Additional review problems

for Exam 1, 350 Honors Section

Fall, 2019

You should go through the material covered in the classes and review homework problems and examples in the book. Here are some additional review problems:

1. State and prove the following theorems and corollaries in the book:

   (a) The existence of a basis of a vector space generated by a finite subset (Theorem 1.9).

   (b) Replacement Theorem (Theorem 1.10).

   (c) All basis of a vector space generated by a finite subset have the same number of finitely many elements (Corollary 1 in Section 1.6).

   (d) Corollary 2 in Section 1.6.

   (e) Dimension Theorem (Theorem 2.3).

   (f) ϕ_β is an isomorphism (Theorem 2.21).

   (g) Φ_β^γ is an isomorphism (Theorem 2.20).

2. Prove or disprove the following statements:

   (a) Let W be the subset of $\mathbb{F}^3$ consisting of all vectors (x_1, x_2, x_3) such that $3x_1x_2 = 4x_2x_3$ (that is, $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 3x_1x_2 = 4x_2x_3\}$). Then W is a subspace of $\mathbb{F}^3$.

   (b) Let W be the subset of $P_2(\mathbb{F})$ consisting of all polynomials $a_2x^2 + a_1x + a_0$ such that $5a_2 - 3a_0 = 2a_1$ (that is, $W = \{a_2x^2 + a_1x + a_0 \in P_2(\mathbb{F}) \mid 5a_2 - 3a_0 = 2a_1\}$). Then W is a subspace of $P_2(\mathbb{F})$.

   (c) Let W be the subset of $P_n(\mathbb{F})$ consisting of all polynomials of the form $ax^{n-1} + (3ab)x^{n-2} + b$ for $a, b \in \mathbb{F}$ such that either $a \neq 0$ or $a = b = 0$ (that is, $W = \{ax^{n-1} + (3ab)x^{n-2} + b \mid a, b \in \mathbb{F}, a \neq 0 or a = b = 0\}$). Then W is a subspace of $P_n(\mathbb{F})$.

3. Let S_1 and S_2 be subsets of a vector space V.

   (a) Prove that if $S_1 \subset S_2$, then $\mathrm{span}(S_1) \subset \mathrm{span}(S_2)$.

   (b) Prove that if $S_1 \subset S_2$ and $\mathrm{span}(S_1) = V$, then $\mathrm{span}(S_2) = V$.

(c) Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$.

4. Let u, v, w be a set of linearly independent vectors of a vector space V.

   a) Determine whether the vectors $3u - v + 2w$, $2u + 4v$, $u + v - w$ are linear independent?

   b) If u, v, w is a basis of V, is $\{3u - v + 2w, 2u + 4v, u + v - w\}$ also a basis of V. Prove your answer.

5. Let V and W be n-dimensional vector spaces, let $T : V \to W$ be a linear transformation from V to W and let $\{v_1, \ldots, v_k\}$ be a subset of V.

   (a) If $\{v_1, \ldots, v_k\}$ is linearly independent and T is an isomorphism, prove that $\{T(v_1), \ldots, T(v_k)\}$ is a linearly independent subset of W.

   (b) If $\{v_1, \ldots, v_k\}$ is linearly dependent, prove that $\{T(v_1), \ldots, T(v_k)\}$ is a linearly dependent subset of W.

6. Let V be vector space of all 2 by 2 upper triangular matrices. Let $A_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

   (a) Prove that $\beta = \{A_1, A_2, A_3\}$ is a basis of V.

   (b) Let $B = \begin{pmatrix} 3 & 5 \\ 0 & 2 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Find $[B]_\beta$ and $[C]_\beta$.

   (c) Find $[2B - 3C]_\beta$.

7. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be the map defined by

$$T(f(x)) = \int_0^x f(t)dt + f'(x).$$

   (a) Show that T a linear transformation.

   (b) Find a basis of the range $R(T)$ of T and justify that your set is indeed a basis.

   (c) What are the nullity and rank of T? Justify your answers.

   (d) Is T one-to-one? Is T Onto? Justify your answers.

   (e) Let $\beta = \{1, x, x^2\}$ (the standard ordered basis of $P_2(\mathbb{R})$) and $\gamma = \{1, x, x^2, x^3\}$ (the standard ordered basis of $P_3(\mathbb{R})$). Calculate $[T]_\beta^\gamma$.

8. Let $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ be the linear transformation defined by

$$T(a_1 x + a_0) = (-a_1 + a_0)x + (-a_1 + 2a_0).$$

   Let $\beta = \{1, x\}$ be the standard ordered basis of $P_1(\mathbb{R})$, and also consider the ordered basis $\beta' = \{1, x + 1\}$ of $P_1(\mathbb{R})$.

(a) Find the matrix $[T]_\beta$ of T. (Note that $[T]_\beta$ can also be written as $[T]_\beta^\beta$.) Justify your answer.

(b) Find the matrix $[T]_{\beta'}^{\beta}$ of T. Justify your answer.

(c) Find the matrix $[T]_{\beta'}(= [T]_{\beta'}^{\beta'})$ of T. Justify your answer.

(d) Find the change of coordinate matrix $Q = [1_{P_1(\mathbb{R})}]_{\beta'}^{\beta}$.