A probabilistic approach to Liouville CFT

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Polyakov's description of random surfaces

Polyakov, *Quantum geometry of bosonic strings* (1981).

Canonical random Riemannian metric on a Riemann surface $\mathcal{S}$.

- Fix $\gamma \in (0, 2)$.
- Write Riemannian metric tensor of $\mathcal{S}$ as $g = e^{\gamma X} \hat{g}_\tau$.
- $X =$ Gaussian free field (or some modification).
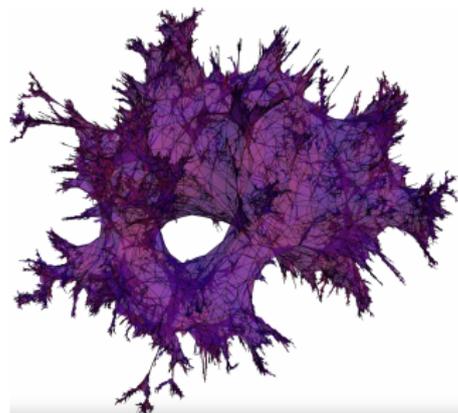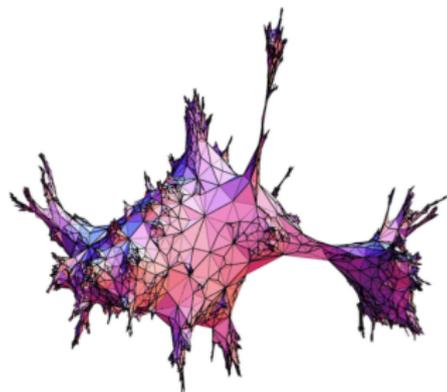- $\hat{g}_\tau$ reference metric of moduli τ, τ random.

More concretely, measure areas/lengths of domains/curves by:

$$\int_D e^{\gamma X} d^2 z, \qquad \int_{\mathcal{C}} e^{\frac{\gamma}{2} X} ds.$$

Random surfaces in probability theory

<u>Idea 1</u> Discretize $\mathcal{S}$: random planar map.

Random graph embedded on a surface modulo homeomorphisms.

Images by N. Curien and J. Bettinelli.

Gaussian free field on a Riemann surface

Idea 2 Make sense of $e^{\gamma X}$ with X GFF directly.

The Gaussian free field (GFF) X on $\mathcal{S}$:

- In physics, using a path integral:

$$\langle F(X) \rangle = \frac{1}{\mathcal{Z}} \int_{\varphi : \mathcal{S} \mapsto \mathbb{R}} F(\varphi) e^{-\frac{1}{4\pi} \int_{\mathcal{S}} |\partial \varphi|^2 d^2 z} D\varphi.$$

- In probability, for λ_i eigenvalues, e_i eigenfunctions of $-\frac{\Delta_g}{2\pi}$:

$$X(z) = \sum_i \frac{\alpha_i}{\sqrt{\lambda_i}} e_i(z), \quad \alpha_i \text{ i.i.d. } \mathcal{N}(0, 1).$$

Gaussian process with covariance:

$$\mathbb{E}[X(x)X(y)] = G(x, y) = \log \frac{1}{|x - y|} + c(x, y).$$

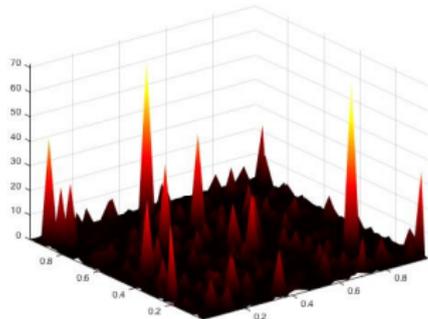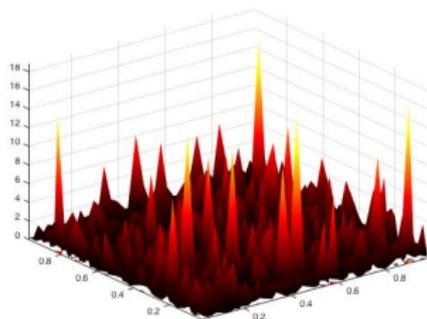Not pointwise defined, lives in the space of distribution.

Gaussian multiplicative chaos (GMC) measure

Idea 2 Make sense of $e^{\gamma X}$ with X GFF directly.

Renormalized measure: $e^{\gamma X_\epsilon - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon^2]} d^2 z$, X_ϵ smoothing of X.

The following limit holds in probability, for a continuous function f:

$$\int_D e^{\gamma X(z)} f(z) d^2 z = \lim_{\epsilon \to 0} \int_D e^{\gamma X_\epsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[X_\epsilon^2(z)]} f(z) d^2 z.$$

Plots by T. Zhu. $\gamma = 1$ $\gamma = 1.8$

Liouville conformal field theory

Polyakov (1981): Liouville field ϕ given by

$$\langle F(\phi)\rangle = \frac{1}{\mathcal{Z}} \int_{\varphi:\mathcal{S}\mapsto\mathbb{R}} D\varphi e^{-S_L(\varphi)} F(\varphi),$$

with energy functional:

$$S_L(\varphi) = \frac{1}{4\pi} \int_{\mathcal{S}} \left(|\partial\varphi|^2 + \mu e^{\gamma\varphi} + QR_g\varphi\right) d\lambda_g.$$

- $\mu > 0$, $\gamma \in (0, 2)$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, central charge $c = 1 + 6Q^2$.
- Corresponds to reweighting the law of the GFF by the volume it assigns to the surface.
- Energy functional is formally conformally invariant.

Framework of Conformal field theory (CFT)

CFT: Belavin, Polyakov, Zamolodchikov (1984).
Solve for correlation functions:

$$\langle \prod_{i=1}^{N} e^{\alpha_i \phi(z_i)} \rangle_{\mathcal{S}}, \quad z_i \in \mathcal{S}, \ \alpha_i \in \mathbb{R}.$$

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Conformal bootstrap program:

- Step 1: Compute three-point function on the sphere
- Step 2: Express any correlation on any surface using:

$$\int \ (\text{Three-point sphere}) \ \times \ (\text{Conformal blocks})$$

- Three-point sphere $=$ DOZZ formula.
- Conformal block: fundamental object of CFT.

Probabilistic construction of Liouville CFT

- Sphere case: David, Kupiainen, Rhodes, Vargas (2014).
- Disk case: Huang, Rhodes, Vargas (2015).
- Other geometries (David, Guillarmou, R., Rhodes, Vargas).

Idea: replace formal path integral by an expectation over the law of the Gaussian free field X on $\mathcal{S}$:

$$\text{``}\frac{1}{\mathcal{Z}} \int_{\varphi:\mathcal{S}\mapsto\mathbb{R}} D\varphi e^{-\int_{\mathcal{S}}|\partial\varphi|^2 d^2z} \hat{F}(\varphi)\text{''} = \int_{\mathbb{R}} dc\, \mathbb{E}\left[\hat{F}(X+c)\right].$$

- X has zero average in the metric g, $\int_{\mathcal{S}} X R_g d\lambda_g = 0$.
- $\int_{\mathbb{R}} dc$ corresponds to integrating over the average (zero mode).

We then choose $\hat{F}(X) = \prod_{i=1}^{N} e^{\alpha_i X(z_i)} e^{-\int_{\mathcal{S}}(QR_g X + \mu e^{\gamma X}) d\lambda_g}$.

Correlations of LCFT on the Riemann sphere

Concrete example: the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$.

Let $z_i \in \mathbb{C}$, $\alpha_i \in \mathbb{R}$. Let X be GFF on $\mathbb{C}$ with $\int_0^{2\pi} X(e^{i\theta}) d\theta = 0$.

For $\alpha_i < Q$, $\sum_i \alpha_i > 2Q$ one defines:

$$\left\langle \prod_{i=1}^N e^{\alpha_i \phi(z_i)} \right\rangle_{\mathbb{S}^2} = \int_{\mathbb{R}} dc\, e^{(\sum_i \alpha_i - 2Q)c} \mathbb{E}\left[\left(\prod_{i=1}^N e^{\alpha_i X(z_i)} \right) e^{-\mu e^{\gamma c} \mathcal{A}(\mathbb{C})} \right],$$

where:

$$\mathcal{A}(\mathbb{C}) = \int_{\mathbb{C}} e^{\gamma X(z)} |z|_+^{-2\gamma Q} d^2 z.$$

- Both $e^{\alpha_i X(z_i)}$ and $e^{\gamma X(z)}$ are defined by the regularization.
- $|z|_+ = \max(1, |z|)$, comes the background metric required to compactify the plane $\mathbb{C}$ into $\mathbb{S}^2$.

Conformal bootstrap for Liouville CFT

Theorem (Kupiainen, Rhodes, Vargas 2017)

$\langle e^{\alpha_1\phi(0)}e^{\alpha_2\phi(1)}e^{\alpha_3\phi(\infty)}\rangle_{\mathbb{S}^2}$ is given by the DOZZ formula $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$ predicted in physics by Dorn-Otto-Zamolodchikov-Zamolodchikov.

Theorem (Guillarmou, Kupiainen, Rhodes, Vargas 2020)

The following holds for the 4 point correlation of Liouville CFT on $\mathbb{S}^2$:

$$\langle\prod_{i=1}^{4} e^{\alpha_i\phi(z_i)}\rangle_{\mathbb{S}^2} = \int_{\mathbb{R}} dP\, C_\gamma(\alpha_1,\alpha_2,Q-iP)C_\gamma(Q+iP,\alpha_3,\alpha_4)|\mathcal{F}_{\alpha_i}(z,P)|^2.$$

Here $\mathcal{F}_{\alpha_i}(z,P)$ is the four-point sphere conformal block depending on $z \in \mathbb{C}$, the cross ratio of the four insertion points.

Full higher genus picture also solved (GKRV 2021).

What is a conformal block ?

Witt algebra $l_n = -z^{n+1}\partial_z$ of holomorphic vector fields:

$$[l_n, l_m] = (n - m)l_{n+m}.$$

Virasoro algebra, Central extension of the Witt algebra:

$$[\mathbf{L}_n, \mathbf{L}_m] = (n - m)\mathbf{L}_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m}\mathbf{1}.$$

Fix parameters $z, P, \alpha_1, \alpha_2, \alpha_3, \alpha_4$.
The 4-point sphere conformal block is defined by

$$\mathcal{F}_{\alpha_i}(z, P) = \sum_{n=0}^{\infty} \beta_n(P, \alpha_i)z^n,$$

where the coefficients $\beta_n(P, \alpha_i)$ are computed using the commutation relations of the Virasoro algebra.

Modular transformation of conformal blocks

Theorem (Ghosal-R.-Sun-Sun 2023+)

Let $\gamma \in (0, 2)$, α_i in an suitable range. The function $\mathcal{F}_{\alpha_i}(z, P)$ obeys

$$z^{\frac{1}{2}P^2} \mathcal{F}_{\alpha_i}(z, P) = C(z) \int_{\mathbb{R}} \mathcal{M}_{\alpha_i}(P, P')(1-z)^{\frac{1}{2}(P')^2} \mathcal{F}_{\alpha_i}(1 - z, P')dP',$$

for an explicit kernel $\mathcal{M}_{\alpha_i}(P, P')$.

As corollary:

- It is jointly analytic in $z \in \mathbb{D}$ and meromorphic in $P \in \mathbb{C}$ with simple poles at $P = \frac{in\gamma}{2} + \frac{2im}{\gamma}$.
- For $z \in (0, 1)$ and $P \in \mathbb{R}$ it admits a probabilistic expression.
- Same result for the 1-point torus block.

Proof requires conformal bootstrap for boundary Liouville CFT.

Boundary Liouville theory

Path integral definition:

$$\langle F(\phi) \rangle = \frac{1}{\mathcal{Z}} \int_{\varphi : \mathcal{S} \mapsto \mathbb{R}} D\varphi \, e^{-S_L(\varphi)} F(\varphi),$$

where for $\mu, \mu_B > 0$ the energy functional is

$$S_L(\varphi) = \int_{\mathcal{S}} \left(|\partial^g \varphi|^2 + \mu e^{\gamma \varphi} + Q R_g \varphi \right) d\lambda_g + \int_{\partial \mathcal{S}} \left(\mu_B e^{\frac{\gamma}{2} \varphi} + Q K_g \varphi \right) ds_g.$$

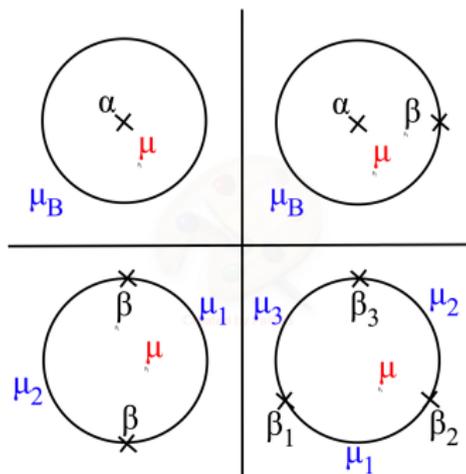Here $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, $Q R_g \varphi$, $Q K_g \varphi$ are curvature terms.

Correlation functions:

$$\left\langle \prod_{i=1}^{N} e^{\alpha_i \phi(z_i)} \prod_{j=1}^{M} e^{\frac{\beta_j}{2} \phi(s_j)} \right\rangle, \quad z_i \in \mathcal{S}, \ s_j \in \partial \mathcal{S}, \ \alpha_i, \beta_j \in \mathbb{R}.$$

Conformal bootstrap for surfaces with boundary

Step 1: compute the basic correlations on $\mathbb{D}$

- $\left\langle e^{\alpha \phi(z)} \right\rangle, \quad z \in \mathbb{D}.$
- $\left\langle e^{\alpha \phi(z)} e^{\frac{\beta}{2} \phi(s)} \right\rangle, z \in \mathbb{D}, s \in \partial \mathbb{D}.$
- $\left\langle e^{\frac{\beta}{2} \phi(s_1)} e^{\frac{\beta}{2} \phi(s_2)} \right\rangle, s_i \in \partial \mathbb{D}.$
- $\left\langle \prod_{j=1}^{3} e^{\frac{\beta_j}{2} \phi(s_j)} \right\rangle, s_i \in \partial \mathbb{D}.$

μ_B is piecewise constant in between boundary insertions.

Step 2: Solve higher order / higher genus correlations using the conformal bootstrap.

Annulus case by Wu (2022), work in progress for general case.

Law of the length of the circle

A very concrete result: mass of the GMC measure on the circle.

> ## Theorem (R. 2017)
>
> Let $\gamma \in (0, 2)$. The following equality in law holds
>
> $$\frac{1}{2\pi} \int_0^{2\pi} e^{\frac{\gamma}{2} X(e^{i\theta})} d\theta \sim \Gamma(1 - \frac{\gamma^2}{4})^{-1} \mathcal{E}(1)^{-\frac{\gamma^2}{4}},$$
>
> where $\mathcal{E}(1)$ is an exponential law of parameter 1.

- Predicted in statistical physics by Fyodorov-Bouchaud (2008).
- Proof: using BPZ equations of boundary Liouville CFT.

More general laws of GMC

We can get explicit descriptions of the following:

- Law of GMC on an interval (R.-Zhu 2018): $\int_0^1 e^{\frac{\gamma}{2} X(x)} dx$.

- Joint law of length and area of the circle (Ang-R.-Sun 2021):

  Conditioning on length $= 1$, area $\sim$ inverse Gamma distribution.

- Joint moments of three boundary arcs of a circle (R.-Zhu 2020).

- Joint moments of three boundary arcs and area of a circle (Ang-R.-Sun-Zhu 2023+)

  Same formula as the fusion kernel $\mathcal{M}_{\alpha_i}(P, P')$.

All part of the exact solvability of boundary Liouville CFT !
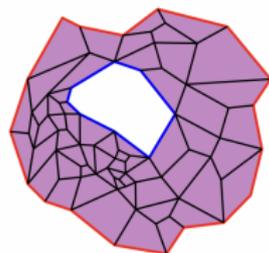
The Brownian annulus

Look at surfaces with the topology of an annulus.

Consider $A_\tau := \{z : |z| \in (e^{-2\pi\tau}, 1)\}$, with $\tau > 0$ the modulus.

Random planar maps of annular topology:

- $a_n, b_n \in \mathbb{N}$: $\lim\limits_{n\infty} \frac{a_n}{3\sqrt{n}} = a$, $\lim\limits_{n\infty} \frac{b_n}{3\sqrt{n}} = b$.
- $\mathcal{Q}_n$: set of annular quadrangulations with bdy length a_n, b_n.
- Sample Q_n from $\mathcal{Q}_n$ with probability proportional to $\sim 12^{-\#vertices}$.

random planar map

Then $\frac{1}{n} Q_n$ converges to the Brownian annulus
= metric-measured space (Bettinelli, Miermont).

Law of the modulus of a random annulus

random planar map · Brownian annulus · $\mathbb{A}_\tau$

scaling limit → conformal embedding →
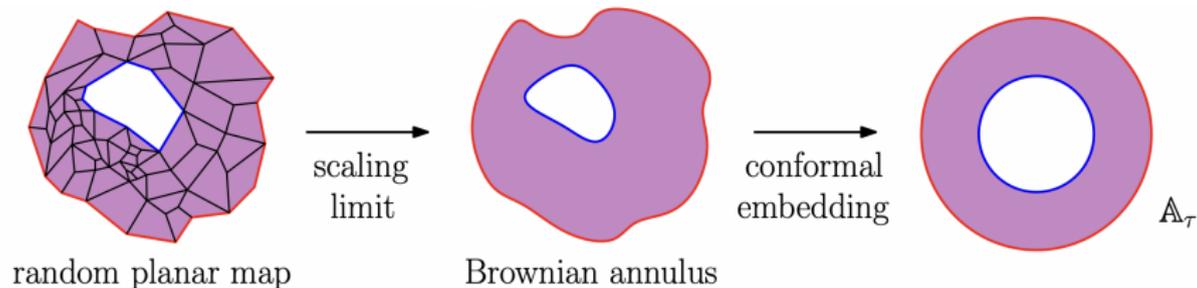
Both arrows above require highly non-trivial work:

Gwynne, Holden, Le Gall, Miermont, Miller, Sheffield, Sun, et al.

Theorem (Ang, R., Sun 2022)

Condition on outer and inner boundary length of $\mathbb{A}_\tau$ being a and b.
The law of τ has density $\propto \mathbf{1}_{\tau>0}\eta(2i\tau)\rho_\tau(b/a)d\tau$, where:
ρ_τ density of $Ze^{\mathcal{N}(0,\frac{4}{3}\pi\tau)}$, $Z \perp \mathcal{N}$, Z log-logistic, $\eta =$ eta function.

Predicted in physics in 80's in string theory / 2D quantum gravity.

Proof strategy

Let L_0 and L_1 denote the boundary length of the annulus.

- When L_0, L_1, τ are all random, the joint law of (L_0, L_1) is known from enumeration of random planar maps (Bernardi-Fusy 2018).

- When τ is fixed, the joint law of (L_0, L_1) is computed using the solvability of the Liouville conformal field theory (Wu 2022).

- By Laplace transform, find the law of τ when $L_0 = a$, $L_1 = b$.

Analogue results for other models of random annuli.

Requires conformal welding of Schramm–Loewner evolution (SLE).

Future directions of research

- Study space of conformal blocks, higher points / genus.

- The law of moduli for any Riemann surface.
  One can also add marked points / related moduli questions.

- Integration over moduli / correlation numbers of 2D gravity.

Longer term perspectives:

- Study 3d bootstrap / apply methods to other QFT's.

- Study related theories in mathematical physics.

Proof strategy

For the DOZZ formula:

- BPZ differential equations: correlations can obey PDEs
- $z \to \langle e^{\chi\phi(z)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle_{\mathbb{S}^2}$ obeys hypergeometric eq for $\chi = -\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$.
- Solution space: functional equations on $\langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle_{\mathbb{S}^2}$

For the conformal bootstrap:

- Canonical Hilbert space given by reflection positivity
- Express 4-pts function as an inner product on this space
- Bootstrap integral is the decomposition of this inner product on the eigenbasis of a self-adjoint operator, the Liouville Hamiltonian.