Natural Construction of Ten Borcherds-Kac-Moody Algebras Associated with Elements in M_{23}

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Section 1

Introduction and Motivation

Borchers-Kac-Moody Algebras

BKMAs are infinite-dimensional generalisations of finite-dimensional simple Lie algebras.

They are defined by generators and relations encoded in a generalised Cartan matrix (not necessarily positive definite, diagonal entries not necessarily positive to allow imaginary simple roots) [Bor88].

They admit Weyl-Kac character formulae and a denominator identity

$$e^{\rho}\prod_{\alpha\in\Phi^+}(1-e^{\alpha})^{\mathrm{mult}(\alpha)}=\sum_{w\in W}\det(w)w\left(e^{\rho}\sum_{\alpha\in\Phi}\varepsilon(\alpha)e^{\alpha}\right).$$

The best-known example is the monster Lie algebra, used in Borcherds' proof of the monstrous moonshine conjecture [Bor92].

General Problem

In many interesting examples the denominator identity of a BKMA is an automorphic form (automorphic product of singular weight) [Bor98].

Classification results for such BKMAs are obtained in [Sch06, GN02, GN18].

Problem A (Borcherds)

Give natural constructions of those BKMAs whose denominator identities are automorphic products of singular weight [Bor01].

Here, "natural" means other than by generators and relations, and in a way that hopefully reveals part of the symmetry group of these algebras.

General Problem

The fake monster Lie algebra $\mathfrak{g}$ [Bor90] is the BKMA obtained naturally as string quantisation of the lattice vertex algebra $V_{II_{25,1}}$.

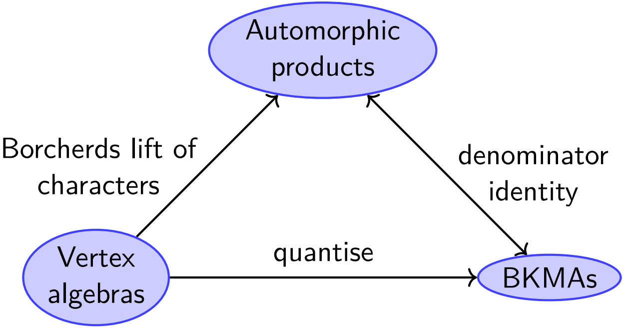
In [Bor92] Borcherds obtained a large family of Borcherds-Kac-Moody (super)algebras $\mathfrak{g}_{\phi_\nu}$ by twisting the fake monster Lie algebra $\mathfrak{g}$ by elements $\nu \in \mathrm{Co}_0$.

Problem B (Borcherds)

Find natural constructions for this family of BKMAs [Bor92].

Goal of this talk: Give partial answer to both problems by giving BRST constructions of a nice subfamily of BKMAs associated with elements of square-free order in $M_{23} < \mathrm{Co}_0$.

Concepts



Examples:

- $V^\natural \otimes V_{II_{1,1}} \quad \to \quad$ monster Lie algebra $\quad \leftrightarrow \quad j(e^{(1,0)}) - j(e^{(0,1)})$,
- $V_{II_{25,1}} \quad \to \quad$ fake monster Lie algebra $\mathfrak{g} \quad \leftrightarrow \quad$ automorphic product Ψ of weight 12 for $O^+(II_{26,2})$.

Specific Problem

Theorem (Classification [Sch06])

There are exactly ten real BKMAs whose whose denominator identities are completely reflective automorphic products of singular weight $-w = k/2 - 1$ *on even lattices* P *of signature* $(k, 2)$ *with* $k \geq 4$*, square-free level* m *and p-ranks of the discriminant form* P'/P *at most* $k + 1$*.*

Moreover, these are exactly the ten real BKMAs $\mathfrak{g}_{\phi_\nu}$ *in [Bor92] obtained by twisting the fake monster Algebra* $\mathfrak{g}$ *by elements* ν *of square-free order* m *in* $M_{23} < \mathrm{Co}_0$*.*

Specific Problem

Give natural constructions (as BRST quantisations from suitable vertex algebras) of these ten BKMAs.

(Done for $m = 1$ [Bor90], $m = 2$ [HS03], $m = 2, 3, 5, 7$ [CKS07].)

Section 2

Vertex Algebras

Vertex Algebras

(Conformal) Vertex Algebras

- $\mathbb{C}$-vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$,
- vacuum vector $\mathbf{1} \in V_0$, Virasoro vector $\omega \in V_2$,
- algebra products $V \otimes V \to V$, $(a, b) \mapsto a_n b$ for $n \in \mathbb{Z}$,
- satisfying generalised associativity and commutativity constraints.

Vertex (super)algebras and their representations give rigorous descriptions of 2-dimensional conformal field theories.

The Moonshine module $V^\natural$ [FLM88] provides the only *natural* construction of the Monster group $M \cong \mathrm{Aut}(V^\natural)$.

The character of a vertex algebra is often a modular form [Zhu96].

Matter Sector

In the following we describe the vertex algebras M_{ϕ_ν} that will serve as the input of the BRST quantisation.

Let V_Λ be the strongly rational, holomorphic VOA of central charge 24 associated with the Leech lattice Λ. Consider the ten automorphisms $\nu \in M_{23} < \mathrm{Co}_0 = \mathrm{O}(\Lambda)$ of square-free order, i.e. their standard lifts $\phi_\nu \in \mathrm{Aut}(V_\Lambda)$.

By the orbifold theory in [EMS20] $V_\Lambda^{\phi_\nu}$ has m^2 irreducible modules $V_\Lambda^{\phi_\nu}(i,j)$ for i,j in the finite quadratic space $\mathbb{Z}_m \times \mathbb{Z}_m$.

Definition (Conformal Vertex Algebra of Central Charge 26)

$$M_{\phi_\nu} := \bigoplus_{\alpha+K\in K'/K} V_\Lambda^{\phi_\nu}(\varphi(\alpha+K)) \otimes V_{\alpha+K}$$

with isometry $\varphi : K'/K \to \overline{\mathbb{Z}_m \times \mathbb{Z}_m}$ and lattice $K = I\!I_{1,1}(m)$.

Matter Sector

There is a well-known lattice decomposition that induces

$$V_\Lambda \cong \bigoplus_{\alpha+\Lambda^\nu\in(\Lambda^\nu)'/\Lambda^\nu} V_{\psi(\alpha+\Lambda^\nu)} \otimes V_{\alpha+\Lambda^\nu}$$

with isometry $\psi\colon (\Lambda^\nu)'/\Lambda^\nu \to \overline{(\Lambda_\nu)'/\Lambda_\nu}$.

Let $\hat{\nu} \in \operatorname{Aut}(V_{\Lambda_\nu})$ be a standard lift of ν restricted to Λ_ν. By the orbifold theory in [Lam20] $V_{\Lambda_\nu}^{\hat{\nu}}$ has $m^2|(\Lambda_\nu)'/\Lambda_\nu|$ irreducible modules $V_{\Lambda_\nu}^{\hat{\nu}}(\alpha+\Lambda_\nu,i,j)$ with fusion described by the finite quadratic space $(\Lambda_\nu)'/\Lambda_\nu \times \mathbb{Z}_m \times \mathbb{Z}_m$.

More generally, the decomposition

$$V_\Lambda^{\phi_\nu}(i,j) \cong \bigoplus_{\alpha+\Lambda^\nu\in(\Lambda^\nu)'/\Lambda^\nu} V_{\Lambda_\nu}^{\hat{\nu}}(\psi(\alpha+\Lambda^\nu),i,j) \otimes V_{\alpha+\Lambda^\nu}$$

holds for all $i,j \in \mathbb{Z}_m$ [Lam20].

Matter Sector

Define the lattice $L := \Lambda^\nu \oplus K$ of signature $(k-1,1)$ and the isometry $\chi := (\psi, \varphi) \colon L'/L \to \overline{(\Lambda_\nu)'/\Lambda_\nu \times \mathbb{Z}_m \times \mathbb{Z}_m}$.

Proposition ([Möl21])

The conformal vertex algebra M_{ϕ_ν} decomposes as

$$M_{\phi_\nu} \cong \bigoplus_{\gamma+L \in L'/L} V_{\Lambda_\nu}^{\hat{\nu}}(\chi(\gamma+L)) \otimes V_{\gamma+L}.$$

This implies that M_{ϕ_ν} has the L'-grading

$$M_{\phi_\nu} = \bigoplus_{\alpha \in L'} V_{\Lambda_\nu}^{\hat{\nu}}(\chi(\alpha+L)) \otimes \pi_\alpha^{(k-1,1)}$$

with Heisenberg modules $\pi_\alpha^{(k-1,1)}$, rather than just a grading by the lattice K' of signature $(1,1)$.

Characters

By [Möl16] the characters

$$\mathrm{ch}_{V^{\hat{\nu}}_{\Lambda_\nu}(\alpha+\Lambda_\nu,i,j)}(\tau)/\eta(\tau)^{\mathrm{rk}(\Lambda^\nu)}$$

form a vector-valued modular form of weight $w = 1 - k/2$ for the dual Weil representation of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{C}[L'/L]$.

There is a procedure (see [Sch06]) to lift modular forms for congruence subgroups to vector-valued modular forms for $\mathrm{SL}_2(\mathbb{Z})$. Let $F(\tau)$ be the lift of $1/\eta_\nu(\tau) = \prod_{t|m} \eta(t\tau)^{-24/\sigma_1(m)}$, a modular form for $\Gamma_0(m)$.

Proposition ([Möl21])

$$\mathrm{ch}_{V^{\hat{\nu}}_{\Lambda_\nu}(\alpha+\Lambda_\nu,i,j)}(\tau)/\eta(\tau)^{\mathrm{rk}(\Lambda^\nu)} = F(\tau).$$

Section 3

Quantisation

BRST Quantisation

Consider the semi-infinite cohomology of graded Lie algebras [Fei84, FGZ86] applied to the Virasoro algebra.

For "positive-energy" Virasoro representation M of central charge 26 define $W = M \otimes V_{\mathrm{gh.}}$ of central charge 0 and BRST operator Q with $Q^2 = 0$. Obtain cohomological spaces $H^p_{\mathrm{BRST}}(M)$.

If M is conformal vertex algebra, then $H^1_{\mathrm{BRST}}(M)$ is a Lie algebra [LZ93] (also inherits invariant bilinear form).

If $M = V \otimes \pi_\alpha^{(k-1,1)}$ (+ some conditions), vanishing theorem [Zuc89] and Euler-Poincaré principle yield:

$$H^1_{\mathrm{BRST}}(V \otimes \pi_\alpha^{(k-1,1)}) \cong \begin{cases} (V \otimes \pi_0^{(k-2,0)})_{1-\langle\alpha,\alpha\rangle/2} & \text{if } \alpha \neq 0, \\ (V \otimes \pi_0^{(k-1,1)})_1 & \text{if } \alpha = 0. \end{cases}$$

Results

Define the L'-graded Lie algebra

$$\mathfrak{g}^{\phi_\nu} := H^1_{\mathrm{BRST}}(M_{\phi_\nu}) = \bigoplus_{\alpha\in L'} H^1_{\mathrm{BRST}}(V^{\hat{\nu}}_{\Lambda_\nu}(\chi(\alpha+L)) \otimes \pi^{(k-1,1)}_\alpha).$$

With the results on the previous slide we show:

Theorem ([Möl21])

$\mathfrak{g}^{\phi_\nu}$ is a complex BKMA of rank $k = \mathrm{rk}(L)$, graded by L', with Cartan subalgebra $\mathfrak{g}(0)$.

$\mathfrak{g}^{\phi_\nu}$ is isomorphic to (the complexification of) $\mathfrak{g}_{\phi_\nu}$.

Moreover, the Borcherds lift [Bor98] of the vector-valued modular form F is an automorphic product Ψ_{ϕ_ν} whose expansion at any cusp is the denominator identity of $\mathfrak{g}^{\phi_\nu}$ and $\mathfrak{g}_{\phi_\nu}$ [Sch04, Sch08].

Properties

For convenience we rescale the quadratic form on L to obtain the even lattice $\Delta := L'(m) \cong \Lambda^\nu \oplus II_{1,1}$.

The dimensions of the root spaces are

$$\dim(\mathfrak{g}^{\phi_\nu}(\alpha)) = [F_{\alpha+L}](-\langle\alpha,\alpha\rangle/2) = \sum_{d|m} \delta_{\alpha\in\Delta\cap d\Delta'}[1/\eta_\nu](-\langle\alpha,\alpha\rangle/2d)$$

for all $\alpha \in \Delta \setminus \{0\}$.

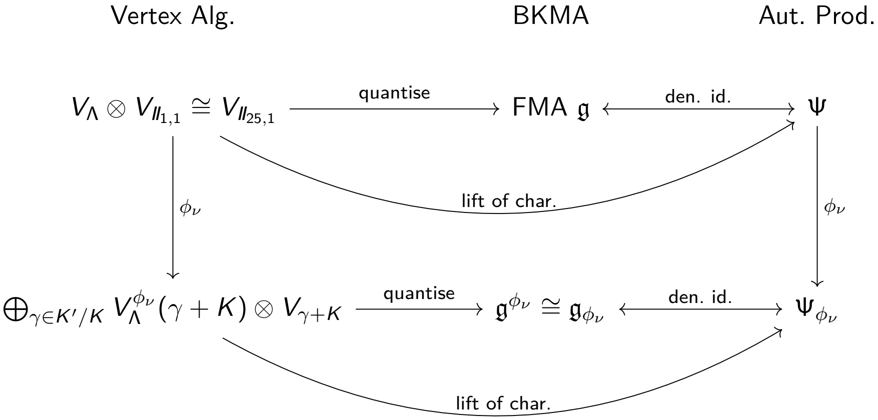
The denominator identity of $\mathfrak{g}^{\phi_\nu}$ may be written as

$$e^\rho \prod_{d|m} \prod_{\alpha\in\Phi^+\cap d\Delta'} (1-e^\alpha)^{[1/\eta_\nu](-\langle\alpha,\alpha\rangle/2d)} = \sum_{w\in W} \det(w) w(\eta_\nu(e^\rho))$$

with Weyl vector ρ. Here, the Weyl group W is the full reflection group of Δ.

The real roots of $\mathfrak{g}^{\phi_\nu}$ are the roots of the lattice Δ. The imaginary roots are $n\rho$, $n \in \mathbb{Z}_{>0}$, with multiplicities $24\sigma_0((m,n))/\sigma_1(m)$.

Summary


Vertex Alg.
BKMA
Aut. Prod.
$V_\Lambda \otimes V_{II_{1,1}} \cong V_{II_{25,1}}$
quantise
FMA $\mathfrak{g}$
den. id.
Ψ
lift of char.
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Thank you for your attention!

Bibliography I

Richard E. Borcherds.
Generalized Kac-Moody algebras.
J. Algebra, 115(2):501–512, 1988.

Richard E. Borcherds.
The monster Lie algebra.
Adv. Math., 83(1):30–47, 1990.

Richard E. Borcherds.
Monstrous moonshine and monstrous Lie superalgebras.
Invent. Math., 109(2):405–444, 1992.

Richard E. Borcherds.
Automorphic forms with singularities on Grassmannians.
Invent. Math., 132(3):491–562, 1998.
(arXiv:alg-geom/9609022v2).

Richard E. Borcherds.
Problems in Moonshine.
In *First international congress of Chinese mathematicians*, volume 20 of *AMS/IP Stud. Adv. Math.*, pages 3–10. Amer. Math. Soc., 2001.

Thomas Creutzig, Alexander Klauer and Nils R. Scheithauer.
Natural constructions of some generalized Kac-Moody algebras as bosonic strings.
Commun. Number Theory Phys., 1(3):453–477, 2007.
(arXiv:0801.1829v1 [math.NT]).

Bibliography II

Jethro van Ekeren, Sven Möller and Nils R. Scheithauer.
Construction and classification of holomorphic vertex operator algebras.
J. Reine Angew. Math., 759:61–99, 2020.
(arXiv:1507.08142v3 [math.RT]).

Boris L. Feigin.
The semi-infinite homology of Kac-Moody and Virasoro Lie algebras.
Russian Math. Surveys, 39(2):155–156, 1984.

Igor B. Frenkel, Howard Garland and Gregg J. Zuckerman.
Semi-infinite cohomology and string theory.
Proc. Nat. Acad. Sci. U.S.A., 83(22):8442–8446, 1986.

Igor B. Frenkel, James I. Lepowsky and Arne Meurman.
Vertex operator algebras and the Monster, volume 134 of *Pure Appl. Math.*
Academic Press, 1988.

Valery A. Gritsenko and Viacheslav V. Nikulin.
On the classification of Lorentzian Kac-Moody algebras.
Russian Math. Surveys, 57(5):921–979, 2002.
(arXiv:math/0201162v2).

Valery A. Gritsenko and Viacheslav V. Nikulin.
Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections.
Proc. Lond. Math. Soc. (3), 116(3):485–533, 2018.
(arXiv:1602.08359v2 [math.AG]).

Bibliography III

Gerald Höhn and Nils R. Scheithauer.
A natural construction of Borcherds' Fake Baby Monster Lie algebra.
Amer. J. Math., 125(3):655–667, 2003.
(arXiv:math/0312106v1 [math.QA]).

Ching Hung Lam.
Cyclic orbifolds of lattice vertex operator algebras having group-like fusions.
Lett. Math. Phys., 110(5):1081–1112, 2020.
(arXiv:1805.10778v2 [math.QA]).

Bong H. Lian and Gregg J. Zuckerman.
New perspectives on the BRST-algebraic structure of string theory.
Comm. Math. Phys., 154(3):613–646, 1993.
(arXiv:hep-th/9211072v1).

Sven Möller.
A Cyclic Orbifold Theory for Holomorphic Vertex Operator Algebras and Applications.
Ph.D. thesis, Technische Universität Darmstadt, 2016.
(arXiv:1611.09843v2 [math.QA]).

Sven Möller.
Natural construction of ten Borcherds-Kac-Moody algebras associated with elements in M_{23}.
Comm. Math. Phys., 383(1):35–70, 2021.
(arXiv:1905.09629v2 [math.QA]).

Nils R. Scheithauer.
Generalized Kac-Moody algebras, automorphic forms and Conway's group. I.
Adv. Math., 183(2):240–270, 2004.

Bibliography IV

Nils R. Scheithauer.
On the classification of automorphic products and generalized Kac-Moody algebras.
Invent. Math., 164(3):641–678, 2006.

Nils R. Scheithauer.
Generalized Kac-Moody algebras, automorphic forms and Conway's group. II.
J. Reine Angew. Math., 625:125–154, 2008.

Yongchang Zhu.
Modular invariance of characters of vertex operator algebras.
J. Amer. Math. Soc., 9(1):237–302, 1996.

Gregg J. Zuckerman.
Modular forms, strings, and ghosts.
In Leon Ehrenpreis and Robert C. Gunning, editors, *Theta Functions Bowdoin 1987*, volume 49 of *Proc. Sympos. Pure Math.*, pages 273–284. Amer. Math. Soc., 1989.