Eigenvalues of Shimura Operators for Lie Superalgebras

Lie Group/Quantum Mathematics Seminar
Rutgers University

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Structure

- Background (S. Sahi and G. Zhang [SZ19])
- Lie Superalgebras
- Super Recipe
- Weyl Groupoid Invariance
- Spherical Representations

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$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+ (= \mathfrak{k} \oplus \mathfrak{p})$$

- $\mathfrak{k}$ is the fixed point subalgebra of an involution θ.
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Schmid Decomposition and Shimura Operators

$\mathscr{H}(n) := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{N}^n : \lambda_i \geq \lambda_{i+1}\}$.

Theorem (Schmid ([Sch70, FK90]))

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\substack{\lambda \in \mathscr{H}(n) \\ |\lambda| = d}} W(\lambda), \; \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\substack{\lambda \in \mathscr{H}(n) \\ |\lambda| = d}} W^*(\lambda)$$

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$D_\lambda \in \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} = \mathfrak{U}(\mathfrak{g})^K \cong \mathbf{D}_K(G)$ (*right* K-invariant differential operators on G). Descends to the Shimura operator in $\mathbf{D}(G/K)$.

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A slight abuse of name: call D_λ the *Shimura operator associated to* λ.

Okounkov Polynomials

$\Lambda := \mathbb{C}[x_1, \ldots, x_n]^{S_n \ltimes \mathbb{Z}_2^n}$ (ring of even symmetric polynomials)
$\rho := (\rho_1, \ldots, \rho_n)$, $\rho_i := \tau(n-i) + \alpha$. τ, α: parameters.

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Theorem ([Oko98], [OO06], c.f.[SZ19])

The Okounkov polynomial $P_\mu(x; \tau, \alpha)$ is the unique polynomial in Λ satisfying

1. $\deg P_\mu = 2|\mu|$;
2. $P_\mu(\lambda + \rho) = 0$ *for* $\lambda \not\supseteq \mu$ [**the vanishing condition**];
3. *Some normalization condition.*

ρ can be specialized to the half sum of positive roots for a root system of Type BC. For the "usual" Type A symmetry, there are Knop–Sahi polynomials [KS96].

An Example

Consider the one-row partition (l) for $l \in \mathbb{N}$.
Then $\rho = \alpha$. We have [SZ19]:

$$P_{(l)}(x;\tau,\alpha) = \prod_{i=0}^{l-1}(x^2 - (i+\alpha)^2) = \prod_{i=0}^{l-1}(x+\alpha-i)(x-\alpha+i).$$

- Even symmetric;
- Degree $2l$;
- Vanishes at $(k+\alpha)$ for $k < l$.

For (l^n), we have

$$P_{(l^n)}(x;\tau,\alpha) = \prod_{i=0}^{l-1}\prod_{j=1}^{n}(x_j^2 - (i+\alpha)^2)$$

Bridge

Iwasawa decomposition: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ ($\mathfrak{a} \subseteq \mathfrak{p}$: a Cartan subspace, $\mathfrak{n}$: sum of positive restricted root spaces)
Let W_0 be the Weyl group of the restricted root system. For Hermitian G/K, $W_0 \cong S_n \ltimes \mathbb{Z}_2^n$.

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$$\begin{aligned} \mathrm{End}_{\mathfrak{k}}(W(\lambda)) \to \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}} &\xrightarrow{\Gamma} \mathfrak{P}(\mathfrak{a}^*)^{W_0} \\ 1 \longmapsto\ &\Gamma(D_\lambda) \end{aligned}$$

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Let $V(\mu)$ be the irreducible $\mathfrak{g}$-mod of highest weight $\sum \mu_i \gamma_i$. Then $V(\mu)$ has a spherical vector $v^{\mathfrak{k}}$, i.e. $\mathfrak{k}.v^{\mathfrak{k}} = 0$. $D_\lambda \in \mathfrak{U}^{\mathfrak{k}}$ acts on $v^{\mathfrak{k}}$ as $\Gamma(D_\lambda)(\mu)$, hence the word EIGENVALUE!

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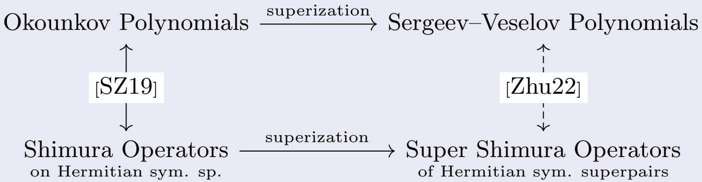
Big Picture

In [Zhu22], I obtained a super analog of this result. Specifically, we show the eigenvalues of the **super Shimura operators** are up to constant equal to **Type** BC **supersymmetric interpolation polynomials** developed by Sergeev and Veselov [SV09].

Big Picture



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- Do Sergeev–Veselov polynomials live in $\operatorname{Im}\Gamma$? *Yes.*

Lie Superalgebras

General Principle of Superization

A (good) $\mathbb{Z}_2$-grading for everything!

$$\mathbb{Z}_2 = \{\overline{0}, \overline{1}\} = \{\texttt{even}, \texttt{odd}\}$$

Lie Superalgebras

Definition

A *vector superspace* V is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. A vector $v \in V_{\bar{0}}$ (resp. $V_{\bar{1}}$) is said to be *even* (resp. *odd*) and write $|v| = 0$ (resp. 1). Denote the vector superspace with even subspace $\mathbb{C}^m$ and odd subspace $\mathbb{C}^n$ as $\mathbb{C}^{m|n}$.

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Definition ([Kac77])

A *Lie superalgebra* is a vector superspace $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with a bilinear map $[-,-] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying

1. $[X, Y] = -(-1)^{|X||Y|}[Y, X]$
2. $[[X, Y], Z] = [X, [Y, Z]] - (-1)^{|X||Y|}[Y, [X, Z]]$

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$\mathrm{Hom}(V, W)$ does NOT contain odd maps, e.g. $V_i \xrightarrow{\Pi} V_{i+1}$. To include odd maps, we write $\underline{\mathrm{Hom}}(V, W)$ (resp. $\underline{\mathrm{End}}(V)$), which consists of ALL linear maps between V, W (resp. V, V).

Super $\mathfrak{gl}$

$\underline{\mathrm{End}}(\mathbb{C}^{m|n})$ is a LSA denoted as $\mathfrak{gl}(m|n)$.

$\mathfrak{gl}_{\overline{0}}$: $\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)$ preserves $|v|$. $\mathfrak{gl}_{\overline{1}}$: $\left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array}\right)$ reverses $|v|$.

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Theorem (We still have PBW)

Let $\mathfrak{g}$ be a f.d. LSA and $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{g}$ consisting of homogeneous elements. Then the following set of monomials is a basis for $\mathfrak{U}(\mathfrak{g})$

$$\left\{x_1^{k_1} \cdots x_n^{k_n}\right\}$$

where $k_i \in \mathbb{N}$ if $|x_i| = 0$ and $k_i \in \{0, 1\}$ if $|x_i| = 1$.

Hermitian Symmetric Superpairs

Now let's introduce the super twins for the superized problem. First, let $(\mathfrak{g}, \mathfrak{k})$ be a pair of complex Lie superalgebras so $\mathfrak{k} \subseteq \mathfrak{g}$.

Definition (Hermitian Symmetric Superpair)

If there is an element J in the center of $\mathfrak{k}$ whose adjoint action gives the $(-1, 0, 1)$-eigenspace decomposition and grading

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+,$$

with $\mathfrak{p} = \mathfrak{p}^- \oplus \mathfrak{p}^+$, then we say $(\mathfrak{g}, \mathfrak{k})$ is a *Hermitian symmetric superpair*.

The $\mathfrak{gl}$-pair

Our discussion focuses on

$$\mathfrak{g} = \mathfrak{gl}(2p|2q), \mathfrak{k} = \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q)$$

with the following embedding of $\mathfrak{k}$ into $\mathfrak{g}$.

$$\left(\left(\begin{array}{c|c} A^{(1)}_{p\times p} & B^{(1)}_{p\times q} \\ \hline C^{(1)}_{q\times p} & D^{(1)}_{q\times q} \end{array} \right), \left(\begin{array}{c|c} A^{(2)}_{p\times p} & B^{(2)}_{p\times q} \\ \hline C^{(2)}_{q\times p} & D^{(2)}_{q\times q} \end{array} \right) \right)$$

$$\mapsto \left(\begin{array}{cc|cc} A^{(1)}_{p\times p} & 0_{p\times p} & B^{(1)}_{p\times q} & 0_{p\times q} \\ 0_{p\times p} & A^{(2)}_{p\times p} & 0_{p\times q} & B^{(2)}_{p\times q} \\ \hline C^{(1)}_{q\times p} & 0_{q\times p} & D^{(1)}_{q\times q} & 0_{q\times q} \\ 0_{q\times p} & C^{(2)}_{q\times p} & 0_{q\times q} & D^{(2)}_{q\times q} \end{array} \right)$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) consists of matrices with non-zero entries only in the upper right (resp. bottom left) sub-blocks in each of the four blocks.

The $\mathfrak{gl}$-pair

The involution θ can be given by $\operatorname{Ad}\exp(i\pi J)$ with

$$J = \frac{1}{2}\left(\begin{array}{cc|cc} I_{p\times p} & & & \\ & -I_{p\times p} & & \\ \hline & & I_{q\times q} & \\ & & & -I_{q\times q} \end{array}\right)$$

Observe that J is central in $\mathfrak{k}$, and the above decomposition is the eigenspace decomposition with respect to J, which also gives the short grading.
Indeed a Hermitian symmetric superpair.

Super Shimura Operators

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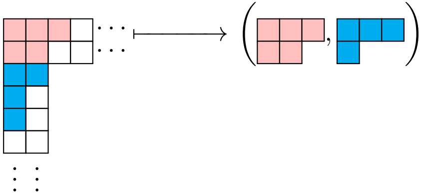
2. $\langle x \rangle := \max\{x, 0\}$ for $x \in \mathbb{Z}$.
3. $\mathscr{H}(p,q) := \{\lambda : \lambda_{p+1} \leq q\}$ (*(p,q)-hooks*)
4. $\lambda^\natural = \left(\lambda_1, \ldots, \lambda_p, \langle \lambda'_1 - p \rangle, \ldots, \langle \lambda'_q - p \rangle\right)$

For example, consider $(3,2,2,1,1) \in \mathscr{H}(2,2)$, then $(3,2,2,1,1)^\natural = (3,2,3,1)$



Super Shimura Operators

For $(\mathfrak{gl}(2p|2q), \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q))$:

Theorem (Cheng–Wang ([CW01]))

$$\mathfrak{S}^d(\mathfrak{p}^+) = \bigoplus_{\substack{\lambda \in \mathscr{H}(p,q) \\ |\lambda| = d}} W\left(\lambda^\natural\right), \quad \mathfrak{S}^d(\mathfrak{p}^-) = \bigoplus_{\substack{\lambda \in \mathscr{H}(p,q) \\ |\lambda| = d}} W^*\left(\lambda^\natural\right)$$

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$D_\lambda \in \mathfrak{U}(\mathfrak{g})^{\mathfrak{k}}$ is called the *super Shimura operator associated with the partition* λ.

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Goal: relate the "Harish-Chandra image" of D_λ with some polynomial.

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  - π depends on a choice of positivity.
- Harish-Chandra homomorphism. $\Gamma(D)(\lambda) = \pi(D)(\lambda - \rho)$.
  - The minus sign is due to the opposite $\mathfrak{n}^-$.
  - DOES NOT DEPEND ON POSITIVITY.

Iwasawa Decomposition and Harish-Chandra Homomorphism

Alldridge ([All12]) introduced certain subalgebra $J_\alpha \subseteq \mathfrak{S}(\mathfrak{a})$, and described the map Γ.

- Σ: (restricted) root system
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We specialize the result to our pair as follows:

Theorem (Alldridge, [All12])

The kernel of Γ *is* $\mathfrak{k}\mathfrak{U} \cap \mathfrak{U}^{\mathfrak{k}}$ *and the image of* Γ *is* $J(\mathfrak{a})$, *where*

$$J(\mathfrak{a}) = \mathfrak{S}(\mathfrak{a})^{W_0} \bigcap_{\alpha \in {}_0\Sigma} J_\alpha.$$

All these J_α can be written out explicitly!

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Groups are used to study symmetries.

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One can use groupoids to capture supersymmetries!

Groupoids

Definition (Groupoid)

A *groupoid* is a small category whose morphisms are invertible.

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Definition (Invariance)

We say a groupoid $\mathcal{G}$ *acts on* a v.s. V if there is a functor $\mathtt{C} : \mathcal{G} \to \mathcal{AF}(V)$. For a function F defined on V, we say F is $\mathcal{G}$*-invariant* if for all $x \xrightarrow{f} y$ in $\mathcal{G}$, we have

$$F|_{\mathtt{C}(x)} = F|_{\mathtt{C}(y)} \circ \mathtt{C}(f) \iff F|_{\mathtt{C}(x)}(x) = F|_{\mathtt{C}(y)}(\mathtt{C}(f)(x))$$

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Provides a better formulation of $\operatorname{Im}\Gamma$.

Weyl Groupoids

Denote the set of isotropic roots as ${}_0\Sigma$. Let W_0 be the Weyl group.

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$$\mathrm{Hom}_{{}_0\mathcal{S}}(\alpha, \beta) = \begin{cases} \varnothing & \text{if } \beta \neq \pm\alpha \\ \{\bar{\tau}_\alpha\} & \text{if } \beta = -\alpha \\ \{\mathrm{id}_\alpha\} & \text{if } \beta = \alpha \end{cases}.$$

One can define the semidirect product of W_0 and ${}_0\mathcal{S}$ via the action of W_0 on ${}_0\Sigma$.

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One can define the semidirect product of W_0 and ${}_0\mathcal{S}$ via the action of W_0 on ${}_0\Sigma$. Weyl groupoid:

$$\mathfrak{W} := W_0 \sqcup W_0 \ltimes {}_0\mathcal{S}$$

Weyl Groupoids

An action ([SV11]) of $\mathfrak{W} = W_0 \sqcup W_0 \ltimes {}_0\mathcal{S}$ on $\mathfrak{a}^*$ is given by (loosely speaking),

1. sending $* \in \mathrm{Obj}(W_0)$ to the entire $\mathfrak{a}^*$, and W_0 acts on $\mathfrak{a}^*$ as usual;
2. sending $\alpha \in \mathrm{Obj}({}_0\mathcal{S}) = {}_0\Sigma$ to $\Pi_\alpha := \{\mu \in \mathfrak{a}^* : (\mu, \alpha) = 0\}$, and $\bar{\tau}_\alpha$ to $\tau_\alpha : \mu \mapsto \mu + \alpha$ in Π_α;
3. making sure that W_0's action and ${}_0\mathcal{S}$'s action are compatible.

Punchline

Proposition ([Zhu22] Weyl Groupoid Formulation of $\operatorname{Im}\Gamma$)

$$\operatorname{Im}\Gamma = \mathfrak{S}\left(\mathfrak{a}\right)^{\mathfrak{W}} \cong \mathfrak{P}(\mathfrak{a}^*)^{\mathfrak{W}}$$

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Theorem (Sergeev and Veselov)

For each $\mu \in \mathscr{H}(p,q)$*, there exists a unique (suitably normalized) polynomial* $I_\mu \in \operatorname{Im}\Gamma \cong \mathfrak{P}(\mathfrak{a}^*)^{\mathfrak{W}}$ *of degree* $2|\mu|$ *such that*

$$I_\mu(\lambda^\natural + \rho) = 0$$

for any $\lambda \not\supseteq \mu$ *(*the **vanishing condition***).*

Result

Theorem ([Zhu22])

Assuming a conjecture, the Harish-Chandra image of the super Shimura operator associate with μ, $\Gamma(D_\mu)$, is equal to some non-zero multiple of I_μ.

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A $\mathfrak{g}$-module V is *spherical* if $V^{\mathfrak{k}} := \{v \in V : \mathfrak{k}.v = 0\}$ is non-zero.
Recall: in the non-super setting, $\Gamma(D_\mu)$ is realized as the eigenvalue of D_μ with eigenvector $v^{\mathfrak{k}}$ (spherical in $V(\lambda)$, the irreducible $\mathfrak{g}$-mod of highest weight $\sum \lambda_i \gamma_i$)

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Which Borel?

This weight is highest w.r.t the Borel subalgebra of $\mathfrak{g}$ extended from the one of $\mathfrak{k}$ by $\mathfrak{p}^+$.

Spherical Representations

For our $(\mathfrak{gl}(2p|2q), \mathfrak{gl}(p|q) \oplus \mathfrak{gl}(p|q))$:

Conjecture

Every irreducible $\mathfrak{g}$-module $V\left(\lambda^\natural\right)$ for $\lambda \in \mathscr{H}(p,q)$ is spherical.

We proved a partial result:

Theorem ([Zhu22])

For $p = q = 1$, all the irreducible $\mathfrak{g}$-modules $V\left(\lambda^\natural\right)$ are spherical.

Spherical Representations

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Cartan–Helgason

Given $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$, we extend $\mathfrak{a}$ to a Cartan $\mathfrak{h}$ of $\mathfrak{g}$. Then

$\lambda \in \mathfrak{h}^*$ is the highest weight of an irr. f.d. spherical $\mathfrak{g}$-rep.

IF AND ONLY IF

$\lambda(i(\mathfrak{h} \cap \mathfrak{k})) = 0$ and λ is integral.

The second condition loosely translates to some "evenness" property. E.g. for $(\mathfrak{sl}(2), \mathfrak{so}(2) = \textit{diag. Cartan } = \mathfrak{t})$, sphericity $\iff$ 0 is a weight. This happens iff the highest $\mathfrak{t}$-weight is *even*!

Super Case

In the super picture:

Theorem (Alldridge, [AS15])

Let V be a $\mathfrak{g}$-module of highest weight $\lambda \in \mathfrak{h} \supseteq \mathfrak{a}$. Then V is spherical if

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1. Sufficient but not necessary;
2. The high enough condition is a purely odd condition. Technical. Involves root multiplicities.
3. The trivial rep ($\lambda = 0$) is spherical, but is not high enough;

For $\lambda \in \mathscr{H}(p, q)$, this is NOT enough to deduce that each $V(\lambda^\natural)$ is spherical!

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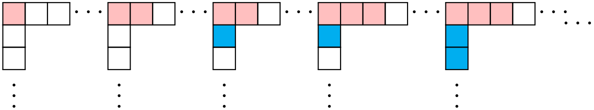

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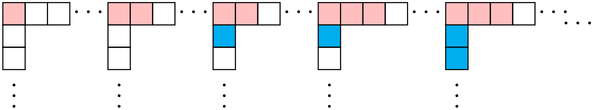

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Spherical Representations

Sketch of the proof (cont.d).

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Punchline 1: $V(\lambda^{\natural})$ is the irreducible quotient of $K(\check{\lambda})$.

Punchline 2: $K(\check{\lambda})$ is indeed quasi-spherical.

□

Spherical Representations

Sketch of the proof (cont.d).

$K(\breve{\lambda}) := \mathrm{Ind}_{\mathfrak{g}_{\bar{0}}+\mathfrak{g}_1}^{\mathfrak{g}} \breve{W}(\breve{\lambda})$

This $\breve{\lambda}$ is a result of changing the Borel of $\mathfrak{g}$ to the *distinguished* one. This step is non-trivial in the super scenario. This is what essentially stops us from generalizing p, q.

Punchline 1: $V(\lambda^\natural)$ is the irreducible quotient of $K(\breve{\lambda})$.

Punchline 2: $K(\breve{\lambda})$ is indeed quasi-spherical.

Only two possibilities for $\lambda = (a, 1^b) \in \mathscr{H}(1,1)$ are present. We show when $b \neq a-1$, $K(\breve{\lambda})$ has a spherical vector that descends to $V\left(\lambda^\natural\right)$. For $b = a-1$, by studying "degree 2" operators in $\mathfrak{U}(\mathfrak{g})$, we prove that $K(\breve{\lambda})$ is indeed quasi-spherical. The proof is computational. □

Thank you!

https://arxiv.org/pdf/2212.09249.pdf
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