. Since A is a subalgebra of $\bar{A}$, U_i is also a left A-module. By the definition of f_i, we have

$$\begin{aligned} f_i(ma \otimes u) &= (0, ma)u_i = ((0, m)(a, 0))u_i = (0, m)(au_i) = f_i(m \otimes au_i), \\ f_i(am \otimes u) &= (0, am)u_i = ((a, 0)(0, m))u_i = a((0, m)u_i) = af_i(m \otimes u), \\ f_i(m \otimes u)p_i &= ((0, m)u_i)p_i = (0, m)(u_ip_i) = f_i(m \otimes u_ip_i) \end{aligned}$$

for $a \in A$, $m \in M$, $u_i \in U_i$ and $p_i \in \bar{P}_i$. So $f_i \in \mathrm{Hom}_{A, \bar{P}_i}(M \otimes_A U_i, U_i)$. Then

$$(\phi_i)_{U_i}(0, m) = (\phi_i)^{f_i}_{U_i}(m).$$

Thus we obtain

$$\phi(m) = \sum_{n=1}^{n} (\phi_i)^{f_i}_{U_i}(m).$$

∎

2.2 Genus-one correlation functions from shifted pseudo-q-traces

In this subsection, we recall the results on genus-one correlation functions constructed from shifted pseudo-q-traces of products of intertwining operators obtained by Fiordalisi in [F1] and [F2].

Let V be a vertex operator algebra. In this section, though some of the results hold for more general vertex operator algebras, we assume that V has no nonzero elements of negative weights (that is, $V_{(n)} = 0$ for $n < 0$) and satisfies the C_2-cofiniteness conditions (that is, $\dim V/C_2(V) < \infty$, where $C_2(V) = \langle \mathrm{Res}_x x^{-2} Y_V(u, x)v \mid u, v \in V \rangle$).

Let P be a finite-dimensional associative algebra equipped with a symmetric linear function ϕ. A grading-restricted (or lower-bounded) generalized (or ordinary) V-module W equipped with a right P-module structure such that the the vertex operators on W commute with the right actions of elements of P is called a grading-restricted (or lower-bounded) generalized (or ordinary) V-P-bimodule.

In this section, we shall consider mostly grading-restricted generalized V-modules and grading-restricted generalized V-P-bimodules. We shall also consider intertwining operators without logarithm and logarithmic intertwining operators. For simplicity, starting from now on, we shall call all these simply intertwining operators, no matter whether they contain or do not contain the logarithm of the variable.

Let W_1, W_2 and W_3 be grading-restricted generalized V-P-bimodules. For an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$ and $w_1 \in W_1$, we say that $\mathcal{Y}(w_1, x)$ is compatible with P or $\mathcal{Y}(w_1, x)$ is P-compatible if

$$(\mathcal{Y}(w_1, x)w_2)p = \mathcal{Y}(w_1, x)(w_2p)$$

for $w_2 \in W_2$ and $p \in P$. An intertwining operator of type $\binom{W_3}{W_1W_2}$ compatible with P or a P-intertwining operator of type $\binom{W_3}{W_1W_2}$ is an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1W_2}$ such that $\mathcal{Y}(w_1, x)$ is compatible with P for every $w_1 \in W_1$. More generally, let $W_1, \dots, W_n$, $\widetilde{W}_1, \dots, \widetilde{W}_{n-1}$ be grading-restricted generalized V-modules and let $\widetilde{W}_0$ and $\widetilde{W}_n$ be grading-restricted generalized V-P-bimodules. Let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be intertwining operators of type $\binom{\widetilde{W}_0}{W_1\widetilde{W}_1}, \dots, \binom{\widetilde{W}_{n-1}}{W_n\widetilde{W}_n}$, respectively. For $w_1 \in W_1, \dots, w_n \in W_n$, we say that the product $\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)$ is compatible with P or is P-compatible if

$$(\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)\tilde{w}_n)p = \mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)(\tilde{w}_n p)$$

for $\tilde{w}_n \in \widetilde{W}_n$ and $p \in P$. We say that the product of $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ is compatible with P if $\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)$ is compatible with P for all $w_1 \in W_1, \dots, w_n \in W_n$. If for $w_1 \in W_1, \dots, w_n \in W_n$, $\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)$ is compatible with P, then the coefficients of $\mathcal{Y}_1(w_1, x_1) \cdots \mathcal{Y}_n(w_n, x_n)$ are elements of $\mathrm{Hom}_P(\widetilde{W}_n, \widetilde{W}_0)$.

From [H5], we know that the conditions to apply the results in [HLZ1] and [HLZ2] are satisfied. In particular, the associativity and commutativity of intertwining operators hold. In [F1] and [F2], Fiordalisi studied these properties in the case of grading-restricted generalized V-P-bimodules and intertwining operators compatible with P. But the main results in [F1] and [F2] also hold in such more general settings with the same proofs except that the associativity and commutativity of intertwining operators compatible with P should be replaced by the versions of the associativity and commutativity below. For these associativity and commutativity of intertwining operators, we give complete proofs.

For simplicity, we use

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

to denote

$$\langle w_4', \mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)w_3\rangle\Big|_{x_1^n = e^{n\log z_1},\ x_2^n = e^{n\log z_2},\ \log x_1 = \log z_1,\ \log x_2 = \log z_2},$$

where for $z \in \mathbb{C}^\times$, $\log z = \log|z| + i\arg z$ for $0 \le \arg z < 2\pi$. Similarly, we have the notation

$$\langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle.$$

We shall use these and similar notations throughout the present paper.

Proposition 2.3 *Let $W_1, W_2, W_3, W_4, W_5, W_6$ be grading-restricted generalized V-modules and $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ intertwining operators of types $\binom{W_4}{W_1W_5}$, $\binom{W_5}{W_2W_3}$, $\binom{W_4}{W_6W_3}$ and $\binom{W_6}{W_2W_3}$, respectively, such that*

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ *and* $w_4' \in W_4'$ *in the region* $|z_1| > |z_2| > |z_1 - z_2| > 0$. *Assume that* W_3 *and* W_4 *are grading-restricted generalized* V-P-*bimodules. Then* $\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)$ *is compatible with* P *if and only if the coefficients of* $\mathcal{Y}_3(\mathcal{Y}_4(w_1, x_0)w_2, x_2)$ *as a formal series in powers of* x_0 *and nonnegative powers of* $\log x_0$ *is compatible with* P. *In particular, in the case that the product of* $\mathcal{Y}_1$ *and* Y_2 *is compatible with* P *and* W_6 *is spanned by the coefficients of* $\mathcal{Y}_4(w_1, x)w_2$ *for* $w_1 \in W_1$ *and* $w_2 \in W_2$, *the product of* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *is compatible with* P *if and only if* $\mathcal{Y}_3$ *is compatible with* P.

Proof. If $\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)$ is compatible with P, then in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have

$$\begin{aligned}
&\langle w_4', (\mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3)p\rangle \\
&\quad = \langle pw_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \\
&\quad = \langle pw_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \\
&\quad = \langle w_4', (\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3)p\rangle \\
&\quad = \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)(w_3p)\rangle \\
&\quad = \langle w_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)(w_3p)\rangle
\end{aligned} \tag{2.1}$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$, $w_4' \in W_4'$ and $p \in P$. Since both sides of (2.1) are convergent in the region $|z_2| > |z_1 - z_2| > 0$, the left-hand and right-hand sides of (2.1) are equal in this larger region. In this region, we can take the coefficients of $\mathcal{Y}_4(w_1, z_1 - z_2)w_2$ in both sides of (2.1). Thus the coefficients of $\mathcal{Y}_3(\mathcal{Y}_4(w_1, x_0)w_2, x_2)$ as a formal series in powers of x_0 and nonnegative powers of $\log x_0$ is compatible with P. If the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ is compatible with P and W_6 is spanned by the coefficients of $\mathcal{Y}_4(w_1, x_0)w_2$ for $w_1 \in W_1$ and $w_2 \in W_2$, then we obtain

$$\langle w_4', (\mathcal{Y}_3(w_6, z_2)w_3)p\rangle = \langle w_4', \mathcal{Y}_3(w_6, z_2)(w_3p)\rangle$$

for $w_3 \in W_3$, $w_6 \in W_6$, $w_4' \in W_4'$ and $p \in P$ in the region $z_2 \neq 0$. This shows that $\mathcal{Y}_3$ is compatible with P.

Conversely, if the coefficients of $\mathcal{Y}_3(\mathcal{Y}_4(w_1, x_0)w_2, x_2)$ as a series in powers of x_0 and nonnegative powers of $\log x_0$ are compatible with P, then in the region $|z_1| > |z_2| > |z_1 - z_2| > 0$, we have

$$\begin{aligned}
&\langle w_4', (\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3)p\rangle \\
&\quad = \langle pw_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \\
&\quad = \langle pw_4', \mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \\
&\quad = \langle w_4', (\mathcal{Y}_3(\mathcal{Y}_4(w_1, z_1 - z_2)w_2, z_2)w_3)p\rangle \\
&\quad = \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)(w_3p)\rangle \\
&\quad = \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)(w_3p)\rangle
\end{aligned} \tag{2.2}$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$, $w_4' \in W_4'$ and $p \in P$. Since both sides of (2.2) are convergent in the region $|z_1| > |z_2| > 0$, the left-hand and right-hand sides of (2.1) are equal in this larger region. This shows that $\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)$ is compatible with P. If $\mathcal{Y}_3$ is compatible with P, then the coefficients of $\mathcal{Y}_3(\mathcal{Y}_4(w_1, x_0)w_2, x_2)$ as a series in powers of x_0 and nonnegative powers of $\log x_0$ are compatible with P. Thus $\mathcal{Y}_1(w_1, x_1)\mathcal{Y}_2(w_2, x_2)$ is compatible with P for all $w_1 \in W_1$ and $w_2 \in W_2$, that is, the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ is compatible with P. ∎

For two analytic functions f and g on two regions, we shall use $f \sim g$ to mean that f and g are analytic extensions of each other.

Proposition 2.4 *Let $W_1, W_2, W_3, W_4, W_5, W_6$ be grading-restricted generalized V-modules and $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_5$ and $\mathcal{Y}_6$ intertwining operators of types $\binom{W_4}{W_1W_5}$, $\binom{W_5}{W_2W_3}$, $\binom{W_4}{W_2W_6}$ and $\binom{W_6}{W_1W_3}$, respectively, such that*

$$\langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \sim \langle w_4', \mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)w_3\rangle$$

for $w_1 \in W_1$, $w_2 \in W_2$, $w_3 \in W_3$ and $w_4' \in W_4'$. Assume that W_3 and W_4 are grading-restricted generalized V-P-bimodules. Then $\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)$ is compatible with P if and only if $\mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)$ is compatible with P. In particular, the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ is compatible with P if and only if the product of $\mathcal{Y}_5$ and $\mathcal{Y}_6$ is compatible with P.

Proof. We need only prove the "if" part; the "only if" part is obtained by symmetry.

If $\mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)$ is compatible with P, then we have

$$\begin{aligned}
&\langle w_4', (\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3)p\rangle\\
&\quad = \langle pw_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle\\
&\quad \sim \langle pw_4', \mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)w_3\rangle\\
&\quad = \langle w_4', (\mathcal{Y}_5(w_2, z_2)\mathcal{Y}_6(w_1, z_1)w_3)p\rangle\\
&\quad = \langle w_4', \mathcal{Y}_5(w_1, z_1)\mathcal{Y}_6(w_2, z_2)(w_3p)\rangle\\
&\quad \sim \langle w_4', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)(w_3p)\rangle.
\end{aligned} \tag{2.3}$$

Since both sides of (2.3) are analytic functions in the same region $|z_1| > |z_2| > 0$, $0 \leq \arg z_1, \arg z_2 < 2\pi$, these two sides must be equal, proving that $\mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)$ is compatible with P. If the product of $\mathcal{Y}_5$ and $\mathcal{Y}_6$ is compatible with P, then $\mathcal{Y}_1(w_1, z_2)\mathcal{Y}_2(w_2, z_2)$ is compatible with P for all $w_1 \in W_1$ and $w_2 \in W_2$. Thus the product of $\mathcal{Y}_1$ and $\mathcal{Y}_2$ is compatible with P. ∎

Let W be a grading-restricted generalized V-P-bimodules which is projective as a right P-module. Then for each $n \in \mathbb{C}$, the homogeneous subspace $W_{[n]}$ of W is a finite-dimensional right projective P-module. Let ϕ be a symmetric linear function on P. We have a pseudo-trace function $\phi_{W_{[n]}}$ on $\mathrm{End}_P W_{[n]}$ (see Subsection 2.1). We also know that W is of finite length. Let K be the length of W. Then we know that $L_W(0)_N^{K+1}$ is 0. For $f \in \mathrm{Hom}_P(W, \overline{W})$, define

$$\mathrm{Tr}_W^{\phi} f q^{L_W(0)} = \sum_{n\in\mathbb{C}}\sum_{k=0}^{K} \phi_{W_{[n]}}\left(\pi_n f \frac{L_W(0)_N^k}{k!}\right)\bigg|_{W_{[n]}} (\log q)^k q^n,$$

where for $n \in \mathbb{C}$, π_n is the projection from W to $W_{[n]}$.

For a lower-bounded generalized V-module W, as in [H2], let

$$\mathcal{U}_W(x) = (2\pi i x)^{L_W(0)} e^{-L_W^+(A)} \in (\mathrm{End}\, W)\{x\}[\log x],$$

where $(2\pi i)^{L_W(0)} = e^{(\log 2\pi + i\frac{\pi}{2})L_W(0)}$, $x^{L_W(0)} = x^{L_W(0)_S} e^{(\log x)L_W(0)_N}$, $L_W^+(A) = \sum_{j\in\mathbb{Z}_+} A_j L_W(j)$ and A_j for $j \in \mathbb{N}$ are given by

$$\frac{1}{2\pi i}\log(1+2\pi i y) = \left(\exp\left(\sum_{j\in\mathbb{Z}_+} A_j y^{j+1}\frac{\partial}{\partial y}\right)\right) y.$$

For $\mathcal{Y} = Y_W$, (1.5) in [H2] gives

$$\mathcal{U}_W(1) Y_W\left(v, \frac{1}{2\pi i}\log(1+x)\right) = Y_W((1+x)^{L_V(0)}\mathcal{U}_W(1)v, x)\mathcal{U}_W(1) \tag{2.4}$$

for $v \in V$. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W W_2}$, where W_2 and W_3 are also lower-bounded generalized V-modules. For $z \in \mathbb{C}$, let $q_z = e^{2\pi i z}$. Then as in [H2], we call $\mathcal{Y}(\mathcal{U}_{W_1}(q_z)w_1, q_z)$ a geometrically-modified intertwining operator.

Let $W_1, \dots, W_n, \widetilde{W}_1, \dots, \widetilde{W}_{n-1}$ be grading-restricted generalized V-modules and let $\widetilde{W}_0 = \widetilde{W}_n$ be a grading-restricted generalized V-P-bimodule which is projective as a right P-module. Let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be intertwining operators of types $\binom{\widetilde{W}_0}{W_1 \widetilde{W}_1}, \dots, \binom{\widetilde{W}_{n-1}}{W_n \widetilde{W}_n}$, respectively. We assume that

$$\mathcal{Y}_1(\mathcal{U}_{W_1}(q_{z_1})w_1, x_1)\cdots\mathcal{Y}_n(\mathcal{U}_{W_n}(q_{z_n})w_n, x_n) \tag{2.5}$$

is compatible with P. If the product of $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ is compatible with P, then this assumption is true for any $w_1 \in W_1, \dots, w_n \in W_n$. Since the product

$$\mathcal{Y}_1(\mathcal{U}_{W_1}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}_{W_n}(q_{z_n})w_n, q_{z_n})$$

of geometrically-modified intertwining operators is absolutely convergent to an element of $\mathrm{Hom}(W, \overline{W})$, it is in fact absolutely convergent to an element of $\mathrm{Hom}_P(W, \overline{W})$. Then we have the pseudo-q_τ-trace shifted by $-\frac{c}{24}$ or simply the shifted pseudo-q_τ-trace

$$\mathrm{Tr}^{\phi}_{\tilde{W}_n}\mathcal{Y}_1(\mathcal{U}_{W_1}(q_{z_1})w_1, q_{z_1})\cdots\mathcal{Y}_n(\mathcal{U}_{W_n}(q_{z_n})w_n, q_{z_n})q_\tau^{L(0)-\frac{c}{24}} \tag{2.6}$$

of products of n geometrically-modified intertwining operators.

We now state several results of Fiordalisi in [F1] and [F2] generalizing the corresponding results in [H2] in the semisimple case. As we discussed above, our statements of these results are slightly more general than those in [F1] and [F2] but the proofs there in fact already gave these results.

Theorem 2.5 (Convergence and analytic extension [F1] [F2]) *For $w_1 \in W_1, \dots, w_n \in W_n$ such that (2.5) is compatible with P, the series (2.6) is absolutely convergent in the region $1 > |q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ and can be analytically extended to a multivalued analytic function in the region $\Im(\tau) > 0$, $z_i \neq z_j + k\tau + l$ for $i \neq j$, $k, l \in \mathbb{Z}$.*

For $w_1 \in W_1, \dots, w_n \in W_n$, we denote the multivalued analytic function in Theorem 2.5 by

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau). \tag{2.7}$$

Note that the multivalued analytic function (2.7) has a particular branch (usually called a preferred branch by the author) in the region $|q_{z_1}| > \cdots > |q_{z_n}| > |q_\tau| > 0$ given by (2.6). Such a function of $z_1, \dots, z_n$ and τ is called a genus-one n-point correlation function from a shifted pseudo-q_τ-trace.

Theorem 2.6 (Genus-one commutativity [F1] [F2]) *For $w_1 \in W_1, \dots, w_n \in W_n$ such that (2.5) is compatible with P and for $1 \le k \le n-1$, there exist grading-restricted generalized V-modules $\widehat{W}_k$ and intertwining operators $\widehat{\mathcal{Y}}_k$ and $\widehat{\mathcal{Y}}_{k+1}$ of types $\binom{\widehat{W}_k}{W_k \widetilde{W}_{k+1}}$ and $\binom{\widetilde{W}_{k-1}}{W_{k+1}\widehat{W}_k}$, respectively, such that*

$$\mathcal{Y}_1(\mathcal{U}_{W_1}(q_{z_1})w_1, x_1) \cdots \mathcal{Y}_{k-1}(\mathcal{U}_{W_{k-1}}(q_{z_{k-1}})w_{k-1}, x_{k-1})\widehat{\mathcal{Y}}_{k+1}(\mathcal{U}_{W_{k+1}}(q_{z_{k+1}})w_{k+1}, x_{k+1})\cdot$$
$$\cdot \widehat{\mathcal{Y}}_k(\mathcal{U}_{W_k}(q_{z_k})w_k, x_k)\mathcal{Y}_{k+2}(\mathcal{U}_{W_{k+2}}(q_{z_{k+2}})w_{k+2}, x_{k+2}) \cdots \mathcal{Y}_n(\mathcal{U}_{W_n}(q_{z_n})w_n, x_n)$$

is compatible with P and

$$\begin{aligned}\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}&(w_1, \dots, w_n; z_1, \dots, z_n; \tau)\\ &= \overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\widehat{\mathcal{Y}}_{k+1},\widehat{\mathcal{Y}}_k,\mathcal{Y}_{k+2}\dots,\mathcal{Y}_n}(w_1, \dots, w_{k-1}, w_{k+1}, w_k, w_{k+2}, \dots, w_n;\\ &\qquad\qquad z_1, \dots, z_{k-1}, z_{k+1}, z_k, z_{k+2}, \dots, z_n; \tau).\end{aligned}$$

More generally, for any $\sigma \in S_n$, there exist grading-restricted generalized V-modules $\widehat{W}_i$ for $i = 1, \dots, n-1$ and intertwining operators $\widehat{\mathcal{Y}}_i$ of types $\binom{\widehat{W}_{i-1}}{W_{\sigma(i)}\widehat{W}_i}$ for $i = 1, \dots, n$ (where $\widehat{W}_0 = \widehat{W}_n = \widetilde{W}_0 = \widetilde{W}_n$), respectively, such that

$$\widehat{\mathcal{Y}}_1(\mathcal{U}_{W_{\sigma(1)}}(q_{z_{\sigma(1)}})w_{\sigma(1)}, x_{\sigma(1)}) \cdots \widehat{\mathcal{Y}}_n(\mathcal{U}_{W_{\sigma(n)}}(q_{z_{\sigma(n)}})w_{\sigma(n)}, x_{\sigma(n)})$$

is compatible with P and

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1, \dots, w_n; z_1, \dots, z_n; \tau) = \overline{F}^{\phi}_{\widehat{\mathcal{Y}}_1,\dots,\widehat{\mathcal{Y}}_n}(w_{\sigma(1)}, \dots, w_{\sigma(n)}; z_{\sigma(1)}, \dots, z_{\sigma(n)}; \tau).$$

Theorem 2.7 (Genus-one associativity [F1] [F2]) *For $w_1 \in W_1, \dots, w_n \in W_n$ such that (2.5) is compatible with P and for $1 \le k \le n-1$, there exist a grading-restricted generalized V-module $\widehat{W}_k$ and intertwining operators $\widehat{\mathcal{Y}}_k$ and $\widehat{\mathcal{Y}}_{k+1}$ of types $\binom{\widehat{W}_k}{W_k W_{k+1}}$ and $\binom{\widetilde{W}_{k-1}}{\widehat{W}_k \widetilde{W}_{k+1}}$, respectively, such that the coefficients as a series in powers of x_0 and nonnegative powers of $\log x_0$ of*

$$\mathcal{Y}_1(\mathcal{U}_{W_1}(q_{z_1})w_1, x_1) \cdots \mathcal{Y}_{k-1}(\mathcal{U}_{W_{k-1}}(q_{z_{k-1}})w_{k-1}, x_{k-1})\widehat{\mathcal{Y}}_{k+1}(\mathcal{U}_{W_{k+1}}(q_{z_{k+1}})\widehat{\mathcal{Y}}_k(w_k, x_0)w_{k+1}, x_{k+1})\cdot$$
$$\cdot \mathcal{Y}_{k+2}(\mathcal{U}_{W_{k+2}}(q_{z_{k+2}})w_{k+2}, x_{k+2}) \cdots \mathcal{Y}_n(\mathcal{U}_{W_n}(q_{z_n})w_n, x_n)$$

is compatible with P *and*

$$\begin{aligned}
&\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\widehat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_n}(w_1,\dots,w_{k-1},\widehat{\mathcal{Y}}(w_k,z_k-z_{k+1})w_{k+1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad w_{k+2},\dots,w_n;z_1,\dots,z_{k-1},z_{k+1},\dots,z_n;\tau)\\
&\quad=\sum_{r\in\mathbb{R}}\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_{k-1},\widehat{\mathcal{Y}}_{k+1},\mathcal{Y}_{k+2},\dots,\mathcal{Y}_n}(w_1,\dots,w_{k-1},\pi_r(\widehat{\mathcal{Y}}(w_k,z_k-z_{k+1})w_{k+1}),\\
&\qquad\qquad\qquad\qquad\qquad\qquad w_{k+2},\dots,w_n;z_1,\dots,z_{k-1},z_{k+1},\dots,z_n;\tau)
\end{aligned}$$

is absolutely convergent in the region $1>|q_{z_1}|>\cdots>|q_{z_{k-1}}|>|q_{z_{k+1}}|>\cdots>|q_{z_n}|>|q_\tau|>0$ *and* $1>|q_{(z_k-z_{k+1})}-1|>0$ *and is convergent to*

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}(w_1,\dots,w_n;z_1,\dots,z_n;\tau)$$

in the region $1>|q_{z_1}|>\cdots>|q_{z_n}|>|q_\tau|>0$*,* $|q_{(z_k-z_{k+1})}|>1>|q_{(z_k-z_{k+1})}-1|>0$*.*

3 Associative algebras, lower-bounded generalized V-modules and intertwining operators

In [H2], for a vertex operator algebra V, an associative algebra $\widetilde{A}(V)$ isomorphic to the Zhu algebra $A(V)$, $\widetilde{A}(V)$-modules and $\widetilde{A}(V)$-bimodules are introduced and used in the proof in the same paper of the modular invariance conjecture of Moore and Seiberg for rational conformal field theories. In [H8] and [H9], the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N\in\mathbb{N}$, their graded modules and their bimodules associated to a lower-bounded generalized V-module W are introduced and studied. In this section, we first prove more results on the $A^\infty(V)$-bimodule $A^\infty(W)$ and the $A^N(V)$-bimodules $A^N(W)$ for $N\in\mathbb{N}$, which will be needed in Section 4. We then introduce associative algebras $\widetilde{A}^\infty(V)$ and $\widetilde{A}^N(V)$ for $N\in\mathbb{N}$ isomorphic to the associative algebras $A^\infty(V)$ and $A^N(V)$ for $N\in\mathbb{N}$. As in [H2], these algebras and their modules can be obtained by using some operators corresponding to a canonical conformal transformation from an annulus to a parallelogram on V and on lower-bounded generalized V-modules, respectively. We then transport the results obtained in [H8] and [H9] using these operators to results on $\widetilde{A}^\infty(V)$, $\widetilde{A}^N(V)$ for $N\in\mathbb{N}$, their modules and their bimodules. We refer the reader to [H8] and [H9] for the basic material and notations on these associative algebras, their modules and bimodules.

In this section, we in general do not assume that V is C_2-cofinite. But we will prove some results needed in later sections when V is C_2-cofinite.

Let W be a lower-bounded generalized V -module. Then we have

$$W=\coprod_{\mu\in\Gamma(W)}\coprod_{n\in\mathbb{N}}W_{[h^\mu+n]}=\coprod_{n\in\mathbb{N}}W_{\llbracket n\rrbracket},$$

where

$$\Gamma(W)=\{\mu\in\mathbb{C}/\mathbb{Z}\mid \text{there exist nonzero elements of } W \text{ of weights in } \mu\},$$

$h^\mu \in \mu$ such that $W_{[h^\mu]} \neq 0$ but $W_{[h^\mu - n]} = 0$ for $n \in \mathbb{Z}_+$, and

$$W_{\llbracket n \rrbracket} = \coprod_{\mu \in \Gamma(W)} W_{[h^\mu + n]}.$$

See [H9].

We first prove some results on $A^\infty(W)$. Recall from [H9] that $U^\infty(W)$ is the space of column-finite infinite matrices with entries in W, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_+$. Recall also that for $w \in W$ and $k, l \in \mathbb{N}$, $[w]_{kl}$ is the matrix with the (k, l)-entry being w and all the other entries being 0. Elements of $U^\infty(W)$ are suitable (possibly infinite) sums of elements of the form $[w]_{kl}$ for $w \in W$, $k, l \in \mathbb{N}$.

Let $O^\infty(W)$ be the subspace of $U^\infty(W)$ spanned by infinite linear combinations of elements of the form

$$\mathrm{Res}_x x^{-k-l-p-2}(1+x)^l [Y_W((1+x)^{L_V(0)}v, x)w]_{kl}$$

for $v \in V$, $w \in W$, $k, l, p \in \mathbb{N}$, with each pair (k, l) appearing in the linear combinations only finitely many times.

Proposition 3.1 *We have $O^\infty(W) \subset Q^\infty(W)$.*

Proof. From the definition of $Q^\infty(W)$, we need to prove $\vartheta_{\mathcal{Y}}(O^\infty(V)) = 0$ for every pair of lower-bounded generalized V-modules W_2, W_3 and every intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{WW_2}$. For

$$\mathrm{Res}_{x_0} x_0^{-k-l-p-2}(1+x_0)^l [Y_W((1+x_0)^{L(0)}v, x_0)w]_{kl} \in O^\infty(W),$$

where $v \in V$, $w \in W$, $k, l, p \in \mathbb{N}$, and for $\mu \in \Gamma(W_2)$, $w_2 \in (W_2)_{[h_2^\mu + l]} \subset (W_2)_{\llbracket l \rrbracket}$, we have

$$\begin{aligned}
&\vartheta_{\mathcal{Y}}(\mathrm{Res}_{x_0} x_0^{-k-l-p-2}(1+x_0)^l [Y_V((1+x_0)^{L(0)}v, x_0)w]_{kl})[w_2]_l \\
&\quad = \sum_{\nu \in \Gamma(W_3)} \mathrm{Coeff}^0_{\log x_2} \mathrm{Res}_{x_2} x_2^{h_2^\mu - h_3^\nu + l - k - 1} \mathrm{Res}_{x_0} x_0^{-k-l-p-2}(1+x_0)^l \cdot \\
&\qquad\qquad \cdot [\mathcal{Y}(x_2^{L_V(0)} Y_W((1+x_0)^{L(0)}v, x_0)w, x_2)w_2]_k \\
&\quad = \sum_{\nu \in \Gamma(W_3)} \mathrm{Coeff}^0_{\log x_2} \mathrm{Res}_{x_2} x_2^{h_2^\mu - h_3^\nu + l - k - 1} \mathrm{Res}_{x_0} x_0^{-k-l-p-2}(1+x_0)^l \cdot \\
&\qquad\qquad \cdot [\mathcal{Y}(Y_W(x_2^{L_V(0)}(1+x_0)^{L(0)}v, x_0x_2) x_2^{L_V(0)}w, x_2)w_2]_k \\
&\quad = \sum_{\nu \in \Gamma(W_3)} \mathrm{Res}_{x_0} \mathrm{Coeff}^0_{\log x_2} \mathrm{Res}_{x_2} x_0^{-k-l-p-2} x_2^{h_2^\mu - h_3^\nu - k - 1} \mathrm{Res}_{x_1} x_1^l x_1^{-1} \delta\left(\frac{x_2 + x_0x_2}{x_1}\right) \cdot \\
&\qquad\qquad \cdot [\mathcal{Y}(Y_W(x_1^{L_V(0)}v, x_0x_2) x_2^{L_V(0)}w, x_2)w_2]_k \\
&\quad = \sum_{\nu \in \Gamma(W_3)} \mathrm{Res}_{x_0} \mathrm{Coeff}^0_{\log x_2} \mathrm{Res}_{x_2} x_0^{-k-l-p-2} x_2^{h_2^\mu - h_3^\nu - k - 1} \mathrm{Res}_{x_1} x_1^l x_0^{-1} x_2^{-1} \delta\left(\frac{x_1 - x_2}{x_0x_2}\right) \cdot \\
&\qquad\qquad \cdot [Y_{W_3}(x_1^{L_V(0)}v, x_1) \mathcal{Y}(x_2^{L_W(0)}w, x_2)w_2]_k
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{\nu\in\Gamma(W_3)}\mathrm{Res}_{x_0}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}x_0^{-k-l-p-2}x_2^{h_2^\mu-h_3^\nu-k-1}\mathrm{Res}_{x_1}x_1^l x_0^{-1}x_2^{-1}\delta\left(\frac{x_2-x_1}{-x_0x_2}\right)\cdot\\
&\qquad\cdot[\mathcal{Y}(x_2^{L_W(0)}w,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2]_k\\
&=\sum_{\nu\in\Gamma(W_3)}\mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}x_1^{-k-p-2}(1-x_1^{-1}x_2)^{-k-l-p-2}x_2^{h_2^\mu-h_3^\nu+l+p}\cdot\\
&\qquad\cdot[Y_{W_3}(x_1^{L_V(0)}v,x_1)\mathcal{Y}(x_2^{L_W(0)}w,x_2)w_2]_k\\
&-\sum_{\nu\in\Gamma(W_3)}\mathrm{Res}_{x_1}\mathrm{Coeff}^0_{\log x_2}\mathrm{Res}_{x_2}(-1+x_1x_2^{-1})^{-k-l-p-2}x_1^l x_2^{h_2^\mu-h_3^\nu-k-2}\cdot\\
&\qquad\cdot[\mathcal{Y}(x_2^{L_W(0)}w,x_2)Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2]_k. \qquad (3.1)
\end{aligned}
$$

Since $w_2\in(W_2)_{[h_2^\mu+l]}$ and the series $(1-x_1^{-1}x_2)^{-k-l-p-2}$ contains only nonnegative powers of x_2,

$$\mathrm{Res}_{x_2}(1-x_1^{-1}x_2)^{-k-l-p-2}x_2^{h_2^\mu-h_3^\nu+l+p}\mathcal{Y}(x_2^{L_V(0)}w,x_2)w_2=0.$$

So the first term in the right-hand side of (3.1) is 0. Since $w_2\in(W_2)_{[h_2^\mu+l]}$ and the series $(-1+x_1x_2^{-1})^{-k-l-p-2}$ contains only nonnegative powers of x_1,

$$\mathrm{Res}_{x_1}(-1+x_1x_2^{-1})^{-k-l-p-2}x_1^l Y_{W_2}(x_1^{L_V(0)}v,x_1)w_2=0.$$

So the second term in the right-hand side of (3.1) is also 0. Thus we have $\vartheta_{\mathcal{Y}}(O^\infty(W))=0$.
∎

Let $U_{NN}(W)$ be the subspace of $U^\infty(W)$ consisting of matrices in $U^\infty(W)$ with the (N,N)-entry to be the only nonzero entry, $Q_{NN}(W)=Q^\infty(W)\cap U_{NN}(W)$ and $O_{NN}(W)=O^\infty(W)\cap U_{NN}(W)$.

Proposition 3.2 *For $N\in\mathbb{N}$, $Q_{NN}(W)=O_{NN}(W)$.*

Proof. By Proposition 3.1, $O_{NN}(W)\subset Q_{NN}(W)$. So we need only prove $Q_{NN}(W)\subset O_{NN}(W)$. We shall use the results on the $A_N(V)$-bimodule $A_N(W)$ introduced in [HY] to prove this proposition. See [HY] for the basic definitions and results on $A_N(V)$, $A_N(W)$ and other structures.

For a lower-bounded generalized V-module W, by Theorem 6.1 in [HY], we have a lower-bounded generalized V-module $S_N(G_N(W))$ such that $G_N(S_N(G_N(W)))$ is equivalent to $G_N(W)$ as modules for $A_N(V)$. In fact, in the construction in [HY], $G_N(W)$ can be an arbitrary $A_N(V)$-module M and we obtain a lower-bounded generalized V-module $S_N(M)$ such that $G_N(S_N(M))$ is equivalent to M as $A_N(V)$-modules and satisfies the following universal property: For any lower-bounded generalized V-module W and any $A_N(V)$-module map $\phi: M\to G_N(W)$, there is a unique V-module map $\bar{\phi}: S_N(M)\to W$ such that $\bar{\phi}|_{G_N(W)}=\phi$. Note that $S_N(M)$ can also be constructed using the method in Section 5 of [H6].

We view $A_N(V)$ as a left $A_N(V)$-module. Then we obtain a lower-bounded generalized V-module $S_N(A_N(V))$. Let $W_2 = S_N(A_N(V))$. From the construction, we have $G_N(W_2) = A_N(V)$.

We now construct a lower-bounded generalized V-module W_3 such that $G_N(W_3) = A_N(W) \otimes_{A_N(V)} A_N(V)$ and an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{WW_2}$. We have

$$W = \sum_{\mu\in\Gamma(W)} \coprod_{n\in\mathbb{N}} W_{[h^\mu+n]}$$

and

$$W_2 = \sum_{\nu\in\Gamma(W_2)} \coprod_{n\in\mathbb{N}} (W_2)_{[h_2^\nu+n]}.$$

Fix $h_3 \in \mathbb{C}$. Consider the space

$$M = \coprod_{\mu\in\Gamma(W),\nu\in\Gamma(W_2)} W \otimes t^{-(h_3-h^\mu-h_2^\nu)}\mathbb{C}[t,t^{-1}] \otimes W_2.$$

We shall write an element of M of the form $w \otimes t^n \otimes w_2$ as $w(n,0)w_2$ for $w \in W$, $n \in -(h_3 - h^\mu - h_2^\nu) + \mathbb{Z}$ and $w_2 \in W_2$. We define the weight of the element $w(n,0)w_2$ to be $\mathrm{wt}\ w - n - 1 + \mathrm{wt}\ w_2$ when w and w_2 are homogeneous. We define $L_M(0)_S$ be the operator on M given by this weight grading. We also define an operator $L_M(0)_N$ on M by

$$L_M(0)_N w(n,0)w_2 = (L_W(0)_N w)(n,0)w_2 + w(n,0)(L_{W_2}(0)_N w_2).$$

Then we have an operator $L_M(0) = L_M(0)_S + L_M(0)_N$.

From M with the grading defined above and the operator $L_M(0)$, $g = 1_V$ and $B = h_3$, we obtain a universal generalized lower-bounded V-module $\widehat{W}_3 = \widehat{M}_{h_3}^{1_V}$ using the construction given in Subsection 4.2 of [H7] based on the construction in Section 5 of [H6]. Let

$$\widehat{\mathcal{Y}}^0(w,x)w_2 = \sum_{n\in-(h_3-h^\mu-h_2^\nu)+\mathbb{N}} w(n,0)w_2 x^{-n-1}$$

for $w \in \coprod_{n\in\mathbb{N}} W_{[h^\mu+n]}$, $w_2 \in \coprod_{n\in\mathbb{N}} W_{[h_2^\nu+n]}$ and

$$\widehat{\mathcal{Y}}(w,x)w_2 = x^{L_{\widehat{W}_3}(0)} \widehat{\mathcal{Y}}^0(x^{-L_W(0)}w, 1)x^{-L_{W_2}(0)}w_2$$

for $w \in W$ and $w_2 \in W_2$. We define $o_{\widehat{\mathcal{Y}}}(w)w_2$ to be $w(\mathrm{wt}\ w + \mathrm{wt}\ w_2 - h_3 - N - 1, 0)w_2$ for homogeneous $w \in W$ and $w_2 \in G_N(W_2) = A_N(V)$ and extend the definition linearly to general $w \in W$ and $w_2 \in A_N(V)$. Let J_1 be the generalized V-submodule of $\widehat{W}_3$ generated by the elements of the forms $o_{\widehat{\mathcal{Y}}}(w)w_2$ for $w \in O_N(W)$ and $w_2 \in A_N(V)$ and $o_{\widehat{\mathcal{Y}}}(w \circ_N v)w_2 - o_{\widehat{\mathcal{Y}}}(w)((v + O_N(V)) \circ_N w_2)$ for $w \in W$, $v \in V$ and $w_2 \in A_N(V)$ and the coefficients of the formal series of the following form

$$\frac{d}{dx}\widehat{\mathcal{Y}}(w,x)w_2 - \widehat{\mathcal{Y}}(L_W(-1)w,x)w_2,$$

$$Y_{\widehat{W}_3}(v,x_1)\widehat{\mathcal{Y}}(w,x_2)w_2-\widehat{\mathcal{Y}}(w,x_2)Y_{W_2}(v,x_1)w_2-\mathrm{Res}_{x_0}x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)\widehat{\mathcal{Y}}(Y_W(v,x_0)w,x_2)w_2$$

for $v \in V$, $w \in W$ and $w_2 \in W_2$. Then the lower-bounded generalized V-module $\widehat{W_3}/J_1$ is generated by the coefficients of formal series of the form $\widehat{\mathcal{Y}}(w,x)w_2 + J_1$ for $w \in W$ and $w_2 \in A_N(V)$. Moreover, elements of $\widehat{W_3}/J_1$ of the form $o_{\widehat{\mathcal{Y}}}(w)(v + O_N(V))$ for $w \in W$ and $v \in V$ can be written uniquely as $o_{\widehat{\mathcal{Y}}}(w \circ_N v)(\mathbf{1} + O_N(V))$. From the definition of J_1, we see that $o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1} + O_N(V))$ is not in J_1 for $w \in W$ but not in $O_N(W)$.

We define a linear map from $A_N(W) = A_N(W) \otimes_{A_N(V)} A_N(V)$ to $\widehat{W_3}/J_1$ by

$$w + O_N(W) \mapsto o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1} + O_N(V)) + J_1.$$

Then by the discussion above, we see that this map is injective. In particular, we can identify the subspace D of $\widehat{W_3}/J_1$ consisting of elements of the form $o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1} + O_N(V)) + J_1$ with $A_N(W)$. Since by construction, elements of the form $o_{\widehat{\mathcal{Y}}}(w)(v + O_N(V)) + J_1$ for $w \in W$ and $v \in V$ is in $G_N(\widehat{W_3}/J_1)$, we have $D \subset G_N(\widehat{W_3}/J_1)$.

Let J_2 be the generalized V-submodule of $\widehat{W_3}/J_1$ generated by the coefficients of the formal series of the form

$$(x_0+x_2)^{\mathrm{wt}\ v+N}\widehat{\mathcal{Y}}(Y_W(v,x_0)w,x_2)w_2-(x_0+x_2)^{\mathrm{wt}\ v+N}Y_{\widehat{W}_3}(v,x_0+x_2)\widehat{\mathcal{Y}}(w,x_2)w_2+J_1$$

for homogeneous $v \in V$, $w \in W$ and $w_2 \in A_N(V)$. Let $W_3 = (\widehat{W_3}/J_1)/J_2$. Then W_3 is a lower-bounded generalized V-module. Let

$$\begin{aligned}\mathcal{Y}: W \otimes W_2 &\to W_3\{x\}[\log x]\\ w \otimes w_2 &\mapsto \mathcal{Y}(w,x)w_2\end{aligned}$$

be a linear map defined by

$$\mathcal{Y}(w,x)w_2 = (\widehat{\mathcal{Y}}(w,x)w_2 + J_1) + J_2$$

for $w \in W$ and $w_2 \in W_2$. By the definitions of J_1 and J_2, we see that $\mathcal{Y}$ satisfies the lower-truncation property, the $L(-1)$ property, the commutator formula for one intertwining operator and the weak associativity for one intertwining operator when acting on $A_N(V)$. The commutativity for one intertwining operator and generalized rationality for one intertwining operator follows from the commutator formula for one intertwining operator. Using this commutativity, we obtain the weak associativity for one intertwining operator acting on W_2. The weak associativity for one intertwining operator gives the associativity for one intertwining operator. Since the lower-truncation property, the $L(-1)$-derivative property, the generalized rationality, commutativity and associativity for one intertwining operator holds for $\mathcal{Y}$, we see that $\mathcal{Y}$ is an intertwining operator of type $\binom{W_3}{WW_2}$.

We want to show that $D \cap J_2 = 0$ and then we can view D as a subspace of $G_N(W_3)$. We consider the graded dual space $(\widehat{W_3}/J_1)'$ with respect to the $\mathbb{N}$-grading of $\widehat{W_3}/J_1$. Given an element $d^* \in D^*$, we extend it to an element of $G_N((\widehat{W}_3/J_1)')$ as follows: For $w \in W$, $v \in V$, $w_2 \in A_N(V)$ and z_1, z_2 satisfying $|z_2| > |z_1 - z_2| > 0$,

$$\langle d^*, \widehat{\mathcal{Y}}(Y_W(v, z_1 - z_2)w, z)w_2 + J_1\rangle$$

is well defined. On the other hand, since the commutator formula for $Y_{\widehat{W}_3}(v, z_1)$ and $\widehat{\mathcal{Y}}(w, z_2)$ holds modulo J_1, $Y_{\widehat{W}_3}(v, z_1)\widehat{\mathcal{Y}}(w, z_2)w_2 + J_1$ is absolutely convergent to an element of the algebraic completion of $\widehat{W}_3/J_1$. We define

$$\langle d^*, Y_{\widehat{W}_3}(v, z_1)\widehat{\mathcal{Y}}(w, z_2)w_2 + J_1\rangle = \langle d^*, \widehat{\mathcal{Y}}(Y_V(v, z_1 - z_2)w, z)w_2 + J_1\rangle$$

for $z_1, z_2 \in \mathbb{C}$ satisfying $|z_1| > |z_2| > |z_1 - z_2| > 0$. Since the homogeneous components of $Y_{\widehat{W}_3}(v, z_1)\widehat{\mathcal{Y}}(w, z_2)w_2 + J_1$ for $v \in V$, $w \in W$ and $w_2 \in A_N(V)$ span $\widehat{W}_3/J_1$, d^* gives an element of $G_N((\widehat{W}_3/J_1)')$. Thus we can identify D^* with a subspace of $G_N((\widehat{W}_3/J_1)')$.

We define a subspace J_3 of $\widehat{W}_3/J_1$ to be the subspace annihilated by D^*, that is

$$J_3 = \{\hat{w}_3 + J_1 \mid \hat{w}_3 \in \hat{W}_3,\ \langle d^*, \hat{w}_3 + J_1\rangle = 0 \text{ for } d^* \in D^*\}.$$

We now show that $J_2 \subset J_3$. The space J_2 is spanned by the coefficients of the formal series

$$\begin{aligned}&(x_0 + x_2)^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u, x)\widehat{\mathcal{Y}}(Y_W(v, x_0)w, x_2)w_2\\ &\quad - (x_0 + x_2)^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u, x)Y_{\widehat{W}_3}(v, x_0 + x_2)\widehat{\mathcal{Y}}(w, x_2)w_2 + J_1\end{aligned} \tag{3.2}$$

for $u, v \in V$ (with v being homogeneous), $w \in W$ and $w_2 \in A_N(V)$. When we substitute z, $z_1 - z_2$ and z_2 for x, x_0 and x_2, where z, z_1 and z_2 are complex numbers satisfying $|z| > |z_1| > |z_2| > |z_1 - z_2| > 0$, (3.2) is absolutely convergent to element

$$z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u, z)\widehat{\mathcal{Y}}(Y_W(v, z_1 - z_2)w, z_2)w_2 - z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u, z)Y_{\widehat{W}_3}(v, z_1)\widehat{\mathcal{Y}}(w, z_2)w_2 + \overline{J}_1 \tag{3.3}$$

of the algebraic completion of $\widehat{W}_3/J_1$, where $\overline{J}_1$ is the algebraic completion of J_1. Moreover, the homogeneous components of these elements in the algebraic completion of $\widehat{W}_3/J_1$ also span J_2. For $d^* \in D^*$, by the definition of d^* as an element of $G_N((\widehat{W}_3/J_1)')$ and the associatitivy for the vertex operator map Y_W and $Y_{\widehat{W}_3}$, we have

$$\begin{aligned}&\langle d^*, z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u, z)\widehat{\mathcal{Y}}(Y_V(v, z_1 - z_2)w, z_2)w_2 + \overline{J}_1\rangle\\ &\quad \sim z_1^{\mathrm{wt}\ v+N}\langle d^*, \widehat{\mathcal{Y}}(Y_W(u, z - z_2)Y_W(v, z_1 - z_2)w, z_2)w_2 + \overline{J}_1\rangle\\ &\quad \sim z_1^{\mathrm{wt}\ v+N}\langle d^*, \widehat{\mathcal{Y}}(Y_W(Y_V(u, z - z_1)v, z_1 - z_2)w, z_2)w_2 + \overline{J}_1\rangle\\ &\quad \sim z_1^{\mathrm{wt}\ v+N}\langle d^*, Y_{\widehat{W}_3}(Y_V(u, z - z_1)v, z_1)\widehat{\mathcal{Y}}(w, z_2)w_2 + \overline{J}_1\rangle\\ &\quad \sim \langle d^*, z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u, z)Y_{\widehat{W}_3}(v, z_1)\widehat{\mathcal{Y}}(w, z_2)w_2 + \overline{J}_1\rangle,\end{aligned} \tag{3.4}$$

where $\sim$ means "can be analytically extended to." Note that z_2 can be fixed for each of the analytic extension step in (3.4). So the analytic extensions in (3.4) do not change the value of $\log z_2$. Since the left-hand side and right-hand side of (3.4) are convergent absolutely in the region $|z| > |z_2| > |z_1 - z_2| > 0$ and $|z| > |z_1| > |z_2| > 0$, respectively, we see that in the region $|z| > |z_1| > |z_2| > |z_1 - z_2| > 0$, we have

$$\begin{aligned}&\langle d^*, z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u,z)\widehat{\mathcal{Y}}(Y_V(v,z_1-z_2)w,z_2)w_2 + \overline{J}_1\rangle\\ &\quad = \langle d^*, z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u,z)Y_{\widehat{W}_3}(v,z_1)\widehat{\mathcal{Y}}(w,z_2)w_2 + \overline{J}_1\rangle. \qquad (3.5)\end{aligned}$$

For such z, z_1 and z_2, we can rewrite (3.5) as

$$\begin{aligned}&\Big\langle d^*, \Big(z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u,z)\widehat{\mathcal{Y}}(Y_V(v,z_1-z_2)w,z_2)\\ &\quad - z_1^{\mathrm{wt}\ v+N} Y_{\widehat{W}_3}(u,z)Y_{\widehat{W}_3}(v,z_1)\widehat{\mathcal{Y}}(w,z_2)w_2 + \overline{J}_1\Big)\Big\rangle = 0. \qquad (3.6)\end{aligned}$$

Since J_2 is spanned by the homogeneous components of elements of the form (3.3), we see from (3.6) that $J_2 \subset J_3$. Then $D \cap J_2 \subset D \cap J_3 = 0$. So we can view D as a subspace of $G_N(W_3)$.

From Proposition 5.7 in [HY], we have an $A_N(V)$-module map

$$\rho(\mathcal{Y}) : A_N(W) \otimes_{A_N(V)} \Omega_N^0(W_2) \to \Omega_N^0(W_3)$$

defined by

$$\begin{aligned}\rho(\mathcal{Y})((w + O_N(W)) \otimes w_2) &= \sum_{n=0}^{N} \mathrm{Res}_x x^{-h_3-n-1}\mathcal{Y}^0(x^{L_W(0)_s}w, x)x^{L_{W_2}(0)_s}w_2\\ &= \sum_{n=0}^{N} \mathcal{Y}_{\mathrm{wt}\ w+\mathrm{wt}\ w_2-h_3-n-1,0}(w)w_2,\end{aligned}$$

for homogeneous $w \in W$, $w_{(2)} \in \Omega_N^0(W_2)$. Since $G_N(W_2)$ is an $A_N(V)$-submodule of $\Omega_N^0(W_2)$ and the projection $\pi_{G_N(W_3)}$ from $\Omega_N^0(W_3)$ to $G_N(W_3)$ is an $A_N(V)$-module map, we obtain an $A_N(V)$-module map

$$f = \pi_{G_N(W_3)} \circ \rho(\mathcal{Y}) \circ (1_{A_N(W)} \otimes_{A_N(V)} e_{G_N(W_2)}) : A_N(W) \otimes_{A_N(V)} G_N(W_2) \to G_N(W_3),$$

where $e_{G_N(W_2)}$ is the embedding map from $G_N(W_2)$ to $\Omega_N^0(W_2)$. Since $G_N(W_2) = A_N(V)$, we see that

$$A_N(W) \otimes_{A_N(V)} G_N(W_2) = A_N(W) \otimes_{A_N(V)} (\mathbf{1} + O_N(V))$$

is equivalent as a left $A_N(V)$-module to $A_N(W)$. In particular, f can be viewed as an $A_N(V)$-module map from $A_N(W)$ to $G_N(W_3)$.

If $f(w+O_N(W))=0$ for homogeneous $w\in W$, by the definitions of $o_{\widehat{\mathcal{Y}}}$, $\widehat{\mathcal{Y}}$, $\mathcal{Y}$ and f, we have in W_3

$$\begin{aligned}
&(o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1}+O_N(V))+J_1)+J_2\\
&\quad=(w(\text{wt } w-\text{wt } (\mathbf{1}+O_N(V))h_3-N-1,0)(\mathbf{1}+O_N(V))+J_1)+J_2\\
&\quad=(\widehat{\mathcal{Y}}_{\text{wt } w-h_3-1,0}(w)(\mathbf{1}+O_N(V))+J_1)+J_2\\
&\quad=\mathcal{Y}_{\text{wt } w-h_3-1,0}(w)(\mathbf{1}+O_N(V))\\
&\quad=\pi_{G_N(W_3)}\sum_{n=0}^{N}(\mathcal{Y}_t)_{\text{wt } w+\text{wt } w_2-h_3-n-1,0}(w)(\mathbf{1}+O_N(V))\\
&\quad=(\pi_{G_N(W_3)}\circ\rho(\mathcal{Y}_t)\circ e_{G_N(W_2)})(w\otimes_{A_N(V)}(\mathbf{1}+O_N(V))\\
&\quad=f(w+O_N(W))\\
&\quad=0.
\end{aligned}$$

Since $o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1}+O_N(V))+J_1\in D$ and $D\cap J_2=0$, we obtain $o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1}+O_N(V))+J_1=0$ in $\widehat{W_3}/J_1$ or equivalently, $o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1}+O_N(V))\in J_1$. We have shown above that $o_{\widehat{\mathcal{Y}}}(w)(\mathbf{1}+O_N(V))\in J_1$ only if $w\in O_N(W)$. In summary, we have shown that $f(w+O_N(W))=0$ implies $w\in O_N(W)$.

By definition, $Q^\infty(W)\subset\ker\vartheta_{\mathcal{Y}}$. In particular, for $[w]_{NN}\in Q_{NN}(W)$,

$$\begin{aligned}
f(w+O_N(W))&=(\mathcal{Y})_{\text{wt } w-h_3-1,0}(w)(\mathbf{1}+O_N(V))\\
&=\text{Res}_x x^{-h_3-1}\mathcal{Y}^0(x^{L_W(0)s}w,x)(\mathbf{1}+O_N(V))\\
&=\vartheta_{\mathcal{Y}}([w]_{NN})(\mathbf{1}+O_N(V))\\
&=0.
\end{aligned}$$

Then we have $w\in O_N(W)$ or equivalently, $[w]_{NN}\in O_{NN}(W)$. Thus $Q_{NN}(W)=O_{NN}(W)$. ∎

Let $A_{NN}(W)=U_{NN}(W)/Q_{NN}(W)$.

Theorem 3.3 *The space $A_{NN}(W)$ is invariant under the left and right actions of $A_{NN}(V)$ and is thus an $A_{NN}(V)$-bimodule. Moreover, the linear map $f_{NN}:U_{NN}(W)\to A_N(W)$ defined by $f_{NN}([w]_{NN})=w+O_N(W)$ for $w\in W$ induces an invertible linear map, still denoted by f_{NN}, sending the $A_{NN}(V)$-bimodule structure on $A_{NN}(W)$ to the $A_N(V)$-bimodule structure on $A_N(W)$.*

Proof. By the definition of the left and right actions of $A^\infty(V)$ on $A^\infty(W)$, we have

$$\begin{aligned}
&([v]_{NN}+Q^\infty(V))\diamond([w]_{NN}+Q^\infty(W))\\
&\quad=\text{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N\left[Y_V((1+x)^{L_V(0)}v,x)w\right]_{NN}+Q^\infty(W)\\
&\quad\in A_{NN}(W)
\end{aligned}$$

and

$$\begin{aligned}
&([w]_{NN}+Q^{\infty}(W))\diamond([v]_{NN}+Q^{\infty}(V))\\
&\quad=\mathrm{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N\left[Y_V((1+x)^{-L_V(0)}v,-x(1+x)^{-1})w\right]_{NN}+Q^{\infty}(W)\\
&\quad\in A_{NN}(W)
\end{aligned}$$

for $v\in V$ and $w\in W$. Thus $A_{NN}(W)$ is invariant under the left and right actions of $A_{NN}(V)$.

Let $f_{NN}:U_{NN}(W)\to A_N(W)$ be defined by $f_{NN}([w]_{NN})=w+O_N(W)$ for $w\in W$. Then by the definition of $O_N(W)$, we have

$$\begin{aligned}
&f_{NN}([\mathrm{Res}_x x^{-2N-p-2}(1+x)^N Y_W((1+x)^{L_V(0)}v,x)w]_{NN})\\
&\quad=\mathrm{Res}_x x^{-2N-p-2}(1+x)^N Y_W((1+x)^{L_V(0)}v,x)w+O_N(V)\\
&\quad=0.
\end{aligned}$$

By the definition of $O^{\infty}(W)$, we obtain $O_{NN}(W)=O^{\infty}(W)\cap U_{NN}(W)\subset\ker f_{NN}$. If $[w]_{NN}\notin O_{NN}(W)$, $w\notin O_N(W)$. In particular, $f_{NN}([w]_{NN})=w+O_N(W)\neq 0$. So $[w]_{NN}\notin\ker f_{NN}$. Thus $\ker f_{NN}=O_{NN}(W)$. By Proposition 3.2, $\ker f_{NN}=Q_{NN}(W)$. It is clear that f_{NN} is surjective. In particular, f_{NN} induces a linear isomorphism, still denoted by f_{NN}, from $A_{NN}(W)$ to $A_N(W)$.

For $u,v\in V$,

$$\begin{aligned}
&f_{NN}([v]_{NN}\diamond([w]_{NN})\\
&\quad=\mathrm{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N f_{NN}([Y_W((1+x)^{L(0)}v,x)w]_{NN})\\
&\quad=\mathrm{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N Y_W((1+x)^{L(0)}v,x)w+O_N(W)\\
&\quad=\mathrm{Res}_x\sum_{m=0}^{N}\binom{-N-1}{m}x^{-N-m-1}(1+x)^N Y_W((1+x)^{L(0)}w,x)w+O_N(W)\\
&\quad=v*_N w+O_N(W)\\
&\quad=(v+O_N(V))*_N(w+O_N(W)).
\end{aligned}$$

Therefore f_{NN} is an isomorphism of associative algebra. ∎

Corollary 3.4 *For $N\in\mathbb{N}$, the space $Q_{NN}(W)=Q^{\infty}(W)\cap U_{NN}(W)$ is spanned by elements of the form*

$$\mathrm{Res}_x x^{-2N-p-2}(1+x)^l[Y_W((1+x)^{L_V(0)}v,x)w]_{NN}$$

for $v\in V$, $w\in W$ and $p\in\mathbb{N}$.

We now introduce the associative algebras $\widetilde{A}^{\infty}(V)$ and $\widetilde{A}^N(V)$ for $N\in\mathbb{N}$. Recall from [H8] that $U^{\infty}(V)$ is the space of column-finite infinite matrices with entries in V, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_+$. Recall also that for $v\in V$, $k,l\in\mathbb{N}$, $[v]_{kl}$ is the element of

$U^\infty(V)$ with the (k,l)-entry being v and all the other entries being 0. Elements of $U^\infty(V)$ are suitable (possibly infinite) sums of elements of the form $[v]_{kl}$ for $v \in V$, $k, l \in \mathbb{N}$.

In [H8], a product $\diamond$ on $U^\infty(V)$ is introduced. Here we define a new product $\blacklozenge$ on $U^\infty(V)$ by

$$[u]_{km} \blacklozenge [v]_{nl} = 0$$

for $k, l, m, n \in \mathbb{N}$ and $u, v \in V$ when $m \neq n$ and

$$\begin{aligned}
&[u]_{kn} \blacklozenge [v]_{nl}\\
&\quad= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right)v\right]_{kl}\\
&\quad= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^l \left[Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right)v\right]_{kl}\\
&\quad= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_y 2\pi i e^{2pi(l+1)i}(e^{2\pi i y}-1)^{-k+n-l-m-1}\left[Y_V(u, y)v\right]_{kl}
\end{aligned}$$

for $k, l, n \in \mathbb{N}$ and $u, v \in V$. Then $U^\infty(V)$ equipped with $\blacklozenge$ is a nonassociative algebra.

For $W = V$, we have the invertible linear map $\mathcal{U}_V(1) = (2\pi i)^{L_V(0)} e^{-L_V^+(A)} : V \to V$ (see [H2] and Subsectipon 2.2). We extend the linear isomorphism $\mathcal{U}_V(1) : V \to V$ to a linear isomorphism $\mathcal{U}_V(1) : U^\infty(V) \to U^\infty(V)$ by

$$\mathcal{U}_V(1)[v]_{kl} = [\mathcal{U}_V(1)v]_{kl}$$

for $k, l \in \mathbb{N}$ and $v \in V$.

Recall the subspace $Q^\infty(V)$ of $U^\infty(V)$ in [H8]. Let $\widetilde{Q}^\infty(V) = \mathcal{U}_V(1)^{-1}Q^\infty(V)$ and $\widetilde{A}^\infty(V) = U^\infty(V)/\widetilde{Q}^\infty(V)$. Recall from [H8] that $A^\infty(V) = U^\infty(V)/Q^\infty(V)$ with the product induced by $\diamond$ is an associative algebra.

Proposition 3.5 *The linear isomorphism $\mathcal{U}_V(1)$ from $U^\infty(V)$ to itself is an isomorphism of nonassociative algebras from $U^\infty(V)$ equipped with the product $\blacklozenge$ to $U^\infty(V)$ equipped with the product $\diamond$ such that $\mathcal{U}_V(1)\widetilde{Q}^\infty(V) = Q^\infty(V)$. In particular, $\blacklozenge$ induces an associative product, still denoted by $\blacklozenge$, on $\widetilde{A}^\infty(V)$ such that $\mathcal{U}_V(1)$ induces an isomorphism of associative algebras from $\widetilde{A}^\infty(V)$ to $A^\infty(V)$.*

Proof. Using (2.4), we obtain

$$\mathcal{U}_V(1)([u]_{km} \blacklozenge [v]_{nl}) = 0 = [u]_{km} \diamond [v]_{nl}$$

for $k, l, m, n \in \mathbb{N}$ and $u, v \in V$ when $m \neq n$ and

$$\begin{aligned}
&\mathcal{U}_V(1)([u]_{kn} \blacklozenge [v]_{nl})\\
&\quad= \mathcal{U}_V(1)\left(\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right)v\right]_{kl}\right)
\end{aligned}$$

$$= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_V((1+x)^{L_V(0)}\mathcal{U}_V(1)u, x)\mathcal{U}_V(1)v\right]_{kl}$$
$$= [\mathcal{U}_V(1)u]_{kn} \diamond [\mathcal{U}_V(1)v]_{nl}$$

for $k, l, n \in \mathbb{N}$, $u, v \in V$. These show that $\mathcal{U}_V(1)$ is an isomorphism of the nonassociative algebras. By the definition of $\widetilde{Q}^\infty(V)$, we have $\mathcal{U}_V(1)\widetilde{Q}^\infty(V) = Q^\infty(V)$. The other conclusions follow immediately. ∎

For a lower-bounded generalized V-module W, we have a graded $A^\infty(V)$-module structure on W given by $\vartheta_W : U^\infty(V) \to \mathrm{End}\, W$ (see [H9]). Let $\tilde{\vartheta}_W : U^\infty(V) \to \mathrm{End}\, W$ be defined by

$$\tilde{\vartheta}_W(\mathfrak{v}) = \vartheta_W(\mathcal{U}_V(1)\mathfrak{v})$$

for $\mathfrak{v} \in U^\infty(V)$.

Proposition 3.6 *Let W be a lower-bounded generalized V-module. Then the linear map $\tilde{\vartheta}_W$ gives W an $\widetilde{A}^\infty(V)$-module structure.*

Proof. For $k, l, m, n \in \mathbb{N}$ and $u, v \in V$,

$$\begin{aligned}
\tilde{\vartheta}_W([u]_{km} \blacklozenge [v]_{nl}) &= \vartheta_W(\mathcal{U}_V(1)([u]_{km} \blacklozenge [v]_{nl})) \\
&= \vartheta_W([\mathcal{U}_V(1)u]_{kn} \diamond [\mathcal{U}_V(1)v]_{nl}) \\
&= \vartheta_W([\mathcal{U}_V(1)u]_{kn})\vartheta_W([\mathcal{U}_V(1)v]_{nl}) \\
&= \tilde{\vartheta}_W([u]_{km})\tilde{\vartheta}_W([v]_{nl}).
\end{aligned}$$

Thus W equipped with $\tilde{\vartheta}_W$ is an $\widetilde{A}^\infty(V)$-module. ∎

From Theorem 5.1 in [H9], we know that the category of lower-bounded generalized V-modules is isomorphic to the category of graded $A^\infty(V)$-modules. We now define a *graded $\widetilde{A}^\infty(V)$-module* to be an $\widetilde{A}^\infty(V)$-module obtained from a lower-bounded generalized V-module as in Proposition 3.6. Or equivalently, a graded $\widetilde{A}^\infty(V)$-module is an $\widetilde{A}^\infty(V)$-module W with the $\widetilde{A}^\infty(V)$-module structure given by a linear map $\tilde{\vartheta}_W : U^\infty(V) \to \mathrm{End}\, W$ such that $\vartheta_W : U^\infty(V) \to \mathrm{End}\, W$ defined by $\vartheta_W(\mathfrak{v}) = \tilde{\vartheta}_W(\mathcal{U}_V(1)^{-1}\mathfrak{v})$ gives a graded $A^\infty(V)$-module structure to W.

Recall the subalgebras $A_{NN}(V)$ for $N \in \mathbb{N}$ in [H8] which are proved in [H8] to be isomorphic to the associative algebras $A_N(V)$ introduced in [DLM]. In this paper, we need the following subalgebras $\widetilde{A}_{NN}(V)$ for $N \in \mathbb{N}$: For $N \in \mathbb{N}$, let

$$\widetilde{A}_{NN}(V) = \{[v]_{NN} + \widetilde{Q}^\infty(V) \mid v \in V\}.$$

Proposition 3.7 *For $N \in \mathbb{N}$, $\widetilde{A}_{NN}(V)$ is a subalgebra of $\widetilde{A}^\infty(V)$ and $\mathcal{U}_V(1)$ induces an isomorphism of associative algebras from $\widetilde{A}_{NN}(V)$ to $A_{NN}(V)$. For a lower-bounded generalized V-module W, $W_{\llbracket N \rrbracket} = \coprod_{\mu\in\Gamma(W)} W_{[h^\mu+N]}$ is invariant under the action of $\widetilde{A}_{NN}(V)$ and thus is a $\widetilde{A}_{NN}(V)$-module.*

Proof. By definition, $\mathcal{U}_V(1)\widetilde{A}_{NN}(V) = A_{NN}(V)$. Since $\mathcal{U}_V(1)$ is an isomorphsim from the associative algebra $\widetilde{A}^\infty(V)$ to the associative algebra $A^\infty(V)$ and $A_{NN}(V)$ is a subalgebra of $A^\infty(V)$, $\widetilde{A}_{NN}(V)$ is a subalgebra of $\widetilde{A}^\infty(V)$ and $\mathcal{U}_V(1)$ restricted to $\widetilde{A}_{NN}(V)$ is an isomorphism from $\widetilde{A}_{NN}(V)$ to $A_{NN}(V)$.

By the definition of $\tilde{\vartheta}_W$, $W_{\llbracket N\rrbracket}$ is invariant under the action of $\widetilde{A}_{NN}(V)$. ∎

We also introduce another associative algebra $\widetilde{A}_N(V)$ generalizing the associative algebra $\widetilde{A}(V)$ in [H2]. Define a product $\bullet_N$ of V by

$$u \bullet_N v = \mathrm{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right) v$$

for $u, v \in V$. Let $\widetilde{O}_N(V)$ be the subspace of V spanned by elements of the form

$$\mathrm{Res}_x x^{-2N-2-n}(1+x)^N Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right) v$$

for $n \in \mathbb{N}$ and $u, v \in V$ and of the form $L_V(-1)v$. Let $\widetilde{A}_N(V) = V/\widetilde{O}_N(V)$.

Proposition 3.8 *The product $\bullet_N$ induces an associative algebra structure on $\widetilde{A}_N(V)$ isomorphic to $A_N(V)$. The operator $\mathcal{U}_V(1)$ on V induces an isomorphism from $\widetilde{A}_N(V)$ to $A_N(V)$. In particular, $\mathcal{U}_V(1)^{-1}\omega + \widetilde{O}^\infty(V)$ is in the center of $\widetilde{A}_N(V)$*

Proof. For $n \in \mathbb{N}$ and $u, v \in V$, by (2.4),

$$\begin{aligned}\mathcal{U}_V(1)&\mathrm{Res}_x x^{-2N-2-n}(1+x)^N Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right) v\\ &= \mathrm{Res}_x x^{-2N-2-n}(1+x)^N Y_V\left((1+x)^{L_V(0)}\mathcal{U}_V(1)u, x\right)\mathcal{U}_V(1)v.\end{aligned}$$

Also for $v \in V$, we have

$$\mathcal{U}_V(1)L_V(-1)v = 2\pi i(L_V(-1) + L_V(0))v$$

(this is (1.15) in [H2]). These shows $\mathcal{U}_V(1)\widetilde{O}_N(V) = O_N(V)$. Therefore $\mathcal{U}_V(1)$ induces a linear isomorphism from $\widetilde{A}_N(V)$ to $A_N(V)$.

By (2.4) again, we have

$$\begin{aligned}\mathcal{U}_V(1)(u \bullet_N v) &= \mathcal{U}_V(1)\mathrm{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N Y_V\left(u, \frac{1}{2\pi i}\log(1+x)\right) v\\ &= \mathrm{Res}_x T_{2N+1}((x+1)^{-N-1})(1+x)^N Y_V\left((1+x)^{L_V(0)}u, x\right) v\\ &= u *_N v\end{aligned}$$

for $u, v \in V$. So $\mathcal{U}_V(1)$ induces an isomorphism of associative algebras.

Since $\omega + O_N(V)$ is in the center of $A_N(V)$ and $\mathcal{U}_V(1)^{-1}$ is an isomorphism from $A_N(V)$ to $\widetilde{A}_N(V)$, $\mathcal{U}_V(1)^{-1}\omega + \widetilde{O}^\infty(V)$ is in the center of $\widetilde{A}_N(V)$. ∎

Proposition 3.9 *The associative algebras $\widetilde{A}_N(V)$ and $\widetilde{A}_{NN}(V)$ are isomorphic. The map given by $v + \widetilde{O}_N(V) \mapsto [v]_{NN} + \widetilde{Q}^\infty(V)$ is an isomorphism from $\widetilde{A}_N(V)$ to $\widetilde{A}_{NN}(V)$.*

Proof. This result follows from Propositions 3.7, 3.8 and Theorem 4.2 in [H2]. ∎

We now introduce the main subalgebras of $\widetilde{A}^\infty(V)$ needed in this paper. Recall from [H8] the subspaces $U^N(V)$ for $N \in \mathbb{N}$ of $U^\infty(V)$ spanned by elements of the form $[v]_{kl}$ for $v \in V$, $k, l = 0, \dots, N$. These subspaces are invariant under the operator $\mathcal{U}_V(1)$ on $U^\infty(V)$. Let

$$\widetilde{A}^N(V) = \{\mathfrak{v} + \widetilde{Q}^\infty(V) \mid \mathfrak{v} \in U^N(V)\}.$$

Note that $\widetilde{A}^N(V)$ is spanned by elements of the form $[v]_{kl} + \widetilde{Q}^\infty(V)$ for $v \in V$ and $k, l = 0, \dots, N$.

Proposition 3.10 *For $N \in \mathbb{N}$, $\widetilde{A}^N(V)$ is a subalgebra of $\widetilde{A}^\infty(V)$ and $\mathcal{U}_V(1)$ induces an isomorphism of associative algebras from $\widetilde{A}^N(V)$ to $A^N(V)$. For a lower-bounded generalized V-module W, $\Omega^0_N(W)$ is invariant under the action of $\widetilde{A}^N(V)$ and thus is a $\widetilde{A}^N(V)$-module.*

Proof. This proposition follows from the invariance of $U^N(V)$ under $\mathcal{U}_V(1)$ and Proposition 3.5. ∎

We now construct an $\widetilde{A}^\infty(V)$-bimodule from a lower-bounded generalized V-module W. We define a left action of $U^\infty(V)$ on $U^\infty(W)$ by

$$[v]_{km} \blacklozenge [w]_{nl} = 0$$

for $v \in V$, $w \in W$ and $k, m, n, l \in \mathbb{N}$ when $m \neq n$ and

$$\begin{aligned}
&[v]_{kn} \blacklozenge [w]_{nl}\\
&\quad = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_W\left(v, \frac{1}{2\pi i}\log(1+x)\right)w\right]_{kl}\\
&\quad = \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^l \left[Y_W\left(v, \frac{1}{2\pi i}\log(1+x)\right)w\right]_{kl}\\
&\quad = \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_y 2\pi i e^{2\pi i(l+1)y}(e^{2\pi i y}-1)^{-k+n-l-m-1}\left[Y_W(v, y)w\right]_{kl}.
\end{aligned}$$

for $v \in V$, $w \in W$ and $k, n, l \in \mathbb{N}$. We define a right action of $U^\infty(V)$ on $U^\infty(W)$ by

$$[w]_{km} \blacklozenge [v]_{nl} = 0$$

for $v \in V$, $w \in W$ and $k, m, n, l \in \mathbb{N}$ when $m \neq n$ and

$$[w]_{kn} \blacklozenge [v]_{nl}$$

$$= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^k \left[Y_W\left(v, -\frac{1}{2\pi i}\log(1+x)\right) w\right]_{kl}$$
$$= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_x x^{-k+n-l-m-1}(1+x)^k \left[Y_W\left(v, -\frac{1}{2\pi i}\log(1+x)\right) w\right]_{kl}$$
$$= \sum_{m=0}^{n} \binom{-k+n-l-1}{m} \mathrm{Res}_y 2\pi i e^{2\pi i(k+1)y}(e^{2\pi i y}-1)^{-k+n-l-m-1} \left[Y_W\left(v, -y\right) w\right]_{kl}. \quad (3.7)$$

We extend the invertible linear map $\mathcal{U}_W(1) : W \to W$ to an invertible linear map $\mathcal{U}_W(1) : U^\infty(W) \to U^\infty(W)$ by

$$\mathcal{U}_W(1)[w]_{kl} = [\mathcal{U}_W(1)w]_{kl}$$

for $k, l \in \mathbb{N}$ and $w \in W$. Let $\widetilde{Q}^\infty(W) = \mathcal{U}_W(1)^{-1}Q^\infty(W)$ and $\widetilde{A}^\infty(W) = U^\infty(W)/\widetilde{Q}^\infty(W)$.

In [H9], a left action and a right action (denoted by $\diamond$), a subspace $Q^\infty(W)$ of $U^\infty(W)$ and an $A^\infty(V)$-bimodule $A^\infty(W)$ are introduced.

Proposition 3.11 *The linear isomorphism $\mathcal{U}_W(1)$ from $U^\infty(W)$ to itself sends the left and right actions of $U^\infty(V)$ on $U^\infty(W)$ given by $\diamond$ to the left and right actions given by $\blacklozenge$ such that $\mathcal{U}_W(1)\widetilde{Q}^\infty(W) = Q^\infty(W)$. In particular, $\blacklozenge$ induces left and right actions, still denoted by $\blacklozenge$, of $\widetilde{A}^\infty(V)$ on $\widetilde{A}^\infty(W)$ such that $\widetilde{A}^\infty(W)$ becomes an $\widetilde{A}^\infty(V)$-bimodule and $\mathcal{U}_W(1)$ induces an invertible linear map sending the $A^\infty(V)$-bimodule structure on $A^\infty(W)$ to the $\widetilde{A}^\infty(V)$-bimodule structure on $\widetilde{A}^\infty(W)$.*

Proof. Using (2.4), we have

$$\mathcal{U}_W(1)([v]_{km} \blacklozenge [w]_{nl}) = 0 = [v]_{km} \diamond [w]_{nl}$$

for $k, l, m, n \in \mathbb{N}$, $v \in V$ and $w \in W$ when $m \neq n$ and

$$\mathcal{U}_W(1)([v]_{kn} \blacklozenge [w]_{nl})$$
$$= \mathcal{U}_W(1)\left(\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_W\left(v, \frac{1}{2\pi i}\log(1+x)\right) w\right]_{kl}\right)$$
$$= \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_W((1+x)^{L_V(0)}\mathcal{U}_V(1)v, x)\mathcal{U}_W(1)w\right]_{kl}$$
$$= [\mathcal{U}_V(1)v]_{kn} \diamond [\mathcal{U}_W(1)w]_{nl}$$

for $k, l, n \in \mathbb{N}$, $u, v \in V$. This shows that $\mathcal{U}_W(1)$ sends the left action of $U^\infty(V)$ on $U^\infty(W)$ given by $\blacklozenge$ to the left action given by $\diamond$.

Using (2.4) again, we have

$$\mathcal{U}_W(1)\mathrm{Res}_x x^{-n} Y_W\left(v, -\frac{1}{2\pi i}\log(1+x)\right) w$$
$$= \mathcal{U}_W(1)\mathrm{Res}_x x^{-n} Y_W\left(v, \frac{1}{2\pi i}\log(1-x(1+x)^{-1})\right) w$$

$$
\begin{aligned}
&= \mathrm{Res}_x x^{-n} Y_W\left((1-x(1+x)^{-1})^{L_V(0)}\mathcal{U}_V(1)v, -x(1+x)^{-1}\right)\mathcal{U}_W(1)w\\
&= \mathrm{Res}_x x^{-n} Y_W((1+x)^{-L_V(0)}\mathcal{U}_V(1)v, -x(1+x)^{-1})\mathcal{U}_W(1)w
\end{aligned}
$$

for $n \in \mathbb{Z}$, $v \in V$ and $w \in W$. Then we have

$$
\mathcal{U}_W(1)([w]_{km} \blacklozenge [v]_{nl}) = 0 = [w]_{km} \diamond [v]_{nl}
$$

for $k, l, m, n \in \mathbb{N}$, $v \in V$ and $w \in W$ when $m \neq n$ and

$$
\begin{aligned}
&\mathcal{U}_W(1)([w]_{kn} \blacklozenge [v]_{nl})\\
&\quad = \mathcal{U}_W(1)\left(\mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_W\left(v, -\frac{1}{2\pi i}\log(1+x)\right)w\right]_{kl}\right)\\
&\quad = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1})(1+x)^l \left[Y_W((1+x)^{-L_V(0)}\mathcal{U}_V(1)v, -x(1+x)^{-1})\mathcal{U}_W(1)w\right]_{kl}\\
&\quad = [\mathcal{U}_W(1)w]_{kn} \diamond [\mathcal{U}_V(1)v]_{nl}
\end{aligned}
$$

for $k, l, n \in \mathbb{N}$, $u, v \in V$. This shows that $\mathcal{U}_W(1)$ sends the right action of $U^\infty(V)$ on $U^\infty(W)$ given by $\blacklozenge$ to the right action given by $\diamond$. By the definition of $\widetilde{Q}^\infty(W)$, we have $\mathcal{U}_W(1)\widetilde{Q}^\infty(W) = Q^\infty(W)$.

The other conclusions follow from these. ∎

For $N \in \mathbb{N}$, let

$$
\begin{aligned}
A_{NN}(W) &= \{[w]_{NN} + Q^\infty(W) \mid w \in W\},\\
\widetilde{A}_{NN}(W) &= \{[w]_{NN} + \widetilde{Q}^\infty(W) \mid w \in W\}.
\end{aligned}
$$

Proposition 3.12 *For $N \in \mathbb{N}$, $A_{NN}(W)$ is closed under the left and right actions of the subalgebra $A_{NN}(V)$ of $A^\infty(V)$ and thus has an $A_{NN}(V)$-bimodule structure. Similarly, $\widetilde{A}_{NN}(W)$ is closed under the left and right actions of the subalgebra $\widetilde{A}_{NN}(V)$ of $\widetilde{A}^\infty(V)$ and thus has an $\widetilde{A}_{NN}(V)$-bimodule structure. The map $\mathcal{U}_W(1)$ induces an invertible linear map sending the $\widetilde{A}_{NN}(V)$-bimodule structure on $\widetilde{A}_{NN}(W)$ to the $A_{NN}(V)$-bimodule structure on $A_{NN}(W)$.*

Proof. This result follows immediately from the definitions of the left and right actions of $A^\infty(V)$ on $A^\infty(W)$, the definitions of the left and right actions of $A^\infty(V)$ on $A^\infty(W)$ and Proposition 3.11. ∎

Corollary 3.13 *The map given by $w + O_N(W) \mapsto [w]_{NN} + \widetilde{Q}^\infty(W)$ is an invertible linear map sending the $A_N(V)$-bimodule structure on $A_N(W)$ to the $\widetilde{A}_{NN}(V)$-bimodule structure on $\widetilde{A}_{NN}(W)$.*

Corollary 3.14 *The space $\widetilde{Q}^\infty(W) \cap U_{NN}(W)$ is spanned by the coefficients of*

$$
\mathrm{Res}_x e^{2\pi i(N+1)x}(e^{2\pi i x} - 1)^{-2N-p-2} Y_W(v, x)w
$$

for $p \in \mathbb{N}$, $v \in V$ and $w \in W$.

Recall from [H9] the subspaces $U^N(W)$ for $N \in \mathbb{N}$ of $U^\infty(W)$ spanned by elements of the form $[w]_{kl}$ for $w \in W$, $k, l = 0, \dots, N$. Let

$$\widetilde{A}^N(W) = \{\mathfrak{w} + \widetilde{Q}^\infty(W) \mid \mathfrak{w} \in U^N(W)\}.$$

Note that $\widetilde{A}^N(W)$ is spanned by elements of the form $[w]_{kl} + \tilde{Q}^\infty(W)$ for $w \in W$ and $k, l = 0, \dots, N$. Also note that $\widetilde{A}^N(W)$ is an $\widetilde{A}^N(V)$-subbimodule of $\widetilde{A}^\infty(W)$ viewed as a $\widetilde{A}^N(V)$-bimodule.

Proposition 3.15 *The subspace $\widetilde{A}^N(W)$ of $\widetilde{A}^\infty(W)$ is in variant under the left and right actions of $\widetilde{A}^N(V) \subset \widetilde{A}^\infty(V)$. In particular, $\widetilde{A}^N(W)$ is an $\widetilde{A}^N(V)$-bimodule. Moreover, $\mathcal{U}_W(1)$ induces an invertible linear map sending the $A^N(V)$-bimodule structure on $A^N(W)$ to the $\widetilde{A}^N(V)$-bimodule structure on $\widetilde{A}^N(W)$.*

Let W_1, W_2 and W_3 be lower-bounded generalized V-modules and $\mathcal{Y}$ an intertwining operator of type $\binom{W_3}{W_1 W_2}$. In [H9], a linear map $\vartheta_\mathcal{Y} : U^N(W_1) \to \mathrm{Hom}(W_2, W_3)$ is defined. Since $Q^\infty(V) \subset \ker \vartheta_\mathcal{Y}$, we can view $\vartheta_\mathcal{Y}$ as a map from $A^\infty(V)$ to $\mathrm{Hom}(W_2, W_3)$. Now we define $\tilde{\vartheta}_\mathcal{Y} : U^N(W_1) \to \mathrm{Hom}(W_2, W_3)$ by

$$\tilde{\vartheta}_\mathcal{Y}(\mathfrak{w}_1)w_2 = \vartheta_\mathcal{Y}(\mathcal{U}_{W_1}(1)\mathfrak{w}_1)w_2$$

for $\mathfrak{w}_1 \in U(W_1)$ and $w_2 \in W_2$. In [H9], for $N \in \mathbb{N}$, a linear map

$$\rho^N : \mathcal{V}_{W_1 W_2}^{W_3} \to \mathrm{Hom}(A^N(W_1) \otimes \Omega_N^0(W_2), \Omega_N^0(W_3))$$

is defined by

$$(\rho^N(\mathcal{Y}))((\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^N(V)} w_2) = \vartheta_\mathcal{Y}(\mathfrak{w}_1)w_2$$

for $\mathfrak{w}_1 \in U^N(W_1)$ and $w_2 \in \Omega_N^0(W_2)$. Here we define a linear map

$$\begin{aligned}\tilde{\rho}^N : \mathcal{V}_{W_1 W_2}^{W_3} &\to \mathrm{Hom}(\widetilde{A}^N(W_1) \otimes \Omega_N^0(W_2), \Omega_N^0(W_3))\\ \mathcal{Y} &\mapsto \tilde{\rho}^N(\mathcal{Y})\end{aligned}$$

by

$$\begin{aligned}&(\tilde{\rho}^N(\mathcal{Y}))(\mathfrak{w}_1 + \widetilde{Q}^\infty(W_1)) \otimes_{A^N(V)} w_2)\\ &\quad = \tilde{\vartheta}_\mathcal{Y}(\mathfrak{w}_1)w_2\\ &\quad = \vartheta_\mathcal{Y}(\mathcal{U}_{W_1}(1)\mathfrak{w}_1)w_2\\ &\quad = (\rho^N(\mathcal{Y}))((\mathcal{U}_{W_1}(1)\mathfrak{w}_1 + Q^\infty(W_1)) \otimes_{A^N(V)} w_2)\end{aligned}$$

for $\mathfrak{w}_1 \in U^N(W_1)$ and $w_2 \in \Omega_N^0(W_2)$. The definition of $\tilde{\rho}^N(\mathcal{Y})$ can also be simply written as

$$\tilde{\rho}^N(\mathcal{Y}) = (\rho^N(\mathcal{Y})) \circ (\mathcal{U}_{W_1}(1) \otimes 1_{\Omega_N^0(W_2)}), \tag{3.8}$$

where we use the same notation $\mathcal{U}_{W_1}(1)$ to denote the map from $\widetilde{A}^N(W_1)$ to $A^N(W_1)$ induced from the operator $\mathcal{U}_{W_1}(1)$ on $U^N(W_1)$.

Proposition 3.16 *The linear map* $\tilde{\rho}^N$ *is in fact from* $\mathcal{V}_{W_1W_2}^{W_3}$ *to* $\mathrm{Hom}_{\widetilde{A}^N(V)}(\widetilde{A}^N(W_1)\otimes_{\widetilde{A}^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3))$.

Proof. Let $\mathcal{Y}$ be an intertwining operator of type $\binom{W_3}{W_1W_2}$. For $\mathfrak{v}\in U^\infty(V)$, $\mathfrak{w}\in U^\infty(W)$ and $w_2\in \Omega_N^0(W_2)$,

$$\begin{aligned}
&(\tilde{\rho}^N(\mathcal{Y}))((\mathfrak{v}\blacklozenge \mathfrak{w}_1+\widetilde{Q}^\infty(W_1))\otimes w_2)\\
&\quad =\tilde{\vartheta}_{\mathcal{Y}}^N(\mathfrak{v}\blacklozenge \mathfrak{w}_1)w_2\\
&\quad =\vartheta_{\mathcal{Y}}(\mathcal{U}_{W_1}(1)(\mathfrak{v}\blacklozenge \mathfrak{w}_1))w_2\\
&\quad =\vartheta_{\mathcal{Y}}((\mathcal{U}_V(1)\mathfrak{v})\diamond(\mathcal{U}_{W_1}(1)\mathfrak{w}_1))w_2\\
&\quad =\vartheta_{W_3}(\mathcal{U}_V(1)\mathfrak{v})\vartheta_{\mathcal{Y}}(\mathcal{U}_{W_1}(1)\mathfrak{w}_1)w_2\\
&\quad =\tilde{\vartheta}_{W_3}(\mathfrak{v})\tilde{\rho}_{\mathcal{Y}}^N((\mathfrak{w}_1+\widetilde{Q}^\infty(W_1))\otimes w_2).
\end{aligned}$$

This shows that the image of $\tilde{\rho}^N$ is in $\mathrm{Hom}_{\widetilde{A}^N(V)}(\widetilde{A}^N(W_1)\otimes \Omega_N^0(W_2), \Omega_N^0(W_3))$.

On the other hand, for $\mathfrak{v}\in U^\infty(V)$, $\mathfrak{w}\in U^\infty(W)$ and $w_2\in \Omega_N^0(W_2)$,

$$\begin{aligned}
&(\tilde{\rho}^N(\mathcal{Y}))((\mathfrak{w}_1\blacklozenge \mathfrak{v}+\widetilde{Q}^\infty(W_1))\otimes w_2)\\
&\quad =\tilde{\vartheta}_{\mathcal{Y}}^N(\mathfrak{w}_1\blacklozenge \mathfrak{v})w_2\\
&\quad =\vartheta_{\mathcal{Y}}(\mathcal{U}_{W_1}(1)(\mathfrak{w}_1\blacklozenge \mathfrak{v}))w_2\\
&\quad =\vartheta_{\mathcal{Y}}((\mathcal{U}_{W_1}(1)\mathfrak{w}_1)\diamond(\mathcal{U}_V(1)\mathfrak{v}))w_2\\
&\quad =\vartheta_{\mathcal{Y}}(\mathcal{U}_{W_1}(1)\mathfrak{w}_1)\vartheta_{W_2}(\mathcal{U}_V(1)\mathfrak{v})w_2\\
&\quad =\tilde{\rho}_{\mathcal{Y}}^N((\mathfrak{w}_1+\widetilde{Q}^\infty(W_1))\otimes\tilde{\vartheta}_{W_2}(\mathfrak{v})w_2).
\end{aligned}$$

This shows that the image of $\tilde{\rho}^N$ is in $\mathrm{Hom}_{\widetilde{A}^N(V)}(\widetilde{A}^N(W_1)\otimes_{\widetilde{A}^N(V)} \Omega_N^0(W_2), \Omega_N^0(W_3))$. ∎

In Proposition 6.3 in [H9], given a graded $A^N(V)$-module M, a lower-bounded generalized V-module $S_{\mathrm{voa}}^N(\Omega_N^0(M)$ satisfying a universal property is constructed.

Theorem 3.17 *Let* V *be a vertex operator algebra. Assume that* W_2 *and* W_3' *are equivalent to* $S_{\mathrm{voa}}^N(\Omega_N^0(W_2))$ *and* $S_{\mathrm{voa}}^N(\Omega_N^0(W_3'))$*, respectively. Then the linear map* $\tilde{\rho}^N$ *is a linear isomorphism.*

Proof. We know that $\mathcal{U}_{W_1}(1)$ from $\widetilde{A}^N(W_1)$ to $A^N(W_1)$ is an isomorphism. By Theorem 6.7 in [H9], ρ^N is a linear isomorphism. Then by (3.8), we see that $\tilde{\rho}^N$ is also a linear isomorphism. ∎

Note that in Theorem 3.17, one condition is that W_3' is equivalent to $S_{\mathrm{voa}}^N(\Omega_N^0(W_3'))$. But in applications, for example in the proof of the modular invariance that we shall give later, what we have is often that W_3 is equivalent to $S_{\mathrm{voa}}^N(\Omega_N^0(W_3))$. In these cases, we need the following result to use Theorem 3.17:

Proposition 3.18 *Let W be a lower-bounded generalized V-module of finite length equivalent to $S^N_{\rm voa}(\Omega^0_N(W))$. Assume that W is of finite length such that the lowest weight vectors of the irreducible V-modules in the composition series are given by cosets containing elements of $\Omega^0_N(W)$. Then W' is equivalent to $S^N_{\rm voa}(\Omega^0_N(W'))$.*

Proof. By definition, $S^N_{\rm voa}(\Omega^0_N(W'))$ is generated by $\Omega^0_N(W')$. By the universal property of $S^N_{\rm voa}(\Omega^0_N(W'))$, there is a unique V-module map $f: S^N_{\rm voa}(\Omega^0_N(W')) \to W'$ such that $f|_{\Omega^0_N(W')} = 1_{\Omega^0_N(W')}$. Since W is of finite length, W' is also of finite length. By Property 6 in Proposition 3.19, W' is also generated by $\Omega^0_N(W')$. So f is surjective.

We still need to prove that f is injective. If $\ker f \neq 0$, then it is a nonzero generalized V-submodule of $S^N_{\rm voa}(\Omega^0_N(W))$. Since W' is of finite length, $\Gamma(W')$ must be a finite set. Then since $S^N_{\rm voa}(\Omega^0_N(W))$ is generated by $\Omega^0_N(W')$, $\Gamma(S^N_{\rm voa}(\Omega^0_N(W)))$ is a finite set. Since $\ker f$ is a generalized V-submodule of $S^N_{\rm voa}(\Omega^0_N(W))$, $\Gamma(\ker f)$ is also a finite set. In particular, there exists a lowest weight vector $w'_0 \neq 0$ of $\ker f$. Let W_0 be the generalized V-module generated by w'_0. Applying Zorn's lemma to generalized V-submodule of W_0 not containing w'_0, we know that W_0 has a maximal generalized V-submodule M_0 not containing w'_0. Thus the quotient W_0/M_0 is an irreducible lower-bounded generalized V-module with a lowest weight vector $w'_0 + M_0$. Since W' is also of finite length, by Property 6 in Proposition 3.19, $\Gamma(W')$ is finite and for each $\mu \in \Gamma(W')$, there exist $h^\mu \in \mu$ equal to a lowest weight of an irreducible V-module such that

$$\Omega^0_N(W') = \coprod_{\mu\in\Gamma(W')} \coprod_{n=0}^{N} W^*_{[h^\mu+n]}.$$

Since $S^N_{\rm voa}(\Omega^0_N(W'))$ is generated by $\Omega^0_N(W')$, the weight of every element of $S^N_{\rm voa}(\Omega^0_N(W'))$ is also congruent to h^μ modulo $\mathbb{Z}$ for some $\mu \in \Gamma(W')$. In particular, wt w'_0 is congruent to h^{μ_0} modulo $\mathbb{Z}$ for some $\mu_0 \in \Gamma(W')$. But h^{μ_0} is also the weight of a lowest weight vector of an irreducible V-module. Since the difference between lowest weights of two irreducible V-modules is less than or equal to N, wt $w'_0 - h^{\mu_0} \leq N$. Hence

$$w'_0 \in \coprod_{n=0}^{N} W^*_{[h^{\mu_0}+n]} \subset \Omega^0_N(W').$$

Since $w'_0 \in \ker f$, we have $f(w'_0) = 0$. On the other hand, since $w'_0 \in \Omega^0_N(W')$, we have $f(w'_0) = 1_{\Omega^0_N(W')}(w'_0) = w'_0$. This contradicts with $w'_0 \neq 0$. Thus $\ker f = 0$ and f is injective. ∎

Finally, we give several results on $A^N(V)$, $A^N(W)$, $\widetilde{A}^N(W)$ and suitable generalized or ordinary V-modules when V has no nonzero elements of negative weights and is C_2-cofinite.

A lower-bounded generalized V-module W is said to be of finite length if there are generalized V-submodules $W = W_1 \supset \cdots \supset W_{l+1} = 0$ such that W_i/W_{i+1} for $i = 1, \dots, l$ are irreducible lower-bounded generalized V-modules. A generalized V-module W is said to be quasi-finite dimensional if $\coprod_{\Re(n)\leq N} W_{[n]}$ is finite dimensional for $N \in \mathbb{Z}$.

Our first proposition is in fact a collection of results from [H5] and [H8].

Proposition 3.19 *Let V be a C_2-cofinite vertex operator algebra without nonzero elements of negative weights. Then we have the following properties:*

1. *For $N \in \mathbb{N}$, $A^N(V)$ is finite dimensional.*

2. *Every lower-bounded generalized V-module W generated by a finite-dimensional subspace of $\Omega_N^0(W)$ is quasi-finite dimensional.*

3. *Every irreducible lower-bounded generalized V-moudle is an ordinary irreducible V-module.*

4. *The set of equivalence classes of (ordinary) irreducible V-modules is in bijection with the set of equivalence classes of irreducible nondegenerate graded $A^N(V)$-modules.*

5. *The category of lower-bounded generalized V-module of finite length, the category of grading-restricted generalized V-modules and the category of quasi-finite-dimensional generalized V-modules are the same.*

6. *There are only finitely many irreducible V-modules up to equivalence.*

Proof. Property 1 is Theorem 4.6 in [H8]. Property 2 is Proposition 3.8 in [H5] in the case that V is C_2-cofinite. Property 3 is Theorem 5.8 in [H8]. Property 4 follows from Theorems 5.6 and 5.8 in [H8].

Property 5 is Proposition 4.3 in [H5]. Note that though Proposition 4.3 in [H5] assume that V satisfies in addition $V_{(0)} = \mathbb{C}\mathbf{1}$, the only paper quoted there that needs this condition in some results is [ABD]. But the result needed in the proof of Proposition 4.3 in [H5] is only Proposition 5.2 in [ABD], which does not need this condition. So Proposition 4.3 in [H5] or Property 5 is true for C_2-cofinite V without nonzero elements of negative weights.

Since $A^N(V)$ is finite dimensional, there are only finitely many inequivalent irreducible $A^N(V)$-modules. In particular, there are only finitely many inequivalent irreducible nondegenerate graded $A^N(V)$-modules. By Property 4, we obtain Property 6 that there are only finitely many inequivalent irreducible V-modules. ∎

The following result is about the relation between a lower-bounded generalized V-module W and $A^N(V)$-module $\Omega_N^0(W)$.

Proposition 3.20 *Let V be a C_2-cofinite vertex operator algebra without nonzero elements of negative weights. Let N be a nonnegative integer such that the differences of the real parts of the lowest weights of the finitely many irreducible V-modules (up to equivalence) are less than or equal to N. Then we have:*

1. *For a lower-bounded generalized V-module W of finite length, $\Gamma(W)$ is a finite set and for each $\mu \in \Gamma(W)$, there exists $h^\mu \in \mu$ equal to a lowest weight of an irreducible V-module such that*

$$W = \coprod_{\mu \in \Gamma(W)} \coprod_{n \in \mathbb{N}} W_{[h^\mu + n]}, \tag{3.9}$$

$$\Omega_N^0(W) = \coprod_{\mu\in\Gamma(W)} \coprod_{n=0}^{N} W_{[h^\mu+n]}. \tag{3.10}$$

2. *A lower-bounded generalized V-module W of finite length is generated by $\Omega_N^0(W)$ and is equivalent to $S_{\rm voa}^N(\Omega_N^0(W))$. In particular, W' is equivalent to $S_{\rm voa}^N(W')$.*

3. *Let $N' \in N+\mathbb{N}$ and*
$$M = \coprod_{n=0}^{N'} M_{[n]}$$
a finite-dimensional graded $A^{N'}(V)$-module. Let
$$M^N = \coprod_{n=0}^{N} M_{[n]}.$$
Then M^N is a graded $A^N(V)$-module and M is equivalent to the graded $A^{N'}(V)$-module $\Omega_{N'}^0(S_{\rm voa}^N(M^N))$.

Proof. Let $W = W_1 \supset \cdots \supset W_{l+1} = 0$ be a composition sereis of a lower-bounded generalized V-module W of finite length. Let $w_i \in W_i$ for $i = 1, \dots, l$ be homogeneous such that $w_i + W_{i+1}$ is a lowest weight vector of W_i/W_{i+1}. Since as a graded vector space, W is isomorphic to $\coprod_{i=1}^{l} W_i/W_{i+1}$, $\Gamma(W)$ is the set of all congruence classes in $\mathbb{C}/\mathbb{Z}$ containing the weights of at least one w_i. For each element $\mu \in \Gamma(W)$, let h^μ be the smallest of all $\mathrm{wt}\, w_i \in \mu$. Then (3.9) and (3.10) hold.

Let h_W be the smallest of all h^μ for $\mu \in \Gamma(W)$. Then h_W is a lowest weight of W and we have
$$\coprod_{\Re(h_W)\le\Re(m)\le\Re(h_W)+N} W_{[m]} \subset \coprod_{\mu\in\Gamma(W)} \coprod_{n=0}^{N} W_{[h^\mu+n]} = \Omega_N^0(W).$$
By Proposition 5.11 in [H8],
$$\coprod_{\Re(h_W)\le\Re(m)\le\Re(h_W)+N} W_{[m]}$$
generates W. Thus $\Omega_N^0(W)$ also generates W. By the universal property of $S_{\rm voa}^N(\Omega_N^0(W))$, there exists a unique V-module map $f : S_{\rm voa}^N(\Omega_N^0(W)) \to W$ such that $f|_{\Omega_N^0(W)} = 1_{\Omega_N^0(W)}$. Since W is generated by $\Omega_N^0(W)$, f is surjective. We now show that f is also injective, that is, the kernel $\ker f$ of f is 0. In fact, W is equivalent to the quotient of $S_{\rm voa}^N(\Omega_N^0(W))$ by $\ker f$. Since $f|_{\Omega_N^0(W)} = 1_{\Omega_N^0(W)}$, we have
$$\ker f \subset \coprod_{\mu\in\Gamma(W)} \coprod_{n\in N+\mathbb{Z}_+} W_{[h^\mu+n]}. \tag{3.11}$$

But $\ker f$ is a lower-bounded generalized V-submodule of W, if $\ker f \neq 0$, its lowest weight must be equal to the lowest weight of an irreducible V-module. But the real part of the lowest weight of the irreducible V-module must be less than or equal to the real part of the lowest weight of W plus N. So a lowest weight vector of $\ker f$ must be in $\Omega_N^0(W)$, contradictory to (3.11). Since f is both injective and surjective, f is an equivalence of generalized V-modules.

For a finite-dimensional graded $A^{N'}(V)$-module M, it is clear that M^N is a graded $A^N(V)$-module. Since M is finite dimensional, by Property 2 in Proposition 3.19, $S^{N'}_{\rm voa}(M)$ is quasi-finite dimensional. By Property 5 in Proposition 3.19, $S^{N'}_{\rm voa}(M)$ is of finite length. By Property 2 above, $S^{N'}_{\rm voa}(M)$ is generated by $\Omega_N^0(S^{N'}_{\rm voa}(M)) = M^N$ and is equivalent to $S^N_{\rm voa}(M^N)$. Since $\Omega^0_{N'}(S^{N'}_{\rm voa}(M)) = M$, we see that M is equivalent to $\Omega^0_{N'}(S^N_{\rm voa}(M^N))$. ∎

In [H8], it is proved that $A^N(V)$ is finite dimensional when V has no nonzero elements of negative weights, $V_{(0)} = \mathbb{C}\mathbf{1}$ and is C_2-cofinite. In fact, the condition $V_{(0)} = \mathbb{C}\mathbf{1}$ is not needed. Below we prove that $A^N(W)$ is finite dimensional without this condition for a grading-restricted generalized V-module W.

Theorem 3.21 *Let V be a C_2-cofinite vertex operator algebra without nonzero elements of negative weights and W a grading-restricted generalized V-module. Then $A^N(W)$ is finite dimensional.*

Proof. By Proposition 5.2 in [ABD], every irreducible V-module is C_2-cofinite. By Proposition 3.19, we see that W is of finite length. Since every irreducible V-module is C_2-cofinite, W as a generalized V-module of finite length must also be C_2-cofinite. If we assume in addition that $V_{(0)} = \mathbb{C}\mathbf{1}$, then by Theorem 11 in [GN] (see Proposition 5.5 in [AbN]), V is also C_n-cofinite for $n \geq 2$. By Proposition 5.1 in [AbN], W is C_n-cofinite for $n \geq 2$.

On the other hand, since Lemma 2.4 in [Mi2] gives a spanning set of W without the condition $V_{(0)} = \mathbb{C}\mathbf{1}$, the arguments in [GN] and [AbN] can also be used to show that W is C_n-cofinite for $n \geq 2$ without this condition. For reader's convenience, here we give a direct proof of this fact observed by McRae using Lemma 2.4 in [Mi2]. In fact, since W is of finite length, it is finitely generated. By Lemma 2.4 in [Mi2], W is spanned by elements of the form

$$(Y_W)_{i_1}(v_1) \cdots (Y_W)_{i_k}(v_k) w_j \tag{3.12}$$

for homogeneous $v_1, \dots, v_k$ in a finite set, homogeneous $w_j \in W$ for $j = 1, \dots, l$ and $i_1, \dots, i_k \in \mathbb{Z}$ satisfying $i_1 < \cdots < i_k$. Using the lower-truncation property of the vertex operator map Y_W and the fact that v_k can change only in a finite set and there are only finitely many w_j, we see that there exists $m \in \mathbb{Z}$ such that for $i_k > m$, $(Y_W)_{i_k}(v_k)w_j = 0$ and hence (3.12) is 0. Since in (3.12), we have $i_1 < \cdots < i_k$, there are only finitely many elements of the form (3.12) satisfying $i_1 > -n$. Since elements of the form 3.12) span W, we see that W is C_n-cofinite for $n \geq 2$.

Take $n = k + l + 2$ for $k, l = 0, \dots, N$. Then W is C_{k+l+2}-cofinite. By definition, $C_{k+l+2}(W)$ are spanned by elements of the form $(Y_W)_{-k-l-2}(v)w$ for $v \in V$ and $w \in W$. Since W is C_{k+l+2}-cofinite, there exists a finite dimensional subspace X_{k+l} of W such that $X_{k+l} + C_{k+l+2}(W) = W$. Let $U^N(X)$ be the subspace of $U^N(W)$ consisting matrices in $U^N(W)$ whose (k, l)-th entries are in X_{k+l} for $k, l = 0, \dots, N$. Since X_{k+l} for $k, l = 0, \dots, N$ are finite dimensional, $U^N(W)$ is also finite dimensional. We now prove $U^N(X) + (O^\infty(W) \cap$

$U^N(W)) = U^N(W)$. To prove this, we need only prove that every element of $U^N(W)$ of the form $[w]_{kl}$ for $w \in W$ and $0 \leq k, l \leq N$, can be written as $[w]_{kl} = [w_1]_{kl} + [w_2]_{kl}$, where $w_1 \in X_{k+l}$ and $w_2 \in V$ such that $[w_2]_{kl} \in O^\infty(W)$. We shall denote the subspace of W consisting of elements w such that $[w]_{kl} \in O^\infty(W)$ by $O^\infty_{kl}(W)$. Then what we need to prove is $W = X_{k+l} + O^\infty_{kl}(W)$.

Since $W = \coprod_{n\in\mathbb{N}} W_{\llbracket n\rrbracket}$ is of finite length, $W_{\llbracket n\rrbracket}$ for $n \in \mathbb{N}$ is finite dimensional. We can always take X_{k+l} to be a subspace of W containing $W_{\llbracket 0\rrbracket}$. We use induction on p for $w \in W_{\llbracket p\rrbracket}$. When $w \in W_{\llbracket 0\rrbracket}$, w can be written as $w = w+0$, where $v \in X_{k+l}$ and $0 \in O^\infty_{kl}(V)$.

Assume that when $v \in W_{\llbracket p\rrbracket}$ for $p < q$, $w = w_1 + w_2$, where $w_1 \in X_{k+l}$ and $w_2 \in O^\infty_{kl}(V)$. Then since W is C_{k+l+2}-cofinite, for $w \in W_{\llbracket q\rrbracket}$, there exists homogeneous $w_1 \in X_{k+l}$ and $v^i \in V$ and $w^i \in W$ for $i = 1, \dots, m$ such that $v = u_1 + \sum_{i=1}^m (Y_W)_{-k-l-2}(v^i)w^i$. Moreover, we can always find such w_1 and $v^i, w^i \in V$ for $i = 1, \dots, m$ such that $w_1, (Y_W)_{-k-l-2}(v^i)w_i \in W_{\llbracket q\rrbracket}$. Since $(Y_W)_{n-k-l-2}(v^i)w^i \in W_{\llbracket q-n\rrbracket}$, where $q - n < q$ for $i = 1, \dots, m$ and $n \in \mathbb{Z}_+$, by induction assumption, $(Y_W)_{n-k-l-2}(v^i)w^i \in X_{k+l} + O^\infty_{kl}(V)$ for $i = 1, \dots, m$ and $k \in \mathbb{Z}_+$. Thus

$$\begin{aligned} w &= w_1 + \sum_{i=1}^m (Y_W)_{-k-l-2}(v^i)w^i \\ &= w_1 + \sum_{i=1}^m \mathrm{Res}_x x^{-k-l-2}(1+x)^l Y_W((1+x)^{L(0)}v^i, x)w^i \\ &\quad - \sum_{i=1}^m \sum_{n\in\mathbb{Z}_+} \binom{\mathrm{wt}\ v^i + l}{n} (Y_W)_{n-k-l-2}(v^i)w^i. \end{aligned}$$

By definition,

$$[\mathrm{Res}_x x^{-k-l-2}(1+x)^l Y_W((1+x)^{L(0)}v^i, x)w^i]_{kl} \in O^\infty(W).$$

Thus

$$\mathrm{Res}_x x^{-k-l-2}(1+x)^l Y_W((1+x)^{L(0)}v^i, x)w^i \in O^\infty_{kl}(W).$$

Thus we have $w = w_1 + w_2$, where $w_1 \in X_{k+l}$ and $w_2 \in O^\infty_{kl}(W)$. By induction principle, we have $W = X_{k+l} + O^\infty_{kl}(W)$.

We now have proved $U^N(X) + (O^\infty(W) \cap U^N(W)) = U^N(W)$. Since $O^\infty(W) \cap U^N(W) \subset Q^\infty(W) \cap U^N(W)$, we also have $U^N(X) + (Q^\infty(W) \cap U^N(W)) = U^N(W)$. Since $U^N(X)$ is finite dimensional, $A^N(W)$ is finite dimensional. ∎

By Theorems 3.15 and 3.21, we obtain immediately the following result:

Corollary 3.22 *Let V be a C_2-cofinite vertex operator algebra without nonzero elements of negative weights and W a grading-restricted generalized V-module. Then $\widetilde{A}^N(W)$ is finite dimensional.*

4 Symmetric linear functions on $\widetilde{A}^N(V)$-bimodules

In this section, we give two constructions of symmetric linear functions on the $\widetilde{A}^N(V)$-bimodule $\widetilde{A}^N(W)$ for $N \in \mathbb{N}$ and a grading-restricted generalized V-module W. We first give symmetric linear functions on $\widetilde{A}^N(W)$ using shifted pseudo-q-traces of intertwining operators. Then we also construct symmetric linear functions on $\widetilde{A}^N(W)$ starting from linear maps from a lower-bounded generalized V-module to $\mathbb{C}\{q\}[\log q]$ satisfying suitable conditions corresponding to the conditions for genus-one 1-point conformal blocks (see Definition 5.1 in the next section). This second construction is the main technically difficult part of the present paper. In fact, the first construction can also be obtained using the second construction. But we still give the first construction to show that it is much easier to obtain such linear symmetric functions on $\widetilde{A}^N(W)$ using the properties of shifted pseudo-q-traces of intertwining operators than using only the properties of genus-one 1-point conformal blocks.

Let W be a grading-restricted generalized V-module, $\widetilde{W}$ a grading-restricted generalized V-P-bimodule which is projective as a right P-module and $\mathcal{Y}$ an intertwining operator of type $\binom{\widetilde{W}}{W\widetilde{W}}$ and compatible with P.

Recall from [H9] and the preceding section that we can write

$$\widetilde{W} = \coprod_{\mu\in\Gamma(\widetilde{W})}\coprod_{n\in\mathbb{N}}\widetilde{W}_{[h^\mu+n]} = \coprod_{\mu\in\Gamma(\widetilde{W})}\widetilde{W}^\mu = \coprod_{n\in\mathbb{N}}\widetilde{W}_{\llbracket n\rrbracket},$$

where $\Gamma(\widetilde{W}) \subset \mathbb{C}/\mathbb{Z}$, $h^\mu \in \mu$ for $\mu \in \Gamma(\widetilde{W})$ and

$$\widetilde{W}^\mu = \coprod_{n\in\mathbb{N}}\widetilde{W}_{[h^\mu+n]}, \quad \widetilde{W}_{\llbracket n\rrbracket} = \coprod_{\mu\in\Gamma(\widetilde{W})}\widetilde{W}_{[h^\mu+n]}.$$

Then from the definition of shifted pseudo-q-traces, for $w \in W$, we have

$$\begin{aligned}
&\mathrm{Tr}^\phi_{\widetilde{W}}\mathcal{Y}(\mathcal{U}_W(q_z)w, q_z)q_\tau^{L(0)-\frac{c}{24}}\\
&\quad= \sum_{k=0}^K\frac{(2\pi i)^k}{k!}\sum_{\mu\in\Gamma(\widetilde{W})}\sum_{n\in\mathbb{N}}\phi_{\widetilde{W}_{[h^\mu+n]}}\pi_{h^\mu+n}\mathcal{Y}(\mathcal{U}_W(q_z)w, q_z)L_{\widetilde{W}}(0)_N^k\Big|_{\widetilde{W}_{[h^\mu+n]}}\tau^k q_\tau^{h^\mu+n-\frac{c}{24}}. \qquad (4.1)
\end{aligned}$$

Note that $[L_{\widetilde{W}}(0)_N, Y_{\widetilde{W}}(v, x)] = 0$ for $v \in V$, that is, $L_{\widetilde{W}}(0)_N$ is in fact a V-module map from W to itself. Assume that there exists $K \in \mathbb{N}$ such that $L_{\widetilde{W}}(0)_N^{K+1}\tilde{w} = 0$ for $\tilde{w} \in \widetilde{W}$. This is always true if $\widetilde{W}$ is of finite length. By Proposition 3.19, this is always true when V has no nonzero elements of negative weights and is C_2-cofinite. In this case, for each $k = 0, \dots, K$, $\mathcal{Y}^k = \mathcal{Y} \circ (1_W \otimes L_{\widetilde{W}}(0)_N^k)$ is an intertwining operators of the type $\binom{\widetilde{W}}{W\widetilde{W}}$. Note that $\mathcal{Y}^0 = \mathcal{Y}$.

Let $\mu \in \Gamma(\widetilde{W})$ and $k = 0, \dots, K$. We define

$$\psi^\mu_{\mathcal{Y}^k,\phi}([w]_{mn}) = 0$$

for $w\in W$ and $m,n\in\mathbb{N}$ such that $m\neq n$ and

$$\begin{aligned}\psi^{\mu}_{\mathcal{Y}^k,\phi}([w]_{nn}) &= \phi_{\widetilde{W}_{[h^\mu+n]}}\pi_{h^\mu+n}\mathcal{Y}^k(\mathcal{U}_W(q_z)w,q_z)\bigg|_{\widetilde{W}_{[h^\mu+n]}}\\ &= \phi_{\widetilde{W}_{[h^\mu+n]}}\pi_{h^\mu+n}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)L_{\widetilde{W}}(0)^k_N\bigg|_{\widetilde{W}_{[h^\mu+n]}}\end{aligned}$$

for $w\in W$ and $n\in\mathbb{N}$. We then define

$$\psi_{\mathcal{Y}^k,\phi}([w]_{mn}) = \sum_{\mu\in\Gamma(\widetilde{W})}\psi^{\mu}_{\mathcal{Y}^k,\phi}([w]_{mn})$$

for $w\in W$ and $m,n\in\mathbb{N}$. For $m,n\in\mathbb{N}$ such that $m\neq n$,

$$\psi_{\mathcal{Y}^k,\phi}([w]_{mn})=0$$

and for $n\in\mathbb{N}$,

$$\begin{aligned}\psi_{\mathcal{Y}^k,\phi}([w]_{nn}) &= \sum_{\mu\in\Gamma(\widetilde{W})}\phi_{\widetilde{W}_{[h^\mu+n]}}\pi_{h^\mu+n}\mathcal{Y}^k(\mathcal{U}_W(q_z)w,q_z)\bigg|_{\widetilde{W}_{[h^\mu+n]}}\\ &= \sum_{\mu\in\Gamma(\widetilde{W})}\phi_{\widetilde{W}_{[h^\mu+n]}}\pi_{h^\mu+n}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)L_{\widetilde{W}}(0)^k_N\bigg|_{\widetilde{W}_{[h^\mu+n]}}.\end{aligned}$$

Using the $L(-1)$-derivative property and $L(-1)$-commutator formula for the intertwining operator $\mathcal{Y}$, we have

$$\begin{aligned}&\frac{\partial}{\partial z}\mathrm{Tr}^{\phi}_{\widetilde{W}}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}\\ &\quad= \mathrm{Tr}^{\phi}_{\widetilde{W}}\mathcal{Y}((2\pi i L_W(0)+2\pi i q_z L_W(-1))\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}\\ &\quad= 2\pi i\mathrm{Tr}^{\phi}_{\widetilde{W}}[L_{\widetilde{W}}(0),\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)]q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}\\ &\quad= 0.\end{aligned}$$

So $\mathrm{Tr}^{\phi}_{\widetilde{W}}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}$ is independent of z. Then the coefficients of this series in powers of q_τ and τ are also independent of z. In particular, for $k=0,\dots,K$, $\mu\in\Gamma(\widetilde{W})$ and $m,n\in\mathbb{N}$ $\psi^{\mu}_{\mathcal{Y}^k,\phi}([w]_{mn})$ and $\psi_{\mathcal{Y}^k,\phi}([w]_{mn})$ are independent of z. Then we obtain linear functions $\psi^{\mu}_{\mathcal{Y}^k,\phi}$ for $\mu\in\Gamma(\widetilde{W})$ and $\psi_{\mathcal{Y}^k,\phi}$ on $U^\infty(W)$.

Since $\mathrm{Tr}^{\phi}_{\widetilde{W}}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}$ is independent of z, we have

$$\begin{aligned}&\mathrm{Tr}^{\phi}_{\widetilde{W}}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}\\ &\quad= \mathrm{coeff}^0_{\log q_z}\mathrm{Res}_{q_z}q_z^{-1}\mathrm{Tr}^{\phi}_{\widetilde{W}}\mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}\end{aligned}$$

$$= \mathrm{Tr}^{\phi}_{\widetilde{W}} \mathrm{coeff}^{0}_{\log x} \mathrm{Res}_{x} x^{-1} \mathcal{Y}(\mathcal{U}_{W}(x)w, x) q_{\tau}^{L_{\widetilde{W}}(0) - \frac{c}{24}}, \tag{4.2}$$

where as in [H9], we use $\mathrm{coeff}^{0}_{\log q_z}$ to denote the operation to take the constant term of a polynomial in $\log q_z$.

Let $n, N \in \mathbb{N}$ such that $n \leq N$. Using (4.2), $\mathcal{U}_W(x) = x^{L_W(0)}\mathcal{U}(1)$, the definition of $\tilde{\vartheta}_{\mathcal{Y}}$ and $\pi_{h^\mu+m}\tilde{\vartheta}_{\mathcal{Y}}([w]_{nn}) = 0$ for $m \neq n$, we obtain

$$\begin{aligned}
\psi^{k}_{\mathcal{Y},\phi}([w]_{nn}) &= \sum_{\mu \in \Gamma(\widetilde{W})} \phi_{\widetilde{W}_{[h^\mu+n]}} \pi_{h^\mu+n} \mathrm{coeff}^{0}_{\log x} \mathrm{Res}_{x} x^{-1} \mathcal{Y}^{k}(\mathcal{U}_{W}(x)w, x)\Big|_{\widetilde{W}_{[h^\mu+n]}} \\
&= \sum_{\mu \in \Gamma(\widetilde{W})} \phi_{\widetilde{W}_{[h^\mu+n]}} \pi_{h^\mu+n} \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nn})\Big|_{\widetilde{W}_{[h^\mu+n]}} \\
&= \sum_{\mu \in \Gamma(\widetilde{W})} \sum_{m=0}^{N} \phi_{\widetilde{W}_{[h^\mu+m]}} \pi_{h^\mu+m} \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nn})\Big|_{\widetilde{W}_{[h^\mu+m]}} \\
&= \phi_{\Omega^{0}_{N}(\widetilde{W})} \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nn})\Big|_{\Omega^{0}_{N}(\widetilde{W})}.
\end{aligned} \tag{4.3}$$

for $n \in \mathbb{N}$.

Lemma 4.1 *For $m, n \in \mathbb{N}$, $k = 0, \ldots, K$, $v \in V$ and $w \in W$, we have*

$$\psi_{\mathcal{Y}^k,\phi}([v]_{mn} \blacklozenge [w]_{nm}) = \psi_{\mathcal{Y}^k,\phi}([w]_{nm} \blacklozenge [v]_{mn}). \tag{4.4}$$

Proof. By definition,

$$[v]_{mn} \blacklozenge [w]_{nm} = \mathrm{Res}_{x} T_{2m+1}((x+1)^{-2m+n-1})(1+x)^{m} \left[Y_{W}\left(v, \frac{1}{2\pi i}\log(1+x)\right) w \right]_{mm}.$$

Let $N \in \mathbb{N}$ be larger than or equal to both m and n. Using (4.3) and

$$\begin{aligned}
\tilde{\vartheta}_{\mathcal{Y}}([v]_{mn} \blacklozenge [w]_{nm}) &= \tilde{\vartheta}_{\widetilde{W}}([v]_{mn}) \tilde{\vartheta}_{\mathcal{Y}}([w]_{nm}), \\
\tilde{\vartheta}_{\mathcal{Y}}([w]_{nm} \blacklozenge [v]_{mn}) &= \tilde{\vartheta}_{\mathcal{Y}}([w]_{nm}) \tilde{\vartheta}_{\widetilde{W}}([v]_{mn}),
\end{aligned}$$

we have

$$\begin{aligned}
&\psi_{\mathcal{Y}^k,\phi}([v]_{mn} \blacklozenge [w]_{nm}) \\
&\quad = \phi_{\Omega^{0}_{N}(\widetilde{W})} \tilde{\vartheta}_{\mathcal{Y}^k}([v]_{mn} \blacklozenge [w]_{nm})\Big|_{\Omega^{0}_{N}(\widetilde{W})} \\
&\quad = \phi_{\Omega^{0}_{N}(\widetilde{W})} \tilde{\vartheta}_{\widetilde{W}}([v]_{mn}) \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nm})\Big|_{\Omega^{0}_{N}(\widetilde{W})} \\
&\quad = \phi_{\Omega^{0}_{N}(\widetilde{W})} \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nm}) \tilde{\vartheta}_{\widetilde{W}}([v]_{mn})\Big|_{\Omega^{0}_{N}(\widetilde{W})}
\end{aligned}$$

$$= \phi_{\Omega_N^0(\widetilde{W})} \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nm} \diamond [v]_{mn})\Big|_{\Omega_N^0(\widetilde{W})}$$
$$= \psi_{\mathcal{Y}^k,\phi}([w]_{nm} \diamond [v]_{mn}).$$

∎

Fix $N \in \mathbb{N}$. Then the restriction of the linear function $\psi_{\mathcal{Y}^k,\phi}$ on $U^\infty(W)$ to $U^N(W)$ is a linear function $\psi^N_{\mathcal{Y}^k,\phi}$ on $U^N(W)$.

Proposition 4.2 *The linear function $\psi^N_{\mathcal{Y}^k,\phi}$ on $U^\infty(W)$ is in fact a symmetric linear function on $\widetilde{A}^N(W)$, that is, $\psi^N_{\mathcal{Y}^k,\phi}(\widetilde{Q}^\infty(W) \cap U^N(W)) = 0$ and*

$$\psi^N_{\mathcal{Y}^k,\phi}(\mathfrak{v} \diamond \mathfrak{w}) = \psi^N_{\mathcal{Y}^k,\phi}(\mathfrak{w} \diamond \mathfrak{v}) \tag{4.5}$$

for $\mathfrak{v} \in U^N(V)$ and $\mathfrak{w} \in U^N(W)$.

Proof. Since $\psi_{\mathcal{Y}^k,\phi}([w]_{mn}) = 0$ for $m \neq n$, to prove $\psi^N_{\mathcal{Y}^k,\phi}(\widetilde{Q}^\infty(W) \cap U^N(W)) = 0$, we need only prove $\psi_{\mathcal{Y}^k,\phi}([w]_{nn}) = 0$ if $[w]_{nn} \in \widetilde{Q}^\infty(W)$. From (4.3), we have

$$\psi_{\mathcal{Y}^k,\phi}([w]_{nn}) = \phi_{\Omega_N^0(\widetilde{W})} \tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nn})\Big|_{\Omega_N^0(\widetilde{W})}.$$

But from the definition of $\widetilde{Q}^\infty(W)$, $\tilde{\vartheta}_{\mathcal{Y}^k}([w]_{nn}) = 0$ if $[w]_{nn} \in \widetilde{Q}^\infty(W)$. This proves $\psi^N_{\mathcal{Y}^k,\phi}(\widetilde{Q}^\infty(W) \cap U^N(W)) = 0$.

For $\mathfrak{v} = [v]_{mn}$ and $\mathfrak{w} = [w]_{kl}$, if $n \neq k$, we have

$$\mathfrak{v} \diamond \mathfrak{w} = [v]_{mn} \diamond [w]_{kl} = 0$$

and

$$\psi_{\mathcal{Y}^k,\phi}(\mathfrak{w} \diamond \mathfrak{v}) = \psi_{\mathcal{Y}^k,\phi}([w]_{kl} \diamond [v]_{mn}) = 0.$$

So (4.5) holds in this case. The same argument shows that if $m \neq l$, (4.5) holds. The case $n = k$ and $m = l$ is given by (4.4). Thus (4.5) holds for all $\mathfrak{v} \in U^N(V)$ and $\mathfrak{w} \in U^N(W)$. ∎

Proposition 4.2 gives the first construction of linear symmetric functions on $\widetilde{A}^N(W)$. We now give the second construction.

We recall some basic facts on the Weierstrass $\wp$-function, denoted by $\wp_2(z;\tau)$ in this paper, the Weierstrass ζ-function, denoted by $\wp_1(z;\tau)$ in this paper, and the Eisenstein series $G_2(\tau)$. For details, see [L] and [K]. The Weierstrass $\wp$-function $\wp_2(z;\tau)$ has the q-expansion

$$\wp_2(z;\tau) = (2\pi i)^2 q_z (q_z - 1)^{-2} + (2\pi i)^2 \sum_{s\in\mathbb{Z}_+} \sum_{l|s} l(q_z^l + q_z^{-l}) q_\tau^s - \frac{\pi^2}{3} - 2(2\pi i)^2 \sum_{l\in\mathbb{Z}_+} \sigma(l) q_\tau^l \tag{4.6}$$

in the region given by $|q_\tau| < |q_z| < |q_\tau|^{-1}$ and $z \neq 0$, where $\sigma(l) = \sum_{n|l} n$ for $l \in \mathbb{Z}_+$. Note that the coefficients of the power series (4.6) in q_τ are holomorphic functions of z on the whole complex plane except for the the coefficient $(2\pi i)^2 q_z (q_z - 1)^{-2}$ of q_τ^0, which has a pole of order 2 at $z = 0$. Let

$$\begin{aligned}\tilde{\wp}_2(x;q) = (2\pi i)^2 e^{2\pi i x}(e^{2\pi i x} - 1)^{-2} + (2\pi i)^2 \sum_{s\in\mathbb{Z}_+}\sum_{l|s} l(e^{2l\pi i x} + e^{-2l\pi i x})q^s \\ - \frac{\pi^2}{3} - 2(2\pi i)^2 \sum_{l\in\mathbb{Z}_+} \sigma(l) q^l, \end{aligned} \tag{4.7}$$

where $e^{2\pi i x}(e^{2\pi i x} - 1)^{-2}$ is understood as the formal Laurent series obtained by expanding $e^{2\pi i z}(e^{2\pi i z} - 1)^{-2}$ as a Laurent series near $z = 0$ and then replacing z by x, $e^{2l\pi i x} = \sum_{k\in\mathbb{N}} \frac{(2l\pi i x)^k}{k!}$ and $e^{-2l\pi i x} = \sum_{k\in\mathbb{N}} \frac{(-2l\pi i x)^k}{k!}$. In terms of only formal variables, the formal Laurent series $e^{2\pi i x}(e^{2\pi i x} - 1)^{-2}$ can also be obtained as follows: Write $(e^{2\pi i x} - 1)^{-2}$ as $(2\pi i x)^{-2}(1 + \sum_{k\in\mathbb{Z}_+ +1} \frac{(2\pi i x)^{k-1}}{k!})^{-2}$ and then expand $(1 + \sum_{k\in\mathbb{Z}_+ +1} \frac{(2\pi i x)^{k-1}}{k!})^{-2}$ using binomial expansion as a power series in $\sum_{k\in\mathbb{Z}_+ +1} \frac{(2\pi i x)^{k-1}}{k!}$. Since the l-th power of $\sum_{k\in\mathbb{Z}_+ +1} \frac{(2\pi i x)^{k-1}}{k!}$ is a power series in x such that the coefficients of x^j for $j = 0, \dots, l-1$ are 0, this expansion of $(1 + \sum_{k\in\mathbb{Z}_+ +1} \frac{(2\pi i x)^{k-1}}{k!})^{-2}$ gives a well-defined formal power series in x. Multiplying $(2\pi i x)^{-2}$ with this formal power series in x, we obtain a formal Laurent series expansion of $(e^{2\pi i x} - 1)^{-2}$. Multiplying this formal Laurent series expansion of $(e^{2\pi i x} - 1)^{-2}$ with the formal power series $e^{2l\pi i x} = \sum_{k\in\mathbb{N}} \frac{(2l\pi i x)^k}{k!}$, we obtain a formal Laurent series expansion of $e^{2\pi i x}(e^{2\pi i x} - 1)^{-2}$ which is the same as the formal Laurent series expansion obtained using complex analysis.

Similarly, the Weierstrass ζ-function $\wp_1(z;\tau)$ minus $G_2(\tau)z$ has the q-expansion

$$\wp_1(z;\tau) - G_2(\tau)z = 2\pi i q_z (q_z - 1)^{-1} - 2\pi i \sum_{s\in\mathbb{Z}_+}\sum_{l|s} (q_z^l - q_z^{-l}) q_\tau^s - \pi i \tag{4.8}$$

in the region given by $|q_\tau| < |q_z| < |q_\tau|^{-1}$ and $z \neq 0$. Let

$$\tilde{\wp}_1(x;q) - \widetilde{G}_2(q)x = 2\pi i e^{2\pi i x}(e^{2\pi i x} - 1)^{-1} - 2\pi i \sum_{s\in\mathbb{Z}_+}\sum_{l|s} (e^{2l\pi i x} - e^{-2l\pi i x}) q^s - \pi i, \tag{4.9}$$

where $e^{2\pi i x}(e^{2\pi i x} - 1)^{-1}$ is understood as the formal Laurent series obtained by expanding $e^{2\pi i z}(e^{2\pi i z} - 1)^{-1}$ as a Laurent series near $z = 0$ and then replacing z by x. The formal Laurent series $e^{2\pi i x}(e^{2\pi i x} - 1)^{-1}$ can also be obtained in terms of only formal variables in a way completely similar to that for $e^{2\pi i x}(e^{2\pi i x} - 1)^{-2}$ above.

We also have the Laurent series expansion in z

$$\begin{aligned}\wp_2(z;\tau) &= \frac{1}{z^2} + \sum_{k\in\mathbb{Z}_+} (2k+1) G_{2k+2}(\tau) z^{2k}, \\ \wp_1(z;\tau) - G_2(\tau)z &= \frac{1}{z} - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau) z^{2k+1}\end{aligned}$$

in the region $0 < |z| < \min(1, |\tau|)$, where $G_{2k+2}(\tau)$ are the Eisenstein series. Let

$$\wp_2(x;\tau) = \frac{1}{x^2} + \sum_{k\in\mathbb{Z}_+} (2k+1)G_{2k+2}(\tau)x^{2k}, \tag{4.10}$$

$$\wp_1(x;\tau) - G_2(\tau)x = \frac{1}{x} - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)x^{2k+1} \tag{4.11}$$

Lemma 4.3 *The formal Laurent series $\tilde{\wp}_2(x;q)$ and $\tilde{\wp}_1(x;q) - \widetilde{G}_2(q)x$ in x and q can be obtained by expanding the coefficients of $\wp_2(x;\tau)$ and $\wp_1(x;\tau) - G_2(\tau)x$, respectively, in powers of x as power series in q_τ and then replacing q_τ by q.*

Proof. Since (4.6) is absolutely convergent in the region given by $|q_\tau| < |q_z| < |q_\tau|^{-1}$ and $z \neq 0$ and it has a pole of order 2 at $z = 0$, the double series (4.7) with x and q replaced by z and q_τ is absolutely convergent in the region given by $|q_\tau| < \epsilon$ and $0 < |z| < \frac{-\log\epsilon}{2\pi}$ for any $\epsilon \in (0,1)$. In particular, after we substitute z and q_τ for x and q, we can first sum over the powers of q_τ in the region $|q_\tau| < \epsilon$ to obtain a Laurent series in z which is absolutely convergent in the region $0 < |z| < \frac{-\log\epsilon}{2\pi}$ to $\wp_2(z;\tau)$. Substituting x for z in this Laurent series, we obtain a formal Laurent series in x which by definition is equal to $\wp_2(x;\tau)$. This is equivalent to the statement for $\tilde{\wp}_2(x;q)$.

The proof for the statement for $\tilde{\wp}_1(x;q) - \widetilde{G}_2(q)x$ is the same. ∎

In the formulations, discussions and proofs below, for $m \in \mathbb{Z}_+$, $(e^{2\pi ix} - 1)^{-m}$ is always understood as the formal Laurent series obtained by expanding $(e^{2\pi iz} - 1)^{-m}$ as a Laurent series near $z = 0$ and then replacing z by x.

The following theorem gives our second construction of symmetric linear functions on $\widetilde{A}^N(W)$ and is the main result of this section:

Theorem 4.4 *Let W be a grading-restricted generalized V-module and*

$$\begin{aligned} S : W &\to \mathbb{C}\{q\}[\log q] \\ w &\mapsto S(w;q) \end{aligned}$$

a linear map satisfying

$$S(w;q) = \sum_{k=0}^{K}\sum_{j=1}^{J}\sum_{m\in\mathbb{N}} S_{k,j,m}(w)(\log q)^k q^{r_j+m},$$

$$S(\mathrm{Res}_x Y_W(v,x)w;q) = 0, \tag{4.12}$$

$$S(\mathrm{Res}_x \tilde{\wp}_2(x;q) Y_W(v,x)w;q) = 0, \tag{4.13}$$

$$(2\pi i)^2 q\frac{\partial}{\partial q}S(w;q) = S(\mathrm{Res}_x(\tilde{\wp}_1(x;q) - \widetilde{G}_2(q)x)Y_W(\omega,x)w;q) \tag{4.14}$$

for $u \in V$ and $w \in W$, where $K \in \mathbb{N}$, $r_1, \ldots, r_J \in \mathbb{C}$ are independent of w. For $N \in \mathbb{N}$, $k = 0, \ldots, K$ and $j = 1, \ldots, J$, let $\psi^N_{S;k,j} : U^N(W) \to \mathbb{C}$ be the linear map defined by

$\psi^N_{S;k,j}([w]_{mn}) = 0$ *for* $w \in W$ *and* $m, n \in \mathbb{N}$ *satisfying* $0 \leq m, n \leq N$ *and* $m \neq n$ *and* $\psi^N_{S;k,j}([w]_{nn}) = S_{k,j,n}(w)$ *for* $w \in W$ *and* $n \in \mathbb{N}$ *satisfying* $n \leq N$*. Then* $\psi^N_{S;k,j}(\widetilde{Q}^\infty(W)) = 0$ *so that* $\psi^N_{S;k,j}$ *for* $k = 0, \dots, K$ *and* $j = 1, \dots, J$ *induce linear maps from* $\widetilde{A}^N(W)$ *to* $\mathbb{C}$*, still denoted by* $\psi^N_{S;k,j}$*. Moreover, these induced linear maps* $\psi^N_{S;k,j}$ *are in fact symmetric linear functions on* $\widetilde{A}^N(W)$*, that is, for* $\mathfrak{v} \in \widetilde{A}^N(V)$ *and* $\mathfrak{w} \in \widetilde{A}^N(W)$*,*

$$\psi^N_{S;k,j}(\mathfrak{v} \blacklozenge \mathfrak{w}) = \psi^N_{S;k,j}(\mathfrak{w} \blacklozenge \mathfrak{v})$$

satisfying

$$\psi^N_{S;k,j}(([\omega]_{nn} - (r_j + n)[\mathbf{1}]_{nn})^{\blacklozenge(K-k+1)} \blacklozenge \mathfrak{w}) = 0 \tag{4.15}$$

for $n = 0, \dots, N$ *and* $\mathfrak{w} \in U(W)$*, where*

$$([\omega]_{nn} - (r_j + n)[\mathbf{1}]_{nn})^{\blacklozenge(K-k+1)} = \overbrace{([\omega]_{nn} - (r_j + n)[\mathbf{1}]_{nn}) \blacklozenge \cdots \blacklozenge ([\omega]_{nn} - (r_j + n)[\mathbf{1}]_{nn})}^{K-k+1}.$$

We will prove this result after we prove a number of lemmas and propositions. We first recall and prove some identities involving binomial coefficients.

Lemma 4.5 *We have the following identities:*

1. *For* $m, n, k, l \in \mathbb{Z}_+$ *satisfying* $n \geq m \geq l$*, we have*

$$\sum_{j=0}^{m} \frac{l}{(n-j+k)} \binom{m}{j} \binom{-l-1}{n-j+k-1} = 0. \tag{4.16}$$

2. *For* $\alpha, \beta \in \mathbb{C}$ *and* $m, n \in \mathbb{N}$*,*

$$\sum_{j=0}^{m} \binom{\alpha}{j} \binom{\beta}{m-j} = \binom{\alpha+\beta}{m}, \tag{4.17}$$

$$\sum_{j=0}^{m} \binom{m}{j} \binom{\alpha}{j+n} = \binom{m+\alpha}{m+n}. \tag{4.18}$$

3. *An identity of Andersen ([An]): For* $\alpha \in \mathbb{C}$*,* $n \in \mathbb{Z}_+$ *and* $k = 0, \dots, n$*,*

$$\sum_{j=0}^{k} \binom{\alpha}{j} \binom{-\alpha}{m-j} = \frac{m-k}{m} \binom{\alpha-1}{k} \binom{-\alpha}{m-k}. \tag{4.19}$$

4. *For* $m, n \in \mathbb{N}$*,*

$$(n+1+m)\binom{n+m}{m} \sum_{p=0}^{n} \frac{(-1)^p}{p+m+1} \binom{n}{p} = 1. \tag{4.20}$$

Proof. For $j = 0, \dots, m$, by the definition of the binomial coefficients, we have

$$
\begin{aligned}
\frac{l}{(n-j+k)}\binom{-l-1}{n-j+k-1} &= \frac{l}{(n-j+k)}\frac{(-l-1)\cdots(-l-1-n+j-k+1+1)}{(n-j+k-1)!} \\
&= (-1)^{n-j+k-1}\frac{(n-j+l+k-1)\cdots(l+1)l(l-1)!}{(l-1)!(n-j+k)!} \\
&= (-1)^{n+k-1}(-1)^j\frac{(n-j+l+k-1)\cdots(n-j+k+1)}{(l-1)!}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{j=0}^{m}\frac{l}{(n-j+k)}\binom{m}{j}\binom{-l-1}{n-j+k-1} \\
&= \sum_{j=0}^{m}(-1)^{n+k-1}(-1)^j\binom{m}{j}\frac{(n-j+l+k-1)\cdots(n-j+k+1)}{(l-1)!} \\
&= \sum_{j=0}^{m}(-1)^{n+k-1}(-1)^j\binom{m}{j}\frac{(n-j+l+k-1)\cdots(n-j+k+1)}{(l-1)!}x^{n-j+k}\bigg|_{x=1} \\
&= \frac{(-1)^{n+k-1}}{(l-1)!}\frac{d^{l-1}}{dx^{l-1}}\sum_{j=0}^{m}(-1)^j\binom{m}{j}x^{n-j+l+k-1}\bigg|_{x=1} \\
&= \frac{(-1)^{n+k-1}}{(l-1)!}\frac{d^{l-1}}{dx^{l-1}}x^{n+l+k-1}(1-x^{-1})^m\bigg|_{x=1} \\
&= \frac{(-1)^{n+k-1}}{(l-1)!}\frac{d^{l-1}}{dx^{l-1}}x^{n-m+l+k-1}(x-1)^m\bigg|_{x=1} \\
&= 0, \qquad (4.21)
\end{aligned}
$$

where in the last step, we have used $l - 1 < m$.

The identity (4.17) is well known and is obtained by taking the coefficient of x^m from both sides of

$$
\sum_{m\in\mathbb{N}}\left(\sum_{j=0}^{m}\binom{\alpha}{j}\binom{\beta}{m-j}\right)x^m = (1+x)^\alpha(1+x)^\beta = (1+x)^{\alpha+\beta} = \sum_{m\in\mathbb{N}}\binom{\alpha+\beta}{m}x^m.
$$

We have

$$
\begin{aligned}
\sum_{j=0}^{m}\binom{m}{j}\binom{\alpha}{j+n} &= \sum_{j=0}^{m}\binom{m}{m-j}\binom{\alpha}{j+n} \\
&= \sum_{j=0}^{m}\binom{m}{(m+n)-(j+n)}\binom{\alpha}{j+n} \\
&= \sum_{k=n}^{m+n}\binom{m}{(m+n)-k}\binom{\alpha}{k}. \qquad (4.22)
\end{aligned}
$$

Using $\binom{m}{(m+n)-k} = 0$ for $k = 0, \ldots, n-1$ and (4.17), we see that the right-hand side of (4.22) is equal to

$$\sum_{k=0}^{m+n} \binom{m}{(m+n)-k}\binom{\alpha}{k} = \binom{m+\alpha}{m+n}. \tag{4.23}$$

Combning (4.22) and (4.23), we obtain (4.18).

For the proof of (4.19), see [An].

Multiplying x^m to both sides of the binomial expansion

$$\sum_{p=0}^{n} \binom{n}{p} x^p = (1+x)^n,$$

we obtain

$$\sum_{p=0}^{n} \binom{n}{p} x^{p+m} = (1+x)^n x^m.$$

Integrating both sides from 0 to x, we obtain

$$\begin{aligned}
&\sum_{p=0}^{n} \binom{n}{p} \frac{x^{p+m+1}}{p+m+1} \\
&\quad = \int_0^x (1+t)^n t^m dt \\
&\quad = \sum_{i=0}^{m} \frac{m\cdots(m-i+1)}{(n+1)\cdots(n+1+i)}(1+x)^{n+1+i}x^{m-i} - (-1)^m \frac{m\cdots 1}{(n+1)\cdots(n+1+m)}.
\end{aligned} \tag{4.24}$$

Substituting -1 for x in both sides of (4.24), we obtain

$$(-1)^{m+1}\sum_{p=0}^{n} \frac{(-1)^p}{p+m+1}\binom{n}{p} = -(-1)^m \frac{m\cdots 1}{(n+1)\cdots(n+1+m)}. \tag{4.25}$$

The identity (4.25) is equivalent to (4.20). ∎

Next we prove a lemma for the vertex operators for a generalized V-module.

Lemma 4.6 *Let W be a generalized V-module. For $m, n \in \mathbb{N}$ satisfying $n \geq m$, we have*

$$\begin{aligned}
&\mathrm{Res}_x e^{2(m+1)\pi i x}(e^{2\pi i x}-1)^{-n-2} Y_{W_1}(v, \pm x)w \\
&\quad = \sum_{j=0}^{m} \frac{1}{(n-j+1)}\binom{m}{j} \mathrm{Res}_x e^{2\pi i x}(e^{2\pi i x}-1)^{-2} Y_W\left(\binom{\pm\frac{1}{2\pi i}L_V(-1)-1}{n-j}v, \pm x\right)w.
\end{aligned} \tag{4.26}$$

Proof. We first prove (4.26) in the special case $m = 0$, that is,

$$\mathrm{Res}_x e^{2\pi i x}(e^{2\pi i x}-1)^{-n-2} Y_{W_1}(v, \pm x)w$$

$$= \frac{1}{n+1}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_W\left(\binom{\pm\frac{1}{2\pi i}L_V(-1)-1}{n}v, \pm x\right)w \tag{4.27}$$

for $n \in \mathbb{N}$. In this case,

$$\begin{aligned}
&\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_W(v,\pm x)w\\
&\quad = \mathrm{Res}_x e^{4\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_W(v,\pm x)w - \mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}Y_W(v,\pm x)w\\
&\quad = -\frac{1}{2\pi i(n+1)}\mathrm{Res}_x\left(\frac{d}{dx}e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}\right)Y_W(v,\pm x)w\\
&\quad\quad + \frac{1}{n+1}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}Y_W(v,\pm x)w\\
&\quad\quad - \mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}Y_W(v,\pm x)w\\
&\quad = \frac{1}{2\pi i(n+1)}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}\frac{d}{dx}Y_W(v,\pm x)w\\
&\quad\quad - \frac{n}{n+1}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}Y_W(v,\pm x)w\\
&\quad = \frac{1}{(n+1)}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-1}Y_W\left(\left(\pm\frac{1}{2\pi i}L_V(-1)-n\right)v,\pm x\right)w.
\end{aligned}\tag{4.28}$$

Using (4.28) repeatedly, we have

$$\begin{aligned}
&\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_W(v,\pm x)w\\
&\quad = \frac{1}{(n+1)!}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}\cdot\\
&\quad\quad\quad \cdot Y_W\left(\left(\pm\frac{1}{2\pi i}L_V(-1)-1\right)\cdots\left(\pm\frac{1}{2\pi i}L_V(-1)-n\right)v,\pm x\right)w\\
&\quad = \frac{1}{(n+1)}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_W\left(\binom{\pm\frac{1}{2\pi i}L_V(-1)-1}{n}v,\pm x\right)w.
\end{aligned}$$

In the general case, using the binomial expansion of $(1+(e^{2\pi x}-1))^m$, we have

$$\begin{aligned}
&\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_W(v,\pm x)w\\
&\quad = \sum_{j=0}^{m}\binom{m}{j}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-(n-j)-2}Y_W(v,\pm x)w.
\end{aligned}\tag{4.29}$$

Since $n \geq m$, we have $n \geq j$ for $j = 0, \dots, m$. Then (4.27) gives

$$\begin{aligned}
&\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-(n-j)-2}Y_W(v,\pm x)w\\
&\quad = \frac{1}{(n-j+1)}\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_W\left(\binom{\pm\frac{1}{2\pi i}L_V(-1)-1}{n-j}v,\pm x\right)w.
\end{aligned}\tag{4.30}$$

Using (4.29) and 4.30), we obtain (4.26). ∎

We now prove a lemma giving some identities for maps from a generalized V-module to the space of power series in a formal variable q satisfying (4.12) and (4.13).

Lemma 4.7 *Let W be a generalized V-module. Assume that a linear map*

$$\begin{aligned} S : W &\to \mathbb{C}[[q]] \\ w \mapsto S(w;q) &= \sum_{m\in\mathbb{N}} S_m(w)q^m \end{aligned}$$

satisfies (4.12) and (4.13). Then for $m, n \in \mathbb{N}$ satisfying $n \geq m$,

$$S_0(\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_{W_1}(u,\pm x)w_1) = 0 \tag{4.31}$$

and

$$\begin{aligned} &S_p(\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_{W_1}(v,\pm x)w_1) \\ &\quad = -\sum_{j=0}^{m}\sum_{s=1}^{p}\sum_{l|s}\frac{l}{(n-j+1)}\binom{m}{j}\cdot \\ &\qquad\qquad \cdot S_{p-s}\left(\mathrm{Res}_x\left(\binom{-l-1}{n-j}e^{2l\pi ix}+\binom{l-1}{n-j}e^{-2l\pi ix}\right)Y_W(v,\pm x)w\right). \end{aligned} \tag{4.32}$$

for $p \in \mathbb{Z}_+$.

Proof. We first prove (4.31) and (4.32) in the special case $m = n = 0$, that is,

$$S_0(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_{W_1}(u,\pm x)w_1) = 0 \tag{4.33}$$

and

$$S_p(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_{W_1}(v,\pm x)w_1) = -\sum_{s=1}^{p}\sum_{l|s} lS_{p-s}(\mathrm{Res}_x(e^{2l\pi ix}+e^{-2l\pi ix}))Y_W(v,\pm x)w) \tag{4.34}$$

for $p \in \mathbb{Z}_+$ From (4.7) and (4.13), we have

$$\begin{aligned} 0 &= S(\mathrm{Res}_x\tilde{\wp}_2(x;q)Y(u,\pm x)w_1;q) \\ &= (2\pi i)^2\sum_{n\in\mathbb{N}} S_n(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_{W_1}(u,\pm x)w_1)q^n \\ &\quad + (2\pi i)^2\sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{Z}_+}\sum_{l|s} lS_n(\mathrm{Res}_x(e^{2l\pi ix}+e^{-2l\pi ix})Y_{W_1}(u,\pm x)w_1)q^{n+s} \\ &\quad - \frac{\pi^2}{3}\sum_{n\in\mathbb{N}} S_n(\mathrm{Res}_x Y_{W_1}(u,\pm x)w_1)q^n \\ &\quad - 2(2\pi i)^2\sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{Z}_+}\sigma(s)S_n(\mathrm{Res}_x Y_{W_1}(u,\pm x)w_1)q^{n+s}. \end{aligned} \tag{4.35}$$

Using (4.12), we see that (4.35) gives

$$\sum_{n\in\mathbb{N}} S_n(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_{W_1}(u,\pm x)w_1)q^n$$

$$= -\sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{Z}_+}\sum_{l|s} lS_n(\mathrm{Res}_x(e^{2l\pi ix}+e^{-2l\pi ix})Y_{W_1}(u,\pm x)w_1)q^{n+s}. \tag{4.36}$$

Taking the coefficients of the power of q^p for $p\in\mathbb{N}$ of both sides of (4.36), we obtain (4.33) and (4.34) for $p\in\mathbb{Z}_+$.

For $m,n\in\mathbb{N}$ satisfying $n\geq m$, using (4.26) and (4.33), we obtain

$$\begin{aligned}
&S_0(\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_{W_1}(u,\pm x)w_1)\\
&\quad=\sum_{j=0}^{m}\frac{1}{(n-j+1)}\binom{m}{j}S_0\left(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-2}Y_W\left(\binom{\pm\frac{1}{2\pi i}L_V(-1)-1}{n-j}v,\pm x\right)w\right)\\
&\quad=0,
\end{aligned}$$

proving (4.31). For $m,n\in\mathbb{N}$ satisfying $n\geq m$ and $p\in\mathbb{Z}_+$, using (4.26) and (4.34), we obtain

$$\begin{aligned}
&S_p(\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix}-1)^{-n-2}Y_{W_1}(v,\pm x)w_1)\\
&\quad=-\sum_{j=0}^{m}\sum_{s=1}^{p}\sum_{l|s}\frac{l}{(n-j+1)}\binom{m}{j}\cdot\\
&\qquad\quad\cdot S_{p-s}\left(\mathrm{Res}_x(e^{2l\pi ix}+e^{-2l\pi ix})Y_W\left(\binom{\pm\frac{1}{2\pi i}L_V(-1)-1}{n-j}v,\pm x\right)w\right)\\
&\quad=-\sum_{j=0}^{m}\sum_{s=1}^{p}\sum_{l|s}\frac{l}{(n-j+1)}\binom{m}{j}\cdot\\
&\qquad\quad\cdot S_{p-s}\left(\mathrm{Res}_x(e^{2l\pi ix}+e^{-2l\pi ix})\binom{\frac{1}{2\pi i}\frac{d}{dx}-1}{n-j}Y_W(v,\pm x)w\right)\\
&\quad=-\sum_{j=0}^{m}\sum_{s=1}^{p}\sum_{l|s}\frac{l}{(n-j+1)}\binom{m}{j}\cdot\\
&\qquad\quad\cdot S_{p-s}\left(\mathrm{Res}_x\left(\binom{-\frac{1}{2\pi i}\frac{d}{dx}-1}{n-j}(e^{2l\pi ix}+e^{-2l\pi ix})\right)Y_W(v,\pm x)w\right)\\
&\quad=-\sum_{j=0}^{m}\sum_{s=1}^{p}\sum_{l|s}\frac{l}{(n-j+1)}\binom{m}{j}\cdot\\
&\qquad\quad\cdot S_{p-s}\left(\mathrm{Res}_x\left(\binom{-l-1}{n-j}e^{2l\pi ix}+\binom{l-1}{n-j}e^{-2l\pi ix}\right)Y_W(v,\pm x)w\right),
\end{aligned}\tag{4.37}$$

proving (4.32). ∎

We are now ready to prove our main technical result.

Proposition 4.8 *Let W be a grading-restricted generalized V-module. Assume that a linear map*

$$\begin{aligned} S : W &\to \mathbb{C}[[q]] \\ w &\mapsto S(w; q) = \sum_{m\in\mathbb{N}} S_m(w)q^m \end{aligned}$$

satisfies (4.12) and (4.13) for $u \in \mathcal{V}$ and $w \in W$. Then we have:

1. *For $m, n, p \in \mathbb{N}$ satisfying $0 \le p \le m$ and $2m \le n$, $v \in V$ and $w \in W$,*

$$S_p(\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix} - 1)^{-n-2} Y_W(v, x)w) = 0. \tag{4.38}$$

2. *For $m, n \in \mathbb{N}$, $v \in V$ and $w \in W$,*

$$\begin{aligned} &\sum_{k=0}^{n} \binom{-2m+n-1}{k} S_m(\mathrm{Res}_x e^{2\pi i(m+1)x}(e^{2\pi ix} - 1)^{-2m+n-k-1} Y_W(v, x)w) \\ &\quad = \sum_{k=0}^{m} \binom{-2n+m-1}{k} S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix} - 1)^{-2n+m-k-1} Y_W(v, -x)w). \end{aligned} \tag{4.39}$$

Proof. Note that (4.38) in the case $m = 0$ is (4.31) in the case $m = 0$ with $\pm$ being $+$. So we need only prove (4.38) in the case $m, n \in \mathbb{Z}_+$. From (4.32), we have

$$\begin{aligned} &S_p(\mathrm{Res}_x e^{2(m+1)\pi ix}(e^{2\pi ix} - 1)^{-n-2} Y_{W_1}(v, x)w_1) \\ &= -\sum_{j=0}^{m}\sum_{s=1}^{p}\sum_{l|s} \frac{l}{(n-j+1)}\binom{m}{j} \cdot \\ &\qquad \cdot S_{p-s}\left(\mathrm{Res}_x \left(\binom{-l-1}{n-j} e^{2l\pi ix} + \binom{l-1}{n-j} e^{-2l\pi ix}\right) Y_W(v, x)w\right) \\ &= -\sum_{s=1}^{p}\sum_{l|s}\sum_{j=0}^{m} \frac{l}{(n-j+1)}\binom{m}{j}\binom{-l-1}{n-j} S_{p-s}\left(\mathrm{Res}_x e^{2l\pi ix} Y_W(v, x)w\right) \\ &= 0, \end{aligned}$$

where in the last two steps, we have used $\binom{l-1}{n-j} = 0$ (since $0 \le l-1 < s \le p \le m \le 2m-j \le n-j$) and (4.16). This proves (4.38).

We divide the proof of (4.39) into three cases: $m = n$, $m > n$ and $m < n$.

We first prove (4.39) in the case $m = n$. In this case, the difference of the left-hand side and the right-hand side of (4.39) is

$$\sum_{k=0}^{n} \binom{-n-1}{k} S_n(e^{2\pi i(n+1)x}(e^{2\pi ix} - 1)^{-n-k-1} Y_W(v, x)w)$$

$$
\begin{aligned}
&\quad - \sum_{k=0}^{n} \binom{-n-1}{k} S_n(e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-n-k-1}Y_W(v,-x)w)\\
&= \sum_{k=0}^{n} \binom{-n-1}{k} S_n(e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-n-k-1}Y_W(v,x)w)\\
&\quad + \sum_{k=0}^{n} \binom{-n-1}{k} S_n(e^{-2\pi i(n+1)x}(e^{-2\pi ix}-1)^{-n-k-1}Y_W(v,x)w)\\
&= \sum_{k=0}^{n} \binom{-n-1}{k} S_n((e^{2\pi i(n+1)x}+(-1)^{-n-k-1}e^{2\pi ikx})(e^{2\pi ix}-1)^{-n-k-1}Y_W(v,x)w). \quad (4.40)
\end{aligned}
$$

Using the identity

$$
\sum_{k=0}^{n} \binom{k+n}{n} \frac{(-1)^k(1+x)^{n+1}-(-1)^n(1+x)^k}{x^{n+k+1}} = 1
$$

given by Proposition 5.2 in [DLM], we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n} \binom{-n-1}{k} (e^{2\pi i(n+1)x}+(-1)^{-n-k-1}e^{2\pi ikx})(e^{2\pi ix}-1)^{-n-k-1}\\
&\quad = \sum_{k=0}^{n} \binom{k+n}{n} \frac{(-1)^k e^{2\pi i(n+1)x}-(-1)^n e^{2\pi ikx}}{(e^{2\pi ix}-1)^{n+k+1}}\\
&\quad = \sum_{k=0}^{n} \binom{k+n}{n} \frac{(-1)^k(1+(e^{2\pi ix}-1))^{n+1}-(-1)^n(1+(e^{2\pi ix}-1))^k}{(e^{2\pi ix}-1)^{n+k+1}}\\
&\quad = 1. \quad (4.41)
\end{aligned}
$$

Using (4.41) and (4.12), we see that the right-hand side of (4.41) is equal to 0, proving (4.39) in this case.

Swapping m and n in (4.39) in the case of $m < n$, we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n} \binom{-2m+n-1}{k} S_m(\mathrm{Res}_x e^{2\pi i(m+1)x}(e^{2\pi ix}-1)^{-2m+n-k-1}Y_W(v,-x)w)\\
&\quad = \sum_{k=0}^{m} \binom{-2n+m-1}{k} S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-2n+m-k-1}Y_W(v,x)w),
\end{aligned}
$$

which differs from (4.39) only by the sign of the variable x in the vertex operators. Thus we can prove (4.39) in the cases $m > n$ and $m < n$ together by proving

$$
\sum_{k=0}^{n} \binom{-2m+n-1}{k} S_m(\mathrm{Res}_x e^{2\pi i(m+1)x}(e^{2\pi ix}-1)^{-2m+n-k-1}Y_W(v,\pm x)w)
$$

$$= \sum_{k=0}^{m} \binom{-2n+m-1}{k} S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi i x}-1)^{-2n+m-k-1} Y_W(v,\mp x)w). \qquad (4.42)$$

in the case of $m > n$.

For $m, n, k \in \mathbb{N}$ satisfying $m > n$, $v \in V$ and $w \in W$, using (4.32) and (4.16) with n there replaced by $2m-n+k-1$ and noting that $2m-n+k-1 \geq m$, we have

$$\sum_{k=0}^{n} \binom{-2m+n-1}{k} S_m(\mathrm{Res}_x e^{2\pi i(m+1)x}(e^{2\pi i x}-1)^{-2m+n-k-1} Y_W(v,\pm x)w)$$
$$= -\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{s=1}^{m}\sum_{l|s} \binom{-2m+n-1}{k} \frac{l}{2m-j-n+k}\binom{m}{j} \cdot$$
$$\cdot S_{m-s}\left(\mathrm{Res}_x\left(\left(\binom{-l-1}{2m-j-n+k-1} e^{2l\pi i x} + \binom{l-1}{2m-j-n+k-1} e^{-2l\pi i x}\right) Y_W(v,\pm x)\, w\right)\right.$$
$$= -\sum_{k=0}^{n}\sum_{j=0}^{m}\sum_{s=1}^{m}\sum_{l|s} \binom{-2m+n-1}{k} \frac{l}{2m-j-n+k}\binom{m}{j}\binom{l-1}{2m-j-n+k-1} \cdot$$
$$\cdot S_{m-s}(\mathrm{Res}_x e^{-2l\pi i x} Y_W(v,\pm x)w). \qquad (4.43)$$

For $k = 0, \dots, n$ and $j, l = 0, \dots, m$, we have

$$\frac{l}{(2m-j-n+k)}\binom{l-1}{2m-j-n+k-1}$$
$$= \frac{l}{(2m-j-n+k)} \frac{(l-1)\cdots(l-1-2m+j+n-k+2)}{(2m-j-n+k-1)!}$$
$$= \frac{l\cdots(l-2m+j+n-k+1)}{(2m-j-n+k)!}$$
$$= \binom{l}{2m-j-n+k}. \qquad (4.44)$$

In the case $\alpha \in \mathbb{N}$ and $\alpha \leq m$, we have $\binom{\alpha}{j} = 0$ for $\alpha < j \leq m$ and (4.17) becomes

$$\sum_{j=0}^{\alpha} \binom{\alpha}{j}\binom{\beta}{m-j} = \binom{\alpha+\beta}{m}. \qquad (4.45)$$

Note that $2m-n+k-1 \geq m \geq s \geq l$. Using (4.44), (4.45) with $\alpha = m$, $q = l$ and m replaced by $2m-n+k$, the fact that $\binom{m+l}{2m-n+k} = 0$ when $l+n-m < 0$, and (4.17) with $\alpha = -2m+n-1$, $\beta = m+l$ and m replaced by $l+n-m$, we have

$$\sum_{k=0}^{n}\sum_{j=0}^{m} \binom{-2m+n-1}{k} \frac{l}{2m-j-n+k}\binom{m}{j}\binom{l-1}{2m-j-n+k-1}$$

$$
\begin{aligned}
&= \sum_{k=0}^{n}\sum_{j=0}^{m}\binom{-2m+n-1}{k}\binom{m}{j}\binom{l}{2m-j-n+k}\\
&= \sum_{k=0}^{n}\binom{-2m+n-1}{k}\binom{m+l}{2m-n+k}\\
&= \begin{cases} 0 & l+n-m<0\\ \displaystyle\sum_{k=0}^{l+n-m}\binom{-2m+n-1}{k}\binom{m+l}{l+n-m-k} & l+n-m\geq 0\end{cases}\\
&= \begin{cases} 0 & l+n-m<0\\ \displaystyle\binom{l+n-m-1}{l+n-m} & l+n-m\geq 0\end{cases}\\
&= \begin{cases} 0 & l+n-m\neq 0\\ 1 & l+n-m=0.\end{cases}\\
&= \delta_{l,m-n} \qquad (4.46)
\end{aligned}
$$

From (4.46), we see that in this case ($m>n$), the right-hand side of (4.43) and thus also the left-hand side of (4.42) are equal to

$$
-\sum_{(m-n)|s,\ 1\leq s\leq m} S_{m-s}(\mathrm{Res}_x e^{-2(m-n)\pi ix}Y_W(v,\pm x)w). \qquad (4.47)
$$

We now prove (4.39) in the case $m>n$. We give the proof in the three cases $m>2n$, $m=2n>0$ and $2n>m>n$ separately.

In the case $m>2n$, we have $p(m-n)\geq 2(m-n)=m+(m-2n)>m$ for $p\in\mathbb{Z}_+ +1$. So the only integer s satisfying $1\leq s\leq m$ and $(m-n)|s$ is $m-n$. Thus in this case, (4.47) is equal to

$$
-S_n(\mathrm{Res}_x e^{-2(m-n)\pi ix}Y_W(v,\pm x)w). \qquad (4.48)
$$

On the other hand, in this case ($m>2n$), we have $m\geq -2n+m-1\geq 0$. Then for $v\in V$ and $w\in W$, we have

$$
\begin{aligned}
&\sum_{k=0}^{m}\binom{-2n+m-1}{k}S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-2n+m-k-1}Y_W(v,\mp x)w)\\
&\quad= \sum_{k=0}^{-2n+m-1}\binom{-2n+m-1}{k}S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-2n+m-k-1}Y_W(v,\mp x)w)\\
&\quad= S_n(\mathrm{Res}_x e^{2\pi i(m-n)x}Y_W(v,\mp x)w)\\
&\quad= -S_n(\mathrm{Res}_x e^{-2\pi i(m-n)x}Y_W(v,\pm x)w). \qquad (4.49)
\end{aligned}
$$

So (4.42) holds when $m>2n$.

In the case $m=2n>0$, we have $m-n=n>0$. Then the only integers s between 1 and $m=2n$ containing a factor n are n and $2n$. So in this case, (4.47) becomes

$$
-S_n(\mathrm{Res}_x e^{-2n\pi ix}Y_W(v,\pm x)w)-S_0(\mathrm{Res}_x e^{-2n\pi ix}Y_W(v,\pm x)w). \qquad (4.50)
$$

In this case, the right-hand side of (4.42) is equal to

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-k-1}Y_W(v,\mp x)w)\\
&\quad=\sum_{k=0}^{2n}(-1)^k S_n(\mathrm{Res}_x e^{2\pi inx}(e^{2\pi ix}-1)^{-k}Y_W(v,\mp x)w)\\
&\qquad+\sum_{k=0}^{2n}(-1)^k S_n(\mathrm{Res}_x e^{2\pi inx}(e^{2\pi ix}-1)^{-k-1}Y_W(v,\mp x)w)\\
&\quad=S_n(\mathrm{Res}_x e^{2\pi inx}Y_W(v,\mp x)w)+S_n(\mathrm{Res}_x e^{2\pi inx}(e^{2\pi ix}-1)^{-2n-1}Y_W(v,\mp x)w)\\
&\quad=-S_n(\mathrm{Res}_x e^{-2\pi inx}Y_W(v,\pm x)w)-S_n(\mathrm{Res}_x e^{-2\pi inx}(e^{-2\pi ix}-1)^{-2n-1}Y_W(v,\pm x)w)\\
&\quad=-S_n(\mathrm{Res}_x e^{-2\pi inx}Y_W(v,\pm x)w)+S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-2n-1}Y_W(v,\pm x)w).
\end{aligned}\tag{4.51}$$

By (4.32), we see that the second term in the right-hand side of (4.51) is equal to

$$\begin{aligned}
&-\sum_{j=0}^{n}\sum_{s=1}^{n}\sum_{l|s}\frac{l}{(2n-j)}\binom{n}{j}\cdot\\
&\qquad\qquad\cdot S_{n-s}\left(\mathrm{Res}_x\left(\binom{-l-1}{2n-1-j}e^{2l\pi ix}+\binom{l-1}{2n-1-j}e^{-2l\pi ix}\right)Y_W(v,\pm x)w\right)\\
&\quad=-\sum_{s=1}^{n}\sum_{l|s}\sum_{j=0}^{n}\frac{l}{(2n-j)}\binom{n}{j}\binom{-l-1}{2n-1-j}S_{n-s}(\mathrm{Res}_x e^{2l\pi ix}Y_W(v,\pm x)w)\\
&\qquad-\sum_{s=1}^{n}\sum_{l|s}\sum_{j=0}^{n}\frac{l}{(2n-j)}\binom{n}{j}\binom{l-1}{2n-1-j}S_{n-s}(\mathrm{Res}_x e^{-2l\pi ix}Y_W(v,\pm x)w).
\end{aligned}\tag{4.52}$$

Taking m and n be n and $2n-1$, respectively, in (4.16), we obtain

$$\sum_{j=0}^{n}\frac{l}{(2n-j)}\binom{n}{j}\binom{-l-1}{2n-1-j}=0$$

and thus the first term in the right-hand side of (4.52) is equal to 0. Note that $\binom{l-1}{2n-1-j}=0$ when $2n-j-1>l-1$ or equivalently when $2n>l+j$. But $l,j\le n$. So only when $l=j=n$, $\binom{l-1}{2n-1-j}$ is not 0. Thus the second term in the right-hand side of (4.52) is equal to

$$-\sum_{s=1}^{n}\sum_{n|s}\frac{n}{n}\binom{n}{n}\binom{n-1}{n-1}S_{n-s}(\mathrm{Res}_x e^{-2n\pi ix}Y_W(v,\pm x)w)=-S_0(\mathrm{Res}_x e^{-2n\pi ix}Y_W(v,\pm x)w).$$

From these calculations, we see that the right-hand side of (4.51) is equal to (4.50), proving (4.42) in the case $m=2n>0$.

In the case $2n > m > n$, we have

$$\begin{aligned}
\sum_{k=0}^{m} & \binom{-2n+m-1}{k} S_n(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi ix}-1)^{-2n+m-k-1} Y_W(v,\mp x)w) \\
&= \sum_{k=0}^{m} \binom{-2n+m-1}{k} S_n(\mathrm{Res}_x e^{2\pi i(m-n)x} e^{2\pi i(2n-m+1)x}(e^{2\pi ix}-1)^{-2n+m-k-1} Y_W(v,\mp x)w) \\
&= \sum_{k=0}^{m} \sum_{j=0}^{2n-m+1} \binom{-2n+m-1}{k}\binom{2n-m+1}{j} \cdot \\
&\qquad\qquad \cdot S_n(\mathrm{Res}_x e^{2\pi i(m-n)x}(e^{2\pi ix}-1)^{-j-k} Y_W(v,\mp x)w) \\
&= -\sum_{k=0}^{m} \sum_{j=0}^{2n-m+1} \binom{-2n+m-1}{k}\binom{2n-m+1}{j} \cdot \\
&\qquad\qquad \cdot S_n(\mathrm{Res}_x e^{-2\pi i(m-n)x}(e^{-2\pi ix}-1)^{-j-k} Y_W(v,\pm x)w) \\
&= -\sum_{k=0}^{m} \sum_{j=0}^{2n-m+1} (-1)^{-j-k}\binom{-2n+m-1}{k}\binom{2n-m+1}{j} \cdot \\
&\qquad\qquad \cdot S_n(\mathrm{Res}_x e^{2\pi i(-m+n+j+k)x}(e^{2\pi ix}-1)^{-j-k} Y_W(v,\pm x)w) \\
&= -\sum_{r=0}^{2n+1}(-1)^{-r} \sum_{k=0}^{\min(m,r)} \binom{-2n+m-1}{k}\binom{2n-m+1}{r-k} \cdot \\
&\qquad\qquad \cdot S_n(\mathrm{Res}_x e^{2\pi i(-m+n+r)x}(e^{2\pi ix}-1)^{-r} Y_W(v,\pm x)w) \\
&= -\sum_{r=0}^{m}(-1)^{-r} \sum_{k=0}^{r} \binom{-2n+m-1}{k}\binom{2n-m+1}{r-k} \cdot \\
&\qquad\qquad \cdot S_n(\mathrm{Res}_x e^{2\pi i(-m+n+r)x}(e^{2\pi ix}-1)^{-r} Y_W(v,\pm x)w) \\
&\quad -\sum_{r=m+1}^{2n+1}(-1)^{-r} \sum_{k=0}^{m} \binom{-2n+m-1}{k}\binom{2n-m+1}{r-k} \cdot \\
&\qquad\qquad \cdot S_n(\mathrm{Res}_x e^{2\pi i(-m+n+r)x}(e^{2\pi ix}-1)^{-r} Y_W(v,\pm x)w). \qquad (4.53)
\end{aligned}$$

Using (4.17), we see that the first term in the right-hand side of (4.53) is equal to

$$-S_n(\mathrm{Res}_x e^{2\pi i(m-n)x} Y_W(v,\pm x)w).$$

Since $m > n \geq l$ and $r \geq m+1$, we have $r-2 \geq -m+n+r-1 \geq n \geq l$. Then by (4.16) with m, n replaced by $-m+n+r-1, r-2$, respectively, we have

$$\sum_{j=0}^{-m+n+r-1} \frac{l}{(r-j-1)}\binom{-m+n+r-1}{j}\binom{-l-1}{r-2-j} = 0. \qquad (4.54)$$

Using (4.32) with m, n replaced by $-m+n+r-1, r-2$, respectively, and (4.54), we see that the second term in the right-hand side of (4.53) is equal to

$$
\begin{aligned}
&\sum_{r=m+1}^{2n+1}(-1)^{-r}\sum_{k=0}^{m}\binom{-2n+m-1}{k}\binom{2n-m+1}{r-k}\cdot\\
&\qquad\cdot\sum_{j=0}^{-m+n+r-1}\sum_{s=1}^{n}\sum_{l|s}\frac{l}{(r-j-1)}\binom{-m+n+r-1}{j}\cdot\\
&\qquad\cdot S_{n-s}\left(\mathrm{Res}_x\left(\left(\binom{-l-1}{r-2-j}e^{2l\pi ix}+\binom{l-1}{r-2-j}e^{-2l\pi ix}\right)Y_W(v,\pm x)w\right)\right.\\
&=\sum_{s=1}^{n}\sum_{l|s}\sum_{r=m+1}^{2n+1}(-1)^{-r}\sum_{k=0}^{m}\sum_{j=0}^{-m+n+r-1}\binom{-2n+m-1}{k}\binom{2n-m+1}{r-k}\cdot\\
&\qquad\cdot\binom{-m+n+r-1}{j}\binom{l}{r-1-j}S_{n-s}\left(\mathrm{Res}_x e^{-2l\pi ix}Y_W(v,\pm x)w\right)\\
&=-\sum_{s=1}^{n}\sum_{l|s}\sum_{p=0}^{2n-m}(-1)^{-p-m}\sum_{k=0}^{m}\binom{-2n+m-1}{k}\binom{2n-m+1}{p+m+1-k}\cdot\\
&\qquad\cdot\sum_{j=0}^{m+p}\binom{n+p}{j}\binom{l}{p+m-j}S_{n-s}\left(\mathrm{Res}_x e^{-2l\pi ix}Y_W(v,\pm x)w\right).
\end{aligned}
\tag{4.55}
$$

Using (4.17) with α, β, m replaced by $n+p, l, p+m$, respectively, we see that the right-hand side of (4.55) is equal to

$$
\begin{aligned}
&-\sum_{s=1}^{n}\sum_{l|s}\sum_{p=0}^{2n-m}(-1)^{-p-m}\sum_{k=0}^{m}\binom{-2n+m-1}{k}\binom{2n-m+1}{p+m+1-k}\cdot\\
&\qquad\cdot\binom{n+p+l}{p+m}S_{n-s}\left(\mathrm{Res}_x e^{-2l\pi ix}Y_W(v,\pm x)w\right).
\end{aligned}
\tag{4.56}
$$

We now prove

$$
\sum_{p=0}^{2n-m}(-1)^{-p-m}\sum_{k=0}^{m}\binom{-2n+m-1}{k}\binom{2n-m+1}{p+m+1-k}\binom{n+p+l}{p+m}=\delta_{l,m-n}. \tag{4.57}
$$

Using the identity (4.19) with α, m, k replaced by $-2n+m-1, p+m+1, m$ and

$$
(-1)^{-p-m}\binom{n+p+l}{p+m}=\binom{m-n-l-1}{p+m},
$$

we see that the left-hand side of (4.57) is equal to

$$
\sum_{p=0}^{2n-m}(-1)^{-p-m}\frac{p+1}{p+m+1}\binom{-2n+m-2}{m}\binom{2n-m+1}{p+1}\binom{n+p+l}{p+m}
$$

$$= \binom{-2n+m-2}{m} \sum_{p=0}^{2n-m} \frac{2n-m+1}{p+m+1} \binom{2n-m}{p} \binom{m-n-l-1}{p+m}. \tag{4.58}$$

When $l \neq m-n$, we have

$$\frac{1}{p+m+1} \binom{m-n-l-1}{p+m} = \frac{1}{m-n-l} \binom{m-n-l}{p+m+1}. \tag{4.59}$$

Using (4.59) and (4.18) with m, n, α replaced by $2n-m, m+1, m-n-l$, respectively, the right-hand side of (4.58) is equal to

$$\begin{aligned} &\frac{2n-m+1}{m-n-l} \binom{-2n+m-2}{m} \sum_{p=0}^{2n-m} \binom{2n-m}{p} \binom{m-n-l}{p+m+1} \\ &\quad = \frac{2n-m+1}{m-n-l} \binom{-2n+m-2}{m} \binom{n-l}{2n+1}. \end{aligned} \tag{4.60}$$

Since $l \leq n < 2n+1$, $\binom{n-l}{2n+1} = 0$. So the right-hand side of (4.60) and also the right-hand side of (4.58) is 0 in this case. Thus (4.57) holds in the case $l \neq m-n$. In the case $l = m-n$, using

$$\binom{-2n+m-2}{m} \binom{-1}{p+m} = (2n+1) \binom{2n}{m} (-1)^p$$

and (4.20) with n replaced by $2n-m$, we see that the right-hand side of (4.58) is equal to

$$\begin{aligned} &\binom{-2n+m-2}{m} \sum_{p=0}^{2n-m} \frac{2n-m+1}{p+m+1} \binom{2n-m}{p} \binom{-1}{p+m} \\ &\quad = (2n+1) \binom{2n}{m} \sum_{p=0}^{2n-m} \frac{(-1)^p}{p+m+1} \binom{2n-m}{p} \\ &\quad = 1. \end{aligned}$$

Thus (4.57) also holds in the case $l = m-n$. Using (4.57), we see that (4.56) is equal to

$$\begin{aligned} &-\sum_{s=1}^{n} \sum_{l|s} \delta_{l,m-n} S_{n-s} \left(\mathrm{Res}_x e^{-2l\pi i x} Y_W(v, \pm x) w \right) \\ &\quad = -\sum_{(m-n)|s,\ 1 \leq s \leq n} S_{n-s} \left(\mathrm{Res}_x e^{-2(m-n)\pi i x} Y_W(v, \pm x) w \right). \end{aligned}$$

From the calculations above, we obtain that the right-hand side of (4.53) is equal to

$$- S_n(\mathrm{Res}_x e^{2\pi i (m-n) x} Y_W(v, \pm x) w) - \sum_{(m-n)|s,\ 1 \leq s \leq n} S_{n-s} \left(\mathrm{Res}_x e^{-2(m-n)\pi i x} Y_W(v, \pm x) w \right)$$

$$= \sum_{(m-n)|s,\ 1\leq s\leq m} S_{m-s}\left(\mathrm{Res}_x e^{-2(m-n)\pi i x} Y_W(v, \pm x)w\right).$$

Thus (4.42) holds in the case $2n > m > n$. This finishes the proof of (4.42) in the case $m > n$. The proof of (4.39) is now complete. ∎

Remark 4.9 The special case $n = m$ and $W = V$ of (4.38) is the same as Proposition 4.4 in [Mi2]. But the proof of Proposition 4.4 in [Mi2] uses some formulas obtained by Zhu for shifted q-traces of vertex operators. It is claimed in [Mi2] that these formulas give some properties of $O_q(V)$. But actually these formulas of Zhu show that such properties hold for the kernels of suitable linear maps constructed from shifted q-traces of vertex operators. Although $O_q(V)$ is a subspace of these kernels, one cannot conclude from only these facts that $O_q(V)$ also satisfies the same properties. The proof of Proposition 4.4 in [Mi2] indeed gives strong evidence that the conclusion of Proposition 4.4 in [Mi2] must be true. But a proof of this proposition was first given by McRae [Mc].

Proposition 4.10 *Let W be a grading-restricted generalized V-module. Assume that a linear map*

$$\begin{aligned} S : W &\to q^r\mathbb{C}[[q]][\log q] \\ w &\mapsto S(w; q) = \sum_{k=0}^{K}\sum_{n\in\mathbb{N}} S_{k,n}(w)(\log q)^k q^{r+n} \end{aligned}$$

satisfies (4.12), (4.13) and (4.14). Then

$$\begin{aligned} &\sum_{m=0}^{n}\binom{-n-1}{m} S_{k,n}(\mathrm{Res}_x e^{2\pi(n+1)ix}(e^{2\pi ix}-1)^{-n-m-1}Y_W(\omega, x)w) - 2\pi i(r+n)S_{k,n}(w) \\ &\quad = 2\pi i(k+1)S_{k+1,n}(w) \end{aligned} \tag{4.61}$$

for $k = 0, \dots, K$, $n \in \mathbb{N}$ and $w \in W$, where $S_{K+1,n}(w) = 0$ for $n \in \mathbb{N}$ and $w \in W$.

Proof. From (4.14), (4.9) and (4.12), we obtain

$$\begin{aligned} &(2\pi i)^2\sum_{k=0}^{K}\sum_{n\in\mathbb{N}} kS_{k,n}(w)(\log q)^{k-1}q^{r+n} + (2\pi i)^2\sum_{k=0}^{K}\sum_{n\in\mathbb{N}}(r+n)S_{k,n}(w)(\log q)^k q^{r+n} \\ &\quad = 2\pi i\sum_{k=0}^{K}\sum_{n\in\mathbb{N}} S_{k,n}(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-1}Y_{W_1}(\omega, x)w)(\log q)^k q^{r+n} \\ &\quad\quad - 2\pi i\sum_{k=0}^{K}\sum_{n\in\mathbb{N}}\sum_{s\in\mathbb{Z}_+}\sum_{l|s} S_{k,n}(\mathrm{Res}_x(e^{2l\pi ix} - e^{-2l\pi ix})Y_{W_1}(\omega, x)w)(\log q)^k q^{r+n+s} \end{aligned}$$

$$= 2\pi i \sum_{k=0}^{K} \sum_{n\in\mathbb{N}} S_{k,n}(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-1} Y_{W_1}(\omega,x)w)(\log q)^k q^{r+n}$$

$$- 2\pi i \sum_{k=0}^{K} \sum_{n\in\mathbb{Z}_+} \sum_{s=1}^{n} \sum_{l|s} S_{k,n-s}(\mathrm{Res}_x (e^{2l\pi ix} - e^{-2l\pi ix}) Y_{W_1}(\omega,x)w)(\log q)^k q^{r+n}. \qquad (4.62)$$

Taking the coefficients of $(\log q)^k q^{r+n}$ in (4.62), we obtain

$$S_{k,0}(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-1} Y_{W_1}(\omega,x)w) - 2\pi i r S_{k,0}(w) = 2\pi i(k+1) S_{k+1,0}(w) \qquad (4.63)$$

for $k = 0, \dots, K$, which is (4.61) in the case of $n = 0$, and

$$S_{k,n}(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-1} Y_{W_1}(\omega,x)w) - \sum_{s=1}^{n} \sum_{l|s} S_{k,n-s}(\mathrm{Res}_x (e^{2l\pi ix} - e^{-2l\pi ix}) Y_{W_1}(\omega,x)w)$$
$$= 2\pi i(k+1) S_{k+1,n}(w) + 2\pi i(r+n) S_{k,n}(w) \qquad (4.64)$$

for $k = 0, \dots, K$ and $m \in \mathbb{Z}_+$.

Note that for $k = 0, \dots, K$, $\sum_{n\in\mathbb{N}} S_{k,n}(w) q^{r+n}$ satisfy (4.12) and (4.13). Thus for fixed k, all the results above hold for $S_{k,n}$ for $n \in \mathbb{N}$. Expanding $e^{2\pi nix} = (1 + (e^{2\pi ix} - 1))^n$ as a polynomial in $e^{2\pi ix} - 1$ and then using (4.32) for $S_{k,n}$, we have

$$\sum_{m=0}^{n} \binom{-n-1}{m} S_{k,n}(\mathrm{Res}_x e^{2\pi(n+1)ix}(e^{2\pi ix}-1)^{-n-m-1} Y_W(\omega,x)w)$$
$$= \sum_{m=0}^{n} \sum_{j=0}^{n} \binom{-n-1}{m} \binom{n}{j} S_{k,n}(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-n+j-m-1} Y_W(\omega,x)w)$$
$$= S_{k,n}(\mathrm{Res}_x e^{2\pi ix}(e^{2\pi ix}-1)^{-1} Y_W(\omega,x)w)$$
$$- \sum_{j=0}^{n-1} \sum_{s=1}^{n} \sum_{l|s} \binom{n}{j} \frac{l}{n-j} \cdot$$
$$\cdot S_{k,n-s}\left(\mathrm{Res}_x \left(\binom{-l-1}{n-j-1} e^{2\pi ilx} + \binom{l-1}{n-j-1} e^{-2\pi ilx}\right) Y_W(\omega,x)w\right)$$
$$- \sum_{m=1}^{n} \sum_{j=0}^{n} \sum_{s=1}^{n} \sum_{l|s} \binom{-n-1}{m} \binom{n}{j} \frac{l}{n-j+m} \cdot$$
$$\cdot S_{k,n-s}\left(\mathrm{Res}_x \left(\binom{-l-1}{n-j+m-1} e^{2\pi ilx} + \binom{l-1}{n-j+m-1} e^{-2\pi ilx}\right) Y_W(\omega,x)w\right). \qquad (4.65)$$

We now calculate the second and third terms in the right-hand side of (4.65).

For $1 \le l \le n$,

$$\sum_{j=0}^{n-1} \binom{n}{j} \frac{l}{n-j} \binom{-l-1}{n-j-1}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} \binom{n}{j} \frac{l}{n-j} \frac{(-l-1)\cdots(-l-1-n+j+1+1)}{(n-j-1)!} \\
&= \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} \frac{(l+n-j-1)\cdots(l+1)l(l-1)!}{(l-1)!(n-j)!} \\
&= \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} \frac{(l+n-j-1)\cdots(n-j+1)}{(l-1)!} \\
&= \sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} \frac{1}{(l-1)!} \frac{d^{l-1}}{dx^{l-1}} x^{l+n-j-1} \bigg|_{x=1}. \qquad (4.66)
\end{aligned}
$$

But

$$
\begin{aligned}
&\sum_{j=0}^{n} (-1)^{n-j-1} \binom{n}{j} \frac{1}{(l-1)!} \frac{d^{l-1}}{dx^{l-1}} x^{l+n-j-1} \bigg|_{x=1} \\
&= (-1)^{n-1} \frac{1}{(l-1)!} \frac{d^{l-1}}{\sum_{j=0}^{n}} \binom{n}{j} (-1)^j dx^{l-1} x^{l+n-j-1} \bigg|_{x=1} \\
&= (-1)^{n-1} \frac{1}{(l-1)!} \frac{d^{l-1}}{dx^{l-1}} x^{l+n-j-1} (1-x^{-1})^n \bigg|_{x=1} \\
&= (-1)^{n-1} \frac{1}{(l-1)!} \frac{d^{l-1}}{dx^{l-1}} x^{l-j-1} (x-1)^n \bigg|_{x=1} \\
&= 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{j=0}^{n-1} (-1)^{n-j-1} \binom{n}{j} \frac{1}{(l-1)!} \frac{d^{l-1}}{dx^{l-1}} x^{l+n-j-1} \bigg|_{x=1} \\
&= \sum_{j=0}^{n} (-1)^{n-j-1} \binom{n}{j} \frac{1}{(l-1)!} \frac{d^{l-1}}{dx^{l-1}} x^{l+n-j-1} \bigg|_{x=1} + 1 \\
&= 1. \qquad (4.67)
\end{aligned}
$$

From (4.66) and (4.67), we obtain

$$
\sum_{j=0}^{n-1} \binom{n}{j} \frac{l}{n-j} \binom{-l-1}{n-j-1} = 1. \qquad (4.68)
$$

For $1 \le l \le n$,

$$
\sum_{j=0}^{n-1} \binom{n}{j} \frac{l}{n-j} \binom{l-1}{n-j-1} = \sum_{j=0}^{n-1} \binom{n}{j} \binom{l}{n-j}
$$

$$= \sum_{j=0}^{n} \binom{n}{j} \binom{l}{n-j} - 1$$

$$= \binom{n+l}{n} - 1, \tag{4.69}$$

where in the last step, we have used (4.17).

For $1 \leq l \leq n$, by (4.16), we have

$$\sum_{m=1}^{n} \sum_{j=0}^{n} \binom{-n-1}{m} \binom{n}{j} \frac{l}{n-j+m} \binom{-l-1}{n-j+m-1}$$

$$= \sum_{m=1}^{n} \binom{-n-1}{m} \sum_{j=0}^{n} \binom{n}{j} \frac{l}{n-j+m} \binom{-l-1}{n-j+m-1}$$

$$= 0. \tag{4.70}$$

For $1 \leq l \leq n$, using the properties of binomial coefficients and (4.17), we have

$$\sum_{m=0}^{n} \sum_{j=0}^{n} \binom{-n-1}{m} \binom{n}{j} \frac{l}{n-j+m} \binom{l-1}{n-j+m-1}$$

$$= \sum_{m=0}^{n} \sum_{j=0}^{n} \binom{-n-1}{m} \binom{n}{j} \binom{l}{n-j+m}$$

$$= \sum_{m=0}^{n} \binom{-n-1}{m} \sum_{j=0}^{n+m} \binom{n}{j} \binom{l}{n-j+m}$$

$$= \sum_{m=0}^{n} \binom{-n-1}{m} \binom{n+l}{n+m}$$

$$= \sum_{m=0}^{l} \binom{-n-1}{m} \binom{n+l}{l-m}$$

$$= \sum_{m=0}^{l} \binom{-n-1}{m} \binom{n+l}{l-m}$$

$$= \binom{l-1}{l}$$

$$= 0.$$

Then

$$\sum_{m=1}^{n} \sum_{j=0}^{n} \binom{-n-1}{m} \binom{n}{j} \frac{l}{n-j+m} \binom{l-1}{n-j+m-1}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n}\sum_{j=0}^{n}\binom{-n-1}{m}\binom{n}{j}\frac{l}{n-j+m}\binom{l-1}{n-j+m-1} - \sum_{j=0}^{n}\binom{n}{j}\frac{l}{n-j}\binom{l-1}{n-j-1}\\
&= -\sum_{j=0}^{n}\binom{n}{j}\binom{l}{n-j}\\
&= -\binom{n+l}{n},
\end{aligned}
\tag{4.71}
$$

where in the last step, we have used (4.17).

Using (4.66)–(4.71), we see that the right-hand side of (4.65) is equal to the left-hand side of (4.64). Then by (4.64), we obtain (4.61). ∎

Remark 4.11 The special case $W = V$ of (4.61) in fact implies immediately Proposition 4.5 in [Mi2]. Note that the proof of Propositions 4.5 in [Mi2] is based on the same arguments as what the proof of Proposition 4.4 in [Mi2] is based on. As we have discussed in Remark 4.9, these arguments indeed give stromg evidence that the conclusion of Proposition 4.5 in [Mi2] must be true. But a proof of this proposition is in fact given by Proposition 4.10 above. We also note that the proof of Proposition 4.6 in [Mi2] also gives only strong evidence that its conclusion must be true. But since in this paper, we do not need anything equivalent to Proposition 4.6 in [Mi2] and thus do not give a proof, it will still be interesting to find a proof of Proposition 4.6 in [Mi2].

We now prove the main result of this section.

Proof of Theorem 4.4. Since $\psi_S^{N;k,j}([v]_{mn}) = 0$ for $m, n \in \mathbb{N}$ satisfying $0 \leq m, n \leq N$ and $m \neq n$, to show $\psi^N_{S;k,j}(\widetilde{Q}^\infty(W)) = 0$, we need only show $\psi^N_{S;k,j}([w]_{nn}) = S_{k,j,n}(w) = 0$ for $w \in W$ and $0 \leq n \leq N$ such that $[w]_{nn} \in \widetilde{Q}^\infty(W) \cap U_{nn}(W)$. But by Corollary 3.14, $\widetilde{Q}^\infty(W) \cap U_{nn}(W)$ is spanned by elements of the form by the coefficients of

$$
[\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi i x} - 1)^{-2n-p-2} Y_W(v, x)w]_{nn}
$$

for $p \in \mathbb{N}$, $v \in V$ and $w \in W$. Since S satisfies (4.12) and (4.13), we see that for $k = 0, \dots, K$ and $j = 1, \dots, J$, the linear map given by $w \mapsto \sum_{m \in \mathbb{N}} S_{k,j,m}(w)$ for $w \in W$ also satisfies (4.12) and (4.13). Then by (4.38), we have

$$
\begin{aligned}
&\psi^N_{S;k,j}([\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi i x} - 1)^{-2n-p-2} Y_W(v, x)w]_{nn})\\
&\quad = S_{k,j,n}(\mathrm{Res}_x e^{2\pi i(n+1)x}(e^{2\pi i x} - 1)^{-2n-p-2} Y_W(v, x)w)\\
&\quad = 0,
\end{aligned}
$$

proving $\psi^N_{S;k,j}(\widetilde{Q}^\infty(W)) = 0$.

To prove $\psi^N_{S;k,j}$ is symmetric, we need only prove

$$
\psi^N_{S;k,j}([v]_{mn} \blacklozenge [w]_{nm}) = \psi^N_{S;k,j}([w]_{nm} \blacklozenge [v]_{mn}) \tag{4.72}
$$

for $v \in V$, $w \in W$ and $0 \leq m, n \leq N$. Since for $k = 0, \dots, K$ and $j = 1, \dots, J$, the linear map given by $w \mapsto \sum_{m\in\mathbb{N}} S_{k,j,m}(w)$ for $w \in W$ satisfies (4.12) and (4.13), we see that (4.39) holds for $S_{k,j,m}$ for $m, n \in \mathbb{N}$. But by the definitions of $[v]_{mn} \blacklozenge [w]_{nm}$, $[w]_{nm} \blacklozenge [v]_{mn}$ and $\psi^N_{S;k,j}$, (4.72) is the same as (4.39).

Since S satisfies (4.14), we see that for $j = 1, \dots, J$, the linear map given by $w \mapsto \sum_{k=0}^{K} \sum_{m\in\mathbb{N}} S_{k,j,m}(w)$ for $w \in W$ also satisfies (4.14). In particular, (4.61) holds for $S_{k,j,n}$ for $j = 1, \dots, J$. By the definition of $\psi^N_{S;k,j}$ and $[\omega]_{nn} \blacklozenge [w]_{nn}$, we see that the first term in the left-hand side of (4.61) multiplied by $2\pi i$ is equal to $\psi^N_{S;k,j}([\omega]_{nn} \blacklozenge [w]_{nn})$ and thus from (4.61) and $[\mathbf{1}]_{nn} \blacklozenge [w]_{nn} = [w]_{nn}$, we obtain

$$\psi^N_{S;k,j}(([\omega]_{nn} - (2\pi i)^2(r_j + n)[\mathbf{1}]_{nn}) \blacklozenge [w]_{nn}) = (2\pi i)^2(k+1)\psi^N_{S;k+1,j}([w]_{nn}). \tag{4.73}$$

Using (4.73) repeatedly and note that $\psi^N_{S;K+1,j}([w]_{nn}) = S_{K+1,j,n}(w) = 0$, we obtain

$$\psi^N_{S;k,j}(([\omega]_{nn} - (2\pi i)^2(r_j + n)[\mathbf{1}]_{nn})^{\blacklozenge\,(K-k+1)} \blacklozenge [w]_{nn}) = 0$$

for $w \in W$ and $n = 0, \dots, N$. For $k, l \in \mathbb{N}$, $k \neq n$ and $w \in W$, we have

$$\psi^N_{S;k,j}(([\omega]_{nn} - (2\pi i)^2(r_j + n)[\mathbf{1}]_{nn})^{\blacklozenge\,(K-k+1)} \blacklozenge [w]_{kl}) = \psi^N_{S;k,j}(0) = 0.$$

If for $l \in \mathbb{N}$ and $l \neq n$, note that

$$([\omega - (2\pi i)^2(r_j + n)\mathbf{1}]_{nn})^{\blacklozenge\,(K-k+1)} \blacklozenge [w]_{nl} = [\tilde{w}]_{nl},$$

for some $\tilde{w} \in W$. Then by the definition of $\psi^N_{S;k,j}$ we have

$$\psi^N_{S;k,j}(([\omega]_{nn} - (2\pi i)^2(r_j + n)[\mathbf{1}]_{nn})^{\blacklozenge\,(K-k+1)} \blacklozenge [w]_{nl}) = \psi^N_{S;k,j}([\tilde{w}]_{nl}) = 0.$$

Thus (4.15) is proved. ∎

5 Modular invariance

In this section, we prove the conjecture on the modular invariance of intertwining operators. See Theorem 5.5. From the genus-one associativity, as in the proof of the modular invariance in the semisimple case in [H2], we see that we need only prove that the space of all genus-one 1-point correlation functions constructed from shifted pseudo-q-traces of geometrically-modified intertwining operators are invariant under the modular transformations. We introduce a notion of genus-one 1-point conformal block as a map from a grading-restricted generalized V-module W to the space of analytic functions on the upper-half plane $\mathbb{H}$ satisfying some properties involving the Weierstrass functions $\wp$-function $\wp_2(z;\tau)$, the Weierstrass ζ-function $\wp_1(z;\tau)$ and the Eisenstein series $G_2(\tau)$. We prove that modular transformations of genus-one 1-point correlation functions constructed from shifted pseudo-q-traces of geometrically-modified intertwining operators give genus-one 1-point conformal blocks. Then we prove that the image of $w \in W$ under a genus-one 1-point conformal block must be a sum of genus-one 1-point correlation functions constructed from shifted pseudo-q-traces of intertwining operators. This in particular proves the modular invariance conjecture.

We now introduce the notion of genus-one 1-point conformal block. Let $\mathbb{H}$ be the open upper-half plane and $H(\mathbb{H})$ the space of analytic functions on $\mathbb{H}$. Recall the formal Laurent series expansion $\wp_1(x;\tau)-G_2(\tau)x$ of $\wp_1(z;\tau)-G_2(\tau)z$ given by (4.10) and the formal Laurent series expansion $\wp_2(x;\tau)$ of $\wp_2(z;\tau)$ given by (4.11).

Definition 5.1 Let W be a grading-restricted generalized V-module. A *genus-one* 1*-point conformal block labeled by* W is a linear map

$$\begin{aligned} F : W &\to H(\mathbb{H}) \\ w &\mapsto F(w;\tau) \end{aligned}$$

satisfying

$$F(\mathrm{Res}_x Y_W(v,x)w;\tau) = 0, \tag{5.1}$$

$$2\pi i \frac{\partial}{\partial \tau} F(w;\tau) = F(\mathrm{Res}_x(\wp_1(x;\tau) - G_2(\tau)x)Y_W(\omega,x)w;\tau), \tag{5.2}$$

$$F(\mathrm{Res}_x \wp_2(x;\tau)Y_W(v,x)w;\tau) = 0 \tag{5.3}$$

for $v \in V$ and $w \in W$.

Let W be grading-restricted generalized V-modules and $w \in W$. Given a finite-dimensional associative algebra P with a symmetric linear function ϕ, a grading-restricted generalized V-P-bimodule $\widetilde{W}$, projective as a right P-module, and an intertwining operator $\mathcal{Y}$ of type $\binom{\widetilde{W}}{W\widetilde{W}}$ compatible with P, we have a genus-one 1-point correlation function $\overline{F}^{\phi}_{\mathcal{Y}}(w;z;\tau)$, which is the analytic extension of the sum of the series

$$F^{\phi}_{\mathcal{Y}}(w;z;\tau) = \mathrm{Tr}^{\phi}_{\widetilde{W}} \mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}.$$

Using the $L(-1)$-derivative property and $L(-1)$-commutator formula for the intertwining operator $\mathcal{Y}$, we have

$$\begin{aligned} &\frac{\partial}{\partial z} \mathrm{Tr}^{\phi}_{\widetilde{W}} \mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}_1}(0)-\frac{c}{24}} \\ &\quad= \mathrm{Tr}^{\phi}_{\tilde{W}} \mathcal{Y}_1((2\pi i L_W(0) + 2\pi i q_z L_W(-1))\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}} \\ &\quad= 2\pi i \mathrm{Tr}^{\phi}_{\tilde{W}} [L_{\widetilde{W}}(0), \mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)]q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}} \\ &\quad= 0. \end{aligned}$$

Since $\overline{F}^{\phi}_{\mathcal{Y}}(w;z;\tau)$ is the analytic extension of $\mathrm{Tr}^{\phi}_{\widetilde{W}} \mathcal{Y}(\mathcal{U}_W(q_z)w,q_z)q_\tau^{L_{\widetilde{W}}(0)-\frac{c}{24}}$, we obtain

$$\frac{d}{dz} \overline{F}^{\phi}_{\mathcal{Y}_1}(w;z;\tau) = 0, \tag{5.4}$$

that is, $\overline{F}^{\phi}_{\mathcal{Y}}(w; z; \tau)$ is independent of z. From now on, we shall write $\overline{F}^{\phi}_{\mathcal{Y}}(w; z; \tau)$ simply as $\overline{F}^{\phi}_{\mathcal{Y}}(w; \tau)$. Then we have a linear map $\overline{F}^{\phi}_{\mathcal{Y}} : W \to H(\mathbb{H})$ given by $w \mapsto \overline{F}^{\phi}_{\mathcal{Y}}(w; \tau)$.

Proposition 5.2 *For*

$$\left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array}\right) \in SL(2, \mathbb{Z}),$$

the linear map from W *to* $H(\mathbb{H})$ *given by*

$$w \mapsto \overline{F}^{\phi}_{\mathcal{Y}}\left((\gamma\tau + \delta)^{-L_W(0)}w; \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right) \tag{5.5}$$

is a genus-one 1*-point conformal block labeled by* W*.*

Proof. The identity (2.19) in [F2] in the case $n = 1$ becomes

$$\begin{aligned}\left(2\pi i\frac{\partial}{\partial\tau} + G_2(\tau)z\frac{\partial}{\partial z}\right) F^{\phi}_{\mathcal{Y}}(w; \tau) + G_2(\tau)F^{\phi}_{\mathcal{Y}}(L_W(0)w; \tau)\\ = F^{\phi}_{\mathcal{Y}}(L_W(-2)w; \tau) - \sum_{k\in\mathbb{Z}_+} G_{2k+2}(\tau)F^{\phi}_{\mathcal{Y}}(L_W(2k)w; \tau),\end{aligned} \tag{5.6}$$

where in the left-hand side, we have used

$$F^{\phi}_{\mathcal{Y}}(L_W(0)w; \tau) = (\mathrm{wt}\ w)F^{\phi}_{\mathcal{Y}}(w; \tau) + F^{\phi}_{\mathcal{Y}}(L_W(0)_N w; \tau).$$

Using (5.4), we see that (5.6) becomes

$$\begin{aligned}2\pi i\frac{\partial}{\partial\tau}F^{\phi}_{\mathcal{Y}}(w; \tau) &= F^{\phi}_{\mathcal{Y}}(L_W(-2)w; \tau) - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)F^{\phi}_{\mathcal{Y}}(L_W(2k)w; \tau)\\ &= F^{\phi}_{\mathcal{Y}}\left(L_W(-2) - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)L_W(2k)w; \tau\right).\end{aligned} \tag{5.7}$$

Let $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$. Then (5.7) holds with τ replaced by τ'. Using the modular transformation properties of $G_{2k+2}(\tau)$ for $k \in \mathbb{N}$ and the commutator formulas between $L_W(0)$ and $L_W(n)$ for $n \in \mathbb{Z}$, we see that the formula obtained from (5.7) with τ and w replaced by τ' and $(\gamma\tau + \delta)^{-L_W(0)}w$ is equivalent to

$$\begin{aligned}&2\pi i\frac{\partial}{\partial\tau}F^{\phi}_{\mathcal{Y}}((\gamma\tau + \delta)^{-L_W(0)}w; \tau')\\ &\quad = F^{\phi}_{\mathcal{Y}}((\gamma\tau + \delta)^{-L_W(0)}L_W(-2)w; \tau')\\ &\quad\quad - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)F^{\phi}_{\mathcal{Y}}((\gamma\tau + \delta)^{-L_W(0)}L_W(2k)w; \tau')\\ &\quad = F^{\phi}_{\mathcal{Y}}\left((\gamma\tau + \delta)^{-L_W(0)}\left(L_W(-2) - \sum_{k\in\mathbb{N}} G_{2k+2}(\tau)L_W(2k)\right)w; \tau'\right).\end{aligned} \tag{5.8}$$

For $u \in V$, from the $n=1$ case of the identity (1.10) in [F2] and the fact that $L_W(0)_N$ commutes with vertex operators for W, we obtain

$$F^{\phi}_{\mathcal{Y}}((\gamma\tau+\delta)^{-L_W(0)}\mathrm{Res}_x Y_W(u,x)w;\tau')=0 \tag{5.9}$$

and from the $n=1$, $l=2$ case of the identity (1.14) in [F2], the modular transformation property of $G_{2k+2}(\tau)$ for $k \in \mathbb{N}$ and the commutator formula between $L_W(0)$ and vertex operators for W, we obtain

$$F^{\phi}_{\mathcal{Y}}\left((\gamma\tau+\delta)^{-L_W(0)}\left(u_{-2}+\sum_{k\in\mathbb{Z}_+}(2k+1)G_{2k+2}(\tau)u_{2k}\right)w;\tau'\right)=0. \tag{5.10}$$

Formulas (7.9) and (7.10) in [H2] still hold, that is, we have

$$\left(L_W(-2)-\sum_{k\in\mathbb{N}}G_{2k+2}(\tau)L_{W_2}(2k)\right)w=\mathrm{Res}_x(\wp_1(x;\tau)-G_2(\tau)x)Y_W(\omega,x)w \tag{5.11}$$

and

$$\left(u_{-2}+\sum_{k\in\mathbb{Z}_+}(2k+1)G_{2k+2}(\tau)u_{2k}\right)w=\mathrm{Res}_x\wp_2(x;\tau)Y_W(u,x)w. \tag{5.12}$$

Using (5.11) and (5.12), we see that (5.8) and (5.10) become

$$\begin{aligned}&2\pi i\frac{\partial}{\partial\tau}F^{\phi}_{\mathcal{Y}}((\gamma\tau+\delta)^{-L_W(0)}w;\tau')\\&\quad=F^{\phi}_{\mathcal{Y}}\left((\gamma\tau+\delta)^{-L_W(0)}\mathrm{Res}_x(\wp_1(x;\tau)-G_2(\tau)x)Y_W(\omega,x)w;\tau'\right)\end{aligned} \tag{5.13}$$

and

$$F^{\phi}_{\mathcal{Y}}\left((\gamma\tau+\delta)^{-L_W(0)}\mathrm{Res}_x\wp_2(x;\tau)Y_W(u,x)w;\tau'\right)=0, \tag{5.14}$$

respectively.

Using analytic extensions, we see that (5.9), (5.13) and (5.14) becomes

$$\begin{aligned}&\overline{F}^{\phi}_{\mathcal{Y}}((\gamma\tau+\delta)^{-L_W(0)}\mathrm{Res}_x Y_W(u,x)w;\tau')=0,\\&2\pi i\frac{\partial}{\partial\tau}\overline{F}^{\phi}_{\mathcal{Y}}((\gamma\tau+\delta)^{-L_W(0)}w;\tau')\\&\quad=\overline{F}^{\phi}_{\mathcal{Y}}\left((\gamma\tau+\delta)^{-L_W(0)}\mathrm{Res}_x(\wp_1(x;\tau)-G_2(\tau)x)Y_W(\omega,x)w;\tau'\right),\\&\overline{F}^{\phi}_{\mathcal{Y}}\left((\gamma\tau+\delta)^{-L_W(0)}\mathrm{Res}_x\wp_2(x;\tau)Y_W(u,x)w;\tau'\right)=0.\end{aligned}$$

These are exactly the conditions for the linear map given by (5.5) to be a genus-one 1-point conformal block. ∎

As in [H2], Theorem 2.5 in [F1] and [F2] is proved by deriving a system of differential equations of regular singular points satisfied by the formal series of shifted pseudo-q_τ-traces of products of geometrically-modified intertwining operators. In particular, the genus-one 1-point correlation functions $\overline{F}^{\phi}_{\mathcal{Y}}(L_W(0)_N^p w;\tau)$ for $k=0,\dots,K$ satisfy a system of differential equations of a regular singular point $q_\tau=0$, where K is the smallest number of $\mathbb{N}$ such that $L_W(0)_N^{K+1}w=0$. In fact, the system of differential equations satisfied by $\overline{F}^{\phi}_{\mathcal{Y}}(L_W(0)_N^p w;\tau)$ are derived using only the C_2-cofiniteness and the q_τ-expansions of (5.1), (5.2) and (5.3). Using this fact, we first show that elements in the image of a genus-one 1-point conformal block satisfy differential equations of a regular singular point $q_\tau=0$.

Proposition 5.3 *Let V be a C_2-cofinite vertex operator algebra and W a grading-restricted generalized V-module. For a genus-one 1-point conformal block F labeled by W and $w\in W$, $F(L_W(0)_N^j w;\tau)$ for $j=0,\dots,K$ satisfy a system of $K+1$ differential equations of a regular singular point at $q_\tau=0$, where K is the smallest number of $\mathbb{N}$ such that $L_W(0)^{K+1}w=0$,*

Proof. As in the case of $n=1$ in [H2], [F1] and [F2], let $R=\mathbb{C}[G_4(\tau),G_6(\tau)]$. Then $G_{2k+2}(\tau)\in R$ for $k\in\mathbb{Z}_+$. Consider the R-module M_F generated by functions of τ of the form $F(w;\tau)$. Then the linear map F in fact induces an R-module map $\widehat{F}$ from the R-module $T=R\otimes W$ to M_F given by $\widehat{F}(f(\tau)\otimes w)=f(\tau)F(w;\tau)$.

As in [H2], [F1] and [F2], let J be an R-submodule of T generated by elements of the form

$$(Y_W)_{-2}(v)w+\sum_{k\in\mathbb{Z}_+}(2k+1)G_{2k+2}(\tau)(Y_W)_{2k}(v)w$$

for $v\in V$ and $w\in W$. Then by (5.12) and (5.3), we see that $\ker\widehat{F}\subset J$. In particular, $\widehat{F}$ can in fact be viewed as an R-module map from T/J to M_F.

Since V is C_2-cofinite, the same proof as those in [H2], [F1] and [F2] shows that T/J is a finitely generated R-module. As in [H2], [F1] and [F2], let $\mathcal{Q}:T\to T$ be the map defined by

$$\mathcal{Q}(f(\tau)\otimes w)=f(\tau)\otimes(\mathrm{Res}_x\wp_1(x;\tau)Y_W(\omega,x)w).$$

Since R is Noetherian, given $w\in W$, the R-submodule of T generated by $\mathcal{Q}^n(1\otimes w)$ for $n\in\mathbb{N}$ is also finitely generated. Then there exist $m\in\mathbb{Z}_+$ and $b_p(\tau)\in R$ for $p=1,\dots,m$ such that

$$\mathcal{Q}^m(1\otimes w)+\sum_{p=1}^{m}b_p(\tau)\mathcal{Q}^{m-p}(1\otimes w)\in J.$$

Then by the definition of $\mathcal{Q}$, we obtain

$$1\otimes(\mathrm{Res}_x\wp_1(x;\tau)Y_W(\omega,x))^m w+\sum_{p=1}^{m}b_p(\tau)\otimes(\mathrm{Res}_x\wp_1(x;\tau)Y_W(\omega,x))^{m-p}w)\in J. \qquad (5.15)$$

For homogeneous $w\in W$, from (5.2), we have

$$\widehat{F}(1\otimes(\mathrm{Res}_x\wp_1(x;\tau)Y_W(\omega,x))w)$$

$$= F(\mathrm{Res}_x \wp_1(x;\tau) Y_W(\omega, x))w)$$
$$= 2\pi i \frac{d}{d\tau} F(w) + G_2(\tau) F(L_W(0)w)$$
$$= \left(2\pi i \frac{d}{d\tau} + (\mathrm{wt}\ w) G_2(\tau)\right) F(w) + G_2(\tau) F(L_W(0)_N w).$$

Then for $m \in \mathbb{N}$,

$$\widehat{F}(1 \otimes (\mathrm{Res}_x \wp_1(x;\tau) Y_W(\omega, x))^m w)$$
$$= \left(2\pi i \frac{d}{d\tau} + (\mathrm{wt}\ w) G_2(\tau)\right)^m F(w) + \sum_{j=1}^{m} D_{m,j}\left(\frac{d}{d\tau}, \tau\right) F(L_W(0)_N^j w),$$

where for $j = 1, \dots, m$, $D_{m,j}(\frac{d}{d\tau}, \tau)$ is a polynomial in $\frac{d}{d\tau}$ of degree less than m with polynomials in $G_2(\tau)$ and its derivatives as coefficients.

Applying $\widehat{F}$ to the element of J in (5.15) and using $\ker \hat{F} \subset J$, we obtain

$$\left(2\pi i \frac{d}{d\tau} + (\mathrm{wt}\ w) G_2(\tau)\right)^m F(w) + \sum_{p=1}^{m} b_p(\tau) \left(2\pi i \frac{d}{d\tau} + (\mathrm{wt}\ w) G_2(\tau)\right)^{m-p} F(w)$$
$$+ \sum_{j=1}^{m} D_{m,j}\left(\frac{d}{d\tau}, \tau\right) F(L_W(0)_N^j w) + \sum_{p=1}^{m} \sum_{j=1}^{m-p} b_p(\tau) D_{m-p,j}\left(\frac{d}{d\tau}, \tau\right) F(L_W(0)_N^j w)$$
$$= 0. \tag{5.16}$$

Note that in (5.16), $b_p(\tau)$ and $D_{m-p,j}\left(\frac{d}{d\tau}, \tau\right)$ for $p = 1, \dots, m$ and $j = 0, \dots, m-p$ are independent of the genus-one 1-point conformal block F. Since $L_W(0)_N$ commutes with vertex operators acting on W, we see that the linear map given by $w \mapsto F(L_W(0)^j w)$ for $w \in W$ is also a genus-one 1-point conformal block. Then for $j = 1, \dots, K$, $F(L_W(0)^j w)$ also satisfy the equation (5.16). So we see that $F(L_W(0)^j w)$ for $j = 0, \dots, K$ satisfy a system of $K+1$ differential equations. Moreover, since $\frac{d}{d\tau} = 2\pi i q_\tau \frac{d}{dq_\tau}$, The singular point $q_\tau = 0$ of this system of differential equations is regular. ∎

We now prove the following main result of this section, which together with Proposition 5.2 implies the modular invariance theorem (Theorem 5.5):

Theorem 5.4 *Let V be a C_2-cofinite vertex operator algebra of without nonzero elements of negative weights, W a grading-restricted generalized V-module and $F : W \to H(\mathbb{H})$ a genus-one 1-point conformal block labeled by W. The for $w \in W$, $F(w;\tau)$ is in $\mathcal{F}_w$.*

Proof. By Proposition 5.3, $F(L_W(0)^j w)$ for $j = 0, \dots, K$ satisfy a system of $K+1$ differential equations and the singular point $q_\tau = 0$ is regular. Using the theory of differential equations of regular singular points, we have an expansion

$$F(L_W(0)_N^p w; \tau) = \sum_{k=0}^{K} \sum_{j=1}^{J} \sum_{m \in \mathbb{N}} C_{k,j,m}^p(w) \tau^k q_\tau^{r_j + m}, \tag{5.17}$$

where r_j for $j = 1, \dots, J$ are complex numbers such that $r_{j_1} - r_{j_2} \notin \mathbb{Z}$ when $j_1 \neq j_2$.

From (5.17), the properties of the genus-one 1-point conformal block F and Lemma 4.3, we obtain

$$\sum_{k=0}^{K}\sum_{j=1}^{J}\sum_{m\in\mathbb{N}} C^0_{k,j,m}(\mathrm{Res}_x Y_W(u,x)w)(\log q)^k q^{r_j+m} = 0, \tag{5.18}$$

$$\begin{aligned} 2\pi i \sum_{k=0}^{K}\sum_{j=1}^{J}\sum_{m\in\mathbb{N}} (k+1)C^0_{k+1,j,m}(w)(\log q)^k q^{r_j+m} + (2\pi i)^2 \sum_{m\in\mathbb{N}} (r_j+m) C^0_{k,j,m}(w)(\log q)^k q^{r_j+m} \\ = \sum_{m\in\mathbb{N}} C^0_{k,j,m}(\mathrm{Res}_x(\tilde{\wp}_1(x;q) - \widetilde{G}_2(q)x)Y_W(\omega,x)w)(\log q)^k q^{r_j+m}, \end{aligned} \tag{5.19}$$

$$\sum_{k=0}^{K}\sum_{j=1}^{J}\sum_{m\in\mathbb{N}} C^0_{k,j,m}(\mathrm{Res}_x \tilde{\wp}_2(x;q)Y_W(u,x)w)(\log q)^k q^{r_j+m} = 0. \tag{5.20}$$

From (5.18)–(5.20), we see that the linear map $S : W \to \mathbb{C}\{q\}[\log q]$ given by

$$S(w) = \sum_{k=0}^{K}\sum_{j=1}^{J}\sum_{m\in\mathbb{N}} C^0_{k,j,m}(w)q^m$$

satisfies the conditions needed in Theorem 4.4.

Since V is C_2-cofinite, there are only finitely many inequivalent irreducible V-modules. Let N be a nonnegative integer larger than all the real parts of the differences between the finitely many lowest weights of the irreducible V-modules. Then any $N' \in N + \mathbb{N}$ is also larger than all the real parts of the differences between the finitely many lowest weights of the irreducible V-modules. By Theorem 4.4, for $k = 0, \dots, K$ and $j = 1, \dots, J$, the linear map $\psi^{N'}_{S;k,j} : U^{N'}(W) \to \mathbb{C}$ defined by

$$\psi^{N'}_{S;k,j}([w]_{mn}) = 0$$

for $0 \leq m, n \leq N'$, $m \neq n$, and $w \in W$ and

$$\psi^{N'}_{S;k,j}([w]_{mm}) = C^0_{k,j,m}(w)$$

for $0 \leq m \leq N'$ and $w \in W$ induces a symmetric linear function, still denoted by $\psi^{N'}_{S;k,j}$, on $\widetilde{A}^{N'}(W)$ satisfying

$$\psi^{N'}_{S;k,j}(([\omega]_{mm} - (2\pi i)^2(r_j+m)[\mathbf{1}]_{mm})^{\blacklozenge\,(K-k+1)} \blacklozenge \mathfrak{w}) = 0. \tag{5.21}$$

By Proposition 3.22, we know that $\widetilde{A}^{N'}(W)$ is finite dimensional.

Note that

$$\mathbf{1}^{N'} + \widetilde{Q}^{\infty}(V) = \sum_{k=0}^{N'}[\mathbf{1}]_{kk} + \widetilde{Q}^{\infty}(V)$$

is the identity of $\widetilde{A}^{N'}(V)$. Let

$$\mathbf{1}^{N'}+\widetilde{Q}^{\infty}(V)=\tilde{e}_1^{N'}+\cdots+\tilde{e}_{n'}^{N'}$$

with the largest n, where $\tilde{e}_1^{N'},\dots,\tilde{e}_{n'}^{N'}\in\widetilde{A}^{N'}(V)$ are orthogonal central idempodents in $\widetilde{A}^{N'}(V)$. Then $\widetilde{A}^{N'}(V)=\widetilde{A}_1^{N'}\oplus\cdots\oplus\widetilde{A}_{n'}^{N'}$ where $\widetilde{A}_i^{N'}=\widetilde{A}^{N'}(V)\blacklozenge\tilde{e}_i^{N'}$ for $i=1,\dots,n'$, is a decomposition of $\widetilde{A}^{N'}(V)$ into a direct sum of indecomposable $\widetilde{A}^{N'}(V)$-bimodules. Let $\widetilde{U}_i^{N'}=\tilde{e}_i^{N'}\blacklozenge\widetilde{A}^{N'}(W)\blacklozenge\tilde{e}_i^{N'}$ for $i=1,\dots,n'$. Then we have the decomposition

$$\widetilde{A}^{N'}(W)=\widetilde{U}_1^{N'}\oplus\cdots\oplus\widetilde{U}_{n'}^{N'}$$

of $\widetilde{A}^{N'}(W)$ as a direct sum of $\widetilde{A}_1^{N'}$-, ..., $\widetilde{A}_{n'}^{N'}$-bimodules. Let $\widetilde{B}^{N'}=\widetilde{A}^{N'}(V)\oplus\widetilde{A}^{N'}(W)$ be the trivial square-zero extension of $\widetilde{A}^{N'}(V)$ by $\widetilde{A}^{N'}(W)$ and $\widetilde{B}_i^{N'}=\widetilde{A}_i^{N'}\oplus\widetilde{U}_i^{N'}$ for $i=1,\dots,n'$ the trivial square-zero extension of $\widetilde{A}_i^{N'}$ by $\widetilde{U}_i^{N'}$. Then

$$\widetilde{B}^{N'}=\widetilde{B}_1^{N'}\oplus\cdots\oplus\widetilde{B}_{n'}^{N'}.$$

Let $\phi^i_{k,j;N'}=\psi^{N'}_{S;k,j}|_{\widetilde{A}_i^{N'}}$ for $i=1,\dots,n'$. By Theorem 2.2,

$$P^i_{k,j;N'}=\tilde{\epsilon}_i^{N'}(\widetilde{B}^{N'}/\mathrm{Rad}(\phi^i_{k,j;N'}))\tilde{\epsilon}_i^{N'}$$

for $i=1,\dots,n'$, where $\tilde{\epsilon}_i^{N'}=\tilde{e}_i^{N'}+\mathrm{Rad}(\phi^i_{k,j;N'})\in\widetilde{B}^{N'}/\mathrm{Rad}(\phi^i_{k,j;N'})$, are basic symmetric algebras equipped with symmetric linear functions given by $\phi^i_{k,j;N'}$, and

$$M^i_{k,j;N'}=(\widetilde{B}^{N'}/\mathrm{Rad}(\phi^i_{k,j;N'}))\tilde{\epsilon}_i^{N'}$$

are $\widetilde{A}^{N'}(V)$-$P^i_{k,j;N'}$-bimodules which are finitely generated and projective as right $P^i_{k,j;N'}$-modules. Moreover, we define

$$f^i_{k,j;N'}\in\mathrm{Hom}_{\widetilde{A}^{N'}(V),P^i_{k,j;N'}}(\widetilde{A}^{N'}(W)\otimes_{\widetilde{A}^{N'}(V)}M^i_{k,j;N'},M^i_{k,j;N'})$$

for $i=1,\dots,n'$ by $f^i_{k,j;N'}(\mathfrak{w}\otimes w_i)=(0,\mathfrak{w})w_i$ for $\mathfrak{w}\in\widetilde{A}^{N'}(W)$ and $w_i\in M^i_{k,j;N'}$ and then we have

$$\psi^{N'}_{S;k,j}(\mathfrak{w})=\sum_{i=1}^{n'}(\phi^i_{k,j;N'})^{f^i_{k,j;N'}}_{M^i_{k,j;N'}}(\mathfrak{w}+\widetilde{Q}^{\infty}(W))\tag{5.22}$$

for $\mathfrak{w}\in A^{N'}(W)$. We use $\tilde{\vartheta}_{M^i_{k,j;N'}}:\widetilde{A}^{N'}(V)\to(\mathrm{End}\,M^i_{k,j;N'})$ to denote the homomorphism of associative algebras giving the $\widetilde{A}^{N'}(V)$-module structure on $M^i_{k,j;N'}$. Then from (5.21) and Theorem 2.2, we have

$$\tilde{\vartheta}_{M^i_{k,j;N'}}(([\omega]_{mm}-(2\pi i)^2(r_j+m)[\mathbf{1}]_{mm})^{\blacklozenge(K-k+1)}+\widetilde{Q}^{\infty}(V))M^i_{k,j;N'}=0\tag{5.23}$$

for $m=0,\dots,N'$.

From Proposition 3.5, the map $\mathcal{U}(1)^{-1}$ induces an isomorphism from $A^{\infty}(V)$ to $\widetilde{A}^{\infty}(V)$ and thus also induces an isomorphism from $A^{N'}(V)$ to $\widetilde{A}^{N'}(V)$. In particular, the map $\vartheta_{M^i_{k,j;N'}} : A^{N'}(V) \to (\mathrm{End}\ M^i_{k,j;N'})$ defined by

$$\vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^{\infty}(V)) = \tilde{\vartheta}_{M^i_{k,j;N'}}([\mathcal{U}(1)^{-1}v]_{kl} + \widetilde{Q}^{\infty}(V))$$

for $v \in V$ and $k, l = 0, \dots, N'$ gives an $A^{N'}(V)$-module structure to $M^i_{k,j;N'}$.

We now show that $M^i_{k,j;N'}$ for $i = 1, \dots, n'$, $k = 0, \dots, K$ and $j = 1, \dots, J$ are graded $A^{N'}(V)$-modules (see Definition 5.1 in [H8]).

Note that

$$\sum_{m=0}^{N'} \vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{mm} + Q^{\infty}(V)) = 1_{M^i_{k,j;N'}}.$$

Also we have

$$([\mathbf{1}]_{mm} + Q^{\infty}(V)) \diamond ([\mathbf{1}]_{mm} + Q^{\infty}(V)) = [\mathbf{1}]_{mm} + Q^{\infty}(V),$$
$$([\mathbf{1}]_{ll} + Q^{\infty}(V)) \diamond ([\mathbf{1}]_{mm} + Q^{\infty}(V)) = 0$$

for $l, m = 0, \dots, N'$, $l \neq m$. So

$$\{\vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{mm} + Q^{\infty}(V))\}_{m=0}^{N'}$$

is a partition of the identity on $M^i_{k,j;N'}$. In particular,

$$M^i_{k,j;N'} = \coprod_{m=0}^{N'} (M^i_{k,j;N'})_{[m]},$$

where for $m = 0, \dots, N'$,

$$(M^i_{k,j;N'})_{[m]} = \vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{mm} + Q^{\infty}(V))M^i_{k,j;N'}.$$

Since $[\omega]_{mm} + Q^{\infty}(V)$ commutes with $[\mathbf{1}]_{mm} + Q^{\infty}(V)$ for $m = 0, \dots, N'$, $(M^i_{k,j;N'})_{[m]}$ is invariant under $\vartheta_{M^i_{k,j;N'}}([\omega]_{mm} + Q^{\infty}(V))$. Since

$$([\omega]_{mm} + Q^{\infty}(V)) \diamond ([\mathbf{1}]_{ll} + Q^{\infty}(V)) = ([\omega]_{ll} + Q^{\infty}(V)) \diamond ([\mathbf{1}]_{mm} + Q^{\infty}(V)) = 0$$

for $l \in \mathbb{N}$ not equal to m, we see that $\vartheta_{M^i_{k,j;N'}}([\omega]_{mm} + Q^{\infty}(V))$ is 0 on $(M^i_{k,j;N'})_{[l]}$. Note also that $\vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{mm} + Q^{\infty}(V))$ on $(M^i_{k,j;N'})_{[m]}$ is the identity on $(M^i_{k,j;N'})_{[m]}$. From (5.23), we have

$$\tilde{\vartheta}_{M^i_{k,j;N'}}((([\omega]_{mm} - (2\pi i)^2(r_j + m)[\mathbf{1}]_{mm} + \widetilde{Q}^{\infty}(V))^{\blacklozenge\,(K-k+1)})(M^i_{k,j;N'})_{[m]} = 0. \tag{5.24}$$

By the definition of $\vartheta_{M^i_{k,j;N'}}$ and (5.24), we obtain

$$\vartheta_{M^i_{k,j;N'}}((([\mathcal{U}(1)\omega]_{mm} - (2\pi i)^2(r_j + m)[\mathcal{U}(1)\mathbf{1}]_{mm} + Q^{\infty}(V))^{\diamond\,(K-k+1)})(M^i_{k,j;N'})_{[m]} = 0. \tag{5.25}$$

Using $\mathcal{U}(1)\omega = (2\pi i)^2(\omega - \frac{c}{24}\mathbf{1})$ and $\mathcal{U}(1)\mathbf{1} = \mathbf{1}$ (see the definition of $\mathcal{U}(1)$ and Lemma 1.1 in [H2]), we see that (5.25) becomes

$$\vartheta_{M^i_{k,j;N'}}\left(\left([\omega]_{mm} - \left(r_j + \frac{c}{24} + m\right)[\mathbf{1}]_{mm} + Q^\infty(V)\right)^{\diamond\,(K-k+1)}\right)(M^i_{k,j;N'})_{[m]} = 0. \quad (5.26)$$

Thus $(M^i_{k,j;N'})_{[m]}$ is the generalized eigenspace of $\vartheta_{M^i_{k,j;N'}}([\omega]_{mm} + Q^\infty(V))$ with eigenvalue $r_j + \frac{c}{24} + m$.

For $v \in V$, $k, l = 0, \dots, N'$ and $w \in (M^i_{k,j;N'})_{[m]}$,

$$\begin{aligned}
\vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))w &= \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))\vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{mm} + Q^\infty(V))w\\
&= \vartheta_{M^i_{k,j;N'}}([v]_{kl} \diamond [\mathbf{1}]_{mm} + Q^\infty(V))w\\
&= \delta_{lm}\vartheta_{M^i_{k,j;N'}}([v]_{km} + Q^\infty(V))w\\
&= \delta_{lm}\vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{kk} \diamond [v]_{km} + Q^\infty(V))w\\
&= \delta_{lm}\vartheta_{M^i_{k,j;N'}}([\mathbf{1}]_{kk} + Q^\infty(V))\vartheta_{M^i_{k,j;N'}}([v]_{km} + Q^\infty(V))w,
\end{aligned}$$

which is 0 when $l \neq m$ and is in $(M^i_{k,j;N'})_{[k]}$ when $l = m$. So Condition 1 in Definition 5.1 in [H8] is satisfied.

We define

$$\begin{aligned}
L_{M^i_{k,j;N'}}(0) &= \sum_{m=0}^{N'} \vartheta_{M^i_{k,j;N'}}([\omega]_{mm} + Q^\infty(V)),\\
L_{M^i_{k,j;N'}}(-1) &= \sum_{m=0}^{N'-1} \vartheta_{M^i_{k,j;N'}}([\omega]_{m+1,m} + Q^\infty(V)).
\end{aligned}$$

Then $(M^i_{k,j;N'})_{[m]}$ for $m = 0, \dots, N'$ are generalized eigenspaces for $L_{M^i_{k,j;N'}}(0)$ and $M^i_{k,j;N'}$ is the direct sum of these generalized eigenspaces. The eigenvalues of $L_{M^i_{k,j;N'}}(0)$ are $r_j + \frac{c}{24} + m$ for $m = 0, \dots, N'$ and the real parts of these eigenvalues has a minimum $\Re(r_j)$. This shows that Condition 2 in Definition 5.1 in [H8] is satisfied. From what we have shown above, we see that $L_{M^i_{k,j;N'}}(-1)$ maps $(M^i_{k,j;N'})_{[m]}$ for $m = 0, \dots, N'-1$ to $(M^i_{k,j;N'})_{[m+1]}$. So Condition 3 in Definition 5.1 in [H8] is also satisfied.

From Remark 4.5 in [H9] with $W = V$, we have

$$\begin{aligned}
[L_{M^i_{k,j;N'}}(0), L_{M^i_{k,j;N'}}(-1)]w &= \vartheta_{M^i_{k,j;N'}}([\omega]_{m+1,m+1}[\omega]_{m+1,m} - [\omega]_{m+1,m}[\omega]_{mm} + Q^\infty(V))w\\
&= \vartheta_{M^i_{k,j;N'}}([(L_V(-1) + L_V(0))\omega]_{m+1,m} + Q^\infty(V))w \quad (5.27)
\end{aligned}$$

for $w \in (M^i_{k,j;N'})_{[m]}$. By Proposition 2.3 in [H8], we know that

$$[(L_V(-1) + L_V(0) - 1)\omega]_{m+1,m} \in Q^\infty(V).$$

Then the right-hand side of (5.27) is equal to

$$\vartheta_{M^i_{k,j;N'}}([\omega]_{m+1,m})w = L_{M^i_{k,j;N'}}(-1)w. \tag{5.28}$$

From (5.27) and (5.28), we obtain the commutator formula

$$[L_{M^i_{k,j;N'}}(0), L_{M^i_{k,j;N'}}(-1)] = L_{M^i_{k,j;N'}}(-1). \tag{5.29}$$

For $k, l = 0, \ldots, N'$ and $v \in \mathbb{N}$, by Remark 4.5 in [H9] with $W = V$ and Proposition 2.3 in [H8],

$$\begin{aligned}
&[L_{M^i_{k,j;N'}}(0), \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))]w\\
&\quad = \vartheta_{M^i_{k,j;N'}}([\omega]_{kk} + Q^\infty(V))\vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))w\\
&\qquad - \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))\vartheta_{M^i_{k,j;N'}}([\omega]_{ll} + Q^\infty(V))w\\
&\quad = \vartheta_{M^i_{k,j;N'}}([\omega]_{kk} \diamond [v]_{kl} - [v]_{kl} \diamond [\omega]_{ll} + Q^\infty(V))w\\
&\quad = \vartheta_{M^i_{k,j;N'}}([(L_V(-1) + L_V(0))v]_{kl} + Q^\infty(V))w\\
&\quad = (k-l)\vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))w
\end{aligned}$$

for $w \in (M^i_{k,j;N'})_{[l]}$. Then we obtain the commutator formula

$$[L_{M^i_{k,j;N'}}(0), \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))] = (k-l)\vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V)). \tag{5.30}$$

For $k = 0, \ldots, N' - 1$, $l = 1, \ldots, N'$ and $v \in V$, also by Remark 4.5 in [H9] with $W = V$,

$$\begin{aligned}
&[L_{M^i_{k,j;N'}}(-1), \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))]w\\
&\quad = \vartheta_{M^i_{k,j;N'}}([\omega]_{k+1,k} + Q^\infty(V))\vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))w\\
&\qquad - \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))\vartheta_{M^i_{k,j;N'}}([\omega]_{l+1,l} + Q^\infty(V))w\\
&\quad = \vartheta_{M^i_{k,j;N'}}([\omega]_{k+1,k} \diamond [v]_{kl} - [v]_{kl} \diamond [\omega]_{l+1,l} + Q^\infty(V))w\\
&\quad = \vartheta_{M^i_{k,j;N'}}([L_V(-1)v]_{kl} + Q^\infty(V))w
\end{aligned}$$

for $w \in (M^i_{k,j;N'})_{[l]}$. Then we obtain the commutator formula

$$[L_{M^i_{k,j;N'}}(-1), \vartheta_{M^i_{k,j;N'}}([v]_{kl} + Q^\infty(V))] = \vartheta_{M^i_{k,j;N'}}([L_V(-1)v]_{kl} + Q^\infty(V)). \tag{5.31}$$

From (5.29), (5.30) and (5.31), we see that Condition 4 in Definition 5.1 in [H8] is satisfied. Thus we have shown that $M^i_{k,j;N'}$ is indeed a graded $A^{N'}(V)$-module.

From Section 6 in [H9], we have the lower-bounded generalized V-module $W^i_{k,j;N'} = S^{N'}_{\mathrm{voa}}(M^i_{k,j;N'})$ constructed from $M^i_{k,j;N'}$. By Proposition 6.2 in [H9], we see that $\Omega^0_{N'}(W^i_{k,j;N'}) = M^i_{k,j;N'}$. Since V has no nonzero elements of negative weights and C_2-cofinite and $M^i_{k,j;N'}$ is finite dimensional, by Property 2 in Proposition 3.19, $W^i_{k,j;N'}$ as a lower-bounded generalized V-module generated by $M^i_{k,j;N'}$ is quasi-finite dimensional and is in particular grading restricted. By Property 5 in Proposition 3.19, $W^i_{k,j;N'}$ is of finite length.

Now we consider the case $N' = N$. Given an element of $P^i_{k,j;N}$, its action on the right $P^i_{k,j;N}$-module $M^i_{k,j;N}$ is in fact an $A^N(V)$-module map from $M^i_{k,j;N}$ to itself. By the universal property of $W^i_{k,j} = S^N_{\text{voa}}(M^i_{k,j;N})$, there is a unique V-module map from $W^i_{k,j;N}$ to itself such that its restriction to $M^i_{k,j;N}$ is the action of the element of $P^i_{k,j;N}$ on $M^i_{k,j;N}$. Thus we obtain a right action of $P^i_{k,j;N}$ on $W^i_{k,j;N}$. Since the action of $P^i_{k,j;N}$ on $W^i_{k,j;N}$ is given by V-module maps, the homogeneous subspaces $(W^i_{k,j;N})_{[r_j+m]}$ of $W^i_{k,j;N}$ for $m \in \mathbb{N}$ also have right $P^i_{k,j;N}$-module structures. We now show that these right $P^i_{k,j;N}$-modules are in fact projective.

Since $\Omega^0_N(W^i_{k,j;N}) = M^i_{k,j;N}$ is a right projective $P^i_{k,j;N}$-module,

$$\Omega^0_N(W^i_{k,j;N}) = \coprod_{m=0}^{N} (W^i_{k,j;N})_{[r_j+m]}$$

and $(W^i_{k,j;N})_{[r_j+m]} = (M^i_{k,j;N})_{[m]}$ for $m = 0, \dots, N$ are right $P^i_{k,j;N}$-modules, $(W^i_{k,j;N})_{[r_j+m]}$ for $m = 0, \dots, N$ as direct summands of $M^i_{k,j;N}$ are also projective as right $P^i_{k,j;N}$-modules. We still need to prove that $(W^i_{k,j;N})_{[r_j+N']}$ for $N' \in N + \mathbb{Z}_+$ are projective as right $P^i_{k,j;N}$-modules.

Using the isomorphisms $\mathcal{U}(1) : \widetilde{A}^{N'}(V) \to A^{N'}(V)$ and the $\mathcal{U}_W(1) : \widetilde{A}^{N'}(W) \to A^{N'}(W)$, all the elements and structures we obtained from $\widetilde{A}^{N'}(V)$ and $\widetilde{A}^{N'}(W)$ are mapped to the corresponding elements and structures that one can obtain from $A^{N'}(V)$ and $A^{N'}(W)$ in the same way. Moreover, these elements and structures have completely the same properties.

Let $e^{N'}_i = \mathcal{U}(1)\tilde{e}^{N'}_i$ for $i = 1, \dots, n'$. Then $\mathbf{1}^{N'} + Q^\infty(V) = e^{N'}_1 + \cdots + e^{N'}_{n'}$ and $N \le N'$, we have

$$\begin{aligned}\mathbf{1}^N + Q^\infty(V) &= (\mathbf{1}^{N'} + Q^\infty(V)) \diamond (\mathbf{1}^N + Q^\infty(V))\\ &= e^{N'}_1 \diamond (\mathbf{1}^N + \widetilde{Q}^\infty(V)) + \cdots + e^{N'}_{n'} \diamond (\mathbf{1}^N + \widetilde{Q}^\infty(V)).\end{aligned}$$

Using the properties of $\mathbf{1}^N$ and $e^{N'}_i$ for $i = 1, \dots, n'$, we see that $e^{N'}_1 \diamond (\mathbf{1}^N + Q^\infty(V))$ for $i = 1, \dots, n'$ are all in $A^N(V)$. Let $e^N_1, \dots, e^N_n$ be the nonzero elements in the set consisting of the elements $e^{N'}_i \diamond (\mathbf{1}^N + Q^\infty(V))$ for $i = 1, \dots, n'$. Then we have

$$\mathbf{1}^N + Q^\infty(V) = e^N_1 + \cdots + e^N_n.$$

From the properties of $\mathbf{1}^N$ and $e^{N'}_i$ for $i = 1, \dots, n'$ again, we see that $e^N_1, \dots, e^N_n$ are orthogonal central idempodents of $A^N(V)$ and n is the largest for such decompositions of the identity $\mathbf{1}^N + Q^\infty(V)$ of $A^N(V)$. Let $A^N_i = A^N(V) \diamond e^N_i$. Then we have

$$A^N(V) = A^N_1 \oplus \cdots \oplus A^N_n.$$

Also we have $A^N(W) = (\mathbf{1}^N + Q^\infty(V)) \diamond A^{N'}(W) \diamond (\mathbf{1}^N + Q^\infty(V))$. Since $A^{N'}(W) = U^{N'}_1 \oplus \cdots \oplus U^{N'}_{n'}$, where $U^{N'}_i = e^{N'}_i \diamond A^{N'}(W_1) \diamond e^{N'}_i$ for $i = 1, \dots, n'$, we have

$$A^N(W) = (\mathbf{1}^N + Q^\infty(V)) \diamond A^{N'}(W) \diamond (\mathbf{1}^N + Q^\infty(V))$$

$$
\begin{aligned}
&= (\mathbf{1}^N + Q^\infty(V)) \diamond U_1^{N'} \diamond (\mathbf{1}^N + Q^\infty(V)) \\
&\quad \oplus \cdots \oplus (\mathbf{1}^N + Q^\infty(V)) \diamond U_{n'}^{N'} \diamond (\mathbf{1}^N + Q^\infty(V)) \\
&= (e_1^{N'} \diamond (\mathbf{1}^N + Q^\infty(V))) \diamond A^N(W) \diamond (e_1^{N'} \diamond (\mathbf{1}^N + Q^\infty(V))) \\
&\quad \oplus \cdots \oplus (e_{n'}^{N'} \diamond (\mathbf{1}^N + Q^\infty(V))) \diamond A^N(W) \diamond (e_{n'}^{N'} \diamond (\mathbf{1}^N + Q^\infty(V))) \\
&= U_1^N \oplus \cdots \oplus U_n^N,
\end{aligned}
$$

where $U_i^N = e_i^N \diamond A^N(W_1) \diamond e_i^N$ is an A_i^N-bimodule for $i = 1, \dots, n'$.

By definition,

$$
B^N = A^N(V) \oplus A^N(W_1) \subset A^{N'}(V) \oplus A^{N'}(W_1) = B^{N'}
$$

as associative algebras. For i and i' such that $e_i^N = e_{i'}^{N'} \diamond (\mathbf{1}^N + Q^\infty(V))$ is nonzero, we have $\mathrm{Rad}(\phi_{k,j;N'}^{i'}) \cap B^N = \mathrm{Rad}(\phi_{k,j;N}^i)$. Then the kernel of the homomorphism of associative algebras from B^N to $B^{N'}/\mathrm{Rad}(\phi_{k,j;N'}^{i'})$ is $\mathrm{Rad}(\phi_{k,j;N'}^i) \cap B^N = \mathrm{Rad}(\phi_{k,j;N}^i)$. In particular, we obtain an injective homomorphism of associative algebras from $B^N/\mathrm{Rad}(\phi_{k,j;N}^i)$ to $B^{N'}/\mathrm{Rad}(\phi_{k,j;N'}^{i'})$. This injective homomorphism maps $\epsilon_i^N = e_i^N + \mathrm{Rad}(\phi_{k,j;N}^i)$ to $\epsilon_{i'}^{N'} = e_{i'}^{N'} + \mathrm{Rad}(\phi_{k,j;N'}^{i'})$. Thus we obtain an injective homomorphism of associative algebras from $P_{k,j;N}^i = \epsilon_i^N \diamond (B^N/\mathrm{Rad}(\phi_{k,j;N}^i)) \diamond \epsilon_i^N$ to $P_{k,j;N'}^{i'} = \epsilon_{i'}^{N'} \diamond (B^{N'}/\mathrm{Rad}(\phi_{k,j;N}^{i'})) \diamond \epsilon_{i'}^N$. We shall view $P_{k,j;N}^i$ as a subalgebra of $P_{k,j;N'}^{i'}$ from now on. In particular, $M_{k,j;N'}^{i'} = (B^{N'}/\mathrm{Rad}(\phi_{k,j;N'}^{i'})) \diamond \epsilon_{i'}^{N'}$ is also a right $P_{k,j;N}^i$-module. Then we also obtain an injective $P_{k,j;N}^i$-module map from $M_{k,j;N}^i = (B^N/\mathrm{Rad}(\phi_{k,j;N}^i)) \diamond \epsilon_i^N$ to $M_{k,j;N'}^{i'} = (B^{N'}/\mathrm{Rad}(\phi_{k,j;N'}^{i'})) \diamond epsilon_{i'}^{N'}$. Moreover, $M_{k,j;N}^i$ and $M_{k,j;N'}^{i'}$ are left B^N-module and $B^{N'}$-module, respectively, and this injective $P_{k,j;N}^i$-module map induces the left B^N-module structure on $M_{k,j;N}^i$ from the left $B^{N'}$-module structure on $M_{k,j;N'}^{i'}$.

We know that $M_{k,j;N'}^{i'}$ as a right $P_{k,j;N'}^{i'}$-module is projective. We now show that $M_{k,j;N'}^{i'}$ as a right $P_{k,j;N}^i$-module is also projective. It is enough to show that the right action of $P_{k,j;N'}^{i'}$ on $M_{k,j;N'}^{i'}$ is in fact determined by the right action of $P_{k,j;N}^i$. In fact, we have proved that $M_{k,j;N}^i$ and $M_{k,j;N'}^{i'}$ are $A^N(V)$-$P_{k,j;N}^i$-bimodule and $A^{N'}(V)$-$P_{k,j;N'}^{i'}$-bimodule, respectively and the left $A^N(V)$-module structure on $M_{k,j;N}^i$ is induced from the left $A^{N'}(V)$-module structure on $M_{k,j;N'}^{i'}$ when we view $M_{k,j;N}^i$ as a right $P_{k,j;N}^i$-submodule of $M_{k,j;N'}^{i'}$. Then we have

$$
M_{k,j;N'}^{i'} = \coprod_{m=0}^{N'} (M_{k,j;N'}^{i'})_{[m]}
$$

and

$$
M_{k,j;N}^i = (M_{k,j;N'}^{i'})^N = \coprod_{m=0}^{N} (M_{k,j;N'}^{i'})_{[m]}.
$$

By Proposition 3.20, $M_{k,j;N'}^{i'}$ is equivalent to the graded $A^{N'}(V)$-module $\Omega_{N'}^0(S_{\mathrm{voa}}^N(M_{k,j;N}^i)) = \Omega_{N'}^0(W_{k,j}^i)$. We know that for $m = 1, \dots, N'$, $(M_{k,j;N'}^{i'})_{[m]}$ are right $P_{k,j;N'}^{i'}$-submodules of

$M^{i'}_{k,j;N'}$. In particular, $M^i_{k,j;N}$ is a right $P^{i'}_{k,j;N'}$-submodule of $M^{i'}_{k,j;N'}$. Note that the action of every element of $P^{i'}_{k,j;N'}$ on $M^i_{k,j;N}$ is an $A^N(V)$-module map. By the universal property of $S^N_{\rm voa}(M^i_{k,j;N})$, such an $A^N(V)$-module map gives a unique V-module map from $S^N_{\rm voa}(M^i_{k,j;N})$ to itself. In particular, such an $A^N(V)$-module map gives a unique $A^{N'}(V)$-module map from $M^{i'}_{k,j;N'}$. Thus we see that the action of the element of $P^{i'}_{k,j;N'}$ on $M^{i'}_{k,j;N'}$ must be the one obtained from its restriction to $M^i_{k,j;N}$. But the restriction to $M^i_{k,j;N}$ of the action of $P^{i'}_{k,j;N'}$ on $M^{i'}_{k,j;N'}$ is exactly the action of $P^i_{k,j;N}$. So we see that the action of $P^{i'}_{k,j;N'}$ on $M^{i'}_{k,j;N'}$ is determined by the action of $P^i_{k,j;N}$ on $M^i_{k,j;N}$. Thus $M^{i'}_{k,j;N'}$ as a right $P^i_{k,j;N}$-module is also projective. In particular, $(W^i_{k,j})_{[r_j+N']} = (M^{i'}_{k,j;N'})_{[N']}$ is projective as a right $P^i_{k,j;N}$-module. Since N' is arbitrary, we see that $W^i_{k,j}$ and its homogeneous subspaces are all projective as a right $P^i_{k,j;N}$-modules.

We know that $\Omega^0_N(W^i_{k,j}) = \Omega^0_N(S^N_{\rm voa}(M^i_{k,j;N})) = M^i_{k,j;N}$. Then by Proposition 3.20, $(W^i_{k,j})'$ is equivalent to $S^N_{\rm voa}(\Omega^0_N((W^i_{k,j})'))$. In particular, Theorem 3.17 can be applied to the case that $W_1 = W$ and $W_2 = W_3 = W^i_{k,j} = S^N_{\rm voa}(M^i_{k,j;N})$. By this theorem, we obtain a unique $P^i_{k,j;N}$-compatible intertwining operator $\mathcal{Y}^i_{k,j}$ of type $\binom{W^i_{k,j}}{WW^i_{k,j}}$ such that

$$\rho^N(\mathcal{Y}^i_{k,j}) = f^i_{k,j;N} \in \mathrm{Hom}_{\widetilde{A}^{N'}(V),P^i_{k,j;N}}(\widetilde{A}^N(W) \otimes_{\widetilde{A}^N(V)} M^i_{k,j;N}, M^i_{k,j;N}).$$

By Proposition 4.2, we have a symmetric linear function $\psi_{\mathcal{Y}^i_{k,j},\phi^i_{k,j;N}}$ on $\widetilde{A}^N(W)$.

We actually need only the intertwining operators $\mathcal{Y}^i_{0,j}$ (that is, the case $k=0$). It is clear that the linear map given by

$$w \mapsto F(w;\tau) - \sum_{j=1}^{J}\sum_{i=1}^{n} \overline{F}^{\phi^i_{0,j;N}}_{\mathcal{Y}^i_{0,j}}(w;\tau) \tag{5.32}$$

for $w \in W$ is also a genus-one 1-point conformal block labeled by W. Then by Proposition 5.3,

$$F(w;\tau) - \sum_{j=1}^{J}\sum_{i=1}^{n} \overline{F}^{\phi^i_{0,j;N}}_{\mathcal{Y}^i_{0,j}}(w;\tau)$$

can be expanded as

$$\sum_{j=1}^{J^{(1)}}\sum_{m\in\mathbb{N}} C^{(1)}_{0,j,m}(w)q_\tau^{r^{(1)}_j+m} + \sum_{k=1}^{K}\tau^k G^{(1)}_k(w;q_\tau),$$

where for $j=1,\dots,J^{(1)}$, there exists j' satisfying $1 \le j' \le J$ such that $r^{(1)}_j - r_{j'} \in \mathbb{Z}_+$ and where $G^{(1)}_k(w;q_\tau)$ for $k=1,\dots,K$ are in $\coprod_{r\in\mathbb{C}} q_\tau^r\mathbb{C}[[q_\tau]]$. Let $s \in \mathbb{Z}_+$ be larger than the maximum of the real parts of the differences of the lowest weights of the (finitely many) irreducible V-modules. Then we repeat the argument above s times to find finite-dimensional associative algebras $P^{(s)}_l$ with symmetric linear functions $\phi^{(s)}_l$, grading-restricted generalized

V-$P^{(s)}$-modules $W_l^{(s)}$ and $P_l^{(s)}$-compatible intertwining operators $\mathcal{Y}_l^{(s)}$ of types $\binom{W_l^{(s)}}{WW_l^{(s)}}$ for $l = 1, \dots, p^{(s)}$ such that the linear map given by

$$w \mapsto F(w; \tau) - \sum_{l=1}^{p^{(s)}} \overline{F}^{\phi_l^{(s)}}_{\mathcal{Y}_l^{(s)}}(w; \tau)$$

is a genus-one 1-point conformal block and for $w \in W$,

$$F(w; \tau) - \sum_{l=1}^{p^{(s)}} \overline{F}^{\phi_l^{(s)}}_{\mathcal{Y}_l^{(s)}}(w; \tau) \tag{5.33}$$

can be expanded as

$$\sum_{j=1}^{J^{(s)}} \sum_{m \in \mathbb{N}} C^{(s)}_{0,i,m}(w) q_\tau^{r_j^{(s)}+m} + \sum_{k=1}^{K} \tau^k G_k^{(s)}(w; q_\tau), \tag{5.34}$$

where for $j = 1, \dots, J^{(s)}$, there exists j' satisfying $1 \leq j' \leq J$ such that $r_j^{(s)} - r_{j'} \in s + \mathbb{N}$ and where $G_k^{(s)}(w; q_\tau)$ for $k = 1, \dots, K$ is in $\coprod_{r \in \mathbb{C}} q_\tau^r \mathbb{C}[[q_\tau]]$. We now show that (5.33) is equal to 0.

We first prove that the first term in (5.34) is 0 for all $w \in W$. Assume that it is not 0 for some $w \in W$. Note that we have shown that $r_j + \frac{c}{24}$ for $j = 1, \dots, J$ are the lowest weights of $M^i_{k,j;N}$ and thus are also the lowest weights of $W^i_{k,j}$. But the lowest weight of a lower-bounded generalized V-module of finite length must also be the lowest weight of an irreducible V-module. In particular, $r_j + \frac{c}{24}$ for $j = 1, \dots, J$ are lowest weights of irreducible V-modules. Since the first term in (5.34) is not 0 for some $w \in W$, the same proof shows that $r_j^{(s)} + \frac{c}{24}$ for $j = 1, \dots, J^{(s)}$ are also lowest weights of irreducible V-modules. But $r_j^{(s)} - r_{j'} \in s + \mathbb{N}$, we have $\Re(r_j^{(s)} - r_{j'}) \geq s$ which is larger than the maximum of the real parts of the differences of the lowest weights of the (finitely many) irreducible V-modules. This is impossible since as we have shown above, $r_j^{(s)} - r_{j'}$ is the difference between the lowest weights $r_j^{(s)}$ and $r_{j'}$ of some irreducible V-modules. Thus the first term in (5.34) must be 0.

In fact, if we write the first term in (5.34) as $G_0(w; q_\tau)$, then (5.34) can be written as

$$\sum_{k=0}^{K} \tau^k G_k^{(s)}(w; q_\tau).$$

We have proved that $G_0(w; q_\tau) = 0$. We now use induction to prove that $G_k(w; q_\tau) = 0$ for $k = 0, \dots, K$. Assume that $G_k(w; q_\tau) = 0$ for $k = 0, \dots, k_0$. Then (5.34) becomes

$$\sum_{k=k_0+1}^{K} \tau^k G_k^{(s)}(w; q_\tau).$$

Note that (5.33) gives a genus-one 1-point conformal block. In particular, the property (5.2) for this genus-one 1-point conformal block gives

$$2\pi i\frac{\partial}{\partial\tau}\sum_{k=k_0+1}^{K}\tau^k G_k^{(s)}(w;q_\tau)=\sum_{k=k_0+1}^{K}\tau^k G_k^{(s)}(\mathrm{Res}_x(\tilde{\wp}_1(x;q_\tau)-\tilde{G}_2(q_\tau)x)Y_W(\omega,x)w;q_\tau) \quad (5.35)$$

(see (4.9) for the definition of $\tilde{\wp}_1(x;q_\tau)-\tilde{G}_2(q_\tau)x$).

The left-hand side of (5.35) is equal to

$$2\pi i\sum_{k=k_0+1}^{K}k\tau^{k-1}G_k^{(s)}(w;q_\tau)+2\pi i\sum_{k=k_0+1}^{K}\tau^k\frac{\partial}{\partial\tau}G_k^{(s)}(w;q_\tau). \quad (5.36)$$

Since the coefficients of τ^{k_0} in the right-hand side of (5.35) and in (5.36) are 0 and $2\pi i(k_0+1)G_{k_0+1}^{(s)}(w;q_\tau)$, respectively, we obtain $G_{k_0+1}^{(s)}(w;q_\tau)=0$. By induction principle, we obtain $G_k(w;q_\tau)=0$ for $k=0,\ldots,K$. Thus (5.34) is 0.

This proves (5.33) is 0, that is,

$$F(w;\tau)=\sum_{j=l}^{p^{(s)}}\overline{F}^{\phi_l^{(s)}}_{\mathcal{Y}_l^{(s)}}(w;q_\tau) \quad (5.37)$$

for $w\in W$. Since the right-hand side of (5.37) is in $\mathcal{F}_w$, the left-hand side of (5.37) is also in $\mathcal{F}_w$, proving our theorem. ∎

Let $W_1,\ldots,W_n$ be grading-restricted generalized V-modules and $w_1\in W_1,\ldots,w_n\in W_n$. Given a finite-dimensional associative algebra P with a symmetric linear function ϕ, grading-restricted generalized V-modules $\widetilde{W}_1,\ldots,\widetilde{W}_{n-1}$, a grading-restricted generalized V-P-bimodule $\widetilde{W}_0=\widetilde{W}_n$ projective as a right P-module and intertwining operators $\mathcal{Y}_1,\ldots,\mathcal{Y}_n$ of types $\binom{\widetilde{W}_0}{W_1\widetilde{W}_1},\ldots,\binom{\widetilde{W}_{n-1}}{W_n\widetilde{W}_n}$, respectively, such that the product of $\mathcal{Y}_1,\ldots,\mathcal{Y}_n$ is compatible with P, we have a genus-one correlation n-point function

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\ldots,\mathcal{Y}_n}(w_1,\ldots,w_n;z_1,\ldots,z_n;\tau).$$

Let $\mathcal{F}_{w_1,\ldots,w_n}$ be the vector space of such genus-one correlation n-point functions for all P, ϕ, $\widetilde{W}_1,\ldots,\widetilde{W}_{n-1}$, $\widetilde{W}_0=\widetilde{W}_n$ and $\mathcal{Y}_1,\ldots,\mathcal{Y}_n$.

We are ready to prove the main theorem of this paper.

Theorem 5.5 *Let V be a C_2-cofinite vertex operator algebra without nonzero negative weight elements. Then for a finite-dimensional associative algebra P with a symmetric linear function ϕ, grading-restricted generalized V-modules $\widetilde{W}_1,\ldots,\widetilde{W}_{n-1}$, a grading-restricted generalized V-P-bimodule $\widetilde{W}_0=\widetilde{W}_n$ projective as a right P-module, intertwining operators $\mathcal{Y}_1,\ldots,\mathcal{Y}_n$ of types $\binom{\widetilde{W}_0}{W_1\widetilde{W}_1}$, ..., $\binom{\widetilde{W}_{n-1}}{W_n\widetilde{W}_n}$, respectively, such that product of $\mathcal{Y}_1,\ldots,\mathcal{Y}_n$ is compatible with P, and*

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in SL(2,\mathbb{Z}),$$

$$\overline{F}^{\phi}_{\mathcal{Y}_1,\dots,\mathcal{Y}_n}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L_{W_1}(0)}w_1,\dots,\left(\frac{1}{\gamma\tau+\delta}\right)^{L_{W_n}(0)}w_n;\frac{z_1}{\gamma\tau+\delta},\dots,\frac{z_n}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is in $\mathcal{F}_{w_1,\dots,w_n}$.

Proof. As in the proof of the modular invariance (Theorem 7.3 in [H2]) in the rational case, using the genus-one associativity proved in [F1] and [F2] (see Theorem 2.7), the proof in the general n case is reduced to the $n = 1$ case. So we need only prove the $n = 1$ case.

In the $n = 1$ case, by Proposition 5.2, the linear map given by

$$w_1 \mapsto \overline{F}^{\phi}_{\mathcal{Y}_1}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L_{W_1}(0)}w_1;\frac{z_1}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is a genus-one 1-point conformal block. Then by Theorem 5.4, for $w_1 \in W_1$,

$$\overline{F}^{\phi}_{\mathcal{Y}_1}\left(\left(\frac{1}{\gamma\tau+\delta}\right)^{L_{W_1}(0)}w_1;\frac{z_1}{\gamma\tau+\delta};\frac{\alpha\tau+\beta}{\gamma\tau+\delta}\right)$$

is in $\mathcal{F}_{w_1}$, proving the theorem in this case. ∎

References

[ABD] T. Abe, G. Buhl, C. Dong, Rationality, Regularity, and C_2-cofiniteness, *Trans. Amer. Math. Soc.* **356** (2004), 3391–3402.

[AbN] T. Abe and K. Nagatomo, Finiteness of conformal blocks over the projective line, in: *Vertex operator algebras in mathematics and physics (Toronto, 2000)*, ed. S. Berman, Y. Billig, Y.-Z. Huang and J. Lepowsky, Fields Inst. Commun., Vol. 39, Amer. Math. Soc., Providence, 2003, 1–12.

[An] E. S. Andersen, Two summation formulae for product sums of binomial coefficients, *Math. Scand.* **1** (1953), 261–262.

[Ar] Y. Arike, Some remarks on symmetric linear functions and pseudotrace maps, *Proc. Japan Acad. Ser. A Math. Sci.* **86** (2010), 119-124.

[ArN] Y. Arike and K. Nagatomo, Some remarks on pseudo-trace functions for orbifold models associated with symplectic fermions. *Int. J. Math.* **24**, (2013) 1350008

[B] G. Buhl, A spanning set for VOA modules, *J. Alg.* **254** (2002), 125–151.

[DLM] C. Dong, H. Li and G. Mason, Vertex operator algebras and associative algebras, *J. Algebra* **206** (1998), 67-96.

[F1] F. Fiordalisi, Logarithmic intertwining operator and genus-one correlation functions, Ph.D. Thesis, Rutgers University, 2015. (Online PDF file of the thesis.)

[F2] F. Fiordalisi, Logarithmic Intertwining Operators and Genus-One Correlation Functions, *Comm. Contemp. Math.* **18** (2016), 1650026.

[GN] M. R. Gaberdiel and A. Neitzke, Rationality, quasirationality and finite W-algebras, *Comm. Math.Phys.* **238** (2003) 305–331.

[H1] Y.-Z. Huang, Differential equations and intertwining operators, *Comm. Contemp. Math.* **7** (2005), 375–400.

[H2] Y.-Z. Huang, Differential equations, duality and modular invariance, *Comm. Contemp. Math.* **7** (2005), 649–706.

[H3] Y.-Z. Huang, Vertex operator algebras and the Verlinde conjecture, *Comm. Contemp. Math.* **10** (2008), 103–154.

[H4] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Comm. Contemp. Math.* **10** (2008), 871–911.

[H5] Y.-Z. Huang, Cofiniteness conditions, projective covers and the logarithmic tensor product theory, *J. Pure Appl. Alg.* **213** (2009), 458–475.

[H6] Y.-Z. Huang, A construction of lower-bounded generalized twisted modules for a grading-restricted vertex (super)algebra, *Comm. Math. Phys.* **377** (2020), 909–945.

[H7] Y.-Z. Huang, Lower-bounded and grading-restricted twisted modules for affine vertex (operator) algebras, *J. Pure Appl. Alg.* **225** (2021), Paper no. 106618.

[H8] Y.-Z. Huang, Associative algebras and the representation theory of grading-restricted vertex algebras, to appear; arXiv:2009.00262.

[H9] Y.-Z. Huang, Associative algebras and intertwining operators, *Comm. Math. Phys.* **396** (2022), 1–44.

[H10] Y.-Z. Huang, Convergence in conformal field theory, *Chin. Ann. Math.* **B43** (2022), 1101–1124.

[HLZ1] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmic tensor category theory, VI: Expansion condition, associativity of logarithmic intertwining operators, and the associativity isomorphisms; arXiv:1012.4202.

[HLZ2] Y.-Z. Huang, J. Lepowsky and L. Zhang, Logarithmi tensor category theory, VII: Convergence and extension properties and applications to expansion for intertwining maps; arXiv:1110.1929.

[HY] Y.-Z. Huang and J. Yang, Logarithmic intertwining operators and associative algebras, *J. Pure Appl. Alg.* **216** (2011), 1467–1492. Corrigendum, *J. Pure Appl. Alg.* **226** (2021), 107020.

[K] N. Koblitz, *Introduction to elliptic curves and modular forms*, Second Edition, Graduate Texts in Mathematics, Vol. 97, Springer-Verlag, New York, 1993.

[L] S. Lang, *Elliptic functions*, Graduate Texts in Mathematics, Vol. 112, Springer-Verlag, New York, 1987.

[Mc] R. McRae, Proof of Miyamoto's Proposition 4.4, private communication.

[MS1] G. Moore and N. Seiberg, Polynomial equations for rational conformal field theories, *Phys. Lett.* **B212** (1988), 451–460.

[MS2] G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Comm. Math. Phys.* **123** (1989), 177–254.

[Mi1] M. Myiamoto, Intertwining operators and modular invariance, arXiv:math/0010180.

[Mi2] M. Myiamoto, Modular invariance of vertex operator algebras satisfying C_2-cofiniteness, *Duke Math. J.* **122** (2004), 51-91.

[T] V. G. Turaev, *Quantum invariants of knots and* 3*-manifolds*, de Gruyter Studies in Math., Vol. 18, Walter de Gruyter, Berlin, 1994.

[V] E. Verlinde, Fusion rules and modular transformations in 2D conformal field theory, *Nucl. Phys.* **B300** (1988), 360–376.

[Z] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–307.

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