A mathematical theory of gapless boundaries of 2+1D topological orders

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- A well known problem in mathematics:

  For a given modular tensor category (MTC) $\mathcal{C}$, find a mathematical structure $\mathcal{D}$ such that its "Drinfeld center" gives $\mathcal{C}$, i.e. $\mathcal{C} \simeq Z(\mathcal{D})$.

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- For an MTC $\mathcal{D}$, its Drinfeld center $\mathcal{C} \simeq Z(\mathcal{D}) \simeq \mathcal{D} \boxtimes \overline{\mathcal{D}}$. Therefore, in some sense, we are trying to make sense of taking the square root of an MTC, i.e. "$\sqrt{\mathcal{C}}$". It certainly reminds us "$\sqrt{-1}$", "$\sqrt{\Delta}$", etc.

Partial results:

- When $\mathcal{C}$ is the Drinfeld center of a spherical fusion category $\mathcal{M}$, i.e. $\mathcal{C} = Z(\mathcal{M})$, the TQFT can be extended to points, to each of which we assign $\mathcal{M}$. Such TQFT is called Turaev-Viro TQFT.

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- For a generic MTC, there is no clue from the mathematical side.

Clues from physics:

In physics, topological orders are certain quantum phases (at zero temperature) whose low energy effective field theories are topological field theories.

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A 2d topological order is described by a pair $(\mathcal{C}, c)$, where

- $\mathcal{C}$ is a unitary modular tensor category (UMTC) $\mathcal{C}$,
    1. objects in $\mathcal{C}$ are topological excitations;
    2. morphisms are observables on 0+1D world line (i.e. instantons).
- c is a real number called chiral central charge.

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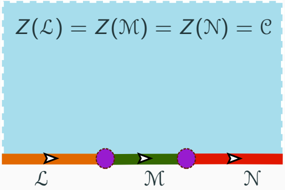
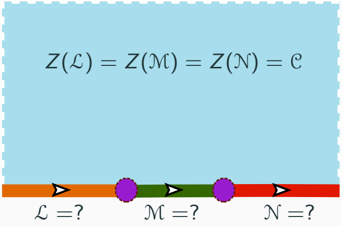
  2. morphisms are observables on 0+1D world line (i.e. instantons).
- Boundary-bulk relation:
  1. $Z(\mathcal{M}) = \mathcal{C}$; [Kitaev-K.:11, K.:13]
  2. Two edges $\mathcal{M}$ and $\mathcal{N}$ share the same bulk as their Drinfeld center iff they are Morita equivalent [Müger:01,Etingof-Nikshych-Ostrik:08].



Without knowing what a gapless boundary is, in 2105, K-Wen-Zheng provide a physical proof of the boundary-bulk relation for topological orders in all dimensions, i.e.



1. Our mother nature provides a solution to the equation $\mathcal{C} = Z(?)$.
2. Only thing remains to do is to read her book.

Now let us start to read mother nature's book.

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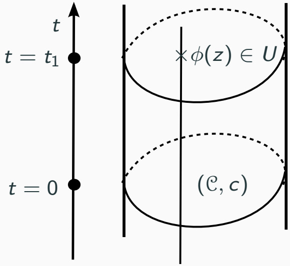
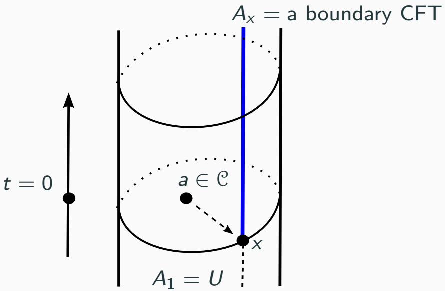
Since the category $\mathcal{C}$ describes the fusion-braiding properties of anyons, which can be viewed as the observables of the 2d phase in the long wave length limit, ? in the equation $\mathcal{C} = Z(?)$ shoud be all the observables on a chiral/non-chiral gapless edge of a 2d topological order $(\mathcal{C}, c)$.

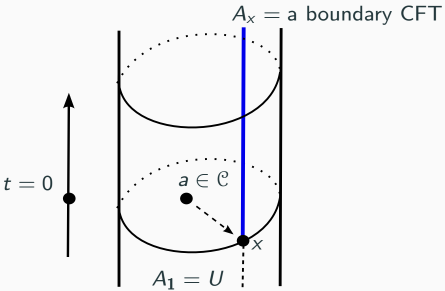
Observables on the 1+1D world sheet of a gapless edge of $(\mathcal{C}, c)$:



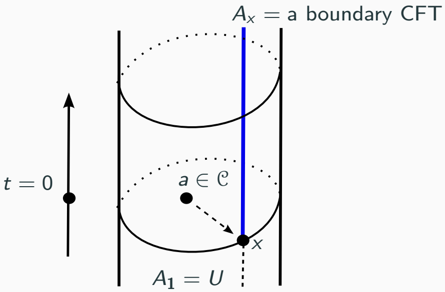
On the edge of 2d bulk topological phase $(\mathcal{C}, c)$, there are gapless chiral edge modes which are states in a chiral algebra (= VOA) U. Chiral fields $\phi(z)$ in U lives on the entire 1+1D world sheet of the 1d boundary.



1. When a bulk excitation $a \in \mathcal{C}$ is moved to the edge, it creates a topological edge excitation x.



1. When a bulk excitation $a \in \mathcal{C}$ is moved to the edge, it creates a topological edge excitation x.
2. The chiral fields living on the world line supported on x are (potentially) different from those in U. We denote the space of all these chiral fields by A_x.



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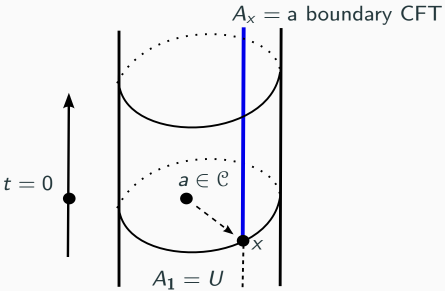
3. These chiral fields has OPE. A_x is an open-string VOA Huang-K.:2004 (actually a boundary CFT).



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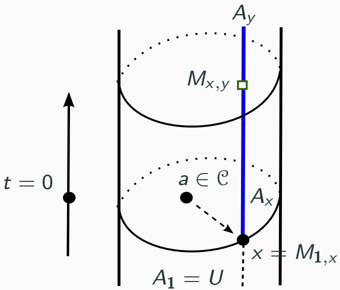
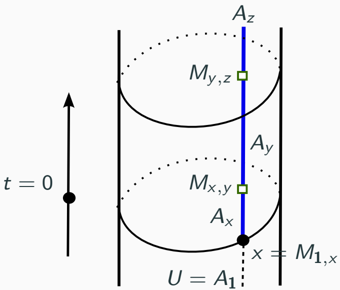
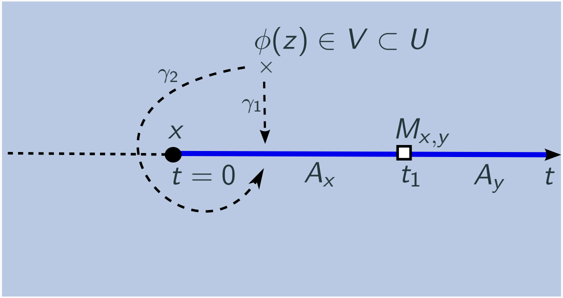
4. $A_{\mathbf{1}} = U$, where $\mathbf{1}$ is the trivial topological edge excitation.



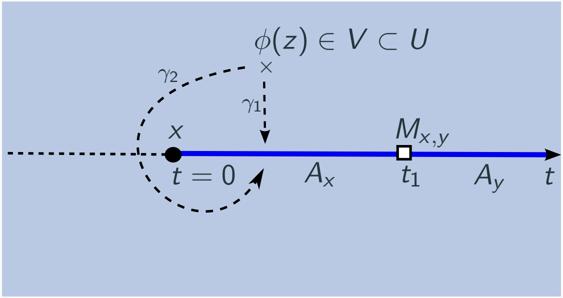
1. It is possible that the topological edge excitation is changed from x to y on the world line at $t = t_1 > 0$.
2. We use $M_{x,y}$ to denote the space of defect fields (boundary condition changing operators) between two boundary CFT's.
3. We have $x = M_{\mathbf{1},x}$ and $A_x = M_{x,x}$.



1. defects fields can be fused (OPE): $M_{y,z} \otimes_{\mathbb{C}} M_{x,y} \to M_{x,z}$.
2. associativity of OPE: $M_{z,w} \otimes_{\mathbb{C}} M_{y,z} \otimes_{\mathbb{C}} M_{x,y} \to M_{x,w}$
3. as a special case: $A_x \otimes_{\mathbb{C}} M_{x,y} \otimes_{\mathbb{C}} A_y \to M_{x,y}$.



Compatibility among $U, A_x, M_{x,y}$. Note that $\iota_{\gamma_i} : U \to A_x$ or $M_{x,y}$,



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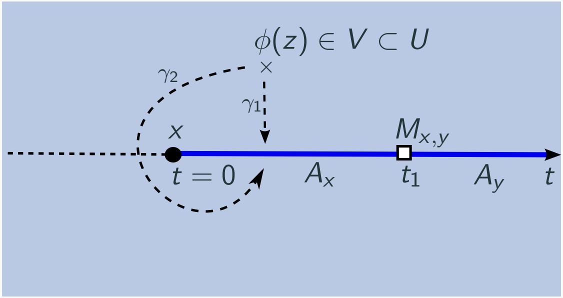

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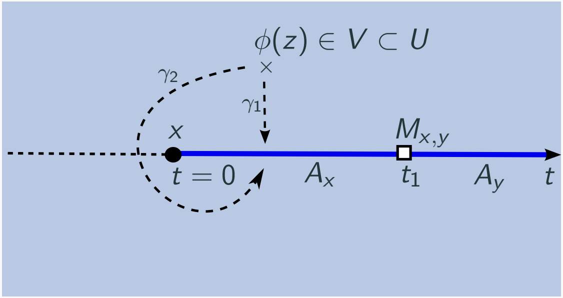
1. ι_{γ_i} preserves OPE's, i.e. an open-string VOA homomorphism.
2. Conformal symmetric boundary condition (minimal requirement): $V = \langle \omega \rangle \subset U$ and $\iota_{\gamma_i}|_V$ are injective and path independent.



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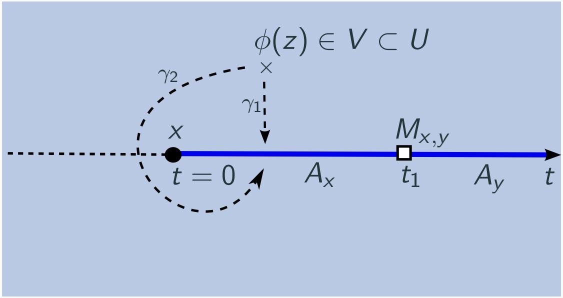
3. V-symmetric boundary condition: $\iota_{\gamma_i}|_V$ are injective and path independent. $V \subsetneq U$ in general.



1. V-symmetric boundary condition: $\iota_{\gamma_i} : V \to A_x$ are injective and path independent. $V \subsetneq U$ in general.
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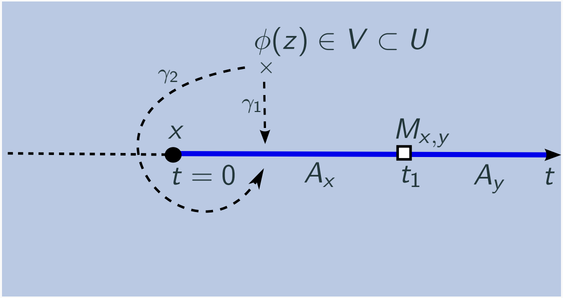

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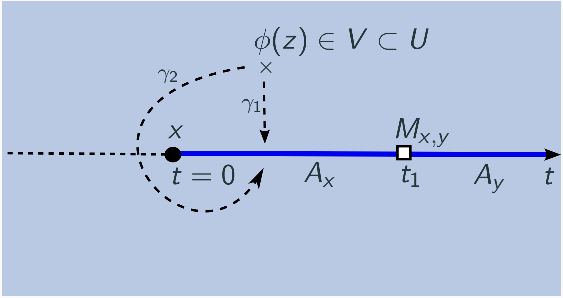
Frenkel-Huang-Lepowsky:1993



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4. $M_{x,y} \in \mathrm{Mod}_V$, the fusion of defect fields is determined by a morphism $M_{y,z} \otimes M_{x,y} \to M_{x,z}$ in Mod_V. Huang-Lepowsky:1995, Huang:1995

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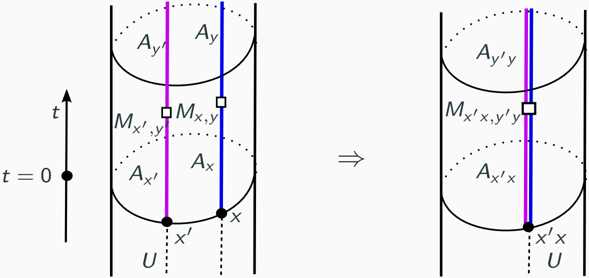
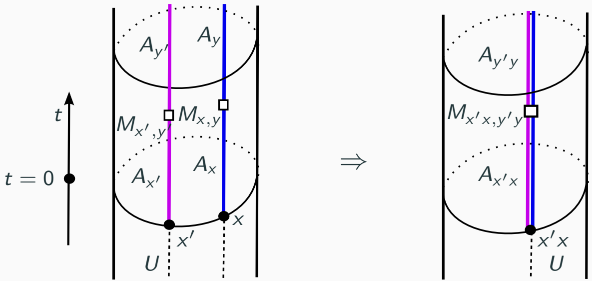
$\mathcal{X}^\sharp$ is a category enriched in Mod_V, or an Mod_V-enriched category.



1. $\otimes : (x', x) \mapsto x'x = x' \otimes x$ and $M_{x',y'} \otimes M_{x,y} \to M_{x' \otimes x, y' \otimes y}$ satisfying some obvious properties.



1. $\otimes : (x', x) \mapsto x'x = x' \otimes x$ and $M_{x',y'} \otimes M_{x,y} \to M_{x'\otimes x, y'\otimes y}$ satisfying some obvious properties.
2. $\otimes$ upgrades $\mathcal{X}^\sharp$ to an Mod_V-enriched monoidal category, a notion which was introduced only recently [Batanin-Markl:12, Morrison-Penneys:17].

Theorem (K.-Zheng, 2017)

All observables on a gapless edge of a 2d topological order $(\mathcal{C}, c)$ *can be described by a pair* $(V, \mathcal{X}^\sharp)$*, where*

1. V *is a VOA (chiral symmetry);*
2. $\mathcal{X}^\sharp$ *is an* Mod_V*-enriched monoidal category.*

Note that $U = A_{\mathbf{1}} = M_{\mathbf{1},\mathbf{1}}$ is a data in $\mathcal{X}^\sharp$.

Now we assume that the chiral symmetry is a (unitary) rational VOA such that Mod_V is a (unitary) modular tensor category (MTC) Huang:2005,2008

1. Rational open-closed CFT over V (($g > 1$)-theories are conjectural):
   - A_{op} = simple symmetric special Frobenius algebra (SSSFA) in Mod_V; Fuchs-Runkel-Schweigert:2003-2007, K.:2008, K.-Runkel:2009
   - A_{cl} = Lagrangian algebra in $Z(\mathrm{Mod}_V)$; K.-Runkel:2009
   - boundary-bulk relation:
     1.1 $A_{\mathrm{cl}} = Z(A_{\mathrm{op}})$; Fuchs-Runkel-Schweigert:2003-2007
     1.2 $Z(A_{\mathrm{op}}^{(1)}) = Z(A_{\mathrm{op}}^{(2)})$ iff $A_{\mathrm{op}}^{(1)}$ and $A_{\mathrm{op}}^{(2)}$ are Morita equivalent K.-Runkel:2008.

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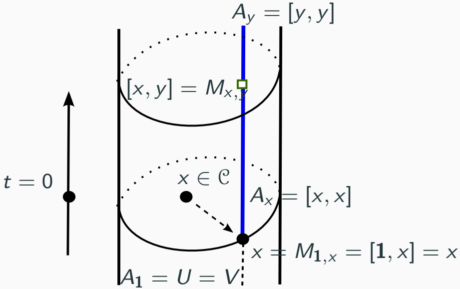
Example: $\mathcal{C} = \mathrm{Mod}_V$, $Z(\mathcal{C}) = \mathcal{C} \boxtimes \overline{\mathcal{C}}$ [Müger:1999]

1. $A = \mathbf{1} \in \mathcal{C}$, $Z(\mathbf{1}) = \oplus_i i^* \boxtimes i$.
2. $\forall x \in \mathcal{C}$, $A = [x, x] = x \otimes x^*$ are SSSFA's in $\mathcal{C}$.
   $Z([x, x]) = Z(\mathbf{1}) = \oplus_i i^* \boxtimes i$.
3. $[x, y]$ defines a V-symmetric defect between boundary CFT's $[x, x]$ and $[y, y]$ [Fröhlich-Fuchs-Runkel-Schweigert:2007].



A canonical gapless edge $(V, \mathcal{C}^\sharp)$ of $(\mathcal{C}, c)$:

1. edge excitations = bulk excitations = $\mathcal{C}$; $U = V$, $\mathrm{Mod}_V = \mathcal{C}$;
2. $M_{x,y} := [x, y] = y \otimes x^*$;
3. $\mathrm{id}_x : \mathbf{1} \to [x, x] = x \otimes x^*$ is given by the duality map;
4. $[y, z] \otimes [x, y] = z \otimes y^* \otimes y \otimes x^* \to z \otimes x^* = [x, z]$.
5. $[x', y'] \otimes [x, y] = y' \otimes x'^* \otimes y \otimes x^* \xrightarrow{1 c_{x'^*, y \otimes x^*} 1} y' \otimes y \otimes x^* \otimes x'^* = [x'x, y'y]$.

Canonical construction: Morrison-Penneys:2017

Let $\mathcal{B}$ be a braided monoidal category and $\mathcal{M}$ a monoidal category. Let $f : \overline{\mathcal{B}} \to Z(\mathcal{M})$ be a braided oplax-monoidal functor. Then we have a functor $\odot : \overline{\mathcal{B}} \times \mathcal{M} \to Z(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$. There is a canonical construction of a $\mathcal{B}$-enriched monoidal category ${}^{\mathcal{B}}\mathcal{M}$ from the pair $(\mathcal{B}, \mathcal{M})$:

- objects in ${}^{\mathcal{B}}\mathcal{M}$ are objects in $\mathcal{M}$, i.e. $Ob(\mathcal{M}^\sharp) := Ob(\mathcal{M})$;
- For $x, y \in \mathcal{M}$, $\hom_{{}^{\mathcal{B}}\mathcal{M}}(x, y) := [x, y]$ in $\overline{\mathcal{B}}$ (or in $\mathcal{B}$);
- $\mathrm{id}_x : \mathbf{1}_{\mathcal{B}} \to [x, x]$ is the morphism in $\mathcal{B}$ canonically induced from the unital action $\mathbf{1}_{\mathcal{B}} \odot x \simeq x$;
- $\circ : [y, z] \otimes [x, y] \to [x, z]$ is the morphism canonically induced from the action $([y, z] \otimes [x, y]) \odot x \to [y, z] \odot y \to z$.
- $\otimes : [x', y'] \otimes [x, y] \to [x' \otimes x, y' \otimes y]$ is the morphism in $\mathcal{B}$ canonically induced from the action

$$\begin{aligned}([x', y'] \otimes [x, y]) \odot x' \otimes x &= \phi_{\mathcal{M}}([x', y'] \otimes [x, y]) \otimes x' \otimes x \\ &\to \phi_{\mathcal{M}}([x', y']) \otimes \phi_{\mathcal{M}}([x, y]) \otimes x' \otimes x \\ &\xrightarrow{\mathrm{Id} \otimes b_{\phi_{\mathcal{M}}([x,y]),x'} \otimes \mathrm{Id}_x} \phi_{\mathcal{M}}([x', y']) \otimes x' \otimes \phi_{\mathcal{M}}([x, y]) \otimes x \to y' \otimes y.\end{aligned}$$

Remark:

1. In the canonical edge $(V, \mathcal{C}^\sharp)$ of $(\mathcal{C}, c)$, $\mathcal{C}^\sharp$ is obtained from the pair $(\mathcal{C}, \mathcal{C})$ (because $\overline{\mathcal{C}} \hookrightarrow \overline{\mathcal{C}} \boxtimes \mathcal{C} = Z(\mathcal{C})$) via the canonical construction. We will also denote the canonical edge $(V, \mathcal{C}^\sharp)$ by $(V, {}^{\mathcal{C}}\mathcal{C})$, i.e. $(V, \mathcal{C}^\sharp) = (V, {}^{\mathcal{C}}\mathcal{C})$.

2. Let $\mathbf{H}$ be the category of finite dimensional Hilbert spaces. A unitary fusion category (UFC) $\mathcal{M}$ can be viewed as an $\mathbf{H}$-enriched monoidal category canonical constructed from $(\mathbf{H}, \mathcal{M})$, i.e. $\mathcal{M} = {}^{\mathbf{H}}\mathcal{M}$. A gapped edge of $(\mathcal{C}, 0)$ can be denoted by $(\mathbb{C}, {}^{\mathbf{H}}\mathcal{M})$ such that $Z(\mathcal{M}) = \mathcal{C}$, where $\mathbb{C}$ is viewed as the trivial VOA.

Question: $Z(\mathcal{C}^\sharp) = Z({}^{\mathcal{C}}\mathcal{C}) = \mathcal{C}$?

Definition (K.-Zheng, 2017)

Let $\mathcal{C}^\sharp$ be a monoidal category enriched over $\mathcal{B}$. A *half-braiding* for an object $x \in \mathcal{C}^\sharp$ is an enriched natural isomorphism $b_x : x \otimes - \to - \otimes x$ between enriched endo-functors of $\mathcal{C}^\sharp$ such that it defines a half-braiding in the underlying monoidal category $\mathcal{C}$. The Drinfeld center of $\mathcal{C}^\sharp$ is a category $Z(\mathcal{C}^\sharp)$ enriched over $\mathcal{B}$ defined as follows:

- an object is a pair (x, b_x), where $x \in \mathcal{C}^\sharp$ and b_x is a half-braiding for x;
- $\hom_{Z(\mathcal{C}^\sharp)}((x, b_x), (y, b_y))$ is the intersection of the equalizers of the diagrams $\hom_{\mathcal{C}^\sharp}(x, y) \rightrightarrows \hom_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y)$ depicted below for all $z \in \mathcal{C}^\sharp$

$$\begin{array}{ccc} \hom_{\mathcal{C}^\sharp}(x, y) & \xrightarrow{\otimes\circ(\mathrm{id}_z \otimes \mathrm{Id})} & \hom_{\mathcal{C}^\sharp}(z \otimes x, z \otimes y) \\ {\scriptstyle \otimes\circ(\mathrm{Id}\otimes\mathrm{id}_z)} \downarrow & & \downarrow {\scriptstyle -\circ b_{x,z}} \\ \hom_{\mathcal{C}^\sharp}(x \otimes z, y \otimes z) & \xrightarrow{b_{y,z}\circ -} & \hom_{\mathcal{C}^\sharp}(x \otimes z, z \otimes y); \end{array}$$

- the composition law $\circ$ is induced from that of $\mathcal{C}^\sharp$.

Remark: This notion satisfies the same universal property of the notion of center but in a new 2-category of enriched categories, where an enriched functor is allowed to change the base categories. K.-Yuan-Zhang-Zheng:2021

Theorem (K.-Zheng, 2017)

$Z({}^{\mathcal{C}}\mathcal{C}) = \mathcal{C}$.

Boundary-bulk duality holds for the canonical gapless edge!

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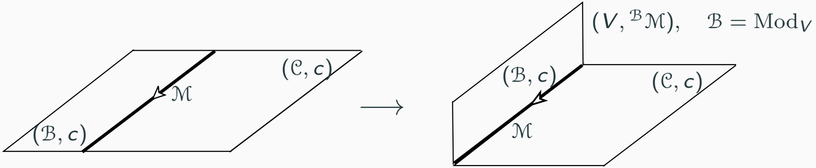
Theorem (Zheng, 2017)

The enriched monoidal category ${}^{\mathcal{C}}\mathcal{C}$ *is a fully dualizable object in a symmetric monoidal (4,3)-category. According to cobordism hypothesis, it defines a 0-1-2-3-4 TQFT. It assigns to the circle* S^1 *an enriched monoidal category that is equivalent to the modular tensor category* $\mathcal{C}$.

For general chiral gapless boundaries obtained from topological Wick rotation of a gapped domain wall $\mathcal{M}$ between $(\mathcal{B}, c)$ and $(\mathcal{C}, c)$, i.e. $\overline{\mathcal{B}} \boxtimes \mathcal{C} \xrightarrow{\simeq} Z(\mathcal{M})$:



Theorem (K.-Zheng, 2017)

$Z({}^{\mathcal{B}}\mathcal{M}) = \mathcal{C}$.

Boundary-bulk duality holds for these gapless boundaries!

Remark: $V \subsetneq U = M_{\mathbf{1},\mathbf{1}} = [\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}] \in \mathcal{B}$ in general.

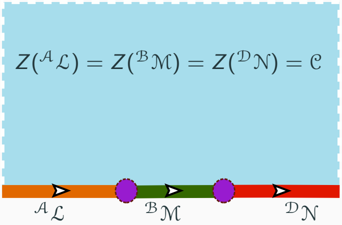
Theorem (K.-Zheng:2017,2019)

Gapped/gapless boundaries of a 2+1D topological order described by a pair $(\mathcal{C}, c)$, where $\mathcal{C}$ is unitary modular tensor category and c is chiral central charge, are classified and described by pairs $(V, {}^{\mathrm{Mod}_V}\mathcal{M})$, where

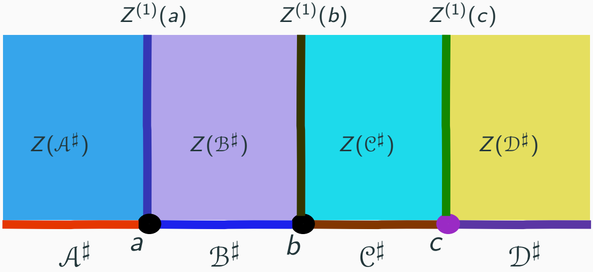
1. *V is called local quantum symmetry. It is a unitary rational VOA Huang:2005-2008,Gui:2017 of central charge c for a chiral gapless boundary; it is unitary rational full field algebra ($c_L - c_R = c$) for a non-chiral gapless boundary Huang-K.:2007; When $V = \mathbb{C}$, it is a gapped boundary.*
2. *$\mathcal{M}$ is a unitary fusion category equipped with a braided equivalence $\phi : \mathcal{C} \boxtimes \overline{\mathrm{Mod}_V} \to Z(\mathcal{M})$.*
3. *${}^{\mathrm{Mod}_V}\mathcal{M}$ is the Mod_V-enriched fusion category obtained from ϕ via the canonical construction Morrison-Penneys:2017, and is called topological skeleton the boundary phase.*
   $\hom_{{}^{\mathrm{Mod}_V}\mathcal{M}}(x, y) = [x, y] \in \mathrm{Mod}_V, \forall x, y \in \mathcal{M}$.

A gapless/gapped edge of $(\mathcal{C}, c)$ is described by a pair $(V, {}^{\mathrm{Mod}_V}\mathcal{M})$. We have boundary-bulk relation:

1. $Z({}^{\mathcal{B}}\mathcal{M}) = \mathcal{C}$; [K.-Zheng, 2017]
2. Two edges ${}^{\mathcal{A}}\mathcal{L}$ and ${}^{\mathcal{B}}\mathcal{M}$ share the same bulk as their Drinfeld center iff they are Morita equivalent [Zheng, 2017].



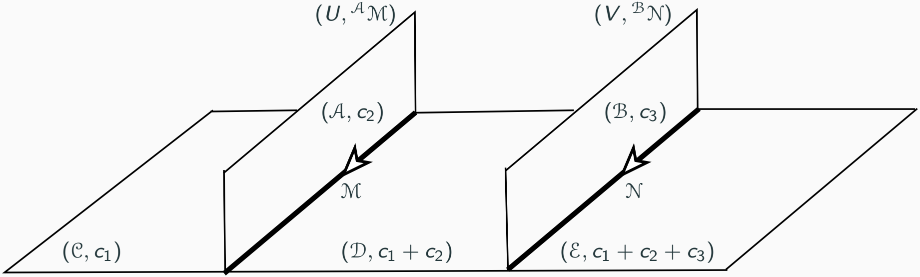
Complete boundary-bulk relation:



Theorem (K.-Zheng:2019)

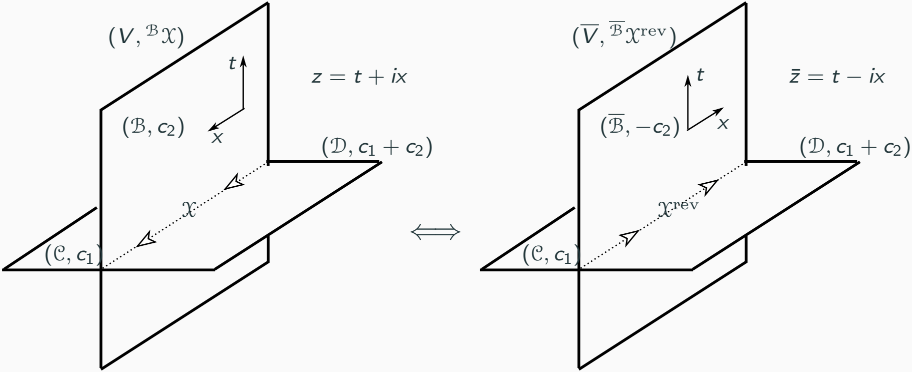
There is well-defined functor Z from the category of gapless/gapped edges to their bulks. This functor is fully faithful.

The fusion of two chiral gapless domain walls is defined as follows:

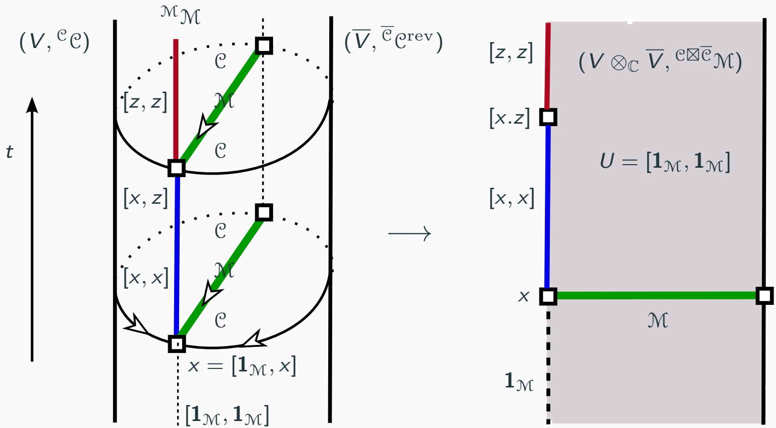


$$(U, {}^{\mathcal{A}}\mathcal{M}) \boxtimes_{(\mathcal{D}, c_1+c_2)} (V, {}^{\mathcal{B}}\mathcal{N}) = (U \otimes_{\mathbb{C}} V, {}^{\mathcal{A}\boxtimes\mathcal{B}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),$$

Physically equivalent $\iff$ flipping orientation + changing chirality

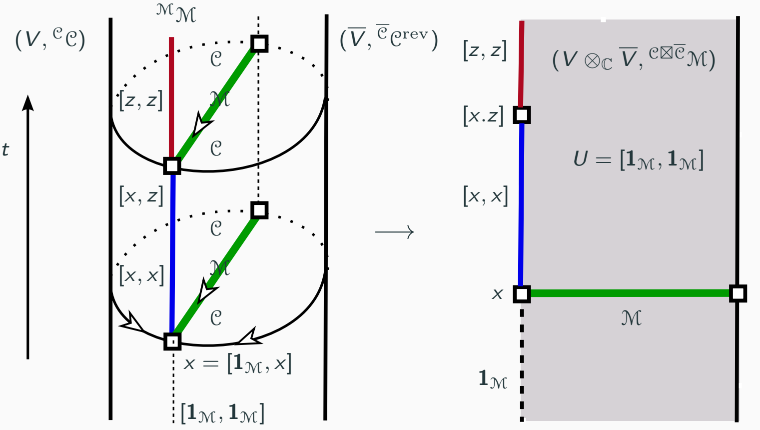


We consider a special case of above formula:

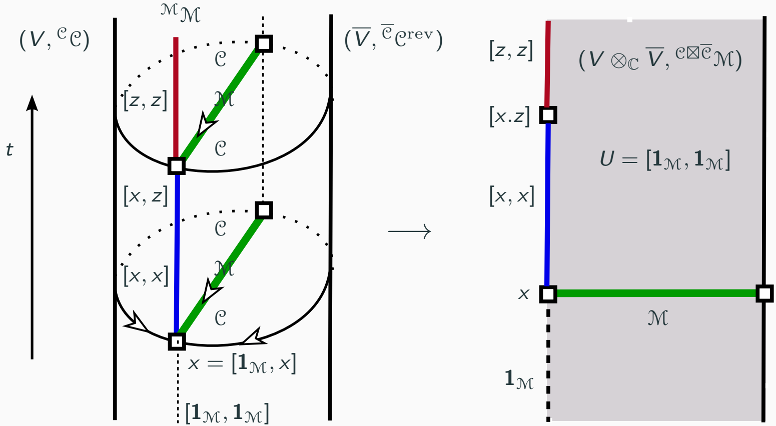


$$(V, {}^{\mathcal{C}}\mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, {}^{\mathbf{H}}\mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, {}^{\overline{\mathcal{C}}}\mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M}).$$

$(V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M})$ is a gapless domain wall between two trivial phases.



1. $U = M_{\mathbf{1},\mathbf{1}} = [\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ gives a modular invariant bulk CFT, and at the same time, gives the Lagrangian algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$, whose condensation defines the gapped boundary $\mathcal{M}$ of $(\mathcal{C} \boxtimes \overline{\mathcal{C}}, 0)$;
2. If $\mathcal{M} = \mathcal{C}$, $[\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}] = \oplus_i \, i \boxtimes i^*$ is the famous charge conjugate modular invariant bulk CFT.
3. If $\mathcal{M} \neq \mathcal{C}$, $[\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}]$ is a different modular invariant bulk CFT.



The observables on the 0+1D world line of the boundary of the domain wall $\mathcal{M}$ are described by an enriched category ${}^{\mathcal{M}}\mathcal{M}$, where $\hom_{{}^{\mathcal{M}}\mathcal{M}}(x,y) := [x,y] = y \otimes x^* \in \mathcal{M}$ are precisely BCFT's that are compatible with the bulk CFT $U = [\mathbf{1}_{\mathcal{M}}, \mathbf{1}_{\mathcal{M}}]$. Moreover, ${}^{\mathcal{M}}\mathcal{M}$ is a module over ${}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M}$. Zheng:2017,K.-Zheng:2021

Conclusion and outlooks:

1. We have found a mathematical description and the classifications of gapless/gapped edges of all 2d topological order.
2. All rational boundary-bulk CFT's can be naturally recovered from 2d topological orders via dimensional reduction.
3. It opens the way to study topological phase transitions and gapless quantum liquid phases in all dimensions.
4. Mathematically, it opens the way to study enriched monoidal (higher) categories.

Thank you !