Completing the A_2 Andrews-Schilling-Warnaar identities

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Based on joint work with Matthew C. Russell

1. Before I begin

1. Talk by Matthew Russell at "Speciality Seminar"

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4. $\mathcal{W}_3(3, \ell + 3)$ with $3, \ell$ coprime

2. Preliminaries

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$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q)_a}{(q)_b (q)_{a-b}}$$

Affine Lie algebras

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$[a \otimes t^n, b \otimes t^m] = [a, b] \otimes t^{m+n} + n\langle a, b \rangle \delta_{n+m,0} c$

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$$\chi(L(a\Lambda_0 + b\Lambda_1))$$
$$= \frac{1}{(q, q^5;\, q^6)_\infty}$$
$$\times \frac{(q^{a+1}, q^{a+b+2}, q^{a+2b+3};\, q^{a+2b+3})_\infty (q^{a+1}, q^{a+4b+5};\, q^{2a+4b+6})_\infty}{(q)_\infty}$$

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$$\chi(\Omega(\lambda)) = \frac{\chi(L(\lambda))}{\chi(L(\Lambda_0))}$$

Odd levels: Andrews–Gordon

Let $k \geq 1$, $1 \leq i \leq k+1$

$$\sum \frac{q^{n_1^2 + \cdots + n_k^2 + n_i + n_{i+1} + \cdots + n_k}}{(q)_{n_1 - n_2}(q)_{n_2 - n_3} \cdots (q)_{n_{k-1} - n_k}(q)_{n_k}}$$

$$= \frac{(q^{2k+3-i}, q^i, q^{2k+3}; q^{2k+3})_\infty}{(q)_\infty}$$

$$= \chi(\Omega((2k + 2 - i)\Lambda_0 + (i - 1)\Lambda_1))$$

Even levels: Andrews–Bressoud

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$$= \frac{(q^{2k+2-i}, q^i, q^{2k+2}; q^{2k+2})_\infty}{(q)_\infty}$$

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$\widehat{\mathfrak{sl}_2}$ at level=∞

Level = ∞

For $i \in \mathbb{Z}_{>0} \cup \{\infty\}$

$$\sum \frac{q^{n_1^2 + n_2^2 \cdots + + n_i + n_{i+1} + \cdots}}{(q)_{n_1 - n_2}(q)_{n_2 - n_3} \cdots}$$
$$= \frac{1 - q^i}{(q)_\infty}$$
$$= \chi(\Omega(\infty \Lambda_0 + (i-1)\Lambda_1))$$

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$$\mathcal{U}(\widehat{\mathfrak{sl}_2}) \otimes_{\mathcal{U}(\mathfrak{h} \oplus \mathfrak{b})} \mathbb{C}_\lambda$$

Garland–Lepowsky resolution

For $\widehat{\mathfrak{sl}}_2$ and integrable modules $L(\lambda)$,

there is a resolution in terms of generalized Verma modules E_j:

$$\cdots \to E_2 \to E_1 \to E_0 \to L(\lambda) \to 0.$$

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Question

How does the Euler–Poincaré character calculated using "infinite-fold" sums close up to give "finite" sums in Andrews–Gordon and Andrews–Bressoud identities?

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2. Sum-sides for principal characters of generalized Verma modules (arising as limits).

3. $A_2^{(1)}$ and higher rank

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Andrews–Schilling–Warnaar

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Start with a A_2 Bailey pair $\rightarrow$ transform to new A_2 Bailey pair

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   then $3k - 1$
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Example: Andrews–Schilling–Warnaar

Level 7, Modulus 10

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$$\sum \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2 + r_1 + r_2 + s_1 + s_2}(1 - q^{r_2 + s_2 + 1})}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2 + s_2 + 1}}$$

$$= \frac{1}{(q)_\infty} \chi(\Omega(7\Lambda_0 + 0\Lambda_1 + 0\Lambda_2))$$

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Level 5, Modulus 8

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$$= \frac{1}{(q)_\infty} \chi(\Omega(6\Lambda_0 + 0\Lambda_1 + 0\Lambda_2))$$

Location of $L(a\Lambda_0 + b\Lambda_1 + c\Lambda_2)$ with $a \geq b \geq c$, $a + b + c = 16$

$16, 0, 0$					
$15, 1, 0$	$14, 1, 1$				
$14, 2, 0$	$13, 2, 1$	$12, 2, 2$			
$13, 3, 0$	$12, 3, 1$	$11, 3, 2$	$10, 3, 3$		
$12, 4, 0$	$11, 4, 1$	$10, 4, 2$	$9, 4, 3$	$8, 4, 4$	
$11, 5, 0$	$10, 5, 1$	$9, 5, 2$	$8, 5, 3$	$7, 5, 4$	$6, 5, 5$
$10, 6, 0$	$9, 6, 1$	$8, 6, 2$	$7, 6, 3$	$6, 6, 4$	
$9, 7, 0$	$8, 7, 1$	$7, 7, 2$			
$8, 8, 0$					

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   Level 6 still remains!

Location of $L(\infty\Lambda_0 + a\Lambda_1 + b\Lambda_2)$:

Infinite level A_2

Location of $L(\infty\Lambda_0 + a\Lambda_1 + b\Lambda_2)$:

$\infty, 0, 0$

$\infty, 1, 0 \quad \infty, 1, 1$

$\infty, 2, 0 \quad \infty, 2, 1 \quad \infty, 2, 2$

$\infty, 3, 0 \quad \infty, 3, 1 \quad \infty, 3, 2 \quad \infty, 3, 3$

$\infty, 4, 0 \quad \infty, 4, 1 \quad \infty, 4, 2 \quad \infty, 4, 3 \quad \infty, 4, 4$

$\infty, 5, 0 \quad \infty, 5, 1 \quad \infty, 5, 2 \quad \infty, 5, 3 \quad \infty, 5, 4 \quad \infty, 5, 5$

$\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \ddots$

Infinite level A_2

Location of $L(\infty\Lambda_0 + a\Lambda_1 + b\Lambda_2)$:

$\infty, 0, 0$

$\infty, 1, 0 \qquad \infty, 1, 1$

$\infty, 2, 0 \qquad \infty, 2, 1 \qquad \infty, 2, 2$

$\infty, 3, 0 \qquad \infty, 3, 1 \qquad \infty, 3, 2 \qquad \infty, 3, 3$

$\infty, 4, 0 \qquad \infty, 4, 1 \qquad \infty, 4, 2 \qquad \infty, 4, 3 \qquad \infty, 4, 4$

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$\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \ddots$

$L(\infty\Lambda_0 + a\Lambda_1 + b\Lambda_2)$ can be viewed as generalized Verma module induced from $L(a\Lambda_1 + b\Lambda_2)$

Warnaar 2021

Corresponding to $\chi(\Omega(\infty\Lambda_0 + 0\Lambda_1 + 0\Lambda_2))$

$$\sum \frac{q^{r_1 - r_1 s_1 + s_1^2 + r_2 - r_2 s_2 + s_2^2 + \cdots + r_1 + r_2 + \cdots + s_1 + s_2 + \cdots}}{(1-q) \cdot (q)_{r_1 - r_2}(q)_{r_2 - r_3} \cdots (q)_{s_1 - s_2}(q)_{s_2 - s_3} \cdots}$$
$$= \frac{(1-q)(1-q)(1-q^2)}{(q)_\infty^3}$$

Also proved identities corresponding to

$\chi(\Omega(\infty\Lambda_0 + \infty\Lambda_1 + 0\Lambda_2))$

$\chi(\Omega(\infty\Lambda_0 + \infty\Lambda_1 + \infty\Lambda_2))$

Kanade–Russell 2022

Conjectures for all a, b.

At finite levels, the identities "above the line" are appropriate truncations.

Kanade–Russell 2022

Conjectures for all a, b.

At finite levels, the identities "above the line" are appropriate truncations.

Kanade–Russell 2022 (Conjecture)

Corresponding to $\chi(\Omega(\infty\Lambda_0 + 2\Lambda_1 + 1\Lambda_2))$

$$\sum \frac{q^{r_1 - r_1 s_1 + s_1^2 + r_2 - r_2 s_2 + s_2^2 + \cdots + r_3 + r_4 + \cdots + s_2 + \cdots}}{(1-q) \cdot (q)_{r_1 - r_2}(q)_{r_2 - r_3} \cdots (q)_{s_1 - s_2}(q)_{s_2 - s_3} \cdots}$$

$$-q \sum \frac{q^{r_1 - r_1 s_1 + s_1^2 + r_2 - r_2 s_2 + s_2^2 + \cdots + r_2 + r_3 + \cdots + s_1 + \cdots}}{(1-q) \cdot (q)_{r_1 - r_2}(q)_{r_2 - r_3} \cdots (q)_{s_1 - s_2}(q)_{s_2 - s_3} \cdots}$$

$$= \frac{(1-q^3)(1-q^2)(1-q^5)}{(q)_\infty^3}$$

Kanade–Russell, 2022

Level 3, Modulus 6

$$\sum_{r,s \geq 0} \frac{q^{r^2-rs+s^2+s}}{(q)_{r+s}(q)_{r+s+1}} \begin{bmatrix} r+s \\ r \end{bmatrix}_{q^3}$$
$$= \frac{1}{(q)_\infty} \frac{1}{\theta(q, q^2; q^6)}$$
$$= \frac{1}{(q)_\infty} \chi(\Omega(2\Lambda_0 + \Lambda_1)).$$

Kanade–Russell, 2022

Level 6, Modulus 9

$$\sum \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2 + r_2 + s_2}(1 - 2q^{1 + r_1 + s_1})}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2 + s_2}(q)_{r_2 + s_2 + 1}} \begin{bmatrix} r_2 + s_2 \\ r_2 \end{bmatrix}_{q^3}$$

$$= \frac{1}{(q)_\infty} \frac{1}{\theta(q, q^2, q^2, q^3, q^3;\, q^9)_\infty}$$

$$= \frac{1}{(q)_\infty} \chi(\Omega(3\Lambda_0 + 3\Lambda_1 + 0\Lambda_2)).$$

Kanade–Russell, 2022

Level 7, Modulus 10

$$\sum \frac{q^{r_1^2 - r_1 s_1 + s_1^2 + r_2^2 - r_2 s_2 + s_2^2}}{(q)_{r_1 - r_2}(q)_{s_1 - s_2}(q)_{r_2}(q)_{s_2}(q)_{r_2 + s_2 + 1}}$$
$$\times \left(q^{-r_1 + s_2} - q^{r_2} + q^{1 + r_1 + r_2 + s_2} - q^{1 - r_1 + r_2 + s_1 + s_2} - q^{s_1 + s_2} \right)$$
$$= \frac{1}{(q)_\infty} \frac{1}{\theta(q, q, q^2, q^3, q^3, q^5; \, q^{10})}$$
$$= \frac{1}{(q)_\infty} \chi(\Omega(3\Lambda_0 + 3\Lambda_1 + \Lambda_2)).$$

1. Cylindric partitions (Gessel–Krattenthaler)

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6. Set $z \mapsto 1$.

Note:

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$$ASW_c(z, q) = \frac{(zq; q)_\infty}{(q)_\infty} F_c(z, q)$$

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Question
What exactly are the z, q ASW sum-sides counting?

5. Cylindric partitions

Partitions

Partitions

A partition such as:

Partitions

A partition such as:

$$7 + 4 + 3 + 2 + 2$$

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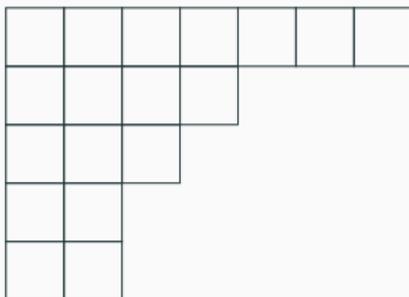
Really really, a linear partition

Really really, a linear partition

| 7 | 4 | 3 | 2 | 2 |

Really really, a linear partition

| 7 | 4 | 3 | 2 | 2 |

is a lattice path:

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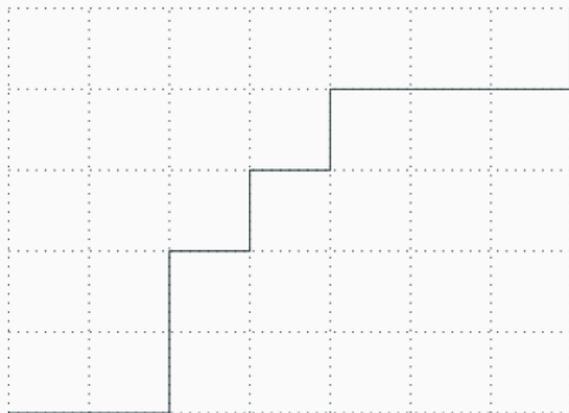
A plane partition is a 2-dimensional partition:

Plane partitions

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7	4	3	2	2
5	4	1		
2				

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Numbers in boxes as heights

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Numbers in boxes as heights $\rightarrow$ 3-dimensional shape.

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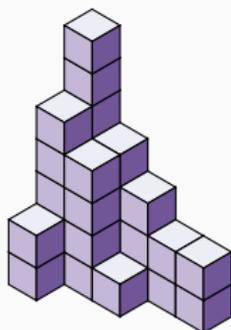
(Credits: Jang Soo Kim)

Really, a collection of non-intersecting lattice paths

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7	4	3	2	2
5	4	1	0	0
2	0	0	0	0

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Counting plane partitions
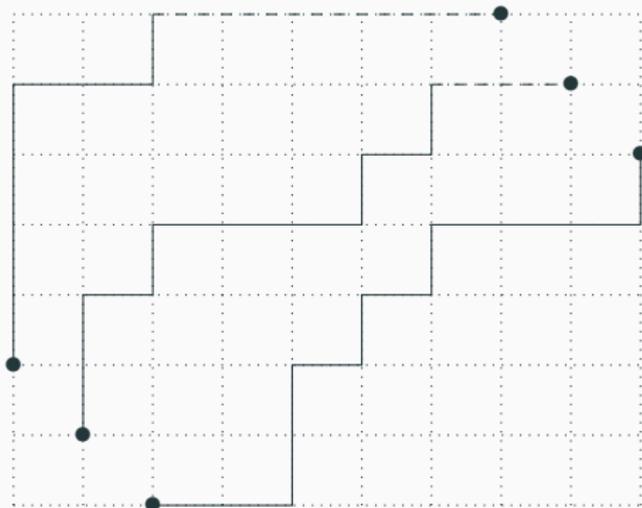
MacMahon (1912)

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Generating function for plane partitions is:

$$\prod_{n \geq 1} \frac{1}{(1 - q^n)^n}$$

Modern approach – Lindström (1973), Gessel and Viennot (1985)

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Count all lattice paths (intersecting or not)

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that start somewhere on SW bullets and end on NE bullets

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We see the Weyl groups S_n controlling the counting.
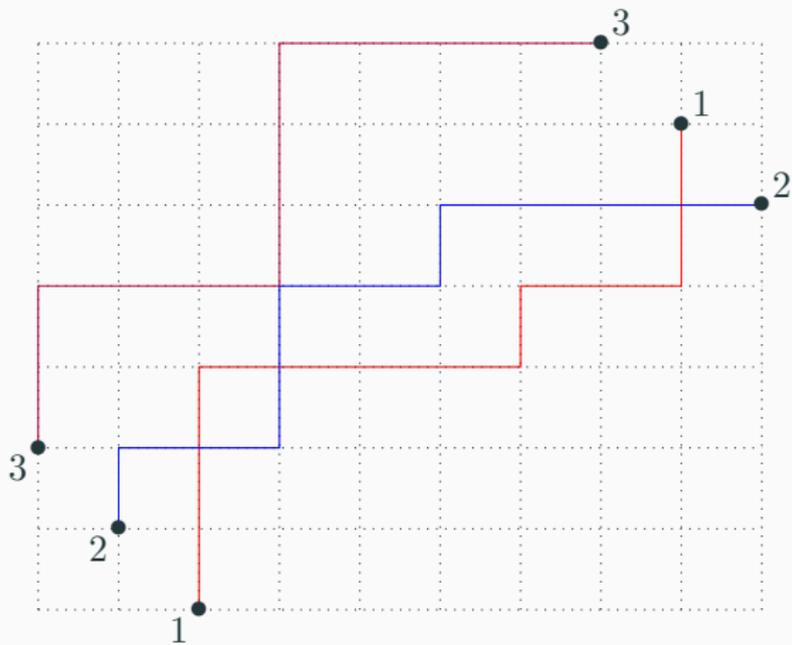
$$\sum \text{sgn}(\pi) \cdot q^{\text{wt(path)}}$$
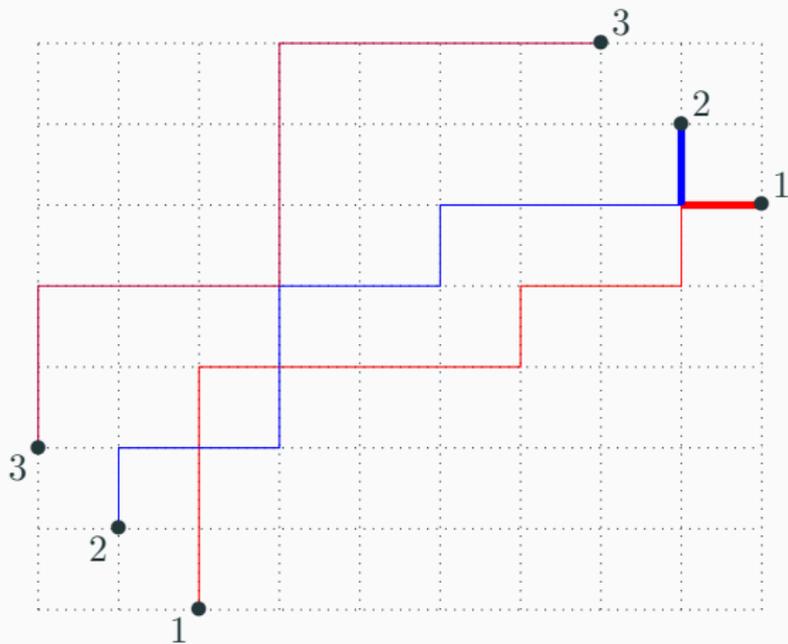
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So, only non-intersecting paths remain.
Now, the sum over S_n is interpreted in terms of determinants.

Cylindric Partitions

First introduced by Gessel and Krattenthaler, 1997

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First introduced by Gessel and Krattenthaler, 1997

First connected to representation theory of $\widehat{\mathfrak{sl}_r}$ by Tingley, 2008.
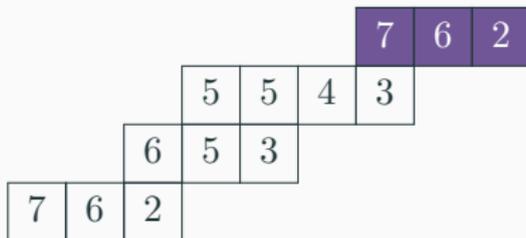
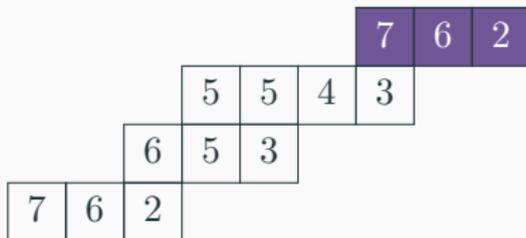Cylindric partition of profile $(3, 1, 2)$

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Connected to $L(3\Lambda_0 + 1\Lambda_1 + 2\Lambda_2)$ for $\widehat{\mathfrak{sl}_3} = A_2^{(1)}$

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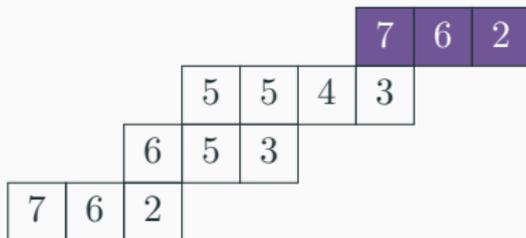
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Number of rows = rank = 3

Total "skew-ness" = level = 6.

Cylindric condition $\rightarrow$ affine translations

Counting is controlled by $\widehat{S_n}$ = affine Weyl group of $\widehat{\mathfrak{sl}_n} = A_{n-1}^{(1)}$.

(See Gessel and Krattenthaler)

Generating Functions

Fix a profile $c = (c_0, c_1, \cdots, c_r)$.

Let $\mathcal{C}_c$ be the set of all cylindric partitions of profile c

$$F_c(z, q) = \sum_{\pi \in \mathcal{C}_c} z^{\max(\pi)} q^{\mathrm{wt}(\pi)}$$

Borodin, Gessel–Krattenthaler, Foda–Welsh

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Let $c = (c_0, c_1, \cdots, c_r)$

$$F_c(1, q)$$
$$= \frac{1}{(q)_\infty} \chi(\Omega(c_0 \Lambda_0 + \cdots + c_r \Lambda_r))$$
$$= \frac{1}{(q^{r+1}; q^{r+1})_\infty} \chi(L(c_0 \Lambda_0 + \cdots + c_r \Lambda_r)).$$

Borodin, Gessel–Krattenthaler, Foda–Welsh

Let $c = (c_0, c_1, \cdots, c_r)$

$$
\begin{aligned}
&F_c(1, q) \\
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&= \frac{1}{(q^{r+1}; q^{r+1})_\infty} \chi(L(c_0 \Lambda_0 + \cdots + c_r \Lambda_r)).
\end{aligned}
$$

Question

Representation-theoretic interpretation of z (max-part statistic)?

By the very definition,

$$F_{(c_0,c_1,\cdots,c_r)}(z,q) = F_{(c_r,c_0,c_1,\cdots,c_{r-1})}(z,q)$$

Symmetries

Outer automorphism / Dynkin diagram symmetry:

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But, in general:

$$F_{(c_0,\cdots,c_r)}(z,q) \neq F_{(c_r,c_{r-1}\cdots,c_0)}(z,q).$$

6. $\mathcal{W}$ algebras

Given a (finite) simple Lie algebra $\mathfrak{g}$

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$$V(\mathfrak{g}, k) \dashrightarrow \mathcal{W}^k(\mathfrak{g}, f) \to \mathcal{W}_k(\mathfrak{g}, f)$$

Principal $\mathcal{W}$ algebras of A

$$\mathfrak{g} = \mathfrak{sl}_n$$

$$f = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & & & \end{bmatrix}$$

$$k = -n + \frac{p}{q}, \quad \frac{p}{q} \in \mathbb{Q}_{>0}, \ (p,q) = 1, \ n \leq p < q$$

Denoted as $\mathcal{W}_n(p,q)$

$n = 2$

$\mathcal{W}_2(p, p') = \text{Vir}(p, p')$

Relations to principal characters

Set of principal characters of $\widehat{\mathfrak{sl}_n}$ at level ℓ

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Andrews–Schilling–Warnaar + Our work

Combinatorics of $\mathcal{W}_3(3, 3 + \ell)$

Relations to cylindric partitions

Let $\lambda = c_0 \Lambda_0 + \cdots + c_{n-1} \Lambda_{n-1}$

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$$\frac{1}{(q)_\infty} \chi \left((\mathcal{W}_n)_{0,\lambda}^{(n,n+\ell)} \right) = F_{(c_0, \cdots, c_{n-1})}(1, q)$$

$\mathcal{W}_\infty$ rigorously constructed by Andy Linshaw (2017)

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$1/(q)_\infty$ = Character of rank 1 Heisenberg VOA.

Rank $\to \infty$

$$\mathcal{H} \otimes \mathcal{W}_\infty = \mathcal{W}_{1+\infty} \quad \longleftrightarrow \quad \text{Plane partitions}$$

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Rank $\to \infty$

Plane partitions

$\longleftrightarrow$

Cylindric partitions with infinitely many rows

Questions

Todo

Why does Heisenberg make life easier?

Questions

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Is there an explicit way to relate combinatorics of $\mathcal{W}$ algebras to cylindric partitions?

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Question

What about the combinatorics of other truncations of $\mathcal{W}_\infty$?

Other types?

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Kanade–Linshaw 2018

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There exists a universal object $\mathcal{W}_\infty^{\text{even}}$ defined over $\mathbb{C}[c, \lambda]$

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Other types?

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Character:

$$\frac{1}{(1-q^2)(1-q^3)(1-q^4)^2(1-q^5)^2(1-q^6)^3\cdots}$$

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What about the combinatorics of $\mathcal{W}_\infty^{\mathrm{even}}$?

Thank you!