Associative algebras and the representation theory of grading-restricted vertex algebras

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Outline

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1. The associative algebra $A^{\infty}(V)$ and modules
2. Subalgebras of $A^{\infty}(V)$
3. Lower-bounded generalized V-modules and graded $A^{\infty}(V)$-modules
4. Lower-bounded generalized V-modules and graded $A^{N}(V)$-modules

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Motivation

- In the representation theory of Lie algebras, the universal enveloping algebra of a Lie algebra plays a crucial role because the module categories for a Lie algebra and for its universal enveloping algebra are isomorphic.
- For a vertex operator algebra, there is also a universal enveloping algebra introduced by Frenkel and Zhu such that the weakest possible module categories for these algebras are isomorphic.
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- But even weak modules for a vertex operator algebra do not involve such infinite sums since the vertex operators on these modules are lower truncated when acting on elements of these modules.
- Weak modules are useful in formulating some notions and results but are not what we are interested in the representation theory of vertex operator algebras.
- The modules that we are interested are lower-bounded and grading-restricted generalized modules.
- In the representation theory of vertex operator algebras, we have the Zhu algebra introduced by Zhu and it's generalizations by Dong, Li and Mason for a vertex operator algebra.

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- These algebras can be used to classify irreducible modules for the vertex operator algebra.
- But the role of these associative algebras played in the representation theory of vertex operator algebras is quite limited.
- For example, the module for one of these associative algebras obtained from a suitable module for the vertex operator algebra in general do not tell us whether the original module is irreducible or completely reducible.
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The setting

- We fix a grading-restricted vertex algebra V, that is, a vertex algebra with a compatible $\mathbb{Z}$-grading satisfying the two grading restriction conditions ($\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$ and $V_{(n)} = 0$ for n sufficiently negative).
- Certainly all the results for a grading-restricted vertex algebra also hold for a vertex operator algebra.
- We are interested lower-bounded generalized V-modules, that is, weak V-modules W with a compatible lower-bounded $\mathbb{C}$-grading ($W_{[n]} = 0$ for $\Re(n)$ sufficiently negative) and operators $L(0)$ and $L(-1)$ of weight 0 and 1, respectively, such that the $\mathbb{C}$-grading is given by the semisimple part of $L(0)$ and $L(0)$ and $L(-1)$ satisfy the usual properties.

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Column-finite infinite matrices with entries in V

- Let $U^{\infty}(V)$ be the space of column-finite infinite matrices with entries in V, but doubly indexed by $\mathbb{N}$ instead of $\mathbb{Z}_{+}$.
- Elements of $U^{\infty}(V)$ are of the form $\mathfrak{v} = [v_{kl}]$ for $v_{kl} \in V$, $k, l \in \mathbb{N}$ such that for each fixed $l \in \mathbb{N}$, there are only finitely many nonzero v_{kl}.
- For fixed $k, l \in \mathbb{N}$ and $v \in V$, let $[v]_{kl}$ be the infinite matrix with the (k, l)-th entry to be v and all the other entries to be 0.
- Elements of $U^{\infty}(\mathbb{C})$ can all be written as $\sum_{k,l\in\mathbb{N}}[v_{kl}]_{kl}$ for $v_{kl} \in V$, $k, l \in \mathbb{N}$.
- We can study $U^{\infty}(V)$ using elements of the form $[v]_{kl}$ for $k, l \in \mathbb{N}$ and $v \in V$.

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A product on $U^{\infty}(V)$

- We define a product $\diamond$ on $U^{\infty}(V)$ by

$$[u]_{kn} \diamond [v]_{nl} = \mathrm{Res}_x T_{k+l+1}((x+1)^{-k+n-l-1}) \cdot (1+x)^{l} \left[Y_V((1+x)^{L_V(0)}u, x)v \right]_{kl}$$

for $u, v \in V$ and $k, l \in \mathbb{N}$, where

$$T_{k+l+1}((x+1)^{-k+n-l-1}) = \sum_{m=0}^{n} \binom{-k+n-l-1}{m} x^{-k+n-l-m-1}.$$

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A subspace $O^{\infty}(V)$ of $U^{\infty}(V)$

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$$\mathrm{Res}_x x^{-k-l-p-2}(1+x)^l[Y_V((1+x)^{L_V(0)}u, x)v]_{kl}$$

for $u, v \in V$, $k, l, p \in \mathbb{N}$ and elements of the form

$$[(L_V(-1) + L_V(0) + l - k)v]_{kl}$$

for $v \in V$ and $k, l \in \mathbb{N}$, with each pair (k, l) appearing in the linear combinations only finitely many times.
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An increasing filtration of a lower-bounded generalized V-module

- We want to find an ideal $Q^{\infty}(V)$ of $U^{\infty}(V)$ containing $O^{\infty}(V)$ such that $U^{\infty}(V)/Q^{\infty}(V)$ is an associative algebra and for each lower-bounded generalized V-module, we can construct a $U^{\infty}(V)/Q^{\infty}(V)$-module.
- Let W be a lower-bounded generalized V-module.
- For $n \in \mathbb{N}$, let $\Omega_n(W)$ be the subspace of W consisting of $w \in W$ such that $(Y_W)_k(v)w = 0$ for homogeneous $v \in V$, $\mathrm{wt}\ v - k - 1 < -n$.
- We have $\Omega_{n_1}(W) \subset \Omega_{n_2}(W)$ for $n_1 \leq n_2$ and $W = \bigcup_{n\in\mathbb{N}} \Omega_n(W)$.
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An action of $U^{\infty}(V)$ on the associated graded space of the filtration of W

- Let $Gr(W) = \sum_{n\in\mathbb{N}} Gr_n(W) = \sum_{n\in\mathbb{N}} \Omega_n(W)/\Omega_{n-1}(W)$ be the associated graded space of the filtration $\{\Omega_n(w)\}_{n\in\mathbb{N}}$.
- For $n \in \mathbb{N}$ and $w \in \Omega_n(W)$, we sometimes use $[w]_n$ to denote the element $w + \Omega_{n-1}(W)$ of $Gr_n(W)$.
- We define a linear map $\vartheta_{Gr(W)} : U^{\infty}(V) \to \mathrm{End}\, Gr(W)$ by

$$\vartheta_{Gr(W)}([v]_{kl})[w]_n = \delta_{ln}[\mathrm{Res}_x x^{l-k-1} Y_W(x^{L_V(0)}v, x)w]_k.$$

  for $k, l, n \in \mathbb{N}$, $v \in V$ and $w \in \Omega_n(W)$.
- When v is homogeneous and $w \in Gr_l(W)$, we have

$$\vartheta_{Gr(W)}([v]_{kl})[w]_l = [(Y_W)_{\mathrm{wt}\ v+l-k-1}(v)w]_k.$$

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An action of $U^{\infty}(V)$ on the associated graded space of the filtration of W

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The definition of $Q^{\infty}(V)$ and $A^{\infty}(V)$

Theorem

Let W be a lower-bounded generalized V-module. Then the linear map

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gives a $U^{\infty}(V)$-module structure on $Gr(W)$ (that is, $\vartheta_{Gr(W)}$ is a homomorphism of (nonassociative) algebras from $U^{\infty}(V)$ to $\mathrm{End}\, Gr(W)$). In particular, $U^{\infty}(V)/\ker \vartheta_{Gr(W)}$ is an associative algebra isomorphic to a subalgebra of $\mathrm{End}\, Gr(W)$.

- Let $Q^{\infty}(V)$ be the intersection of $\ker \vartheta_{Gr(W)}$ for all lower-bounded generalized V-modules W.
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The product $\diamond$ on $U^{\infty}(V)$ induces a product, denoted still by $\diamond$, on $A^{\infty}(V) = U^{\infty}(V)/Q^{\infty}(V)$ such that $A^{\infty}(V)$ equipped with $\diamond$ is an associative algebra with $\mathbf{1}^{\infty} + Q^{\infty}(V)$ as an identity. Moreover, the associated graded space $Gr(W)$ of the increasing filtration $\{\Omega_n(W)\}_{n\in\mathbb{N}}$ of a lower-bounded generalized V-module W is an $A^{\infty}(V)$-module.

- The $A^{\infty}(V)$-module $Gr(W)$ has an $\mathbb{N}$-grading and two operators $L_W(0)$ and $L_W(-1)$ satisfying natural conditions.
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Outline

Zhu algebra

- Let
$$U_{00}(V) = \{[v]_{00} \mid v \in V\} \subset U^{\infty}(V).$$
- $U_{00}(V)$ can be canonically identified with V through the map $i_{00} : U_{00}(V) \to V$ given by $i_{00}([v]_{00}) = v$ for $v \in V$.
- Let $A_{00}(V) = \{[v]_{00} + Q^{\infty}(V) \mid v \in V\}$.

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The subspace $A_{00}(V)$ of $A^{\infty}(V)$ is closed under $\diamond$ and is thus a subalgebra of $A^{\infty}(V)$ with $[\mathbf{1}]_{00} + Q^{\infty}(V)$ as its identity. The associative algebra $A_{00}(V)$ is isomorphic to the Zhu algebra $A(V)$ and, in particular, $[\omega]_{00} + Q^{\infty}(V)$ is in the center of $A_{00}(V)$ if V is a vertex operator algebra with the conformal vector ω.

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New associative algebras from finite matrices

- For $N \in \mathbb{N}$, let $U^N(V)$ be the space of all $(N+1) \times (N+1)$ matrices with entries in V.
- We view $U^N(V)$ as a subspace of $U^\infty(V)$.
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Graded $A^N(V)$-modules and finiteness of $A^N(V)$

- Let $Gr^N(W) = \coprod_{n=0}^{N} Gr_n(W) \subset Gr(W)$.
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Assume that V is of positive energy (CFT type) and C_2-cofinite. Then for $N \in \mathbb{N}$, $A^N(V)$ is finite dimensional.

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Assume that V is of positive energy (CFT type) and C_2-cofinite. Then for $N \in \mathbb{N}$, $A^N(V)$ is finite dimensional.

Graded $A^N(V)$-modules and finiteness of $A^N(V)$

- Let $Gr^N(W) = \coprod_{n=0}^{N} Gr_n(W) \subset Gr(W)$.
- Then $Gr^N(W)$ is an $A^N(W)$-module.
- Moreover, $Gr^N(W)$ has a grading and additional operators $L_W(0)$ and $L_W(-1)$ satisdying natural conditions.
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Irreducible and complete reducible lower-bounded generalized W-modules

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A lower-bounded generalized V-module W is irreducible or completely reducible if and only if the graded $A^{\infty}(V)$-module $Gr(W)$ is irreducible or completely reducible, respectively.

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The set of the equivalence classes of irreducible lower-bounded generalized V-modules is in bijection with the set of the equivalence classes of irreducible graded $A^{\infty}(V)$-modules.

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Irreducible and complete reducible lower-bounded generalized W-modules of finite lengths

- A *Möbius vertex algebra* is a grading-restricted vertex algebra with an operator $L(1)$ such that the usual conditions for the vertex operators, $L(0)$ and $L(-1)$ and $L(1)$ hold.

Theorem

Let V be a Möbius vertex algebra. Assume that the differences between the real parts of the lowest weights of the irreducible lower-bounded generalized V-modules are all less than or equal to $N \in \mathbb{N}$. Then a lower-bounded generalized V-module W of finite length is irreducible or completely reducbible if and only if the graded $A^N(V)$-module $Gr^N(W)$ is irreducible or completely reducible, respectively.

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Lower-bounded generalized W-modules as direct sums of irreduicble ordinary V-modules

Theorem

Let V be a Möbius vertex algebra. Assume that $A_{N'}(V)$ for all $N' \in \mathbb{N}$ are finite dimensional (for example, when V is C_2-cofinite and of positive energy). Let $N \in \mathbb{N}$ such that the differences between the real parts of the lowest weights of the finitely many (inequivalent) irreducible ordinary V-modules are less than or equal to N. Then a lower-bounded generalized V-module W of finite length or a grading-restricted generalized V-module W is a direct sum of irreducible ordinary V-modules if and only if the graded $A^N(V)$-module $Gr^N(W)$ is completely reducible.

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