C_1-cofiniteness and vertex tensor categories

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Conformal field theories and tensor categories

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- This semisimple tensor category theory applies to the minimal models (H., 1995), the moonshine module (H. 1995), Wess-Zumino-Witten models (H.-Lepowsky, 1997), and $N = 1$ and $N = 2$ superconformal minimal models (H.-Milas, 1999, 2000), and a semisimple category of modules for affine Lie algebras at admissible levels (Creutzig-H.-Yang, 2017).

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- V is C_2-cofinite if V as a V-module is C_2-cofinite.
- If V is C_2-cofinite and of positive energy (meaning $V_{(0)} = 0$ for $n < 0$ and $V_{(0} = \mathbb{C}\mathbf{1}$), then every irreducible V-module is C_2-cofinite (Abe, Buhl and Dong, 20024).

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C_2-cofinite reductive vertex operator algebras

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- Nahm in 1994 argued that if a suitable fusion product W_{12} of two V-modules W_1 and W_2 exists, then

$$\dim(W_{12}/C_1(W_{12})) \leq \dim(W_1/C_1(W_1)) \dim(W_2/C_1(W_2)).$$

- If the fusion product is constructed and Nahm's inequality is proved mathematically, we see that the fusion product of two C_1-cofinite modules must be C_1-cofinite.
- Miyamoto in 2014 proved a weak version of Nahm's inequality $\dim(W_3/C_1(W_3)) \leq d_{W_1,W_2}$, where $d_{W_1,W_2} \in \mathbb{N}$ depends only on W_1 and W_2 and W_3 is the image of an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1 W_2}$.
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Category of C_1-cofinite grading-restricted generalized V-modules

- By verifying the assumptions to use the logarithmic tensor category theory of H.-Lepowsky-Zhang, Creutzig-Jiang-Orosz Hunziker-Ridout-Yang (2020), Creutzig-Yang (2021), McRae (2023), and Creutzig-McRae-Orosz Hunziker-Yang (2024) proved that if the category of C_1-cofinite grading-restricted generalized V-modules is closed under the contragredient functor, then this category has a natural structure of vertex and braided tensor category structures.
- Problem: If the category of C_1-cofinite grading-restricted generalized V-modules is not closed under the contragredient functor, does it still have vertex and braided tensor category structures?

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A generalization of the logarithmic tensor category theory of H.-Lepowsky-Zhang

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- In this case, $W_1 \boxtimes_{P(z)} W_2$ is not an object of the category. Instead, $W_1 \boxtimes_{P(z)} W_2$ is grading-restricted generalized module whose contragredient is an object of the category.
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Differential equations for products of intertwining operators among C_1-cofinite grading-restricted generalized modules

- The differential equations derived by H. in 2002 should also be generalized since in that paper, we need the contragredient of an object in the category to be C_1-cofinite.
- But the results and proofs in that paper also works when two modules placed at ∞ and 0 are quasi-finite-dimensional but not necessarily C_1-cofinite.
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- More precisely, let W_1, W_2 be C_1-cofinite grading-restricted generalized modules, W_0, W_3 quasi-finite-dimensional, and W_4 a lower-bounded generalized module, and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be intertwining operators of types $\binom{W_0'}{W_1 W_4}$ and $\binom{W_4}{W_2 W_3}$, respectively, . Then the series

$$\langle w_0, \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle$$

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The main theorem

By verifying the assumptions in the generalization above using the results discussed above and some other results, we obtain the following general result:

Theorem

Let V be a vertex operator algebra, or more generally, a grading-restricted Möbius vertex algebra and $\mathcal{C}$ the category of C_1-cofinite grading-restricted generalized V-modules. Then the associativity of (logarithmic) intertwining operators among objects of $\mathcal{C}$ holds and the category $\mathcal{C}$ has a natural vertex tensor category structure. In particular, the category $\mathcal{C}$ has a natural braided tensor category structure with a twist.

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Beyond C_1-cofiniteness

- For universal Virasoro vertex operator algebras, grading-restricted generalized modules in general are not C_1-cofinite. For example, the Verma modules for the Virasoro algebra.
- In this case, we need to further generalize the tensor category theory.
- Kupiainen, Rhodes, and Vargas in 2020 and Guillarmou, Kupiainen, Rhodes, and Vargas in 2020 and 2021 constructed the Liouville copnformal field thelry.
- The Liouville theory corresponds to a conformal field theory described by the Verma modules for the Virasoro algebra and their intertwining operators.
- We conjecture that we might be able to reformulate some part of the infinite-dimensional analysis used in the construction of Liouville theory as a theory of differential equations in infinite-dimensional spaces.

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- If such a theory can be developed in the future, non-C_1-cofinite modules for vertex operator algebras can also be studied using the representation theory of vertex operator algebras.
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THANK YOU!