Quantum Weyl algebras and a new FFT for $U_q(\mathfrak{gl}_n)$.

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Joint work with G. Letzter and S. Sahi

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The First Funadamental Theorem of Invariant Theory

- $G := \mathrm{GL}_n(\mathbb{C})$, $V := \mathbb{C}^n$, G naturally acts on V (hence on V^*).
- $\mathcal{E}_{k,l} := V^{\oplus k} \oplus (V^*)^{\oplus l}$.
- G acts on the polynomial algebra $\mathcal{P}\left(\mathcal{E}_{k,l}\right)$:

$$g \cdot \phi(x) := \phi(g^{-1} \cdot x) \quad \text{for } g \in G,\ x \in \mathcal{E}_{k,l}.$$

- **Classical Problem:** Find concrete generators and relations for $\mathcal{P}(\mathcal{E}_{k,l})^G$.
- Define $\phi_{i,j}(v_1, \ldots, v_k, v_1^*, \ldots, v_l^*) := \langle v_j^*, v_i \rangle$ for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant l$.

  In coordinates: $\phi_{i,j} = \sum_{r=1}^{n} v_{j,r}^* v_{i,r}$.

Theorem (Schur, Weyl, ...)

$\mathcal{P}(\mathcal{E}_{k,l})^G$ is generated by the $\phi_{i,j}$ for $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant l$.

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- $\mathcal{P} := \mathcal{P}_{m\times n} := \mathcal{P}(\mathrm{Mat}_{m\times n})$.
- $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ naturally acts on $\mathrm{Mat}_{m\times n}$:

$$(g, g') \cdot X := (g^{-1})^T X g'$$

- Passing to Lie algebras, we obtain an action of $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ on $\mathrm{Mat}_{m\times n}$:

$$(A, B) \cdot X := -A^T X + XB.$$

- $\mathrm{GL}_m \times \mathrm{GL}_n$ and $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ naturally act on $\mathcal{P}$.
- **Observation:** $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ act by *polarization* operators:

$$E_{ij} \rightsquigarrow \sum_{r=1}^{n} x_{ir} \frac{\partial}{\partial x_{jr}} \quad \text{for } 1 \leqslant i, j \leqslant m$$

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Weyl algebra on $\mathrm{Mat}_{m\times n}$

- $\mathcal{PD} := \mathcal{PD}_{m\times n}$: the $\mathbb{C}$-algebra generated by the $x_{i,j}$ and the $\partial_{i,j}$ for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, modulo the relations
$$\partial_{i,j}x_{k,l} - x_{k,l}\partial_{i,j} = \delta_{i,k}\delta_{j,l}.$$
- $\mathcal{P}$ is a $\mathcal{PD}$-module.

- Set $U(\mathfrak{g}) := T(\mathfrak{g})/I$, where I is the ideal generated by the elements
$$x \otimes y - y \otimes x - [x, y] \quad , \quad x, y \in \mathfrak{g}.$$
Every Lie theorist knows: $\quad \mathfrak{g} - \mathrm{mod} \cong U(\mathfrak{g}) - \mathrm{mod}.$
- Set $U_{m,n} := U(\mathfrak{gl}_m) \otimes U(\mathfrak{gl}_n)$. The polarization operators result in an algebra homomorphism
$$\phi_{m,n} : U_{m,n} \to \mathcal{PD},$$
such that the following diagram commutes:

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Theorem (Howe, 1995)

(i) The subalgebras $\phi_{m,n}(U(\mathfrak{gl}_m) \otimes 1)$ and $\phi_{m,n}(1 \otimes U(\mathfrak{gl}_n))$ of $\mathcal{PD}$ are mutual centralizers in $\mathcal{PD}$.

(ii) The FFT for $\mathrm{GL}_n(\mathbb{C})$ follows readily from (i).

Remark (trivial)

The mutual centralizer property does not hold in $\mathrm{End}(\mathcal{P})$.

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Eigenvalues of Capelli operators

- $\mathcal{PD} \cong \mathcal{P} \otimes \mathcal{D} \cong \mathcal{P}\left(V^{\oplus m} \oplus (V^*)^{\oplus m}\right)$ as GL_n-modules.

  $\mathcal{P} \cong \bigoplus_\lambda F_\lambda \qquad F_\lambda \cong V_\lambda \otimes V_\lambda$ as $\mathrm{GL}_m \times \mathrm{GL}_n$-modules.

  V_λ: GL_m-module (or GL_n-module) indexed by the Young diagram λ.
- $\mathcal{PD}^{\mathrm{GL}_m \times \mathrm{GL}_n}$ has a distinguished basis $\{D_\lambda : \ell(\lambda) \leqslant \min\{m, n\}\}$.
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The Hopf algebra $U_q(\mathfrak{gl}_n)$

$\Bbbk := \mathbb{C}(q)$.

The quantized enveloping algebra $U_q(\mathfrak{gl}_n)$

$Q := \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ (weight lattice of $\mathfrak{gl}_n$).

$$\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i,\ \mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \ \Rightarrow\ (\lambda, \mu) := \sum \lambda_i \mu_i.$$

$\Phi := \{\varepsilon_i - \varepsilon_j\}_{1 \leqslant i \neq j \leqslant n}$: root system of type A_n , $\alpha_i := \varepsilon_i - \varepsilon_{i+1}$.

$U_q(\mathfrak{gl}_n)$: $\Bbbk$-algebra generated by $\langle E_i, F_i, K_\lambda \;:\; 1 \leqslant i \leqslant n-1, \lambda \in Q \rangle$ modulo:

- $K_\lambda K_\mu = K_{\lambda+\mu}$.
- $K_\lambda E_i = q^{(\lambda, \alpha_i)} E_i K_\lambda$, $K_\lambda F_i = q^{-(\lambda, \alpha_i)} F_i K_\lambda$.
- $[E_i, F_j] := \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}$, $K_i := K_{\alpha_i}$.
- $E_i^2 E_{i\pm1} - (q + q^{-1}) E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0.$
  $F_i^2 F_{i\pm1} - (q + q^{-1}) F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0.$

$U_q(\mathfrak{gl}_n)$ is a Hopf algebra

- $\Delta(E_i) := E_i \otimes K_i + 1 \otimes E_i$, $\Delta(F_i) := F_i \otimes 1 + K_i^{-1} \otimes F_i$, $\Delta(K_\lambda) := K_\lambda \otimes K_\lambda$.
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$\Bbbk := \mathbb{C}(q)$.

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$Q := \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_n$ (weight lattice of $\mathfrak{gl}_n$).

$$\lambda = \sum_{i=1}^{n} \lambda_i \varepsilon_i,\ \mu = \sum_{i=1}^{n} \mu_i \varepsilon_i \ \Rightarrow\ (\lambda, \mu) := \sum \lambda_i \mu_i .$$

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$U_q(\mathfrak{gl}_n)$: $\Bbbk$-algebra generated by $\langle E_i, F_i, K_\lambda \ : \ 1 \leqslant i \leqslant n-1, \lambda \in Q \rangle$ modulo:

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Quantized coordinate ring of $\mathrm{Mat}_{m\times n}$

The FRT bialgebra

- $R = \sum R_{ij}^{kl} E_{ki} \otimes E_{lj} \in \mathrm{End}_{\Bbbk}(\Bbbk^n \otimes \Bbbk^n)$.
- $\mathcal{A}(R) := \langle t_{ij}, 1 \leqslant i, j \leqslant n \ : \ RT_1T_2 = T_2T_1R \rangle$

$$T := [t_{ij}] \quad , \quad T_1 = T \otimes I \quad , \quad T_2 = I \otimes T$$

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- $\mathscr{P}_{n\times n} \hookrightarrow U_q(\mathfrak{gl}_n)^\circ$ (matrix coefficients of the "standard module")
- $\mathscr{P}_{n\times n}$ is a bialgebra:
  $$\Delta(t_{i,j}) = \sum t_{i,k} \otimes t_{k,j}.$$
- $U_q(\mathfrak{gl}_n)$ acts on $\mathscr{P}_{n\times n}$ by left and right translation:
  $$x \cdot_L u := \sum \langle u_1, x \rangle u_2 \quad , \quad x \cdot_R u = \sum u_1 \langle u_2, x^\natural \rangle,$$
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Formulas for right translation of $U_q(\mathfrak{gl}_n)$ on $\mathscr{P}_{n\times n}$

$$E_k \cdot t_{i,j} = -\delta_{k,j} q^{-1} t_{i,k+1} \quad , \quad F_k \cdot t_{i,j} = -\delta_{k+1,j} q t_{i,k} \quad , \quad K_{\varepsilon_k} \cdot t_{i,j} = q^{-\delta_{k,j}} t_{i,j}$$

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- $\mathscr{P}_{m\times n}$: The subalgebra of $\mathscr{P}_{N\times N}$, $N := \max\{m, n\}$, generated by the t_{ij} satisfying $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$.
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- $\mathscr{P}_{m\times n}$ is a $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$-module algebra.

The algebra $\mathscr{D}_{m\times n}$

- Set $\mathscr{D}_{n\times n} := \mathcal{A}(R)$ for the R-matrix given by:
  $R^{ii}_{ii} = q$, $R^{ij}_{ij} = 1$ for $i \neq j$, $R^{ji}_{ij} = (q - q^{-1})$ for $i > j$.
- $\mathscr{D}_{n\times n} \cong \mathscr{P}^{\mathrm{op,cop}}_{n\times n}$.
- $\mathscr{D}_{n\times n} \rightsquigarrow \mathscr{D}_{m\times n}$
- We denote the generators of $\mathscr{D}_{m\times n}$ by ∂_{ij} where $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant n$.
- $\mathscr{D}_{m\times n}$ is also a $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_n)$-module algebra.

Quantized coordinate ring of $\mathrm{Mat}_{m\times n}$

From $\mathscr{P}_{n\times n}$ to $\mathscr{P}_{m\times n}$

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The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

$$\mathscr{P}\mathscr{D}_{m\times n} := \mathscr{P}_{m\times n} \otimes \mathscr{D}_{m\times n} \quad \text{as vector spaces.}$$

Universal $\mathcal{R}$-matrix

- $\mathcal{C}_n$: Category of finite dimensional $U_q(\mathfrak{gl}_n)$-modules (with some mild restrictions).
- **Fact:** $\mathcal{C}_n$ is a braided monoidal category.

$$\begin{tikzcd}
V \otimes W \arrow[rr, "\cong"] \arrow[dr, dotted, "\mathcal{R}"'] & & W \otimes V \\
& V \otimes W \arrow[ur, dotted, "\sigma : v\otimes w \mapsto w\otimes v"'] &
\end{tikzcd}$$

- $\mathcal{R} := \left(e^{h\sum_{i=1}^{n} H_i\otimes H_i}\right) \prod_{i=1}^{n(n-1)/2} \mathrm{Exp}_q\left((q-q^{-1})E_{\beta_i}\otimes F_{\beta_i}\right),$

$E_{\varepsilon_i-\varepsilon_j} := \pm[E_i,[\dots,E_{j-1}]_-]_-$ and $F_{\varepsilon_i-\varepsilon_j} := \pm[F_{j-1},[\dots,F_i]_+]_+$ where $1 \leqslant i < j \leqslant n$ and $[x,y]_\pm := xy - q^{\pm 1}yx$.

$e^{hH_i} := K_{\varepsilon_i}$; $e^h := q$; $\mathrm{Exp}_q(x) := \sum_{r\geqslant 0} q^{\binom{r}{2}} \frac{x^r}{[r]_q!}$;

$\beta_{i+j(j-1)/2} := \varepsilon_i - \varepsilon_{j+1}$ for $1 \leqslant i \leqslant j \leqslant n-1$.

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The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

$$\Delta^{\mathrm{cop}} = \mathcal{R}\Delta\mathcal{R}^{-1} \quad , \quad (\Delta\otimes\mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23} \quad , \quad (\mathrm{id}\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}R_{12}.$$

Proposition

Suppose we are given:

- H: any Hopf algebra with a universal R-matrix $\mathcal{R}\in H\otimes H$.
- A, B : H-module algebras.

Then the following hold:

- $A\otimes B$ can be equipped with an associative algebra structure with multiplication defined by

  $$(a\otimes b)\bullet(a'\otimes b') := (a\otimes 1)\left(\check{\mathcal{R}}(b\otimes a')\right)(1\otimes b'),$$

  where $\check{\mathcal{R}} := \sigma\circ\mathcal{R}$.
- The new algebra $A\otimes_{\check{\mathcal{R}}} B$ is an H-module algebra.

The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

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The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

Theorem (Noumi–Yamada–Mimachi '93; Baumann '98, R. Zhang '02, . . .)

$$\mathscr{P}_{m\times n} \cong \bigoplus_{\ell(\lambda)\leqslant \min\{m,n\}} V_\lambda \otimes V_\lambda \quad \text{and} \quad \mathscr{D}_{m\times n} \cong \bigoplus_{\ell(\lambda)\leqslant \min\{m,n\}} V_\lambda^* \otimes V_\lambda^*$$

as $U_q(\mathfrak{gl}_m)\otimes U_q(\mathfrak{gl}_n)$-modules.

$$V_\lambda \rightsquigarrow q^{-\sum_i \lambda_i \varepsilon_i} \quad \text{(lowest weight module).}$$

$$\mathscr{P}\mathscr{D}^{\mathrm{gr}}_{m\times n} := \mathscr{P}_{m\times n} \otimes_{\check{\mathcal{R}}} \mathscr{D}_{m\times n}$$

Towards *deformed* twisted tensor product

- H: any Hopf algebra with a universal R-matrix $\mathcal{R}\in H\otimes H$.
- A, B : H-module algebras.
- $E_A \subseteq A$ and $E_B \subseteq B$: H-stable generating subspaces.

$$A \cong T(E_A)/I_A \quad \text{and} \quad B \cong T(E_B)/I_B.$$

- **Proposition.** $A\otimes_{\check{\mathcal{R}}} B \cong T(E_A\oplus E_B)/I_{A,B}$, where $I_{A,B}$ is generated by I_A, I_B, and the relations

$$ba - \check{\mathcal{R}}(a\otimes b), \qquad a\in A, b\in B.$$

The quantized Weyl algebra $\mathscr{PD}_{m\times n}$

Theorem (Noumi–Yamada–Mimachi ’93; Baumann ’98, R. Zhang ’02, . . .)

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The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

The *deformed* twisted tensor product

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- $\psi : E_B \otimes E_A \to \Bbbk$ such that

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where $\Delta(x) = \sum x_1 \otimes x_2$.

- **Definition.** We set

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$$\mathscr{P}\mathscr{D}^{\text{gr}}_{m\times n} \quad \rightsquigarrow \quad \mathscr{P}\mathscr{D}_{m\times n}.$$

The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

Explicit mixed relations of $\mathscr{P}\mathscr{D}_{m\times n}$

$\mathscr{P}\mathscr{D}_{m\times n}$ is generated by t_{ij} and ∂_{ij} for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ satisfying the relations in $\mathscr{P}_{m\times n}$, in $\mathscr{D}_{m\times n}$, and

$$\partial_{le} t_{df} = \sum_{j,a,n,b} \hat{R}^{fb}_{ea} \hat{R}^{dn}_{lj} t_{ja} \partial_{nb} + \delta_{l,d}\delta_{e,f}.$$

(i) $\partial_{cb} t_{da} = t_{da}\partial_{cb} \quad$ if $b \neq a$ and $c \neq d$.

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Remark

The algebra $\mathscr{P}\mathscr{D}_{m\times n}$ was also introduced by Shklyarov–Sinelschchikov–Vaksman (circa 1998) and Bershtein (2007), with a completely different construction.

The quantized Weyl algebra $\mathscr{P}\mathscr{D}_{m\times n}$

Explicit mixed relations of $\mathscr{P}\mathscr{D}_{m\times n}$

$\mathscr{P}\mathscr{D}_{m\times n}$ is generated by t_{ij} and ∂_{ij} for $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$ satisfying the relations in $\mathscr{P}_{m\times n}$, in $\mathscr{D}_{m\times n}$, and

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A no-go theorem

$$U_L := U_q(\mathfrak{gl}_m)\ ,\ U_R := U_q(\mathfrak{gl}_n)\ ,\ U_{LR} := U_L \otimes U_R.$$

$$U_{LR} \xrightarrow{\phi_U} \mathrm{End}(\mathscr{P}_{m\times n}) \xleftarrow{\phi_{PD}} \mathscr{P}\mathscr{D}_{m\times n}$$

- Unfortunately, $\phi_U(U_{LR}) \nsubseteq \phi_{PD}(\mathscr{P}\mathscr{D}_{m\times n})$.

Proposition (Letzter–Sahi–S.)

There does not exist a $\Bbbk$-algebra $\widetilde{\mathscr{P}\mathscr{D}}_{m\times n}$ with the following properties:

(i) $\widetilde{\mathscr{P}\mathscr{D}}_{m\times n}$ is a locally finite U_{LR}-module.

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$$\mathsf{L}_{i,j} := \sum_{r=1}^{n} t_{ir}\partial_{jr}\ ,\ 1 \leqslant i,j \leqslant m \quad \text{and} \quad \mathsf{R}_{i,j} := \sum_{r=1}^{m} t_{ri}\partial_{rj}\ ,\ 1 \leqslant i,j \leqslant n.$$

Theorem A (Letzter–Sahi–S.)

(i) $\mathcal{L}$ and $\mathcal{R}$ are mutual centralizers in $\mathscr{P}\mathscr{D}_{m\times n}$.

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Theorem B (Letzter–Sahi–S.)

Let $\mathscr{A}_{k,l,n}$ denote the algebra generated by $kn + ln$ generators t_{ij} and $\partial_{i'j}$, where $1 \leqslant i \leqslant k, 1 \leqslant i' \leqslant l, 1 \leqslant j \leqslant n$, subject to "same" relations as $\mathscr{P}\mathscr{D}_{m\times n}$. Then:

(i) $\left(\mathscr{A}_{k,l,n}\right)^{U_q(\mathfrak{gl}_n)}$ is generated by the $\mathsf{L}_{i,j}$ for $\leqslant i \leqslant k$, $1 \leqslant j \leqslant l$.

(ii) A similar assertion holds for $\mathrm{gr}(\mathscr{A}_{k,l,n})$.

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FFT : the q-version

Comparing with the FFT by Lehrer–Zhang–Zhang (2010)

Algebra proposed by L–Z–Z:

- Generators: t_{ij} for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant n$; $\partial_{i,j}$ for $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant n$.
- Relations between the t_{ij} (or between the ∂_{ij}): same as those of $\mathrm{gr}(\mathscr{A}_{k,l,n})$.
- Mixed relations:
  - (i) $\partial_{cb} t_{da} = t_{da} \partial_{cb} \quad$ if $b \neq a$.
  - (ii) $\partial_{ca} t_{ba} = q t_{ba} \partial_{ca} + (q - q^{-1}) \sum_{a' > a} t_{ba'} \partial_{ca'}$

Mixed relations of $\mathrm{gr}(\mathscr{A}_{k,l,n})$

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Strategy of proof

The map $\Gamma_{k,l,n}$

$$\Gamma_n : \mathscr{P}_{n\times n} \xrightarrow{\Delta} \mathscr{P}_{n\times n} \otimes \mathscr{P}_{n\times n} \xrightarrow{1\otimes\iota} \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}$$

where $\iota(t_{ij}) := \partial_{ji}$.

We define $\Gamma_{k,l,n}$ by restricting Γ_n:

$$\begin{tikzcd}
\mathscr{P}_{k\times l} \arrow[r] \arrow[dr, dotted, "\Gamma_{k,l,n}"'] & \mathscr{P}_{n\times n} \arrow[r, "\Gamma_n"] & \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n} \\
& \mathrm{gr}(\mathscr{A}_{k,l,n}) \arrow[ur, hook] &
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The $\star$ product on $\mathscr{P}_{k\times l}$

$$u \star v := \sum \langle \iota(v_1)^\natural \otimes \iota(u_3)^\natural, \mathcal{R}\rangle \langle \iota(v_3) \otimes \iota(u_2), \mathcal{R}\rangle u_1 v_2.$$

Proposition

(i) $\Gamma_{k,l,n}$ is a bijection onto $\mathrm{gr}(\mathscr{A}_{k,l,n})^{U_q(\mathfrak{gl}_n)}$.

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(i) $\Gamma_{k,l,n}$ is a bijection onto $\mathrm{gr}(\mathscr{A}_{k,l,n})^{U_q(\mathfrak{gl}_n)}$.

(ii) $\Gamma_{k,l,n}(u \star v) = \Gamma_{k,l,n}(u)\Gamma_{k,l,n}(v)$.

Strategy of proof

The map $\Gamma_{k,l,n}$

$$\Gamma_n : \mathscr{P}_{n\times n} \xrightarrow{\Delta} \mathscr{P}_{n\times n} \otimes \mathscr{P}_{n\times n} \xrightarrow{1\otimes\iota} \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n}$$

where $\iota(t_{ij}) := \partial_{ji}$.

We define $\Gamma_{k,l,n}$ by restricting Γ_n:

$$\begin{tikzcd}
\mathscr{P}_{k\times l} \arrow[r] \arrow[dr, dotted, "\Gamma_{k,l,n}"'] & \mathscr{P}_{n\times n} \arrow[r, "\Gamma_n"] & \mathscr{P}\mathscr{D}^{\mathrm{gr}}_{n\times n} \\
& \mathrm{gr}(\mathscr{A}_{k,l,n}) \arrow[ur, hook] &
\end{tikzcd}$$

The $\star$ product on $\mathscr{P}_{k\times l}$

$$u \star v := \sum \langle \iota(v_1)^\natural \otimes \iota(u_3)^\natural, \mathcal{R} \rangle \langle \iota(v_3) \otimes \iota(u_2), \mathcal{R} \rangle u_1 v_2.$$

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Elements of Weyl type

$$U_L := U_q(\mathfrak{gl}_m)\ ,\ U_R := U_q(\mathfrak{gl}_n)\ ,\ U_{LR} := U_L \otimes U_R.$$

$$\begin{tikzcd} U_{LR} \arrow[r, "\phi_U"] & \mathrm{End}(\mathscr{P}_{m\times n}) & \mathscr{P}\mathscr{D}_{m\times n} \arrow[l, "\phi_{PD}"'] \end{tikzcd}$$

Problem. Set $\mathscr{P}\mathscr{D} := \mathscr{P}\mathscr{D}_{m\times n}$. Characterize the subalgebras

$$\overline{U}_L := \{x \in U_L \,:\, \phi_U(x \otimes 1) \in \mathscr{P}\mathscr{D}\} \text{ and } \overline{U}_R := \{x \in U_R \,:\, \phi_U(1 \otimes x) \in \mathscr{P}\mathscr{D}\}.$$

Cartan subalgebra of $U_q(\mathfrak{gl}_n)$: For $U := U_q(\mathfrak{gl}_n)$ we set

$$U^\circ := \Bbbk[K_{\varepsilon_i}, K_{\varepsilon_i}^{-1} \,:\, 1 \leqslant i \leqslant n].$$

Let U_L° and U_R° be the Cartan subalgebras of U_L and U_R. Set

$$\mathcal{L}_\circ := \mathscr{P}\mathscr{D} \cap \phi_U(U_L^\circ) \text{ and } \mathcal{R}_\circ := \mathscr{P}\mathscr{D} \cap \phi_U(U_R^\circ).$$

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Elements of Weyl type

q-Capelli operators in $\mathscr{PD}_{m\times n}$

$$\begin{bmatrix} \star_{1,1} & \cdots & \star_{1,n-k} & \star_{1,n-k+1} & \cdots & \star_{1,n} \\ \star_{2,1} & \cdots & \star_{2,n-k} & \star_{2,n-k+1} & \cdots & \star_{2,n} \\ \vdots & \cdots & & & \cdots & \vdots \\ \star_{m,1} & \cdots & \star_{m,n-k} & \star_{m,n-k+1} & \cdots & \star_{m,n} \end{bmatrix}$$

$$\mathbf{D}_{k,r} := \textstyle\sum_{X_{r\times r}} \det_q(X_t) \det_q(X_\partial) \qquad 0 \leqslant r \leqslant k \leqslant n$$

- X_t : replace $\star_{i,j}$ by $t_{i,j}$.
- X_∂ : replace $\star_{i,j}$ by $\partial_{i,j}$.
- $\det_q(X_t) := \sum_\sigma (-q)^{\ell(\sigma)} t_{i_{\sigma(1)}j_1} \cdots t_{i_{\sigma(r)}j_r}$
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- We define $\mathbf{D}'_{k,r}$ for $0 \leqslant r \leqslant k \leqslant m$ similarly (w.r.t. last k rows).

Theorem C (Letzter–Sahi–S.)

(i) $\mathcal{L}_\circ$ is generated by $\mathbf{L}_1, \ldots, \mathbf{L}_m$, where $\mathbf{L}_i := \sum_{r=0}^{i} (q^2-1)^r \mathbf{D}'_{i,r}$.

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(iii) $U_L^\circ \cap \overline{U}_L$ is generated by $K_{-2(\varepsilon_a+\cdots+\varepsilon_m)}$ for $1 \leqslant a \leqslant m$.

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Quantized Capelli operators

- $\mathfrak{g}$: a reductive Lie algebra – θ: involution of $\mathfrak{g}$ – $\mathfrak{k} := \mathfrak{g}^{\theta}$.

$$(\mathfrak{g}, \mathfrak{k}) \quad \rightsquigarrow \quad (U_q(\mathfrak{g}), \mathcal{B}_\theta)$$

Three symmetric spaces for $\mathfrak{gl}_n$

- Assume $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ for $\mathfrak{g}_{\mathbb{R}} := \mathfrak{gl}_n(\mathbb{F})$, where $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.
- Set $\mathfrak{k} := \mathfrak{k}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ where $\mathfrak{k}_{\mathbb{R}}$ is the maximal compact of $\mathfrak{g}_{\mathbb{R}}$.

$\mathbb{F}$	$\mathfrak{g}$	$\mathfrak{k}$	N
$\mathbb{R}$	$\mathfrak{gl}_n$	$\mathfrak{o}_n$	n
$\mathbb{C}$	$\mathfrak{gl}_n \oplus \mathfrak{gl}_n$	$\mathfrak{gl}_n$	n
$\mathbb{H}$	$\mathfrak{gl}_{2n}$	$\mathfrak{sp}_{2n}$	$2n$

$$(U_q(\mathfrak{g}), \mathcal{B}_\theta) \;\leftrightsquigarrow\; R_{\mathfrak{g}} J_1 R_{\mathfrak{g}}^{t_1} J_2 = J_2 R_{\mathfrak{g}}^{t_1} J_1 R_{\mathfrak{g}}$$

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Quantized Capelli operators

- $\mathbb{F} = \mathbb{R}$: $\widehat{\mathscr{PD}}_{\mathfrak{g}} := \mathscr{PD}$.
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- $\mathbb{F} = \mathbb{H}$: $\widehat{\mathscr{PD}}_{\mathfrak{g}} := \mathscr{PD}$.

$$x_{ij} := \sum_{r,s} J_{rs} t_{ir} t_{js} \quad , \quad d_{ij} := \sum_{r,s} q^{-2\hat{s}} J_{rs} \partial_{ir} \partial_{js} \qquad \text{for } 1 \leqslant i,j \leqslant N$$

- The relations between the x_{ij} are given by: $R_{\mathfrak{g}} X_1 R_{\mathfrak{g}}^{t_1} X_2 = X_2 R_{\mathfrak{g}}^{t_1} X_1 R_{\mathfrak{g}}$ and other relations of the form $x_{ij} = \gamma x_{ji}$, $\gamma \in \{q, -q^{-1}, 0\}$.
- The relations between the d_{ij} are similar.
- The mixed relations are:

$$d_{ab} x_{ef} = \sum_{w,r,x,q,p,m,y,l} (R_{\mathfrak{g}}^{t_2})_{xq}^{wr} (R_{\mathfrak{g}}^{t_2})_{ma}^{pq} (R_{\mathfrak{g}}^{t_2})_{fl}^{xy} (R_{\mathfrak{g}}^{t_2})_{eb}^{ml} x_{pw} d_{ry} + q^{-\delta_{ef}} \delta_{ae} \delta_{bf}$$

- The x_{ij} and the ∂_{ij} are right $\mathcal{B}_\theta$-invariant.

Quantized Capelli operators

- $\mathbb{F}=\mathbb{R}$: $\widehat{\mathscr{P}\mathscr{D}}_{\mathfrak{g}} := \mathscr{P}\mathscr{D}$.
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- The x_{ij} and the ∂_{ij} are right $\mathcal{B}_{\theta}$-invariant.

Quantized Capelli operators

- $\mathbb{F} = \mathbb{R}$: $\widehat{\mathscr{P}\mathscr{D}}_{\mathfrak{g}} := \mathscr{P}\mathscr{D}$.
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Notation

- $\mathscr{P}_\theta$: subalgebra of $\widehat{\mathscr{P}\mathscr{D}}_{\mathfrak{g}}$ generated by the x_{ij}.
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$$\mathscr{P}_\theta \leftrightsquigarrow \mathcal{O}(G/K) \quad , \quad \mathscr{P}\mathscr{D}_\theta \leftrightsquigarrow \mathcal{D}(G/K)$$

- In all three cases $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we have a $U_q(\mathfrak{g})$-module decomposition

$$\mathscr{P}_\theta \cong \bigoplus_{\ell(\lambda)\leqslant n} F_\lambda.$$

- Thus, we can construct a basis $\{D_\lambda : \ell(\lambda) \leqslant n\}$ for $\mathscr{P}\mathscr{D}_\theta$.
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Interpolation Macdonald polynomials (Sahi, 1996)

Let λ be a partition satisfying $\ell(\lambda) \leqslant n$. Then $\mathsf{R}_\lambda \in \mathbb{Q}(\mathsf{q},\mathsf{t})[\mathsf{x}_1,\ldots,\mathsf{x}_n]$ is the unique polynomial satisfying the following:

- $\deg(\mathsf{R}_\lambda) = |\lambda|$.
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- $\mathsf{R}_\lambda(\mathsf{q}^{-\mu}\tau, \mathsf{q}, \mathsf{t}) = 0$, where $\tau = (\tau_1, \ldots, \tau_n)$ with $\tau_i := \mathsf{t}^{-n+i}$, for μ such that $|\mu| \leqslant |\lambda|$ and $\mu \neq \lambda$.
- **Theorem.** The top degree homogeneous component of R_λ is the (usual) Macdonald polynomial P_λ.

Theorem D (Letzter–Sahi–S.)

The eigenvalue of $D_\lambda \in \mathscr{P}\mathscr{D}_\theta$ on the irreducible component F_μ of $\mathscr{P}_\theta$ is equal to $\mathsf{R}_\lambda(\mathsf{q}^{-\mu}\tau, \mathsf{q}, \mathsf{q}^{2/d})$ where $d := \dim \mathbb{F}$.

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