Positivity of interpolation polynomials

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Interpolation Polynomials

- Inhomogenous symmetric polynomials: $P_\lambda^\rho(x) = P_\lambda^\rho(x_1, \ldots, x_n)$.
- Their coefficients depend on n parameters $\rho = (\rho_1, \ldots, \rho_n)$.
- They are indexed by partitions

$$\Lambda_n = \{\lambda \in \mathbb{Z}^n : \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}.$$

- The total degree of $P_\lambda^\rho(x)$ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$
- $P_\lambda^\rho(x)$ vanishes at points of the form

$$\{\mu + \rho : \mu \in \Lambda_n,\ |\mu| \leq |\lambda|,\ \mu \neq \lambda\}.$$

- These properties characterize P_λ^ρ up to a scalar multiple.
- They were introduced by S. Sahi in 1994; motivated by earlier work with B. Kostant on generalizations of the Capelli identity.

Connection with Jack polynomials

- The special case $\rho = r\delta$, $\delta_i = n - i$, is of particular interest.
- This was studied by F. Knop and S. Sahi in [KS1].
- The main result of [KS1] is that

$$P_\lambda^{r\delta} = P_\lambda^{(\alpha)} + \text{ terms of degree} < |\lambda|.$$

- Here $P_\lambda^{(\alpha)}$ is the Jack polynomial with parameter $\alpha = 1/r$.
- Thus $P_\lambda^{r\delta}$ is sometimes called an interpolation Jack polynomial, or shifted Jack polynomial, or even a Knop-Sahi polynomial.

Jack polynomials

- Jack polynomials play a prominent role in Macdonald's remarkable monograph [M] on symmetric functions.
- They are eigenfunctions of Debiard-Sekiguchi differential operators
- Macdonald introduced a normalized version $J_\lambda^{(\alpha)} = c_\lambda(\alpha) P_\lambda^{(\alpha)}$ and considered its coefficients with respect to symmetric monomials m_μ.
- He conjectured ([M], VI.10.26?) that these coefficients belong to $\mathbb{N}[\alpha]$, *i.e.* they are positive integral polynomials in the parameter α.
- Knop and Sahi proved this conjecture in [KS2] and obtained a combinatorial formula in terms of certain "admissible" tableaux.

Positivity of interpolation polynomials

- We prove a conjecture of Knop and Sahi ([KS1]).
- This generalizes the positivity result for Jack polynomials.
- It involves the normalized interpolation polynomials

$$J_\lambda^{r\delta} := (-1)^{|\lambda|} c_\lambda(\alpha) P_\lambda^{r\delta}(-x),$$

- and their monomial coefficients

$$J_\lambda^{r\delta} = \sum_\mu \alpha^{|\mu|-|\lambda|} a_{\lambda,\mu}(\alpha) m_\mu.$$

Theorem ([Naqvi-Sahi-Sergel 2020])

The coefficients $a_{\lambda,\mu}(\alpha)$ belong to $\mathbb{N}[\alpha]$.

Example

- $$J^{r\delta}_{(2,0)} = (\tfrac{1}{r}+1)\Big((x_1+r+1)(x_1+r)+r(x_2)\Big)$$
  $$+\,2\,x_1x_2 \;+\; (\tfrac{1}{r}+1)(x_2+r+1)x_2.$$
- This vanishes at the points
  $$-(0+r,0), \quad -(1+r,0), \quad -(1+r,1)$$
- When $x_1=-1-r$ and $x_2=-1$, we have
  $$(\tfrac{1}{r}+1)\Big(0-r\Big) \;+\; 2(1+r) \;+\; (\tfrac{1}{r}+1)(r)(-1).$$

Nonsymmetric Jack polynomials

- The Jack positivity results of [KS2] are proved in greater generality.
- Jack polynomials have nonsymmetric analogs $E_\eta^{(\alpha)}$, which are indexed by compositions $\eta \in \mathbb{N}^n$.
- They are eigenfunctions of the "trigonometric Dunkl operators" due to Cherednik, and were first considered by Heckman and Opdam [O].
- Knop and Sahi defined a certain normalized version $F_\eta^{(\alpha)} = d_\eta(\alpha) E_\eta^{(\alpha)}$
- The main result of [KS2] is that $F_\eta^{(\alpha)}$ has $\mathbb{N}[\alpha]$-coefficients with respect to ordinary monomials $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$.

Nonsymmetric interpolation polynomials

- The interpolation polynomials admit nonsymmetric analogs $E^{\rho}_{\eta}(x)$.
- $E^{\rho}_{\eta}(x)$ has degree $|\eta| = \eta_1 + \cdots + \eta_n$ and vanishes at the points

$$\{\overline{\gamma} : \gamma \in \mathbb{N}^n,\ |\gamma| \le |\eta|,\ \gamma \neq \eta\},$$

- Here $\overline{\gamma} = \gamma + w_{\gamma}(\rho)$ where w_{γ} is the shortest permutation such that $w_{\gamma}^{-1}(\gamma)$ is a partition.
- These properties characterize E^{ρ}_{η} up to a multiple.
- In the special case $\rho = r\delta$ we have

$$E^{r\delta}_{\eta} = E^{(\alpha)}_{\eta} + \text{ terms of degree } < |\eta|.$$

- For these and other properties of E^{ρ}_{η} see [Kn1, S2].

Positivity of nonsymmetric interpolation polynomials

- We now formulate the nonsymmetric generalization of our result.
- This involves the normalized nonsymmetric interpolation polynomial

$$F_\eta^{r\delta} := (-1)^{|\eta|} d_\eta(\alpha) E_\eta^{r\delta}(-x),$$

- and its ordinary monomial coefficients

$$F_\eta^{r\delta} = \sum_\gamma \alpha^{|\gamma|-|\eta|} b_{\eta,\gamma}(\alpha) x^\gamma.$$

Theorem ([Naqvi-Sahi-Sergel 2020])

The coefficients $b_{\eta,\gamma}(\alpha)$ belong to $\mathbb{N}[\alpha]$.

- As we explain below, we actually prove a stronger result.

Examples

- $F^{r\delta}_{(2,0)} = (\frac{2}{r}+1)(\frac{1}{r}+1)\Big((x_1+1+r)(x_1+r)+r(x_2)\Big)$
$$+(\tfrac{2}{r}+1)\,x_1x_2+(\tfrac{1}{r}+1)(x_2+1+r)(x_2)$$
- Check: It vanishes at these points:

$$-\overline{(0,0)}=(-r,0),\quad -\overline{(1,0)}=(-1-r,0),\quad -\overline{(0,1)}=(0,-1-r).$$
$$-\overline{(1,1)}=(-1-r,-1),\quad -\overline{(0,2)}=(0,-2-r).$$

- Similarly,
  - $F^{r\delta}_{(1,1)} = (\frac{1}{r}+2)(\frac{1}{r}+1)\,x_1x_2$
  - $F^{r\delta}_{(0,2)} = (\frac{2}{r}+2)\,x_1x_2+(\frac{2}{r}+2)(\frac{1}{r}+1)(x_2+1+r)(x_2)$

The dehomogenization operator

- The homogenous $F_\eta^{(\alpha)}$ and the inhomogeneous $F_\eta^{r\delta}$ are both linear bases for the polynomial algebra $\mathbb{F}[x_1, \ldots, x_n]$ over the field $\mathbb{F} = \mathbb{Q}(\alpha) = \mathbb{Q}(r)$.
- Thus there is a unique $\mathbb{F}$-linear operator $\Psi = \Psi_r$, which we call the *dehomogenization* operator, that satisfies

$$\Psi(F_\eta^{(\alpha)}) = F_\eta^{r\delta} \text{ for all } \eta \in \mathbb{N}^n.$$

- In fact, Ψ preserves the space of symmetric polynomials; moreover it maps $J_\lambda^{(\alpha)}$ to $J_\lambda^{r\delta}$ and hence $P_\lambda^{(\alpha)}(-x)$ to $P_\lambda^{r\delta}(-x)$.
- Thus Ψ is a generalization of the symmetric dehomogenization operator of [KS1] that maps $P_\lambda^{(\alpha)}$ to $P_\lambda^{r\delta}$.

Bar monomials and their positivity

- We consider the application of Ψ to an ordinary monomial.
- We call this a *bar-monomial*

$$x^{\underline{\eta}} = \Psi(x^{\eta})$$

- For $n = 1$ it is the rising factorial $x^{\underline{k}} = x(x+1)\cdots(x+k-1)$
- In general, one has a monomial expansion of the form

$$x^{\underline{\eta}} = x^{\eta} + \sum_{\gamma\,:\,|\gamma|<|\eta|} c_{\eta,\gamma}(r) x^{\gamma}.$$

Theorem ([Naqvi-Sahi-Sergel 2020])

The coefficient $c_{\eta,\gamma}(r)$ is a polynomial in $\mathbb{N}[r]$ of degree $\leq |\eta| - |\gamma|$.

- This implies the positivity result for interpolation polynomials.

Vanishing and Examples

Proposition

Let f be any homogeneous polynomial of degree k. Then $\Psi(f)$ is the unique polynomial with top component f that vanishes at all points $-\overline{\gamma}$ for all $|\gamma| < k$.

- $x^{\underline{(2,0)}} = (x_1 + 1 + r)(x_1 + r) + r(x_2)$
- The top component is x_1^2 and it vanishes at the points
  $$-\overline{(0,0)} = (-r, 0), \quad -\overline{(1,0)} = (-1-r, 0), \quad -\overline{(0,1)} = (0, -1-r).$$
- Similarly,
  - $x^{\underline{(1,1)}} = (x_1)(x_2)$
  - $x^{\underline{(0,2)}} = (x_2 + 1 + r)(x_2)$
- So, how can we construct these things? Why are they positive?

Intertwiners and recursions

- Monomials can be generated recursively from $x^0 = 1$ by

$$\Phi\left(x^{\gamma}\right) = x^{\Phi\gamma}, \quad s_i\left(x^{\gamma}\right) = x^{s_i\gamma}$$

- Here $\Phi f\left(x\right) = x_n f\left(x_n, x_1, \ldots, x_{n-1}\right)$ and $\Phi\gamma = \left(\gamma_2, \ldots, \gamma_n, \gamma_1 + 1\right)$.
- Also s_i interchanges x_i and x_{i+1}, or γ_i and γ_{i+1} for compositions.

- It turns out that there is an analogous recursion for bar-monomials.
- Let $\Phi_1 f\left(x\right) = x_n f\left(x_n + 1, x_1, \ldots, x_{n-1}\right)$ and $\sigma_i = s_i - \frac{r}{x_i - x_{i+1}}\left(1 - s_i\right)$.

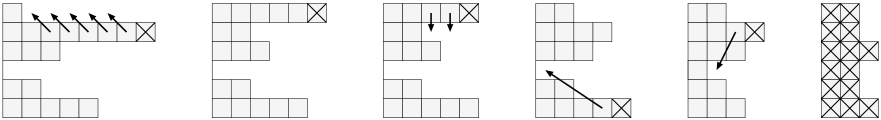
Lemma

We have $\Phi_1\left(x^{\underline{\gamma}}\right) = x^{\underline{\Phi\gamma}}$ *and* $\sigma_i\left(x^{\underline{\gamma}}\right) = x^{\underline{s_i\gamma}}$

- Unfortunately σ_i does *not* preserve positivity. We overcame this issue by solving the recursion to find an explicit combinatorial formula!

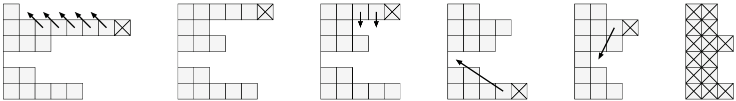
Bar games

- We play a solitaire game on the diagram of η, shrinking it gradually.
- A valid move at an intermediate composition γ has two steps:
  1. Of the rightmost boxes of γ, delete the one in the highest row, i, say
  2. Then move some boxes, possibly none, from the end of row i
     - either to up and strictly left of their original positions
     - or down and weakly left of their original positions.
- A game G is over when there are no more boxes left.
- Here is an example of a game on $(1, 8, 3, 0, 2, 5)$



Weighted sums of bar games

- The *weight* of a move is
  - r if any boxes are moved in step 2;
  - $x_i + (\gamma_i - 1) + r\,|\{j > i : \gamma_j < \gamma_i\} \cup \{j < i : \gamma_j < \gamma_i - 1\}|$ if not.
- The weight $w(G)$ of G is the product of its move weights.
- The game



has weight $r \cdot (x_1 + 5 + 5r) \cdot r \cdot r \cdot r \cdot (x_3 + 2 + 2r) \cdot (x_6 + 2)$
$\cdot (x_1 + 1) \cdot (x_2 + 1) \cdots (x_6 + 1) \cdot x_1 x_2 \cdots x_6.$

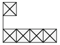
Theorem ([Naqvi-Sahi-Sergel 2020])

We have $x^{\underline{\eta}} = \sum_G w(G)$, summed over all possible G that start with η.

Proof: The operators σ_i and Φ_1 act nicely on weighted sums of games.

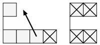
Computing a bar monomial

- Now we give an example of the full computation of $x^{\underline{\eta}}$.
- For brevity, when we delete a box without moving anything else, we record this with an $\times$ and continue working with the same figure.
- Below are *all* games on $(1, 0, 4)$. We use them to compute $x^{\underline{1,0,4}}$.

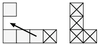


$$(x_3 + 3 + 2r) \cdot (x_3 + 2 + 2r) \cdot (x_3 + 1 + r) \cdot (x_1 + r) \cdot x_3$$

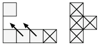
$$+ (x_3 + 3 + 2r) \cdot r \cdot x_1 \cdot x_2 \cdot x_3$$



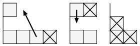
$$+ r \cdot (x_1 + 1 + r) \cdot (x_3 + 1 + r) \cdot (x_1 + r) \cdot x_3$$



$$+ r \cdot (x_3 + 1) \cdot x_1 \cdot x_2 \cdot x_3$$



$$+ r \cdot (x_2 + 1 + r) \cdot x_1 \cdot x_2 \cdot x_3$$



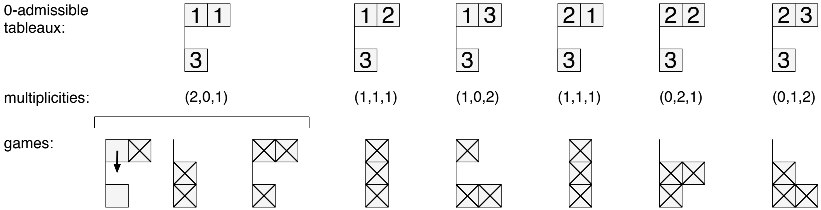
$$+ r^2 \cdot (x_3 + 1 + r) \cdot x_2 \cdot x_3.$$

Computing an interpolation polynomial

- The dehomogenization operator Ψ is linear, so

$$F_\lambda^\alpha = \sum_T d_\lambda^0(T)\, x^T \quad \xrightarrow{\Psi} \quad F_\lambda^{r\delta} = \sum_T d_\lambda^0(T)\, x^{\underline{T}}$$

- This gives a positive, combinatorial expansion for the interpolation polynomials in terms of admissible tableaux and games.
- Each game in the bottom row of the figure below gives a term in the expansion of $F_{(2,0,1)}^{r\delta}$.



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