Deforming vertex algebras by module and comodule actions of vertex bialgebras

Haisheng Li

Rutgers University-Camden

April 9, 2021

This talk is based on a joint work with Naihuan Jing, Fei Kong, Shaobin Tan

In the theory of vertex (operator) algebras, often we start

from (abstract or concrete) "nice" VOAS,

$\downarrow$

to obtain various important results.

We have been interested in the question:

"How to associate vertex algebras or their likes to algebras such as quantum affine algebras and Yangians? "

A closely related problem is to develop quantum vertex algebra theories and associate quantum vertex algebras to various algebras.

Quantum Vertex Algebra Theories

- The first quantum vertex algebra theory—deformed Chiral algebras was introduced by E. Frenkel and N. Reshetikhin [FR97].
- Then Etingof and Kazhdan [EK00] introduced a theory of quantum vertex operator algebras over $\mathbb{C}[[\hbar]]$, including a notion of braided vertex algebras.
- Borcherds [B01] also introduced a theory of quantum vertex algebras.
- Motivated by Etingof-Kazhdan's theory, we [L05-] introduced notions of (weak) quantum vertex algebra and $\hbar$-adic quantum vertex algebra.
- Anguelova and Bergvelt [AB09] introduced a notion of H_D-quantum vertex algebra, based on the work of Borcherds and the work of Frenkel-Reshetikhin.

An overview (layout):

Vertex algebras over $\mathbb{C}$	$\longrightarrow$	(Weak) Quantum VAs over $\mathbb{C}$
$\downarrow$		$\downarrow$
Quantum VOAs over $\mathbb{C}[[\hbar]]$	$\subset$	$\hbar$-adic quantum vertex algebras

Notes:

- (Weak) quantum vertex algebras over $\mathbb{C}$ are generalizations of vertex algebras and vertex (color-)superalgebras over $\mathbb{C}$.
- EK's quantum vertex operator algebras over $\mathbb{C}[[\hbar]]$ are $\hbar$-deformations of vertex algebras.
- $\hbar$-adic (weak) quantum vertex algebras are $\hbar$-deformations of (weak) quantum vertex algebras over $\mathbb{C}$.
  If V is an $\hbar$-adic (weak) quantum vertex algebra, $V/\hbar^n V$ is a weak quantum vertex algebra over $\mathbb{C}$ for every positive integer n.

Vertex Algebras

A vertex algebra is a vector space equipped with infinitely many multiplicative operations parametrized by all integers,

which are presented by a linear (adjoint representation) map

$$\begin{aligned} Y(\cdot, x): \quad & V \to (\mathrm{End} V)[[x, x^{-1}]] \\ & v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \end{aligned}$$

where $v_n \in \mathrm{End} V$. One technical condition is

$$Y(u, x)v = \sum_{n \in \mathbb{Z}} u_n v \, x^{-n-1} \in V((x)) \quad \text{for } u, v \in V. \tag{1}$$

Main axioms:

Weak commutativity: For $u, v \in V$, there exists $k \geq 0$ such that

$$(x-z)^k Y(u,x)Y(v,z) = (x-z)^k Y(v,z)Y(u,x),$$

and

Weak associativity: For $u, v, w \in V$, there exists $l \geq 0$ such that

$$(x+z)^l Y(u,x+z)Y(v,z)w = (x+z)^l Y(Y(u,z)v,x)w.$$

Remark: Vertex algebras are analogues and generalizations of commutative and associative algebras.

Nonlocal vertex algebras

- Nonlocal vertex algebras are analogues and generalizations of (noncommutative) associative unital algebras.
- The notion of nonlocal vertex algebra is defined by using all the axioms that define the notion of vertex algebra except the weak commutativity.

Note: A nonlocal vertex algebra=a field algebra [Bakalov-Kac2003] =an axiomatic G_1-vertex algebra [L2003].

(Weak) quantum vertex algebras

The notion of weak quantum vertex algebra, which was introduced in [L05], generalizes that of vertex super-algebra.

The main defining properties of a weak quantum vertex algebra are the weak associativity and the S-locality: For any $u, v \in V$, there exist (finitely many)

$$u^{(i)}, \;\; v^{(i)} \in V, \;\; f_i(x) \in \mathbb{C}((x)) \;\; (i = 1, \ldots, r)$$

and a nonnegative integer k such that

$$(x_1 - x_2)^k Y(u, x_1) Y(v, x_2)$$

$$= (x_1 - x_2)^k \sum_{i=1}^{r} f_i(x_2 - x_1) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1).$$

Quantum vertex algebras

The notion of quantum vertex algebra was motivated by Etingof-Kazhdan's notion of quantum vertex operator algebra.

A quantum vertex algebra over $\mathbb{C}$ is a weak quantum vertex algebra V equipped with a "unitary rational quantum Yang-Baxter operator $S(x)$" on V, such that for $u, v \in V$, the S-locality holds with

$$S(x)(v \otimes u) = \sum_i v^{(i)} \otimes u^{(i)} \otimes f_i(x) \in V \otimes V \otimes \mathbb{C}((x)),$$

and such that a hexagon identity (a compatibility condition between S-locality and associativity) holds.

Notes:

- Weak quantum vertex algebras are not too “wild;” very close to vertex algebras in various aspects.

- The category of weak quantum vertex algebras (and their $\hbar$-adic counterparts) is large enough for various purposes; quantum vertex algebras can be associated to a large variety of algebras through modules, quasi modules, or ϕ-coordinated quasi modules.

Why: The Approach We Took:

Recall: Vertex algebras and modules are associated to Lie algebras such as affine Lie algebras and the Virasoro algebra via their highest weight modules.

Vacuum modules (special highest weight modules) have a canonical vertex algebra structure,

while general highest weight modules have a canonical module structure for the corresponding vertex algebra.

This association can be better described via representation:
Use generating functions as building blocks and use weak associativity (=Operator Product Expansion) as a tool, to generate the desired vertex algebras.

Note: In the classical case, composition of operators and associativity are intrinsically connected.

This is the approach we have been taking—- Use generating functions (on highest weight modules) as building blocks.

Let W be any vector space and consider space

$$\mathcal{E}(W) := \mathrm{Hom}(W, W((x))).$$

It was proved that any "compatible" subset of $\mathcal{E}(W)$ generates via the weak associativity a nonlocal vertex algebra;

$$Y_{\mathcal{E}}(a(x), z)b(x) \sim a(x_1)b(x)|_{x_1=x+z},$$

where $Y_{\mathcal{E}}(a(x), z)b(x) := \sum_{n\in\mathbb{Z}} a(x)_n b(x)\, z^{-n-1}$ is the generating function.

However, for certain algebras such as quantum affine algebras, to get a better nonlocal vertex algebra with some commutativity we shall need to replace the usual tool (weak associativity) with a new one.

Recall the weak associativity:

$$Y(u, x_1)Y(v, x_2)|_{x_1=x_2+z} \sim Y(Y(u, z)v, x_2),$$

$$Y_{\mathcal{E}}(a(x), z)b(x) \sim a(x_1)b(x)|_{x_1=x+z}.$$

Note: The 1-dimensional additive formal group (law) is $F(x, y) = x + y$.

We then consider replacing $F(x, y)$ with a general formal series $\phi(x, z) \in \mathbb{C}((x))[[z]]$ and introduce new ways to generate:

$$Y_{\mathcal{E}}^{\phi}(a(x), z)b(x) \sim (a(x_1)b(x))|_{x_1=\phi(x,z)}.$$

Formal series $\phi(x, z)$ is required to satisfy

$$\phi(x, 0) = x, \quad \phi(\phi(x, y), z) = \phi(x, y + z).$$

Indeed, we get nonlocal vertex algebras this way.

Interestingly, all such $\phi(x,z)$ can be obtained by

$$\phi(x,z) = e^{zp(x)\frac{d}{dx}}(x),$$

where $p(x) \in \mathbb{C}((x))$. This happens to have deep connections with previous results in [FLM], [FHL], [Huang], [Lepowsky].

Two particular cases are:
1) $p(x) = 1$, $\phi(x,z) = e^{z\frac{d}{dx}}(x) = x + z$ (the formal group itself).

2) $p(x) = x$, $\phi(x,z) = e^{zx\frac{d}{dx}}(x) = xe^{z}$.

On the other hand, for any nonlocal vertex algebra V, correspondingly we introduce a notion of ϕ-coordinated V-module by using weak ϕ-associativity:

$$Y(u,x_1)Y(v,x_2)|_{x_1=\phi(x_2,z)} \sim Y(Y(u,z)v,x_2).$$

- $\hbar$-adic quantum vertex algebras have been successfully associated to the double Yangian of sl_2 at generic level.
- Weak quantum vertex algebras have been associated to quantum affine algebras conceptually.

It remains to construct the desired quantum vertex algebras explicitly.

In the theory of Hopf algebras including quantum groups, smash product, Heisenberg double, Drinfeld double, and Drinfeld twist, are effective tools to construct important (quasi) Hopf algebras.

One of our main goals is to establish vertex-algebra analogs of these well known constructions and then to associate quantum vertex algebras to various algebras, especially quantum affine algebras and double Yangians.

Classical bialgebras

A coalgebra is a vector space C with two linear maps

$$\Delta : A \to A \otimes A \quad \text{(comultiplication)};$$
$$\varepsilon : A \to \mathbb{C} \quad \text{(counit)}$$

such that

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad \text{(coassociativity)};$$
$$(\varepsilon \otimes 1)\Delta = 1, \ \ (1 \otimes \varepsilon)\Delta = 1 \quad \text{(counity)}.$$

A bialgebra is an associative unital algebra A which at the same time is also equipped with a coalgebra structure (Δ, ε) such that Δ and ε are homomorphisms of associative algebras.

Typical Examples: 1. Group algebras $\mathbb{C}[G]$, where

$$\Delta(g) = g \otimes g; \quad \varepsilon(g) = 1 \quad \text{for } g \in G.$$

2. Universal enveloping algebras $U(\mathfrak{g})$, where

$$\Delta(a) = a \otimes 1 + 1 \otimes a; \quad \varepsilon(a) = 0 \quad \text{for } a \in \mathfrak{g}.$$

3. Tensor products $U(\mathfrak{g}) \otimes \mathbb{C}[G]$.

Differential bialgebras and vertex bialgebras

By a differential algebra we mean an associative algebra with 1 together with a derivation.

A differential bialgebra is a bialgebra H together with a derivation ∂ such that

$$\Delta \circ \partial = (\partial \otimes 1 + 1 \otimes \partial)\Delta, \quad \varepsilon \circ \partial = 0.$$

That is, Δ and ε are homomorphisms of differential algebras.

Examples 1. Let $\mathfrak{g}$ be a Lie algebra. Consider Lie algebra $t^{-1}\mathfrak{g}[t^{-1}]$ (a subalgebra of the loop Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$). The universal enveloping algebra $U(t^{-1}\mathfrak{g}[t^{-1}])$ is naturally a differential bialgebra with $\partial = 1\otimes d/dt$.

2. Let L be an abelian group. Denote by $\mathbb{C}[L]$ the group algebra. Set $\mathbf{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$, an abelian Lie algebra. We have a bialgebra

$$B_L = S(t^{-1}\mathbf{h}[t^{-1}])\otimes\mathbb{C}[L],$$

which is also a differential bialgebra with ∂ determined by

$$\partial(e^{\alpha}) = \bar{\alpha}(-1)\otimes e^{\alpha}, \quad \partial(\bar{\alpha}(-n)) = n\bar{\alpha}(-n-1)$$

for $\alpha\in L,\ n\geq 1$, where $\bar{\alpha} = 1\otimes\alpha\in\mathbf{h}$.

Let A be a a commutative associative algebra with 1 together with a derivation ∂.

For $a, b \in A$, define

$$Y(a, x)b = (e^{x\partial}a)b := \sum_{n \geq 0} \frac{1}{n!}(\partial^n a)bx^n \in A[[x]].$$

Then A together with the linear map $Y(\cdot, x)$ and vector 1 is a vertex algebra. This construction is known as the Borcherds construction.

Fact: The vertex algebras obtained by Borcherds' construction also exhaust all commutative vertex algebras, which are also characterized by the property

$$Y(u, x)v \in V[[x]] \quad \text{for al } u, v \in V.$$

A nonlocal vertex bialgebra is a nonlocal vertex algebra V equipped with a classical coalgebra structure (Δ, ε) such that Δ and ε are both homomorphisms of nonlocal vertex algebras.

Fact: If A is a differential bialgebra, then (A, d) is naturally a nonlocal vertex bialgebra.

Then $U(t^{-1}\mathfrak{g}[t^{-1}])$ and B_L are naturally nonlocal vertex bialgebras.

Let B be a nonlocal vertex bialgebra. A B-module nonlocal vertex algebra is a nonlocal vertex algebra V equipped with a B-module structure $Y_V^B(\cdot, x)$ on V such that

$$Y_V^B(b,x)v \in V \otimes \mathbb{C}((x)),$$
$$Y_V^B(b,x)\mathbf{1}_V = \varepsilon(b)\mathbf{1}_V,$$
$$Y_V^B(b,x)Y(u,z)v = Y(Y_V^B(b_{(1)}, x-z)u, z)Y_V^B(b_{(2)}, x)v$$

for $b \in B,\ u, v \in V$.

Theorem (L)

Let B be a vertex bialgebra and let V be a B-module nonlocal vertex algebra. We have a smash product vertex algebra $V\sharp B$, where $V\sharp B = V \otimes B$ as a vector space and

$$Y_\sharp(u\sharp b, x)(v\sharp c) = Y(u,x)Y_V^B(b_{(1)}, x)v\sharp Y(b_{(2)}, x)c$$

for $u, v \in V,\ b, c \in B$.

Lattice vertex algebras V_L

Let L be a non-degenerate even lattice (a free abelian group with $\mathbb{Z}$-valued bilinear form).

Let $\epsilon : L \times L \to \mathbb{C}^{\times}$ be a 2-cocycle such that

$$\epsilon(\alpha, 0) = 1 = \epsilon(0, \alpha) \quad \text{for } \alpha \in L.$$

Denote by $\mathbb{C}_{\epsilon}[L]$ the ϵ-twisted group algebra of L, which by definition has a basis $\{e_{\alpha} \mid \alpha \in L\}$ with

$$e_{\alpha} \cdot e_{\beta} = \epsilon(\alpha, \beta) e_{\alpha+\beta} \quad \text{for } \alpha, \beta \in L.$$

Set

$$B_{L,\epsilon} = S(t^{-1}\mathbf{h}[t^{-1}]) \otimes \mathbb{C}_{\epsilon}[L],$$

which is a differential algebra, and hence a nonlocal vertex algebra.

Recall $\mathbf{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ with the natural symmetric bilinear form $\langle \cdot, \cdot \rangle$. Then we have a (Heisenberg) Lie algebra $\widehat{\mathbf{h}} = \mathbf{h} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c$. There is an irreducible $\widehat{\mathbf{h}}$-module structure on $S(t^{-1}\mathbf{h}[t^{-1}])$ on which c acts as identity.

Set

$$V_L = S(t^{-1}\mathbf{h}[t^{-1}]) \otimes \mathbb{C}_\epsilon[L] \quad (= B_{L,\epsilon}),$$

an $\widehat{\mathbf{h}}$-module. For $\alpha \in L$, define $x^{\alpha(0)} : V_L \to V_L[x, x^{-1}]$ by

$$x^{\alpha(0)}(u \otimes e_\beta) = (u \otimes e_\beta) x^{\langle \alpha, \beta \rangle}$$

for $u \in S(t^{-1}\mathbf{h}[t^{-1}])$, $\beta \in L$.

Associated to non-degenerate even lattice L, we have a vertex algebra V_L such that for $\alpha \in L$,

$$Y(e_\alpha, x) = E^-(-\alpha, x) e_\alpha E^+(-\alpha, x) x^{\alpha(0)},$$

where

$$E^\pm(-\alpha, x) = \exp\left(-\sum_{n \geq 1} \frac{\alpha(\pm n)}{\pm n} x^{\mp n} \right).$$

Recall that $B_L = S(t^{-1}\mathbf{h}[t^{-1}]) \otimes \mathbb{C}[L]$ is a vertex bialgebra and $B_{L,\epsilon} = S(t^{-1}\mathbf{h}[t^{-1}]) \otimes \mathbb{C}_\epsilon[L]$ is a differential algebra, hence a nonlocal vertex algebra.

Theorem (L)

For $B_{L,\epsilon}$ viewed as a nonlocal vertex algebra, we have

$$Y_{B_{L,\epsilon}}(e_\alpha, x) = E^-(-\alpha, x)e_\alpha \quad \text{for } \alpha \in L.$$

On $B_{L,\epsilon}$, there is a B_L-module structure, which is uniquely determined by

$$Y_{B_{L,\epsilon}}^{B_L}(e^\alpha, x) = E^+(-\alpha, x)x^{\alpha(0)} \quad \text{for } \alpha \in L.$$

Furthermore, $B_{L,\epsilon}$ is a B_L-module nonlocal vertex algebra.

Recall the comultiplication: $\Delta : B_L \to B_L \otimes B_L$, a homomorphism of differential algebras.

With $B_{L,\epsilon} = B_L$ as a vector space, we have a linear map

$$\Delta_\epsilon : \ B_L \to B_{L,\epsilon} \otimes B_L.$$

Theorem (L)

The image of Δ_ϵ is a vertex subalgebra of $B_{L,\epsilon} \sharp B_L$ and this gives an isomorphism of the lattice vertex algebra V_L onto $\Delta_\epsilon(B_L)$.

Right H-comodule nonlocal vertex algebras

The following is an analogue of the notion of right comodule algebra in the theory of Hopf algebras.

Definition (Jing-Kong-L-Tan)

Let H be a nonlocal vertex bialgebra. A right H-comodule nonlocal vertex algebra is a nonlocal vertex algebra V equipped with a homomorphism $\rho : V \to V \otimes H$ of nonlocal vertex algebras such that

$$(\rho \otimes 1)\rho = (1 \otimes \Delta)\rho, \quad (1 \otimes \epsilon)\rho = \mathrm{Id}_V, \tag{2}$$

i.e., ρ is also a right comodule structure for H viewed as a coalgebra.

Compatibility:

Let H be a nonlocal vertex bialgebra and let V be a nonlocal vertex algebra.

Suppose that (V, Y_V^H) is an H-module nonlocal vertex algebra with the module vertex operator map

$$Y_V^H(\cdot, x): \; H \to \mathrm{Hom}(V, V \otimes \mathbb{C}((x)))$$

and (V, ρ) is a right H-comodule nonlocal vertex algebra with the comodule map $\rho: V \to V \otimes H$.

We say Y_V^H and ρ are compatible if ρ is an H-module homomorphism with $V \otimes H$ viewed as an H-module on which H acts on the first factor, i.e.,

$$\rho(Y_V^H(h,x)v) = (Y_V^H(h,x) \otimes 1)\rho(v) \quad \text{for } h \in H, \; v \in V. \tag{3}$$

We have:

Theorem (Jing-Kong-L-Tan)

Let H be a cocommutative nonlocal vertex bialgebra. Suppose that (V, Y_V^H) is an H-module nonlocal vertex algebra and (V, ρ) is a right H-comodule nonlocal vertex algebra such that Y_V^H and ρ are compatible. For $a \in V$, set

$$\mathfrak{D}^{\rho}_{Y_V^H}(Y)(a, x) = \sum Y(a_{(1)}, x) Y_V^H(a_{(2)}, x) \tag{4}$$

on V, where $\rho(a) = \sum a_{(1)} \otimes a_{(2)} \in V \otimes H$. Then $(V, \mathfrak{D}^{\rho}_{Y_V^H}(Y), \mathbf{1})$ carries the structure of a nonlocal vertex algebra. Denote this nonlocal vertex algebra by $\mathfrak{D}^{\rho}_{Y_V^H}(V)$. Furthermore, ρ is a nonlocal vertex algebra homomorphism from $\mathfrak{D}^{\rho}_{Y_V^H}(V)$ to $V \sharp H$.

Furthermore, we have:

Theorem (Jing-Kong-L-Tan)

Assume that H is a commutative and cocommutative vertex bialgebra. Let V be a vertex algebra with a right H-comodule vertex algebra structure ρ. Let (V, Y_M) be a compatible H-module vertex algebra structure with Y_M "invertible." Define a linear map $S(x) : V \otimes V \to V \otimes V \otimes \mathbb{C}((x))$ by

$$S(x)(v \otimes u) = \sum Y_M(u_{(2)}, -x)v_{(1)} \otimes Y_M^{-1}(v_{(2)}, x)u_{(1)} \tag{5}$$

for $u, v \in V$. Then $S(x)$ is a unitary rational quantum Yang-Baxter operator and the nonlocal vertex algebra $\mathfrak{D}^{\rho}_{Y_M}(V)$ with $S(x)$ is a quantum vertex algebra.

Examples

Let L be a non-degenerate even lattice. Recall that we have a commutative and cocommutative vertex bialgebra B_L and a vertex algebra V_L.

Theorem (Jing-Kong-L-Tan)

There exists a right B_L-comodule vertex algebra structure ρ on vertex algebra V_L, which is uniquely determined by

$$\rho(h) = h \otimes 1 + 1 \otimes h \quad \text{for } h \in \mathfrak{h},$$
$$\rho(e_\alpha) = e_\alpha \otimes e^\alpha \quad \text{for } \alpha \in L.$$

For $a \in \mathfrak{h},\ f(x) \in \mathbb{C}((x))$, define

$$\Phi(a, f(x)) = \sum_{n \geq 0} \frac{(-1)^n}{n!} f^{(n)}(x) a_n$$

on V_L. Using linearity we define $\Phi(G(x))$ for $G(x) \in \mathfrak{h} \otimes \mathbb{C}((x))$.

Theorem (Jing-Kong-L-Tan)

Let $f : \mathfrak{h} \to \mathfrak{h} \otimes x\mathbb{C}[[x]]$ be any linear map. Then there exists a B_L-module structure $Y_M^f(\cdot, x)$ on V_L, uniquely determined by

$$Y_M^f(e^\alpha, x) = \exp(\Phi(f(\alpha, x)))$$

for $\alpha \in L$, and V_L becomes a B_L-module vertex algebra. Furthermore, $Y_M^f(\cdot, x)$ is compatible with ρ and Y_M^f is invertible.

As an application, we have:

Theorem (Jing-Kong-L-Tan)

Let $f : \mathfrak{h} \to \mathfrak{h} \otimes x\mathbb{C}[[x]]$ be any linear map. Then there exists a quantum vertex algebra structure on V_L, whose vertex operator map, denoted by Y^f, is uniquely determined by

$$Y^f(e_\alpha, x) = Y(e_\alpha, x)\exp(\Phi(f(\alpha, x))),$$
$$Y^f(h, x) = Y(h, x) + \Phi(f'(h, x))$$

for $\alpha \in L,\ h \in \mathfrak{h}$. Denote this quantum vertex algebra by V_L^f. Then the linear map $\rho : V_L \to V_L \otimes B_L$ is an embedding of quantum vertex algebra V_L^f into $V_L \sharp B_L$.

Final Remark: Currently, we use an $\hbar$-adic version to associate $\hbar$-adic quantum vertex algebras to quantum affine algebras.

THANK YOU!