Braid Rigidity for Path Algebras

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February 5, 2021

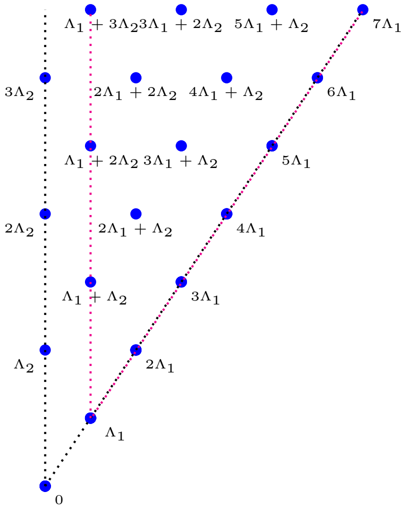
Rutgers Lie Groups Quantum Math Seminar

The Lie algebra G_2

- We have $\mathfrak{g} = \mathfrak{g}(G_2)$ and V its simple 7-dimensional representation.
- We first recall some basic facts about its roots and weights.
- With respect to the orthonormal unit vectors ε_1, ε_2, ε_3 of $\mathbb{R}^3$, the roots of $\mathfrak{g}$ can be written $\Phi = \pm\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, 2\varepsilon_1 - \varepsilon_2 - \varepsilon_3, 2\varepsilon_2 - \varepsilon_1 - \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\}$.
- The base can be chosen $\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = -\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3\}$.
- The Weyl vector is given by $\rho = 2\varepsilon_1 + \varepsilon_2 - 3\varepsilon_3$ and the Weyl group is D_6.
- The fundamental dominant weights are given by $\{\Lambda_1 = \varepsilon_1 - \varepsilon_3, \Lambda_2 = \varepsilon_1 + \varepsilon_2 - 2\varepsilon_3\}$.

The dominant Weyl chamber G_2

The following describes the dominant Weyl chamber for $\mathfrak{g}$:



Tensor product rules

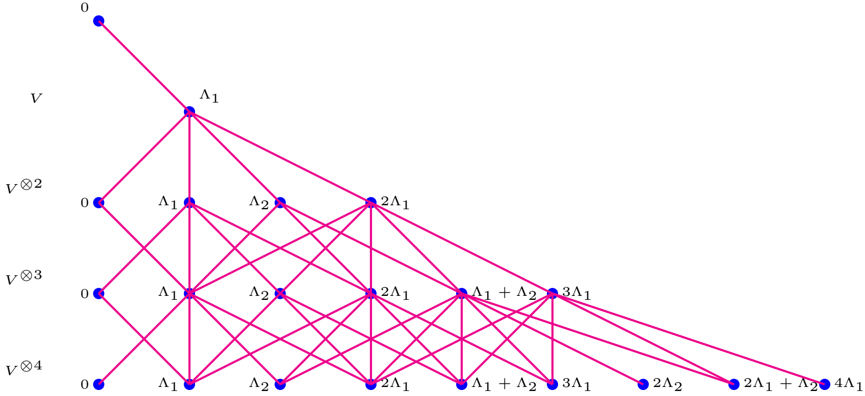
- For the smallest nontrivial $\mathfrak{g}$-module $V = V_{\Lambda_1}$, we will need to know how to tensor irreducible representations with V.
- The representation V has dimension 7, with its weights being the short roots of $\mathfrak{g}$ together with the zero weight.
- The decomposition of the tensor product

$$V_\lambda \otimes V \;\cong\; \bigoplus_{\mu} V_\mu \tag{1}$$

  with V_λ a simple module with highest weight $\lambda = a\Lambda_1 + b\Lambda_2$ can be described as follows :
- Consider the hexagon centered at λ and with corners $\lambda + \omega$, with ω running through the short roots of $\mathfrak{g}$. If this hexagon is contained in the dominant Weyl chamber C, then $V_\lambda \otimes V$ decomposes into the direct sum of irreducibles $\mathfrak{g}$-modules whose highest weights are given by the corners and the center of the hexagon.
- If it is not contained in C, leave out all the corners of the hexagon which are not in C ; moreover, if $\lambda = b\Lambda_2$, also leave out λ itself.

Tensor product rules

Using this, we can draw the Bratteli diagram for $V^{\otimes n}$.



Braided tensor categories

The following result has first been shown by G. I. Lehrer and R. B. Zhang, with different proofs also given by S. Morrison and separately by L. M. and H. Wenzl :

Theorem (First Fundamental Theorem)

Let V be the 7-dimensional representation of $\mathbf{U}=U_q(\mathfrak{g}(G_2))$ with highest weight Λ_1. Then $\mathrm{End}_{\mathbf{U}}(V^{\otimes n})$ is generated by the image of the braid group B_n in $\mathrm{End}_{\mathbf{U}}(V^{\otimes n})$ for q not a root of unity.

More generally, it was shown by L. M. and H. Wenzl

Theorem

Let V be the 7-dimensional irreducible representation of G_2, and let V_λ be an irreducible representation with highest weight λ. Then the image of the affine braid group AB_n generates $\mathrm{End}_{\mathbf{U}}(V_\lambda \otimes V^{\otimes n})$ for all $n \in \mathbb{N}$.

Path Algebras

- Let Λ be a set of labels with distinguished label 0 together with a not necessarily symmetric relation $\rightarrow$.
- A path of length n is a map $t : \{0, 1, \ldots, n\} \rightarrow \Lambda$ such that $t(0) = 0$ and $t(i) \rightarrow t(i+1)$ for $0 \leq i < n$.

- We denote by $\mathcal{P}_n$ the set of all paths of length n.
- We define algebras C_n by

$$C_n = \bigoplus_{\nu} M_{m(\nu,n)}, \tag{2}$$

where $m(\nu, n)$ is the number of paths t of length n with $t(n) = \nu$ and M_m are the $m \times m$ matrices.

Path Algebras

- We can define an embedding of C_{n-1} into C_n.
- Let $W(\nu, n)$ be a simple C_n-module labeled by the label ν.
- It has a basis labeled by the paths in $\mathcal{P}_n$ which end in ν.
- Its decomposition into simple C_{n-1} modules is given by the map $t \mapsto t'$, where t' is the restriction of t to $\{0, 1, \dots, n-1\}$.
- Hence we have the following isomorphism of C_{n-1}-modules :

$$W(\nu, n) \cong \oplus_{\mu} W(\mu, n-1), \tag{3}$$

where μ runs through all labels μ which are endpoint of a path of length $n-1$ such that $\mu \to \nu$.

Path Algebras

Definition

The **path algebra** $\mathcal{P}$ corresponding to the label set Λ with the relation $\to$ is given by the sequence of algebras C_n with the embeddings $C_{n-1} \subset C_n$ defined by 3.

Example

The standard example for a path algebra is given by the labeling set Λ consisting of all Young diagrams with 0 being the empty Young diagram, and $\mu \to \nu$ if $\mu \subset \nu$ and $|\nu| = |\mu| + 1$, i.e. ν is obtained by adding a box to μ . Then each path corresponds to a Young tableau, and $C_n \cong \mathbb{C}S_n$.

Tensor categories

- We let $\mathcal{C}$ be a semisimple tensor category, whose Hom spaces are complex vector spaces.
- Someone not familiar with tensor categories can safely think of $\mathcal{C}$ being the representation category of a Drinfeld-Jimbo quantum group, or just of the corresponding semisimple Lie algebra.
- Let Λ be a labeling set for the simple objects of $\mathcal{C}$, where 0 is the label for the trivial object.
- We also assume that V is a simple object of $\mathcal{C}$ with the *multiplicity 1 property*, i.e $V_\mu \otimes V$ is a direct sum of mutually non-isomorphic simple objects, for any simple object V_μ in $\mathcal{C}$.
- We then define the relation $\mu \to \nu$ if $V_\nu \subset V_\mu \otimes V$.
- This allows us to give a fairly simple description of $\mathrm{End}_{\mathcal{C}}(V^{\otimes n})$ via paths.

Theorem

We have a direct sum decomposition of objects in $\mathcal{C}$ given by

$$V^{\otimes n} = \bigoplus_{\nu} m(\nu, n) V_{\nu},$$

where the multiplicity $m(\nu, n)$ is given by the number of paths in $\mathcal{P}_n$ which end in ν. In particular, we have

$$C_n = \bigoplus_{\nu} M_{m(\nu,n)} \cong \mathrm{End}_{\mathcal{C}}(V^{\otimes n}). \tag{4}$$

Corollary

- *There exists an assignment* $t \in \mathcal{P}_n \mapsto p_t \in C_n = \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$ *such that* $p_t V^{\otimes n}$ *is an irreducible* $\mathcal{C}$*-object labeled by* $t(n)$*, and such that* $p_t p_s = \delta_{ts} p_t$.
- *The idempotents* p_t *are uniquely defined by the properties above and the following one : If* $s \in \mathcal{P}_{n-1}$*, we have*

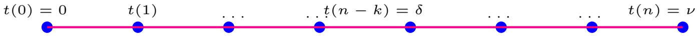
$$p_s \otimes id = \sum_{t,\ t'=s} p_t.$$

Path representations

- We denote by $\mathcal{P}_n(\nu)$ all paths of length n in $\mathcal{P}_n$ which end in ν.
- One checks easily that $z_\nu^{(n)} = \sum_{t\in\mathcal{P}_n(\nu)} p_t$ is a central idempotent in $C_n = \mathrm{End}_{\mathcal{C}}(V^{\otimes n})$.
- By definition, we can define a basis $(v_t)_{t\in\mathcal{P}_n(\nu)}$ for the simple C_n-module $W(\nu, n)$;
- Here the vector v_t spans the image of p_t for each $t \in \mathcal{P}_n(\nu)$ and it is uniquely determined up to scalar multiples.

Path representations

- Let δ, ν be dominant weights for which $V_\delta \subset V^{\otimes n-k}$ and $V_\nu \subset V^{\otimes n}$, and let $\mathcal{P}_k(\delta, \nu)$ be the set of all paths of length k from δ to ν, with paths as defined on slide 2.
- Let $W_k(\delta, \nu)$ be the vector space spanned by these paths and let t be a fixed path in $\mathcal{P}_{n-k}(\delta)$.



- Then we obtain a representation of $\text{End}_\mathcal{C}(V^{\otimes k})$ on $W_k(\delta, \nu)$ by

$$a \in \text{End}_\mathcal{C}(V^{\otimes k}) \mapsto (p_t \otimes a) z_\nu^{(n)}; \tag{5}$$

  here we used the obvious bijection between elements $s \in \mathcal{P}_k(\delta, \nu)$ and paths $\tilde{s} \in \mathcal{P}_n(\nu)$ for which $\tilde{s}_{|[0,n-k]} = t$, i.e. $\tilde{s}$ is the extension of t by s.

Braided tensor categories

We recall a few basic facts about braided and ribbon tensor categories :

- A braided tensor category $\mathcal{C}$ has canonical isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ for any objects V, W in $\mathcal{C}$.
- They satisfy the condition
$$c_{U,V\otimes W} \;=\; (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W), \tag{6}$$
and a similar identity for $c_{U\otimes V,W}$.
- Let B_n be Artin's braid groups, given by generators $\sigma_i,\ 1 \leq i \leq n-1$ and relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ as well as $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$.
- One can show that we obtain a representation of the braid group B_n into $\mathrm{End}(V^{\otimes n})$ for any object V in $\mathcal{C}$ via the map
$$\sigma_i \;\mapsto 1_{i-1} \otimes c_{V,V} \otimes 1_{n-1-i}, \tag{7}$$
where 1_k is the identity morphism on $V^{\otimes k}$.

Braided tensor categories

- Using a path basis (t) as above, we can express the action of σ_i via a matrix A_i such that

$$\sigma_i \mapsto A_i : t \to \sum_s a_{st}^{(i)} s, \qquad (8)$$

  where the summation goes over paths s for which $s(j) = t(j)$ for $j \neq i$; this follows from Eq 5 with $n = i + 1$ and $k = 2$.

- As each vector t is uniquely determined up to rescaling, it also follows that the matrix entries of A_i are uniquely determined up to conjugation by a diagonal matrix.

Braided tensor categories

- An associated ribbon braid structure is given by maps $\Theta_W : W \to W$ satisfying

$$\Theta_{V\otimes W} \; = \; c_{W,V}c_{V,W}(\Theta_V \otimes \Theta_W). \tag{9}$$

- Let $\Delta_n \in B_n$ be defined inductively by $\Delta_2 = \sigma_1$ and $\Delta_n = \Delta_{n-1}\sigma_{n-1}\sigma_{n-2} \dots \sigma_1$.
- Then it is well-known that $\Delta_n^2 = (\sigma_1\sigma_2 \dots \sigma_{n-1})^n$ generates the center of B_n. One can then prove by induction on n that

$$\Theta_{V^{\otimes n}} = \Delta_n^2 \Theta_V^{\otimes n}. \tag{10}$$

Braided tensor categories

- If V_λ is a simple object, the ribbon map just acts via a scalar, which we will denote by Θ_λ.
- Then the central element Δ_n^2 acts in the simple component labeled by α via a scalar denoted by $z_{\alpha,n}$.
- If the B_n-representation labeled by α acts nontrivially on the $\mathrm{End}(V^{\otimes n})$-module $W_\lambda^{(n)}$, then it follows from Eq 10 that

$$\Theta_\lambda \;=\; z_{\alpha,n}\Theta_V^n, \tag{11}$$

where we identified Θ_V with the scalar via which it acts on V.

Braided Rigidity

Definition

Let $\mathcal{P}$ be a path algebra.

1. We call a system of representations of braid groups *representations of type* $\mathcal{P}$ if the braid generators act on paths as in 8. Moreover, we also require that the central element $\Delta_n^2 \in B_n$ acts via a fixed scalar $z_{\lambda,n}$ on every path of length n which ends in λ.

2. We call a path algebra $\mathcal{P}$ *braid rigid* if any non-trivial braid representation of type $\mathcal{P}$ is uniquely determined by the image of σ_1.

Rigidity of path representations

- We consider *path representations of the braid groups* B_n *of type* G_2, i.e. braid representations for the path algebra $\mathcal{P}$ generated by the 7-dimensional irreducible representation V of the Lie algebra $\mathfrak{g}(G_2)$.
- Recall that if $\rho_{\nu,m}$ denotes the representation of B_m on the path space $W(\nu, m)$, we have the restriction rule

$$(\rho_{\lambda,n})_{|B_{n-1}} \cong \bigoplus_{\mu \leftrightarrow \lambda} \rho_{\mu,n-1}, \tag{12}$$

  where the summation goes over all μ for which $\lambda - \mu$ is a weight of $V = V_{\Lambda_1}$, with exceptions for the zero weight.

Rigidity of path representations

- The dominant integral weights λ are of the form $\lambda = (\lambda_1, \lambda_2, -\lambda_1 - \lambda_2)$ with $\lambda_1 \geq \lambda_2 \geq 0$.
- We write $\lambda = (\lambda_1, \lambda_2)$ for brevity.
- So $\Lambda_1 = (1, 0)$ and $\Lambda_2 = (1, 1)$.
- It follows from the decomposition of $V^{\otimes 2}$, that $c_{V,V}$ has four eigenvalues corresponding to the subrepresentations $\mathbf{1} = V_{(0,0)}$, $V = V_{(1,0)}$, $V_{(1,1)}$ and $V_{(2,0)}$ respectively.
- We will refer to the eigenvalue belonging to $V_\lambda \subset V^{\otimes 2}$ by α_λ. We let $\alpha_{(2,0)} = q^2$, it follows $\alpha_{(1,0)} = -q^{-6}$, $\alpha_{(1,1)} = -1$ and $\alpha_{(0,0)} = q^{-12}$.

Rigidity of path representations

1. The eigenvalues of $\rho(\sigma_1)$ are as above for q not a root of unity.
2. The representations of B_3 are irreducible for all modules $W(\nu, 3)$.
3. The braid Δ_n^2 acts as a scalar on each module $W(\nu, n)$ compatible with a ribbon braid structure, see Eq 10 and 11.

Theorem (L.M. and H. Wenzl)

Any path representation of braid groups of type G_2 satisfying the conditions stated above is uniquely determined by the eigenvalues of σ_1. More precisely, if we have two such path representations ρ_1 and ρ_2 such that their restrictions to B_2 coincide, then also their representations of B_n on any module $W(\nu, n)$ are isomorphic.

Application to tensor categories

Let $\mathcal{C}$ be a semisimple ribbon tensor category whose fusion rules are the ones of $\mathfrak{g}(G_2)$, and let $\mathcal{U}$ be equal to $Rep(U_q\mathfrak{g}(G_2))$. The following theorem has been previously obtained by S. Morrison, E. Peters, N. Snyder and it was shown by L.M and H. Wenzl using different methods :

Theorem

Let V be the object in $\mathcal{C}$ corresponding to the 7-dimensional representation of $U_q\mathfrak{g}(G_2)$. Assume that the image of B_3 generates $\mathrm{End}_{\mathcal{C}}(V^{\otimes 3})$. Then $\mathcal{C} \cong Rep(U_q\mathfrak{g}(G_2))$ for some q not a root of unity.