YAMABE TYPE EQUATIONS ON THREE DIMENSIONAL RIEMANNIAN MANIFOLDS

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A theorem of Escobar and Schoen asserts that on a positive three dimensional smooth compact Riemannian manifold which is not conformally equivalent to the standard three dimensional sphere, a necessary and sufficient condition for a $C^2$ function $K$ to be the scalar curvature function of some conformal metric is that $K$ is positive somewhere. We show that for any positive $C^2$ function $K$, all such metrics stay in a compact set with respect to $C^3$ norms and the total Leray-Schauder degree of all solutions is equal to $-1$. Such existence and compactness results no longer hold in such generality in higher dimensions or on manifolds conformally equivalent to standard three dimensional spheres. The results are also established for more general Yamabe type equations on three dimensional manifolds.

0. Introduction

Let $(M, g)$ be an $n$ dimensional compact, smooth, Riemannian manifold. For $n = 2$, we know from the Uniformization Theorem of Poincaré that there exist constant Gauss curvature metrics which are pointwise conformal to $g$. Through the work of Yamabe [57], Trudinger [55], Aubin [2], and Schoen [50], the following Yamabe conjecture has been proved: For $n \geq 3$, there exist constant scalar curvature metrics which are pointwise conformal to $g$. The Yamabe conjecture can be viewed as a generalization of the Uniformization Theorem to higher dimensions. See Lee and Parker [41] for a survey on the Yamabe conjecture. See also Bahri and Brezis [8] and Bahri [6] for works on the problem and its variants.

More recently, Schoen obtained compactness results for the Yamabe problem in [51] where he proved that when $(M, g)$ is locally conformally flat but not conformally equivalent to the standard sphere, all solutions to the Yamabe problem stay in a compact set with respect to $C^3$ norms and the total Leray-Schauder degree of all

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solutions is equal to $-1$. In the same paper, he also announced, with indication of the proof, the same result for general manifolds. See [52] for more details.

The heart of the proof of the above-mentioned results of Schoen is some fine analysis of possible blowup behavior of solutions to the Yamabe problem. More specifically, Schoen obtained in [51] energy independent estimates of solutions to

$$-L_g u = n(n-2)u^p, \quad u > 0, \quad \text{in } M,$$

where $1 < 1 + \epsilon_0 \leq p \leq (n+2)/(n-2)$. $(M,g)$ is $n$ dimensional ($n \geq 3$) compact locally conformally flat Riemannian manifolds, $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ is the conformal Laplacian operator, and $R_g$ is the scalar curvature function of $g$. For $n = 3$, Schoen and Zhang established in [53] such energy independent estimates for solutions to

$$-L_g u = Ku^p, \quad u > 0, \quad \text{in } M,$$

where $1 < 1 + \epsilon_0 \leq p \leq 5$, $K > 0$ is a positive function in $C^2(M)$, and $M$ is locally conformally flat.

Along the approach initiated by Schoen, the first author extended in [42–43] the above-mentioned results of Schoen and Zhang to dimension $n = 4$, as well as to dimension $n \geq 5$ under suitable $(n-2)$-flatness hypothesis on $K$ near critical points of $K$. He also established in [44] such energy independent estimates on three dimensional locally conformally flat Riemannian manifolds $M$ with umbilic boundary $\partial M$ for solutions to

$$\begin{cases}
-L_g u = Ku^p, & \text{in } M^\circ, \\
B_{g} u = 0, & \text{on } \partial M,
\end{cases}$$

where $1 < 1 + \epsilon_0 \leq p \leq 5$, $B_g = \frac{\partial}{\partial \nu} + \frac{n-2}{2} h_g$, $h_g$ is the mean curvature function of $\partial M$, $\nu$ is the unit outer normal of $\partial M$, and $K > 0$ is some positive function in $C^2(M)$. We recall that $\partial M$ is called umbilic if at each point of $\partial M$ the second fundamental form is a constant multiple of the metric $g$.

See Brezis, Li and Shafrir [15], Chang, Gursky and Yang [19], Chen and Lin [23–24], and Han and Li [35] for related results.

Instead of looking for conformal metrics with scalar curvature being a given constant as in the Yamabe problem, one may also look for conformal metrics with scalar curvature being a given function $K$. When dimension $n \geq 3$, this is, up to some harmless constant, equivalent to solving the following semilinear elliptic equation with critical exponent:

$$-\Delta_g u + \frac{n-2}{4(n-1)} R_g u = Ku^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{in } M. \quad (1)$$

Extensive studies have been made on this prescribed scalar curvature problem following the pioneer work of Moser [46], see, for example, [4, 5, 7, 9, 10, 12–14, 18–29, 33, 34, 36, 38–40, 42, 43, 47, 52, 53, 56], and the references therein.
In this paper we give some compactness results on this prescribed scalar curvature problem on three dimensional manifolds. For \( n = 3 \), Eq. (1) takes the form
\[
-\Delta_g u + \frac{1}{8} R_g u = Ku^5, \quad u > 0, \quad \text{on } M.
\]
(2)

In fact, we also consider, for \( 1 < p \leq 5 \),
\[
-\Delta_g u + \frac{1}{8} R_g u = Ku^p, \quad u > 0, \quad \text{on } M,
\]
(3)
which includes subcritical exponent \( p \) as well.

**Definition 0.1.** A smooth compact Riemannian manifold \((M, g)\) is said to be positive if the first eigenvalue of \(-L_g \equiv -\Delta_g + \frac{n-2}{4(n-1)} R_g\) is positive.

It is well known that the Yamabe problem as well as the prescribed scalar curvature problem is much more difficult on positive manifolds. We define on a positive manifold a compact operator \( T : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M) \) by
\[
Tu = \left( -\Delta_g + \frac{1}{8} R_g \right)^{-1} (Ku^5),
\]
where \( 0 < \alpha < 1 \) and \( C^{2,\alpha}(M)^+ \) denotes the set of positive functions in \( C^{2,\alpha}(M) \). If we can prove that all solutions of (2) stay in a bounded set of \( C^{2,\alpha}(M)^+ \), then for \( \Lambda \) large enough 0 does not belong to \((I - T)(\partial D_\Lambda)\) where
\[
D_\Lambda = \left\{ v \in C^{2,\alpha}(M) : \|v\|_{C^{2,\alpha}(M)} \left( \Lambda, \min_M v \right)^{1/\Lambda} \right\}.
\]

Consequently, the Leray-Schauder degree \( \text{deg}((I - T), D_\Lambda, 0) \) is well defined. Moreover \( \text{deg}((I - T), D_\Lambda, 0) \neq 0 \) yields the existence of at least one solution to (2). See, for example, Nirenberg [48] for the definition of the Leray-Schauder degree and its various properties.

A theorem of Escobar and Schoen [29] asserts that on a positive three dimensional smooth compact Riemannian manifold which is not conformally equivalent to the standard three dimensional sphere, every positive \( C^2 \) function \( K \) on \( M \) can be realized as the scalar curvature function of some metric conformal to \( g \). Our first theorem says that for any positive \( C^2 \) function \( K \), all such conformal metrics stay bounded with respect to strong norms (say, \( C^3 \)) and the Leray-Schauder degree \( \text{deg}((I - T), D_\Lambda, 0) = -1 \) for large \( \Lambda \).

**Theorem 0.1.** Let \((M, g)\) be a positive three dimensional smooth compact Riemannian manifold which is not conformally equivalent to the standard three dimensional sphere. Then for any \( 1 < p \leq 5 \) and positive function \( K \in C^2(M) \), there exists some constant \( C \) depending only on \( M, g, \|K\|_{C^2(M)} \), and the positive lower bound of \( K \) and \( p - 1 \) such that
\[
1/C \leq u \leq C \quad \text{and} \quad \|u\|_{C^1(M)} \leq C,
\]
for all solution $u$ of (3). Moreover $\deg((I - T), D_\Lambda, 0) = -1$ for large $\Lambda$. Consequently, Eq. (2) has at least one solution.

We remark that the hypothesis $(M, g)$ which is not conformally equivalent to the standard three dimensional sphere is necessary since (2) may not have a solution in this case due to Kazdan and Warner’s condition for the solvability. We also remark that in dimension greater than three, results in such generality no longer hold. This can be seen as follows. First it has been shown in corollary 0.24 of [43] that, for all $n \geq 4$, there exist $C^2$ converging functions $K_i \geq 1$ on $\mathbb{R}P^n$, the real projective space, such that the corresponding solutions $u_i$ of (1) blows up in $L^\infty$ norm. Next it follows from the work of Bianchi [12] and Bianchi–Egnell [13] that, for all $n \geq 4$, there exist positive $C^2$ functions $K$ on $\mathbb{R}P^n$ such that (1) is not solvable.

Equations of Yamabe type have been studied by Brezis and Nirenberg in [16] where the framework for a functional analysis and variational attack, beyond the minimization techniques used in more geometric methods, is set. Much work follows. In particular, Bahri and Brezis considered in [8] the following:

$$-\Delta_g u + ku = u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{on } M, \quad (4)$$

where $k \in L^\infty(M)$ and $\Delta_g$ is the Laplace–Beltrami operator. They proved, among other things, that for $3 \leq n \leq 5$ Eq. (4) has at least one solution under the necessary hypothesis that the first eigenvalue of $-\Delta_g + k$ is positive. For $k = \frac{1}{8} R_g$, Eq. (4) is the Yamabe equation.

It is natural to ask whether or not one can carry out energy independent estimates to solutions of (4). This is what we address in the first six sections of this paper. In Secs. 7 and 8, we treat more general equations $-\Delta_g u + ku = Ku^p$ which include prescribed scalar curvature equations. In particular, Theorem 0.1 will be established in Sec. 7.

Let $(M, g)$ be a three dimensional smooth compact Riemannian manifold. As before we use $\Delta_g$ to denote the Laplace–Beltrami operator with respect to the metric $g$ and $R_g$ to denote the scalar curvature of $g$. Consider, for $k \in C^1(M)$ and $1 < p \leq 5$, the following semilinear equation

$$-\Delta_g u + ku = u^p, \quad u > 0, \quad \text{on } M. \quad (5)$$

We use $M_{k,p}$ to denote the set of solutions of (5) in $C^2(M)$. Our next theorem gives a priori estimates of solutions of (5) in $H^1(M)$ norm.

**Theorem 0.2.** Let $(M, g)$ be a three dimensional smooth compact Riemannian manifold. Then for all $\epsilon_0 > 0$, 

$$\|u\|_{H^1(M)} \leq C \quad \forall u \in \bigcup_{1+\epsilon_0 \leq p \leq 5} M_{k,p},$$

where $C$ depends only on $M$, $g$, $\epsilon_0$, and $\|k\|_{C^1(M)}$. 

Though the above theorem asserts a priori estimates in $H^1$ norm to solutions of (5), estimates in $L^\infty$ norm does not hold in such generality as will be seen shortly. To give a priori estimates in strong norms under suitable hypotheses, we introduce a function on $M$, denoted as $A(y) = A_{k,g}(y)$, defined as follows.

Suppose that the first eigenvalue $\lambda_1$ of $-\Delta_g + k$ is positive. For $y \in M$, let $G_y$ denote the Green's function of $-\Delta_g + k$ with the pole at $y$, namely,

$$(-\Delta_g + k)G_y = \delta_y \quad \text{in} \ M.$$

The existence and uniqueness of $G_y$ is well known, see for example [3]. Let $x = (x^1, x^2, x^3)$ be some geodesic normal coordinate system centered at $y$, so the metric $g$ is locally given by $g_{ij}(x)dx^idx^j$ with $g_{ij}(0) = \delta_{ij}$ and $\partial_x^ig_{ij}(0) = 0$ for all $i, j$ and $l$. It follows from Lemma 9.2 in the Appendix that, for some real number $A_{k,g}(y)$,

$$G_y(x) = \frac{1}{3\omega_3|x|} + A_{k,g}(y) + O(|x|^\alpha), \quad \text{for} \ |x| \ \text{close to} \ 0,$$

for all $0 < \alpha < 1$, where $\omega_3$ is the volume of the unit ball in $\mathbb{R}^3$. It is not difficult to check that the value of $A_{k,g}(y)$ is independent of the choice of geodesic normal coordinate systems. It is also clear that $A_{k,g}(y)$ thus defined is a continuous function on $M$.

As pointed out earlier, a necessary condition for (5) to have a solution is that the first eigenvalue $\lambda_1$ of $-\Delta_g + k$ is positive.

**Theorem 0.3.** Let $(M, g)$ be a three dimensional smooth compact Riemannian manifold and $k \in C^1(M)$ with positive $\lambda_1$. Assume that

$$\min_{y \in M} A_{k,g}(y) > 0.$$

Then for all $\epsilon_0 > 0$,

$$\frac{1}{C} \leq u \leq C \quad \text{in} \ M \quad \text{and} \quad \|u\|_{C^2(M)} \leq C, \quad \forall u \in \bigcup_{1+\epsilon_0 \leq \rho \leq 5} M_{k,\rho}, \quad (6)$$

where $C$ is some positive constant depending only on $M, g, \epsilon_0, \|k\|_{C^1(M)}$, and the positive lower bound of $\min_M A_{k,g}$ and $\lambda_1$.

The following corollary follows from the above theorem and the positive mass theorem of Schoen and Yau [54].

**Corollary 0.1.** Let $(M, g)$ be a positive three dimensional smooth compact Riemannian manifold which is not conformally equivalent to the standard three dimensional sphere. Then for all $\epsilon_0 > 0$ and all $k \in C^1(M)$ satisfying $k \leq \frac{1}{2}R_g$ on $M$, we have

$$\frac{1}{C} \leq u \leq C \quad \text{and} \quad \|u\|_{C^2(M)} \leq C, \quad \forall u \in \bigcup_{1+\epsilon_0 \leq \rho \leq 5} M_{k,\rho},$$

where $C$ is some positive constant depending only on $M, g, \epsilon_0$, and $\|k\|_{C^1(M)}$. 
To prove Corollary 0.1, we can assume that the first eigenvalue $\lambda_1$ of $-\Delta_g + k$ is positive since otherwise $M_{k, \phi} = \emptyset$. Making a conformal change of the metric by using a positive eigenfunction associated with $\lambda_1$, we can assume without loss of generality that $k > 0$ on $M$. Applying the maximum principle we see that the Green's function of $-\Delta_g + k$ is greater than or equal to the Green's function of $-\Delta_g + \frac{1}{8} R_g$. It follows that $A_{k, \phi} \geq A_{\frac{1}{8} R_g, \phi} > 0$, while the last inequality follows from the positive mass theorem of Schoen and Yau. Corollary 0.1 then follows from Theorem 0.3.

Once we have the compactness results in Theorem 0.3, we can derive existence results and multiplicity results (counting multiplicity).

Consider the following minimization problem

$$\min \left\{ E(v) : v \in H^1(M), \int_M v^6 = 1 \right\},$$  
(7)

where $E(v) = \int_M (|\nabla v|^2 + kv^2)$.

**Theorem 0.4.** Under the hypothesis of Theorem 0.3, (7) is attained at some positive function $v \in C^2(M)$.

Theorem 0.4 can be derived from Theorem 0.3 as follows. Since $\lambda_1 > 0$, $\sqrt{E(v)}$ is an equivalent $H^1(M)$ norm. For $2 < q < 6$, the embedding from $H^1(M)$ to $L^q(M)$ is compact, so we can find some nonnegative minimizer $v_q$ to $\min \{ E(v) : v \in H^1(M), \int_M |v|^q = 1 \}$. The Euler-Lagrange equation of $v_q$ is

$$-\Delta_g v_q + kv_q = E(v_q) v_q^{q-1}, \quad \text{in } M.$$

It is easy to see that $E(v_q)$ is bounded away from zero and infinity. We derive from the strong maximum principle that $v_q$ is positive in $M$. Applying Theorem 0.3 and some standard elliptic estimates to $E(v_q)^{1/(q-2)} v_q$, we have, along a subsequence, that $v_q \to v$ in $C^2(M)$ as $q \to 6$. It is then easy to see that $v$ is a positive minimizer of (7).

Let $v$ be the positive minimizer obtained in Theorem 0.4, then $u = E(v)^{1/4} v \in M_{k, \phi}$. The existence of $u$ in $M_{k, \phi}$ under the necessary hypothesis $\lambda_1 > 0$ has been established in Bahri and Brezis [8]. It is interesting to note that (7) has no minimizer if $k$ is sufficiently large, though $M_{k, \phi} \neq \emptyset$ according to the result in [8]. The nonexistence of minimizers for large $k$ can be seen as follows. We first observe that

$$\min \left\{ E(v) : v \in H^1(M), \int_M v^6 = 1 \right\} \leq \frac{1}{S_1},$$

where

$$\frac{1}{S_1} = \inf \left\{ \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{\left( \int_{\mathbb{R}^3} u^6 \right)^{1/3}} : u \in L^6(\mathbb{R}^3) \setminus \{0\}, \nabla u \in L^2(\mathbb{R}^3) \right\}.$$
It follows from a result of Hebey and Vaugon [37] that for some constant $C = C(M, g)$,

$$
\int_M |\nabla_g v|^2 + C \int_M v^2 \geq \frac{1}{S_1}, \quad \forall v \in H^1(M), \int_M v^6 = 1.
$$

Therefore if $k > C$ on $M$, (7) has no minimizer. In view of the above derivation of Theorem 0.4 from Theorem 0.3, we also know that the compactness result (6) no longer holds if $k > C$ on $M$. The reason is that the $v_q$ constructed before does not have any subsequence converging in $H^1(M)$ due to the nonexistence of minimizer of (7).

To state the multiplicity result, we introduce, as in Schoen [51], an operator $F : C^{2, 0}(M) \to C^{2, 0}(M)$ by

$$
F(v) = v - (-(\Delta_g + k)^{-1}(E(v)v^5),
$$

where $0 < \alpha < 1$, and $E(v) = \int_M (|\nabla_g v|^2 + kv^2)$.

It is clear that the operator $F$ is of the form $I + \text{compact}$ and therefore the Leray-Schauder degree $\text{deg}(F, D_\Lambda, 0)$ is well defined provided $0$ does not belong to $F(\partial D_\Lambda)$.

**Theorem 0.5.** Under the hypothesis in Theorem 0.3, we have, for $\Lambda$ large enough, that $0$ does not belong to $F(\partial D_\Lambda)$ and

$$
\text{deg}(F, D_\Lambda, 0) = -1.
$$

Consequently, $M_{k, 5} \neq \emptyset$.

Using Theorem 0.3, we can prove Theorem 0.5 in the same way as in Schoen [51]. We omit the details.

We will extend in Secs. 7–8 Theorem 0.2–0.5 to include the following generalization of (5):

$$
-\Delta_g u + ku = Ku^p, \quad u > 0, \quad \text{on } M,
$$

where $1 < p \leq 5$ and $K \in C^2(M)$ is some positive function in $M$. In particular, Theorem 0.1 will be established in Sec. 7.

The paper is organized as follows. In Sec. 1, we give the definitions of isolated blowup points and isolated simple blowup points for a sequence of solutions. In Sec. 2, we derive a Pohozaev type identity. In Sec. 3, we give some properties of solutions near isolated and isolated simple blowup points. The main result in this section is Proposition 3.1 which gives sharp pointwise estimates to a sequence of solutions near isolated simple blowup points. In Sec. 4, we prove that an isolated blowup point is in fact an isolated simple blowup point, which rules out the possibility of bubbles on top of bubbles. In Sec. 5, we rule out the possibility of bubble accumulations. Theorem 0.2 is established by the end of Sec. 5. The analysis in the
first five sections is local in nature. In Sec. 6, we study compactness of solutions and establish Theorem 0.3. Here, global properties are used. In Secs. 7–8, we extend results in first six sections to cover more general Eqs. (8) which include the prescribed scalar curvature equations. In particular, Theorem 0.1 follows from results in Sec. 7. In the appendix, we provide, for readers' convenience, some well known descriptions of singular behavior of positive solutions to linear elliptic equations in punctured balls.

1. Definitions and Notations

We start by introducing some notations and definitions. Let \((M, g)\) be a three dimensional smooth compact Riemannian manifold. In local coordinates, the Laplace–Beltrami operator can be written as \(\Delta_g u = g^{\alpha \beta} (\partial_\alpha \partial_\beta u - \Gamma^\gamma_{\alpha \beta} \partial_\gamma u)\) where \(\Gamma^\gamma_{\alpha \beta}\) denotes the Christoffel symbol.

Let \(\Omega \subset M\) be an open set, \(\{k_i\}\) be a sequence of functions converging to \(k\) in \(C^1(M)\), \(\{p_i\}\) be a sequence of numbers satisfying \(2 \leq p_i \leq 5\) and \(p_i \to 5\), and \(\{u_i\}\) be a sequence of \(C^2\) functions satisfying

\[
-\Delta_g u_i + k_i u_i = u_i^{p_i}, \quad u_i > 0, \quad \text{in } \Omega \subset M.
\]

(9)

A point \(\bar{y} \in \Omega\) is called a blowup point of \(\{u_i\}\) if \(u_i(y_i) \to \infty\) for some \(y_i \to \bar{y}\).

Following Schoen, we make the following definitions concerning blowup points of \(\{u_i\}\).

**Definition 1.1.** Let \(\{u_i\}\) satisfy (9). A point \(\bar{y} \in \Omega\) is called an isolated blowup point of \(\{u_i\}\) if there exist \(0 < \bar{r} < \text{dist}(\bar{y}, \partial \Omega), \bar{C} > 0\), and a sequence \(y_i\) tending to \(\bar{y}\), such that, \(y_i\) is a local maximum of \(u_i\), \(u_i(y_i) \to +\infty\) and, for large \(i\),

\[
u_i(y) \leq \bar{C} d(y, y_i)^{-2/(p_i - 1)} \quad \forall d(y, y_i) < \bar{r},
\]

(10)

where \(d(y, y_i)\) denotes the geodesic distance between \(y\) and \(y_i\).

Let \(y_i \to \bar{y}\) be an isolated blowup point of \(\{u_i\}\), we set

\[
\bar{u}_i(r) = \frac{1}{|\partial B_r(y_i)|} \int_{\partial B_r(y_i)} u_i ds_g, \quad 0 < r < \bar{r},
\]

and

\[
\bar{u}_i(r) = r^{2/(p_i - 1)} \bar{u}_i(r), \quad 0 < r < \bar{r},
\]

where \(B_r(y_i)\) denotes the geodesic ball of radius \(r\) centered at \(y_i\), \(ds_g\) is the surface area element and \(|\partial B_r(y_i)|\) denotes the area of \(\partial B_r(y_i)\) with respect to \(g\).

**Definition 1.2.** \(\bar{y} \in \Omega\) is called an isolated simple blowup point, if \(\bar{y}\) is an isolated blowup point of \(\{u_i\}\), such that, for some \(\rho \in (0, \bar{r})\) (independent of \(i\)),

\[
\bar{u}_i \text{ has precisely one critical point in } (0, \rho)
\]

(11)

for large \(i\).
2. A Pohozaev Type Identity

In this section, we provide a Pohozaev type identity which plays important roles in subsequent estimates. Since we will only apply the Pohozaev type identity in small neighborhoods of some point, we write it in a local coordinate chart. Let \( \Omega \subset M \) be some open set in one local coordinate chart \( x = (x^1, x^2, x^3) \) with \( g_{ij}(0) = \delta_{ij} \) and \( \Gamma^k_{ij}(0) = 0 \). In this section we use notation \( \Delta = \sum_{i=1}^n \partial_i^2 \), \( \nabla = (\partial_1, \partial_2, \partial_3) \), \( dv = dx^2 \wedge dx^3 \wedge dx^3, |x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2, B_\sigma = \{(x^1, x^2, x^3) : |x| < \sigma\} \), and \( ds \) to denote the surface area element with respect to the flat metric.

Considering, for \( k \in C^1(\Omega) \) and \( p > 0 \), the following equation

\[ -\Delta \sigma u + ku = u^p, \quad u > 0, \quad \text{in } \Omega, \quad (12) \]

we have

\[ -\int_{B_{\sigma}} \left( k(x) + \frac{x \cdot \nabla k(x)}{2} \right) u^2 dv + \left( \frac{3}{p + 1} - \frac{1}{2} \right) \int_{B_{\sigma}} u^{p+1} dv = A(g, u) \]

\[ = \int_{\partial B_{\sigma}} B(\sigma, x, u, \nabla u) ds - \int_{\partial B_{\sigma}} \sigma \left( \frac{ku^2}{2} - \frac{u^{p+1}}{p+1} \right) ds, \quad (13) \]

where

\[ B(\sigma, x, u, \nabla u) = \sigma \left( \frac{\partial u}{\partial \nu} \right)^2 - \sigma |\nabla u|^2 + \frac{1}{2} \frac{\partial u}{\partial \nu}, \]

\( \nu \) denotes the unit outer normal of \( \partial B_{\sigma} \) with respect to the flat metric, and

\[ A(g, u) = \int_{B_{\sigma}} (x^\alpha \partial_{\alpha} u)(g^{\beta\gamma} - \delta^{\beta\gamma}) \partial_{\beta\gamma} u dv - \int_{B_{\sigma}} (x^\alpha \partial_{\alpha} u)(g^{\beta\gamma} \Gamma^u_{\beta\gamma} \partial_{\beta\gamma} u dv \]

\[ + \frac{1}{2} \int_{B_{\sigma}} u (g^{\alpha\beta} - \delta^{\alpha\beta}) \partial_{\beta\gamma} u dv - \frac{1}{2} \int_{B_{\sigma}} u (g^{\alpha\beta} \Gamma^u_{\beta\gamma} \partial_{\beta\gamma} u dv. \]

**Proof.** It is elementary to check (see e.g., [38]) that for any \( f \in C^2(B_{\sigma}) \), we have

\[ 2\Delta u(\nabla u \cdot \nabla f) = \text{div}[2(\nabla u \cdot \nabla f) \nabla u - |\nabla u|^2 \Delta f] + |\nabla u|^2 \Delta f - 2\nabla u \cdot \nabla^2 f \cdot \nabla u, \quad (14) \]

where \( \nabla^2 f = \left( \frac{\partial^2 f}{\partial x^2} \right) \), thus the last term is a quadratic form.

Let \( f = \frac{1}{2} |x|^2 \), then \( \nabla f = x, \Delta f = 3, \nabla u \cdot \nabla^2 f \cdot \nabla u = |\nabla u|^2 \). Integrating both sides of (14) on \( B_{\sigma} \), we have

\[ \int_{B_{\sigma}} 2\Delta u(\nabla u \cdot x) = \int_{B_{\sigma}} \text{div}[2(\nabla u \cdot x) \nabla u - |\nabla u|^2 x] + \int_{B_{\sigma}} |\nabla u|^2. \]
It is clear that
\[
\int_{B_+} \text{div}[2(\nabla u \cdot x) \nabla u - |\nabla u|^2 x] = \sigma \int_{\partial B_+} \left( 2 \frac{\partial u}{\partial \nu}^2 - |\nabla u|^2 \right),
\]
and
\[
\int_{B_+} |\nabla u|^2 = - \int_{B_+} \Delta u u + \int_{\partial B_+} u \frac{\partial u}{\partial \nu}.
\]
Thus
\[
\int_{B_+} 2\Delta u \nabla u \cdot x + \int_{B_+} \Delta u u = \int_{\partial B_+} \left\{ 2 \left( \frac{\partial u}{\partial \nu} \right)^2 \sigma - |\nabla u|^2 \sigma + u \frac{\partial u}{\partial \nu} \right\}. \tag{15}
\]
Let \( g = g_{\alpha\beta} dx^\alpha dx^\beta \) be the metric, we know
\[
\Delta u = \Delta_g u - (g^{\alpha\beta} - \delta^{\alpha\beta}) \partial_{\alpha\beta} u + g^{\alpha\beta} \Gamma_{\alpha\beta}^\gamma \partial_\gamma u.
\]
Plugging the above into (15), we have
\[
- \int_{B_+} (x^\alpha \partial_\alpha u) \Delta_g u dv - \frac{1}{2} \int_{B_+} u \Delta_g u dv
\]
\[
+ \int_{B_+} (x^\alpha \partial_\alpha u) (g^{ab} - \delta^{ab}) \partial_{ab} u dv - \int_{B_+} (x^\alpha \partial_\alpha u) (g^{ab} \Gamma_{ab}^\gamma) \partial_\gamma u dv
\]
\[
+ \frac{1}{2} \int_{B_+} u (g^{ab} - \delta^{ab}) \partial_{ab} u dv - \frac{1}{2} \int_{B_+} u (g^{ab} \Gamma_{ab}) \partial_\gamma u dv
\]
\[
= - \int_{\partial B_+} \left[ \left( \frac{\partial u}{\partial \nu} \right)^2 \sigma - \frac{1}{2} |\nabla u|^2 \sigma + \frac{1}{2} u \frac{\partial u}{\partial \nu} \right] ds. \tag{16}
\]
Now, using the fact that \( u \) satisfies (12), we have
\[
- \int_{B_+} (x^\alpha \partial_\alpha u) \Delta_g u dv = \int_{B_+} (x^\alpha \partial_\alpha u) (u^p - ku) dv
\]
\[
= - \frac{3}{p+1} \int_{B_+} u^{p+1} dv + \frac{1}{p+1} \int_{\partial B_+} u^{p+1} \sigma ds
\]
\[
+ \frac{3}{2} \int_{B_+} ku^2 dv + \frac{1}{2} \int_{B_+} (x \cdot \nabla k) u^2 dv - \frac{1}{2} \int_{\partial B_+} ku^2 \sigma ds,
\]
and
\[
- \frac{1}{2} \int_{B_+} u \Delta_g u dv = \frac{1}{2} \int_{B_+} u (u^p - ku) dv.
\]
Replacing the first line in (16) by using the above two identities, we have (13). \( \square \)

3. Properties of Isolated and Isolated Simple Blowup Points

In this section, we give some properties of isolated and isolated simple blowup points. We follow [42] and [53] with refinement.
Lemma 3.1. Let $u_i$ satisfy (9) and $y_i \to \bar{y} \in \Omega$ be an isolated blowup point. Then for any $0 < r < \bar{r}/3$, we have
\begin{equation}
\max_{y \in B_{2r}(y_i) \setminus B_{\frac{3}{2}}(y_i)} u_i(y) \leq C_1 \min_{y \in B_{2r}(y_i) \setminus B_{\frac{3}{2}}(y_i)} u_i(y),
\end{equation}
where $B_s(y_i)$ denotes the geodesic ball of radius $s$ centered at $y_i$, and $C_1$ is some positive constant independent of $i$ and $r$.

**Proof.** Let $x = (x^1, x^2, x^3)$ be some geodesic normal coordinates centered at $y_i$ given by $\exp_{y_i}(x)$ and $h = h_{\alpha \beta}(x)dx^\alpha dx^\beta = g_{\alpha \beta}(rx)dx^\alpha dx^\beta$ be the scaled metric. Set $v_i(x) = r^{-\frac{x_i}{2}}u_i(\exp_{y_i}(rx))$ for $|x| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} < 3$, then $v_i(x)$ satisfies
\begin{align*}
-\Delta_h v_i(x) + r^2 \hat{k}_i(x)v_i(x) &= v_i(x)^{p_i} & |x| < 3, \\
0 &< v_i(x) \leq C|x|^{-\frac{x_i}{2}} & |x| < 3,
\end{align*}
where $\hat{k}_i(x) = k_i(\exp_{y_i}(rx))$. Clearly $v_i(x) \leq C$ for all $1/4 \leq |x| \leq 9/4$. It follows from the Harnack inequality (see for example [30]) that
\[\max_{1/2 \leq |x| \leq 2} v_i(x) \leq C_1 \min_{1/2 \leq |x| \leq 2} v_i(x),\]
which yields (17). \qed

Lemma 3.2. Let $u_i$ satisfy (9) and $y_i \to \bar{y} \in \Omega$ be an isolated blowup point. Then for any $R_i \to +\infty$, $\epsilon_i \to 0^+$, we have, after passing to a subsequence $\{u_j\}$ (still denoted as $\{u_i\}$, $\{y_i\}$, etc.), that
\begin{align*}
\left\| u_i(y_i) u_i \left( \exp_{y_i} \left( u_i(y_i)^{-\frac{R_i}{2}} x \right) \right) \right\|_{C^2(B_{2R_i})} &\leq \epsilon_i, \\
\left\| u_i(y_i) u_i \left( \exp_{y_i} \left( u_i(y_i)^{-\frac{R_i}{2}} x \right) \right) \right\|_{H^1(B_{2R_i})} &\leq \epsilon_i,
\end{align*}
and
\[\frac{R_i}{\log u_i(y_i)} \to 0 \quad \text{as} \quad i \to \infty,\]
where $x = (x^1, x^2, x^3)$ denotes some geodesic normal coordinates given by $\exp_{y_i}(x)$.

**Proof.** Let $h = h_{\alpha \beta}(x)dx^\alpha dx^\beta = g_{\alpha \beta}(u_i(y_i)^{-\frac{R_i}{2}} x)dx^\alpha dx^\beta$ denote the scaled metric. Set $\xi_i(x) = u_i(y_i)^{-\frac{R_i}{2}} u_i \left( \exp_{y_i} \left( u_i(y_i)^{-\frac{R_i}{2}} x \right) \right)$, $|x| < \bar{r} u_i(y_i)^{\frac{R_i}{2}}$. It follows from the Harnack inequality (see for example [30]) that
\[\max_{1/2 \leq |x| \leq 2} u_i(y_i) u_i \left( \exp_{y_i} \left( u_i(y_i)^{-\frac{R_i}{2}} x \right) \right) \leq C_1 \min_{1/2 \leq |x| \leq 2} v_i(x),\]
Then $\xi_i(x)$ satisfies
\begin{equation}
\begin{cases}
-\Delta_h \xi_i(x) = \xi_i(x)^{p_i} - u_i(y_i)^{1-p_i} \tilde{k}_i(x)\xi_i(x), & |x| < \frac{\tilde{r}u_i(y_i)^{\frac{p_i-1}{2}}}{2}, \\
\xi_i(0) = 1, & \nabla \xi_i(0) = 0, \\
0 < \xi_i(x) \leq \frac{C}{|x|^{-\frac{p_i-1}{2}}} & |x| < \frac{\tilde{r}u_i(y_i)^{\frac{p_i-1}{2}}}{4}.
\end{cases}
\tag{20}
\end{equation}

where $\tilde{k}_i(x) = k_i(\exp_{y_i}(u_i(y_i)^{\frac{p_i-1}{2}} x))$.

It follows from Lemma 3.1 that for all $0 < r \leq 1$,
\begin{equation}
\max_{|x|=r} \xi_i(x) \leq C \min_{|x|=r} \xi_i(x),
\tag{21}
\end{equation}

where $C$ is independent of $i$ and $r$.

Since $u_i(y_i)^{1-p_i} \tilde{k}_i(x) \to 0$ uniformly in $|x| \leq 2$, we have from (20) that
\[ (-\Delta_h + o(1))\xi_i(x) \geq 0, \quad \forall |x| \leq 2. \]

Consequently, $\eta_i(x) := (1 + |x|^2)^{-1} \xi_i(x)$ satisfies, for some $b_\alpha(x)$, the following inequality for large $i$:
\[ \Delta_h \eta_i(x) + \sum_{\alpha=1}^{n} b_\alpha(x) \partial_\alpha \eta_i(x) \leq 0 \quad \forall |x| \leq 2. \]

It then follows from the maximum principle that
\[ \eta_i(0) \geq \inf_{|x|=r} \eta_i(x) \quad \forall 0 < r \leq 1 \]

which, together with (21) and the fact $\xi_i(0) = 1$, yields
\[ 1 = \xi_i(0) = \eta_i(0) \geq \inf_{|x|=r} \eta_i(x) \geq \frac{1}{2} \inf_{|x|=r} \xi_i(x) \geq \frac{1}{2C} \max_{|x|=r} \xi_i(x) \quad \forall 0 < r \leq 1. \]

Namely,
\[ \max_{|x| \leq 1} \xi_i(x) \leq C. \]

Combining this with the third line in (20), we have
\[ \xi_i(x) \leq C \quad \forall |x| < \frac{\tilde{r}u_i(y_i)^{\frac{p_i-1}{2}}}{2}, \]

where $C$ is independent of $i$ and $r$.

Applying standard elliptic estimates to $\{\xi_i\}$, we have, after passing to a subsequence $\{\xi_i\}$ (still denoted as $\{\xi_i\}$), $\xi_i \to \xi$ in both $C^2_{\text{loc}}(\mathbb{R}^3)$ and $H^1_{\text{loc}}(\mathbb{R}^3)$ for some $\xi$ satisfying
\begin{equation}
\begin{cases}
-\Delta \xi(x) = \xi(x)^{p}, & \xi(x) > 0 \quad \text{in } \mathbb{R}^3, \\
\xi(0) = 1, & \nabla \xi(0) = 0.
\end{cases}
\end{equation}
It follows from [17] that
\[
x(x) = \left(1 + \frac{1}{3|x|^2}\right)^{-\frac{1}{2}}.
\]
Lemma 3.2 follows from the above after passing to another subsequence.

Before stating our main estimate on isolated simple blowup points, we introduce the Green's functions of \(-\Delta_\rho + k\) in small balls. Let \(\bar{y} \in \Omega\) and \(k \in C^1(\Omega)\), it is well known (see Appendix A) that for \(\delta_0 > 0\) small, there exists a unique \(\bar{G}(:\bar{y}) \in C^2(B_{\delta_0}(\bar{y}) \setminus \{\bar{y}\})\) satisfying
\[
\begin{aligned}
-\Delta_\rho \bar{G} + k \bar{G} &= 0 & \text{in } B_{\delta_0}(\bar{y}) \setminus \{\bar{y}\}, \\
\bar{G} &= 0 & \text{on } \partial B_{\delta_0}(\bar{y}), \\
\lim_{y \to \bar{y}} d(y, \bar{y}) \bar{G}(y) &= 1.
\end{aligned}
\tag{22}
\]

\[\Box\]

**Proposition 3.1.** Let \(\{u_i\}\) satisfy (9) and \(y_i \to \bar{y} \in \Omega\) be an isolated simple blowup point, with (10) and (11) for all \(i\). Then for some constant \(C\) depending only on \(\rho, \bar{C}\), and \(\|k_i\|_{C^1(\Omega)}\), we have
\[
u_i(y) \leq Cu_i(y_i)^{-1}d(y, y_i)^{-1}, \quad \forall d(y, y_i) \leq \rho/2.
\tag{23}
\]
where \(\rho, \bar{C}\) are given in Definitions 1.1 and 1.2.

Furthermore, after passing to a subsequence, for some positive constant \(a > 0\) and \(k(x) = \lim_{i \to \infty} k_i(x)\),
\[
u_i(y)u_i \to auG(\cdot, \bar{y}) + b \quad \text{in } C^2_{\text{loc}}(B_{\bar{\rho}}(\bar{y}) \setminus \{\bar{y}\}),
\]
where \(\bar{\rho} = \min\{\delta_0, \rho/2\}\), \(a\bar{G}\) is given by (22) and \(b \in C^2(B_{\bar{\rho}}(\bar{y}))\) satisfies \(-\Delta_\rho b + k(\bar{y})b = 0\) in \(B_{\bar{\rho}}(\bar{y})\).

We will establish this proposition through a series of lemmas.

**Lemma 3.3.** Let \(u_i\) satisfy (9) and \(y_i \to \bar{y} \in \Omega\) be an isolated simple blowup point. Assume \(R_i \to \infty\) and \(0 < c_i \leq e^{-R_i}\) are sequences with which (18) and (19) hold. Then for any given \(0 < \delta < 1/100\), there exists \(\rho_1 \in (0, \rho)\) which is independent of \(i\) (but depending on \(\delta\)), such that
\[
u_i(y) \leq C_2u_i(y_i)^{-\lambda_i}d(y, y_i)^{-1+\delta}, \quad \forall R_iu_i(y_i)^{-\frac{E_i-1}{2}} \leq d(y, y_i) \leq \rho_1,
\tag{24}
\]
\[
|\nabla_y u_i(y)| \leq C_2u_i(y_i)^{-\lambda_i}d(y, y_i)^{-2+\delta}, \quad \forall R_iu_i(y_i)^{-\frac{E_i-1}{4}} \leq d(y, y_i) \leq \rho_1,
\tag{25}
\]
and
\[
|\nabla^2_y u_i(y)| \leq C_2u_i(y_i)^{-\lambda_i}d(y, y_i)^{-3+\delta}, \quad \forall R_iu_i(y_i)^{-\frac{E_i-1}{2}} \leq d(y, y_i) \leq \rho_1,
\tag{26}
\]
where \(\lambda_i = (1 - \delta)^{E_i-1}/2\) and \(C_2\) is some positive constant independent of \(i\).
Proof. Let \( r_i = R_i u_i(y_i)^{-\frac{p_i-1}{2}} \), it follows from Lemma 3.2 that
\[
    u_i(y) \leq C u_i(y_i) R_i^{-1} \quad \text{for } d(y, y_i) = r_i.
\] (27)

We know from the assumption of isolated simple blowup and Lemma 3.2 that
\[
    r_i^{\frac{p_i-1}{2}} u_i(r) \quad \text{is strictly decreasing for } r_i < r < \rho .
\] (28)

We then derive from Lemma 3.1, (28) and (27) that for all \( r_i \leq d(y, y_i) < \rho ,
\]
\[
d(y, y_i)^{\frac{p_i-1}{2}} u_i(y) \leq C d(y, y_i)^{\frac{p_i-1}{2}} u_i(d(y, y_i))
\]
\[
\leq C r_i^{\frac{p_i-1}{2}} u_i(r_i)
\]
\[
\leq C r_i^{\frac{p_i-1}{2}} u_i(y_i) R_i^{-1}
\]
\[
= CR_i^{1+\frac{p_i-1}{2}} = CR_i^{\frac{1}{4}+o(1)} .
\]

Therefore, for all \( r_i \leq d(y, y_i) \leq \rho , \) we have
\[
u_i(y)^{p_i-1} \leq CR_i^{-2+o(1)} d(y, y_i)^{-2} .
\]

Now we consider the operator:
\[
    \ell_i \varphi = \Delta_g \varphi + u_i(y)^{p_i-1} \varphi - k_i(y) \varphi .
\]

Since \( \ell_i u_i = 0 \) and \( u_i > 0 \), the maximum principle holds for \( \ell_i \) (see e.g. [11]). It follows from a straightforward calculation that \( \Delta \varphi (d(y, y_i)^{-\mu}) = [-\mu(1-\mu) + O(1)d(y, y_i)^2]d(y, y_i)^{-\mu - 2} \). It follows that for any given \( 0 < \delta < 1/100 \), we can choose \( \rho_i < \rho \) such that for \( i \) large enough and all \( r_i \leq d(y, y_i) < \rho_i , \) we have
\[
\ell_i (d(y, y_i)^{-\delta}) = -\delta (1-\delta) d(y, y_i)^{-\delta - 2} + u_i(y)^{p_i-1} d(y, y_i)^{-\delta} + O(1)d(y, y_i)^{-\delta}
\]
\[
\leq -\delta (1-\delta) d(y, y_i)^{-\delta - 2} + CR_i^{2+o(1)} d(y, y_i)^{-\delta - 2} + O(1)d(y, y_i)^{-\delta}
\]
\[
\leq -\frac{1}{2} \delta (1-\delta) d(y, y_i)^{-\delta - 2} ,
\] (29)

and, similarly, that
\[
\ell_i (d(y, y_i)^{1+\delta}) \leq -\frac{1}{2} \delta (1-\delta) d(y, y_i)^{-3+\delta} ,
\] (30)

Let \( \lambda_i = (1-\delta) \frac{p_i-1}{2} - 1 \) and \( M_i = \max_{\partial B_{\rho_i}(y_i)} u_i \), where \( B_{\rho_i}(y_i) \) denotes the geodesic ball of radius \( \rho_i \) centered at \( y_i \). Set
\[
\varphi_i(y) = M_i \rho_i^{\lambda_i} d(y, y_i)^{-\delta} + A u_i(y_i)^{-\lambda_i} d(y, y_i)^{-1+\delta} \quad \text{for } r_i \leq d(y, y_i) \leq \rho_i ,
\]

where \( A \) is some large constant to be chosen later. By (29) and (30), \( \ell_i \varphi_i(y) \leq 0 \) for \( r_i \leq d(y, y_i) \leq \rho_i \). On \( \partial B_{\rho_i}(y_i) \), \( \varphi_i(y) \geq M_i \geq u_i(y) \); On \( \partial B_{r_i}(y_i) \), \( \varphi_i(y) \geq
$Au_i(y_i)^{-\lambda_i} r_i^{-1+\delta} = Au_i(y_i) R_i^{-1+\delta}$. Choosing $A$ large enough, we know from (27) that $\varphi_i(y) \geq u_i(y)$ on $\partial B_{r_i}(y_i)$. Thus, in view of $\xi u_i = 0$, we have, using the maximum principle, that

$$\varphi_i(y) \geq u_i(y) \quad \text{in} \quad r_i \leq d(y, y_i) \leq \rho_1. \quad (31)$$

In turn, we have, for all $r_i < \theta < \rho_1$ and large $i$, that

$$\rho_1^{-\frac{2}{n-1}} M_i \leq C \rho_1^{-\frac{2}{n-1}} u_i(\rho_1) \leq C \theta^{-\frac{2}{n-1}} u_i(\theta) \leq C \theta^{-\frac{2}{n-1}} \{ M_0 \theta^{-\delta} + Au_i(y_i)^{-\lambda_i} \theta^{-1+\delta} \} \quad (28) \quad (31).$$

Since $2/(\rho_i - 1) - \delta \geq 1/5$ for large $i$, we can fix $\theta$ to be small enough (independent of $i$) so that we can derive from the above that

$$M_i \leq C u_i(y_i)^{-\lambda_i}.$$

Using (31) and the above, we have

$$u_i(y) \leq C u_i(y_i)^{-\lambda_i} (d(y, y_i)^{-\delta} + d(y, y_i)^{-1+\delta}) \leq C_2 u_i(y_i)^{-\lambda_i} d(y, y_i)^{-1+\delta}.$$ Estimate (24) is established.

To derive (25) from (24), we assume, for simplicity, that $g$ is the flat metric. The general case can be derived essentially in the same way. For any $R_i u_i(y - i)^{-\frac{p_i - 1}{2}} \leq |\hat{y}| \leq \rho_1/2$, we consider

$$v_i(z) = |\hat{y}|^{1-\delta} u_i(y_i)^{\lambda_i} u_i(|\hat{y}| z), \quad \frac{1}{2} \leq |z| \leq 2.$$ It follows from (9) that $v_i$ satisfies, for $\frac{1}{2} \leq |z| \leq 2$,

$$-\Delta v_i(z) + |\hat{y}|^2 k_i(|\hat{y}| z) v_i(z) = |\hat{y}|^{3-\delta-(1-\delta) p_i} u_i(y_i)^{(1-p_i) \lambda_i} v_i(z)^{p_i}. \quad (32)$$ We derive from $|\hat{y}| \geq R_i u_i(y_i)^{-\frac{p_i - 1}{2}}$ and the definition of $\lambda_i$ that

$$|\hat{y}|^{3-\delta-(1-\delta) p_i} u_i(y_i)^{(1-p_i) \lambda_i} \leq R_i^{3-\delta-(1-\delta) p_i} = o(1). \quad (33)$$ In view of (24) (with $R_i$ replaced by $R_i/2$), we have $v_i(z) \leq C \forall \frac{1}{2} \leq |z| \leq 2$. We can then derive from (32), (33), and standard elliptic estimates that

$$|\nabla v_i(z)| \leq C \quad \forall |z| = 1,$$

which implies

$$|\nabla u_i(\hat{y})| \leq C|\hat{y}|^{-2+\delta} u_i(y_i)^{-\lambda_i}.$$ This establishes (25) in a smaller range $2 R_i u_i^{-\frac{p_i - 1}{2}} \leq d(y, y_i) \leq \frac{1}{2} \rho_1$. However it is clear that we can actually establish (24) in a larger range and the above argument
will yield (25). Estimate (26) can be derived in a similar way after differentiating (32) once. We omit the details. Lemma 3.3 is established.

\[ \square \]

**Remark 3.1.** Later on we will fix \( \delta \) close to 0, hence fix \( \rho_1 \). Our goal is to obtain (24) with \( \delta = 0 \) for \( r_i \leq d(y_i, \gamma_i) \leq \rho_1 \), which, together with Lemma 3.1, yields Proposition 3.1.

**Lemma 3.4.** Let \( u_i \) satisfy (9) and \( y_i \to \gamma \in \Omega \) be an isolated simple blowup point. Assume \( R_i \to \infty \) and \( 0 < \epsilon_i < e^{-R_i} \) are sequences with which (18) and (19) hold. Fixing \( 0 < \delta < 1/100 \). Then for all fixed \( 0 < \sigma < \rho_1 \), we have for large \( i \) that

\[
\int_{d(y_i, \gamma_i) \leq \sigma} d(y_i, \gamma_i)^2 u_i^{p_i+1} = O(u_i(y_i)^{-4+\sigma(1)}) \text{,} \tag{34}
\]

\[
\int_{d(y_i, \gamma_i) \leq \sigma} u_i^2 \leq C_3 \sigma u_i(y_i)^{-2\lambda_i} \text{,} \tag{35}
\]

\[
\int_{d(y_i, \gamma_i) \leq \sigma} d(y_i, \gamma_i)^2 |\nabla u_i|^2 \leq O(u_i(y_i)^{-4+\sigma(1)}) + C_3 \sigma u_i(y_i)^{-2\lambda_i} \text{,} \tag{36}
\]

where \( C_3 \) is some constant independent of \( i \) and \( \sigma \).

**Proof.** To derive these estimates, we use (18) to estimate \( u_i \) in the region \( d(y_i, \gamma_i) \leq r_i \), and use Lemma 3.3 to estimate \( u_i \) in the region \( r_i \leq d(y_i, \gamma_i) \leq \sigma \).

We still use notation \( y = \exp_{y_i}(u_i(y_i)^{-(p_i-1)/2}) \). It follows from (18) that

\[
\int_{d(y_i, \gamma_i) \leq r_i} d(y_i, \gamma_i)^2 u_i^{p_i+1} \leq C u_i(y_i)^{p_i+1} \int_{|z| < R_i} |z|^2 \left( 1 + \frac{1}{3} |z| \right)^{-\frac{p_i+1}{3}} u_i(y_i)^{-\frac{p_i-1}{2}} \text{d}x
\]

\[
\leq C u_i(y_i)^{-4+\frac{3p_i}{2}} \text{,}
\]

Also, it follows from (24) that

\[
\int_{r_i \leq d(y_i, \gamma_i) \leq \sigma} d(y_i, \gamma_i)^2 u_i^{p_i+1} \leq C \int_{r_i \leq d(y_i, \gamma_i) \leq \sigma} d(y_i, \gamma_i)^2 (u_i(y_i)^{-\lambda_i} d(y_i, \gamma_i)^{-1+\delta})^{p_i+1}
\]

\[
\leq C R_i^{5-(p_i+1)(1-\delta)} u_i(y_i)^{-\lambda_i(p_i+1)}
\]

\[
= CR_i^{5-(p_i+1)(1-\delta)} u_i(y_i)^{-4+\frac{3p_i}{2}} \text{,}
\]

where \( r_i = 5 - p_i \) and \( r_i = R_i u_i(y_i)^{-\frac{p_i-1}{2}} \). Estimate (34) follows from the above immediately.
The proof of (35) is similar to that of (34). To prove (36), we multiply (9) by \( d(y, y_i)^2 u_i \) and integrate by parts, to obtain, in view of (34), that

\[
\int_{d(y, y_i) \leq \sigma} \nabla_g^2 (d(y, y_i)^2 u_i) \nabla_g u_i \\
= \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 u_i \nabla_g u_i - \int_{d(y, y_i) \leq \sigma} k_{ij} d(y, y_i)^2 u_i^2 + O(\sigma u_i (y_i)^{-2\lambda_i}) \\
= O(u_i (y_i)^{-4+\epsilon}) + O(\sigma u_i (y_i)^{-2\lambda_i}).
\]

It follows that

\[
\int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 |\nabla_g u_i|^2 \\
= -\int_{d(y, y_i) \leq \sigma} u_i \nabla_g (d(y, y_i)^2) \nabla_g u_i + O(\sigma u_i (y_i)^{-2\lambda_i}) \\
\leq C \int_{d(y, y_i) \leq \sigma} d(y, y_i) u_i |\nabla_g u_i| + O(\sigma u_i (y_i)^{-2\lambda_i}) \\
\leq \frac{1}{2} \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 |\nabla_g u_i|^2 + C \int_{d(y, y_i) \leq \sigma} u_i^2 + O(\sigma u_i (y_i)^{-2\lambda_i}).
\]

This, together with (35), yields (36).

In the following, let \((x^1, x^2, x^3)\) be some geodesic normal coordinate system centered at \(y_i\). We use the notation explained at the beginning of Sec. 2.

Lemma 3.5. Let \(u_i\) satisfy (9) and \(y_i \to \tilde{y} \in \Omega\) be an isolated simple blowup point. Assume \(R_i \to \infty\) and \(0 < \epsilon_i < e^{-R_i}\) are sequences with which (18) and (19) hold. Then for \(0 < \sigma < \rho_i\), we have

\[
|A(g, u_i)| \leq C_3 \sigma u_i (y_i)^{-2\lambda_i},
\]

where \(C_3\) is some constant independent of \(i\) and \(\sigma\).

Proof. Recall that

\[
A(g, u_i) = \int_{B_{\sigma}} (x^\alpha \partial_\alpha u_i) (g^{\beta \gamma} - \delta^{\beta \gamma}) \partial_{\beta \gamma} u_i \, dv - \int_{B_{\sigma}} (x^\alpha \partial_\alpha u_i) (g^{\beta \gamma} \Gamma_{\beta \gamma}^\mu) \partial_\mu u_i \, dv \\
+ \frac{1}{2} \int_{B_{\sigma}} u_i (g^{\beta \gamma} - \delta^{\beta \gamma}) \partial_{\beta \gamma} u_i \, dv - \frac{1}{2} \int_{B_{\sigma}} u_i (g^{\beta \gamma} \Gamma_{\beta \gamma}^\mu) \partial_\mu u_i \, dv
\]

\[
= I + II + III + IV.
\]
It follows from Lemma 3.2 and Lemma 3.3 that
\[ |I| \leq C_3 \int_{B_{\varepsilon}} |x|^3 |\nabla u_i| |\nabla^2 u_i| dv \leq C_3 \sigma u_i(y_i)^{-2\lambda_i}, \]
\[ |II| \leq C_3 \int_{B_{\varepsilon}} |x|^2 |\nabla u_i|^2 dv \leq C_3 \sigma u_i(y_i)^{-2\lambda_i}, \]
and
\[ |III| + |IV| \leq C_3 \int_{B_{\varepsilon}} |x| |u_i| |\nabla u_i| dv \leq C_3 \sigma u_i(y_i)^{-2\lambda_i}. \]

Lemma 3.5 is established.

Lemma 3.6. Let \( u_i \) satisfy (9) and \( y_i \rightarrow \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \rightarrow \infty \) and \( 0 < \varepsilon_i < e^{-R_i} \) are sequences with which (18) and (19) hold. Then
\[ \tau_i = O(u_i(y_i)^{-2\lambda_i}). \]

Consequently,
\[ u_i(y_i)^{\tau_i} \rightarrow 1. \]

Proof. We derive from (18) that
\[ \int_{d(y,y_i) \leq \tau_i} u_i^{p_i+1} \geq C^{-1} u_i(y_i)^{\frac{3}{2}} \geq C^{-1}, \]
where \( \tau_i = R_i u_i(y_i)^{-\frac{p_i-1}{2}} \).

In view of the above estimate, we can derive from the Pohozaev identity of \( u_i \) given by Lemma 2.1, using Lemma 3.3 and Lemma 3.4, that
\[ \tau_i \leq C \left( \frac{3}{p_i + 1} - \frac{1}{2} \right) \int_{B_{\varepsilon}} u_i^{p_i+1} \leq C u_i(y_i)^{-2\lambda_i}. \]

Lemma 3.6 follows from the above immediately.

Lemma 3.7. Let \( u_i \) satisfy (9) and \( y_i \rightarrow \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \rightarrow \infty \) and \( 0 < \varepsilon_i < e^{-R_i} \) are sequences with which (18) and (19) hold. Then for all \( 0 < \sigma < \frac{p}{2} \), we have
\[ \limsup_{i \to \infty} \max_{y \in \partial B_{\varepsilon}(y_i)} u_i(y) u_i(y_i) \leq C(\sigma). \]

Proof. In view of Lemma 3.1, we only need to establish the lemma for some sufficiently small \( \sigma > 0 \). Without loss of generality, we assume \( \sigma = 2 \). We distinguish into two cases.

Case 1: \( k_i(x) \geq 0 \ \forall x \in B_1(\bar{y}). \)
Pick any \( y_\sigma \) with \( d(y_\sigma, y_i) = \sigma \) and set \( \xi_i(y) = u_i(y_\sigma)^{-1}u_i(y) \). Then \( \xi_i(y) \) satisfies
\[
-\Delta_g \xi_i(y) = u_i(y_\sigma)^{p_i-1}\xi_i(y)^{p_i} - k_i(y)\xi_i(y).
\]
It follows from Lemma 3.1 that for all compact set \( K \subset B_1(\tilde{y}) \setminus \{\tilde{y}\} \), there exists some constant \( C(K) \) such that
\[
C(K)^{-1} \leq \xi_i \leq C(K) \quad \text{on} \ K.
\]
We also know from (24) that \( u_i(y_\sigma) \to 0 \) as \( i \to \infty \). Then by elliptic theories, we have, after passing to a subsequence,
\[
\xi_i \to \xi \quad \text{in} \ C^2_\text{loc}(B_1(\tilde{y}) \setminus \{\tilde{y}\}),
\]
where \( \xi \) satisfies
\[
-\Delta_g \xi + k\xi = 0, \quad \xi > 0, \quad \text{in} \ B_1(\tilde{y}) \setminus \{\tilde{y}\}
\]
with \( k(y) = \lim_{i \to \infty} k_i(y) \). Notice that for any fixed \( 0 < r < \sigma \),
\[
\lim_{i \to \infty} u_i(y_\sigma)^{-1}r^{\frac{2}{p_i-1}}w_i(r) = r^{\frac{1}{p_i}}\xi(r),
\]
where \( \xi(r) = \int_{\partial B_r(\tilde{y})} \int_{\partial B_r(y_i)} \xi_i ds_g \). From the assumption that \( y_i \to \tilde{y} \) is an isolated simple blowup point of \( \{u_i\} \), we deduce that \( r^{\frac{1}{p_i}}\xi(r) \) is non-increasing for \( 0 < r < \rho \), which is impossible if \( \xi \) were regular near \( \{\tilde{y}\} \). It follows from Corollary 9.1 in our appendix that as \( \sigma \) small enough, there exists some constant \( m > 0 \) independent of \( i \) such that for \( i \) large, we have
\[
-\int_{B_\rho(y_i)} \Delta_g \xi_i = -\int_{\partial B_\rho(y_i)} \nabla_g \xi_i \cdot \nu = -\int_{\partial B_\rho(y)} \nabla_g \xi \cdot \nu + c(1) > m. \tag{38}
\]
On the other hand, since \( k_i(x) \geq 0 \), we have
\[
\int_{B_\rho(y_i)} \Delta_g \xi_i(y) = \int_{B_\rho(y_i)} |u_i(y_\sigma)^{-1}u_i(y)|^{p_i} - k_i(y)\xi_i(y)|dy
\]
\[
\leq u_i(y_\sigma)^{-1}\int_{B_\rho(y_i)} u_i(y)^{p_i}dy. \tag{39}
\]
Similar to (37), we have
\[
\int_{d(y, y_i) \leq r_i} u_i^{p_i} \leq C u_i(y_i)^{-1 + \frac{\delta}{2}}.
\]
Applying Lemma 3.3 and noticing \( 0 < \delta < 1/100 \), we have
\[
\int_{r_i \leq d(y, y_i) \leq \sigma} u_i^{p_i} \leq C_2 \int_{r_i \leq d(y, y_i) \leq \sigma} (u_i(y_i)^{-\lambda_i}d(y, y_i)^{-1+\delta})^{p_i}
\]
\[
\leq Cr_i^{3-p_i(1-\delta)}u_i(y_i)^{-\lambda_i p_i}
\]
\[
= CR_i^{3-p_i(1-\delta)}u_i(y_i)^{-1+\frac{\delta}{2}}
\]
\[
= o(1) u_i(y_i)^{-1+\frac{\delta}{2}}.
\]
Using Lemma 3.6, we have
\[ \int_{B_\rho(y_i)} u_i^{p_i} \leq C u_i(y_i)^{-1}. \]  
(40)

Lemma 3.7 in this case follows from (38)-(40).

**Case 2:** \( k_i(x) < 0 \) for some \( x \in B_1(y) \).

For \( \sigma > 0 \), let \( \varphi \) denote the first eigenfunction of \(-\Delta_g \) in \( B_{2\sigma}(\tilde{y}) \) with respect to the Dirichlet boundary condition, i.e.
\[
\begin{cases}
-\Delta_g \varphi = \lambda_1 \varphi, & \varphi > 0, \quad \text{in } B_{2\sigma}(\tilde{y}) \\
\varphi = 0 & \text{on } \partial B_{2\sigma}(\tilde{y}),
\end{cases}
\]
where \( \lambda_1 \) denotes the first eigenvalue of \(-\Delta_g \). It is well known that we can choose \( \sigma > 0 \) small enough (independent of \( i \)), so that \( \lambda_1 > \|k_i\|_{L^\infty(B_{2\sigma}(\tilde{y}))} + 1 \) for all \( i \). Fix such a \( \sigma \), we know that
\[ -\Delta_g \varphi + k_i \varphi > 0 \quad \text{in } B_{\sigma}(\tilde{y}). \]  
(41)

Let \( \tilde{g} = \varphi^4 g \), \( L_{\tilde{g}} \) and \( L_g \) denote the conformal Laplacian operators of \( g \) and \( \tilde{g} \) respectively. It is well known that
\[ L_{\tilde{g}} \varphi = \varphi^{-5} L_g (\varphi \varphi), \quad \forall \varphi \in C^\infty(B_{\sigma}(\tilde{y})). \]  
(42)

Set \( \tilde{u}_i = \varphi^{-1} u_i \), we derive from (9) and the above with \( \psi = \tilde{u}_i \) that
\[ -\Delta_{\tilde{g}} \tilde{u}_i + \tilde{k}_i \tilde{u}_i = \varphi^{-\gamma} \tilde{u}_i^{p_i}, \]
where \( \tilde{k}_i = \frac{1}{8} R_{\tilde{g}} - \left( \frac{1}{4} R_g - k_i \right) \varphi^{-4} \). Taking \( \psi \equiv 1 \) in (42), we have
\[ -\frac{1}{8} R_{\tilde{g}} = \left( \Delta_g \varphi - \frac{1}{8} R_g \varphi \right) \varphi^{-5}. \]

Using (41) and the above we have \( \tilde{k}_i = (-\Delta_g \varphi + k_i \varphi) \varphi^{-5} > 0 \) in \( B_{\sigma}(\tilde{y}) \). Now, we choose small \( \sigma \in (0, \sigma) \) and set \( \tilde{\xi}_i(y) = \tilde{u}_i(y_i)^{-1} \tilde{u}_i(y) = \varphi(y_i)^{-1} \varphi^{-1}(y) \tilde{\xi}_i(y) \). Clearly (38) and (40) still hold when we substitute \( \tilde{\xi}_i \) for \( \xi_i \) and \( \tilde{u}_i \) for \( u_i \). Using the positivity of \( \tilde{k}_i \), we derive as in (39) that
\[ -\int_{B_{\sigma}} \Delta_g \tilde{\xi}_i \leq \tilde{u}_i(y_i)^{-1} \int_{B_{\rho}} \tilde{u}_i^{p_i}. \]

Lemma 3.7 in this case then follows as in Case 1. \( \square \)

**Proof of Proposition 3.1.** We first establish (23) by contradiction argument. Suppose the contrary, then after passing to a subsequence (still denoted as \( \{u_i\} \)) we can find \( \{\tilde{y}_i\} \) such that \( d(\tilde{y}_i, y_i) \leq \rho/2 \) and
\[ u_i(\tilde{y}_i) u_i(y_i) d(\tilde{y}_i, y_i) \to \infty. \]  
(43)

**Proof of Proposition 3.1.** We first establish (23) by contradiction argument. Suppose the contrary, then after passing to a subsequence (still denoted as \( \{u_i\} \)) we can find \( \{\tilde{y}_i\} \) such that \( d(\tilde{y}_i, y_i) \leq \rho/2 \) and
\[ u_i(\tilde{y}_i) u_i(y_i) d(\tilde{y}_i, y_i) \to \infty. \]  
(43)
After passing to another subsequence \( \{u_{i_k}\} \) (still denoted as \( \{u_i\}, \{y_i\} \), etc.), estimates (18) and (19) hold for some \( R_i \rightarrow \infty \) and \( 0 < \varepsilon_i < e^{-R_i} \). It follows from (18) that \( r_i := R_i u_i(y_i)^{-(p_i-1)/2} \leq d(\bar{y}_i, y_i) \leq \rho/2 \). Let \((x^1, x^2, x^3)\) be some geodesic normal coordinate system centered at \( y_i \) and given by \( \exp_{\bar{y}_i} \). Set \( \bar{r}_i = d(\bar{y}_i, y_i) \) and
\[
\bar{u}_i(x) = (\bar{r}_i)^{2/(p_i-1)} u_i(\exp_{\bar{y}_i}(\bar{r}_i x)) , \quad |x| < 2 .
\]
Clearly \( \bar{u}_i \) satisfies
\[
-\Delta_{g_i} \bar{u}_i(x) + \bar{k}_i(x) \bar{u}_i(x) = \bar{u}_i(x)^{p_i} , \quad |x| < 2 ,
\]
where \( \bar{k}_i(x) = r_i^2 k_i(\exp_{\bar{y}_i}(\bar{r}_i x)) \) and \( (g_i)_{\alpha\beta}(x) = g_{\alpha\beta}(\bar{r}_i x) dx^\alpha dx^\beta \). Here the metrics \( g_i \) depend on \( i \), but clearly all previous results still hold uniformly with respect to this sequence of metrics since the sequence stays in compact sets with respect to strong norms. It is easy to see that \( \{\bar{u}_i\} \) satisfies the hypotheses of Lemma 3.7 and therefore
\[
\max_{|x|=1} \bar{u}_i(0) \bar{u}_i(x) \leq C
\]
from which we can deduce (using also Lemma 3.6) that
\[
u_i(\bar{y}_i) u_i(y_i) d(\bar{y}_i, y_i) \leq C .
\]
This contradicts (43). We have thus established (23).

We see from (9) that \( u_i(y_i) u_i \) satisfies
\[
-\Delta_{g_i} (u_i(y_i) u_i) + k_i(u_i(y_i) u_i) = u_i(y_i)^{1-p_i} (u_i(y_i) u_i)^{p_i} .
\]
In view of (23), we can apply Harnack inequality and standard elliptic estimates to obtain, after passing to a subsequence, that
\[
u_i(y_i) u_i \to v \quad \text{in } C^2_{\text{loc}}(\mathbb{B}_\rho(\bar{y}) \setminus \{\bar{y}\}) ,
\]
where \( v > 0 \) satisfies
\[
-\Delta_g v + kv = 0 \quad \text{in } \mathbb{B}_\rho(\bar{y}) \setminus \{\bar{y}\}
\]
with \( k(x) = \lim_{t \to \infty} k_i(x) \).

It is also clear that for all \( 0 < r < \rho \),
\[
\lim_{i \to \infty} u_i(y_i) r^2/(p_i-1) \bar{u}_i(r) = r^{1/2} \bar{v}(r) ,
\]
where \( \bar{u}_i(r) \) is defined as in Definition 1.1 and \( \bar{v}(r) \) is defined similarly. Therefore it follows from Definition 1.2 (using also (18)) that \( r^{1/2} \bar{v}(r) \) is non-increasing in the interval \( 0 < r < \rho \), which implies that \( v \) is singular at the origin. In turn, it follows from Proposition 9.1 in our appendix that \( v = G(\cdot, \bar{y}) + h \) as stated in Proposition 3.1. Proposition 3.1 is established. Another way to show that \( v \) is singular at the origin is to argue as in the derivation of (47). \[\square\]
Once we have established Proposition 3.1 we can strengthen Lemmas 3.3–3.5 by substituting (23) for (24) in their proofs. We will state these improved versions as three corollaries without repeating the proofs.

**Corollary 3.1.** Let \( u_i \) satisfy (9) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \to \infty \) and \( 0 < \epsilon_i \leq e^{-R_i} \) are sequences with which (18) and (19) hold. Then there exists \( \rho_1 \in (0, \rho) \) which is independent of \( i \) such that

\[
|\nabla \phi u_i(y)| \leq C_2 u_i(y_i)^{-1} d(y, y_i)^{-2} \quad \text{for all } R_i u_i(y_i)^{-\frac{R_i - 1}{2}} \leq d(y, y_i) \leq \rho_1,
\]

and

\[
|\nabla^2 \phi u_i(y)| \leq C_2 u_i(y_i)^{-1} d(y, y_i)^{-3} \quad \text{for all } R_i u_i(y_i)^{-\frac{R_i - 1}{2}} \leq d(y, y_i) \leq \rho_1,
\]

where \( C_2 \) is some positive constant independent of \( i \).

**Corollary 3.2.** Let \( u_i \) satisfy (9) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \to \infty \) and \( 0 < \epsilon_i < e^{-R_i} \) are sequences with which (18) and (19) hold. Then for all fixed \( 0 < \sigma < \rho_1 \), we have for large \( i \) that

\[
\int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 u_i^{2i+1} = O(u_i(y_i)^{-4 + o(1)}),
\]

\[
\int_{d(y, y_i) \leq \sigma} u_i^2 \leq C_3 \sigma u_i(y_i)^{-2} + o(u_i(y_i)^{-2}),
\]

and

\[
\int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 |\nabla u_i|^2 \leq O(u_i(y_i)^{-4 + o(1)}) + C_3 \sigma u_i(y_i)^{-2},
\]

where \( C_3 \) is some constant independent of \( i \) and \( \sigma \).

**Corollary 3.3.** Let \( u_i \) satisfy (9) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \to \infty \) and \( 0 < \epsilon_i < e^{-R_i} \) are sequences with which (18) and (19) hold. Then

\[
|A(g, u_i)| \leq C_3 \sigma u_i(y_i)^{-2},
\]

where \( A(g; u_i) \) is defined as in Lemma 2.1 and \( C_3 \) is some constant independent of \( i \) and \( \sigma \).

We derive easily from (23), Corollaries 3.1–3.3 the following
Corollary 3.4. Let $u_i$ satisfy (9) and $y_i \to \bar{y} \in \Omega$ be an isolated simple blowup point. After passing to some subsequence of $\{u_i\}$, the following estimates hold:

$$\lim_{i \to \infty} u_i(y_i)^2 \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 u_i^{\nu_i+1} = 0,$$

$$\lim_{\sigma \to 0^+} \limsup_{i \to \infty} u_i(y_i)^2 \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 |\nabla u_i|^2 = 0,$$

$$\lim_{\sigma \to 0^+} \limsup_{i \to \infty} u_i(y_i)^2 \int_{d(y, y_i) \leq \sigma} u_i^2 = 0,$$

and

$$\lim_{\sigma \to 0^+} \limsup_{i \to \infty} u_i(y_i)^2 |A(g, u_i)| = 0,$$

where $A(g, u_i)$ is given in Lemma 2.1.

The following proposition will be used repeatedly in later sections.

Proposition 3.2. Let $u_i$ satisfy (9), $y_i \to \bar{y}$ be an isolated simple blowup point with, for some $\bar{\rho} > 0$,

$$u_i(y_i) u_i \to h \quad \text{in } C^2_{\text{loc}}(B_{\bar{\rho}}(\bar{y}) \setminus \{\bar{y}\}).$$

Assume, for some $a > 0$, in some geodesic normal coordinate system $x = (x^1, x^2, x^3)$, that

$$h(x) = \frac{a}{|x|} + A + o(1) \quad \text{as } |x| \to 0.$$

Then $A \leq 0$.

Proof. For $\sigma > 0$ small, the Pohozaev identity of $u_i$ is given by Lemma 2.1:

$$- \int_{B_{\sigma}} \left( k_i(x) + \frac{x \cdot \nabla k_i(x)}{2} \right) u_i(x)^2 dv + \left( \frac{3}{\nu_i + 1} - \frac{1}{2} \right) \int_{B_{\sigma}} u_i(x)^{\nu_i+1} dv$$

$$= \int_{\partial B_{\sigma}} B(x, \sigma, u_i, \nabla u_i) ds + A(g, u_i) - \int_{\partial B_{\sigma}} \left( \frac{k_i u_i^2}{2} - \frac{u_i^{\nu_i+1}}{\nu_i + 1} \right) ds.$$

Multiplying the above by $u_i(0)^2$ and then applying $\lim_{\sigma \to 0^+} \limsup_{i \to \infty}$ to both sides, we have, using Corollary 3.4, that

$$\lim_{\sigma \to 0^+} \int_{\partial B_{\sigma}} B(x, \sigma, u_i, \nabla h) = \lim_{\sigma \to 0^+} \limsup_{i \to \infty} u_i(0)^2 \int_{\partial B_{\sigma}} B(x, \sigma, u_i, \nabla u_i) \geq 0.$$

On the other hand, a direct calculation yields

$$\lim_{\sigma \to 0^+} \int_{\partial B_{\sigma}} B(x, \sigma, h, \nabla h) = -\frac{aA}{2} |[g^2]|.$$
where $|\mathbb{S}^2|$ denotes the area of the standard two dimensional sphere. Proposition 3.2 follows from the above two formulas.

\[ \square \]

**Remark 3.2.** Proposition 3.2 still holds when substituting the metric $g$ by a sequence of good enough metrics.

At the end of this section we point out that the analogue of (23) does not hold in higher dimensions. It was shown by Adimurthi and Yadava [1] that in dimension $4 \leq n \leq 6$ there exists a sequence of radially symmetric solutions to the following Neumann problem:

\[
\begin{align*}
-\Delta u_i + \frac{1}{i} u_i &= u_i^{n+2 \over n-2}, \quad u_i > 0, \quad \text{in } B_1(0), \\
\frac{\partial u_i}{\partial \nu} &= 0 \\ &\quad \text{on } \partial B_1(0),
\end{align*}
\]

with $u_i(0) = \max_{\overline{B}_1} u_i \to \infty$.

In connection with (44), we have

**Lemma 3.8.** For $n \geq 3$, let $u_i$ be a radially symmetric solution of (44). Then, after passing to a subsequence, either $u_i \to 0$ uniformly on $\overline{B}_1$ or $u_i(0) u_i(x) \to \infty$.

**Proof.** As pointed out in remark 4.4 of [43], $\{u_i\}$ can have at most one isolated blowup point at $\{0\}$. More precisely, we have

Step 1. For some constant $C$ independent of $i$,

$$|x|^{n-2 \over 2} u_i(x) \leq C, \quad \forall |x| \leq 1.$$  

This can be proved by contradiction argument. Suppose the contrary, we have, for some $0 < |x_i| \leq 1$,

$$f_i(x_i) = \max_{0 \leq |x| \leq 1} f_i(x) \to \infty,$$

where $f_i(x) := |x|^{n-2 \over 2} u_i(x)$. Let $\sigma_i = |x_i| / 2$, clearly we have

$$\sigma_i^{n-2 \over 2} \max_{|x-x_i| \leq \sigma_i} u_i \geq 2^{n-2 \over 2} f_i(x_i) \to \infty,$$

$$f_i \geq \sigma_i^{n-2 \over 2} \max_{|x-x_i| \leq \sigma_i} u_i.$$  

Consequently,

$$\sigma_i^{n-2 \over 2} u_i(x_i) \to \infty,$$
and

$$u_t(x_t) \geq 2^{-\frac{n-2}{2}} \max_{|z-x_t| \leq \sigma_t} u_t.$$  \hspace{1cm} (45)$$

Set

$$w_t(z) = u_t(x_t)^{-1} u_t(\eta(x_t)^{-\frac{n-2}{2}} z + x_t), \quad |z| < \eta(x_t)^{-\frac{n-2}{2}} \sigma_t \to \infty.$$ 

Clearly, $w_t$ satisfies, in view of (45),

$$
\begin{cases}
-\Delta w_t(z) = w_t(z)^{\frac{n+2}{n-2}} - i^{-1} u_t(\eta(x_t)^{-\frac{n-2}{2}} w_t(z), & u_t(x_t)^{-\frac{n-2}{2}} z + x_t \in B_1(0), \\
2^{-\frac{n-2}{2}} w_t(z) \leq w_t(0) = 1, & u_t(x_t)^{-\frac{n-2}{2}} z + x_t \in B_1(0),
\end{cases}
$$

and also the zero Neumann boundary condition. After passing to a subsequence, $w_t$ strongly converges to some $w$ on any compact subset with $w \geq 0$ satisfying $w(0) = 1$ and $-\Delta w = w^{(n+2)/(n-2)}$ in either the whole space $\mathbb{R}^n$ or some half space containing the origin. In the second case, we also know that $w$ satisfies the zero Neumann boundary condition on the boundary of the half space. Due to the Liouville type theorem of Caffarelli, Gidas and Spruck, $w$ is given explicitly. The explicit form of $w$ and the strong convergence of $w_t$ to $w$ on any compact sets apparently violate the radial symmetry of $u_t$. Step 1 is thus established.

We know from Step 1 that $\{u_t\}$ is uniformly bounded away from the origin, so we can derive from the Harnack inequalities (including the one with boundary conditions, see for example, Lemma A.1 in [35]) the following

**Step 2.** For any $0 < \epsilon < 1$, there exists $C(\epsilon)$ independent of $i$ such that

$$\max_{0 \leq |z| \leq 1} u_t \leq C(\epsilon) \min_{0 \leq |z| \leq 1} u_t.$$ 

Next we show

**Step 3.** After passing to a subsequence, either $u_t$ goes to zero uniformly on $B_1$, or $u_t(0) \to \infty$.

After passing to a subsequence, either $\{u_t(0)\}$ stays bounded or $u_t(0) \to \infty$. If $\{u_t(0)\}$ stays bounded, then, with Step 1, we can show the same way as in the proof of Lemma 3.2 that $\{u_t\}$ is uniformly bounded on $B_1$. After passing to another subsequence, using standard elliptic estimates, $u_t \to u$ in $C^2(B_1)$ for some radially symmetric function $u$ satisfying

$$
\begin{cases}
-\Delta u = u^{\frac{n+2}{n-2}}, & u \geq 0, \quad \text{in } B_1(0), \\
\partial u / \partial v = 0 & \text{on } \partial B_1(0).
\end{cases}
$$

Since the only nonnegative radially symmetric solution is $u \equiv 0$, we have established Step 3. Now we are ready to show that
Step 4. After passing to a subsequence, either \( u_i \) goes to zero uniformly on \( 
abla_1 \), or
\[
u_i(0)u_i(1) \to \infty. \tag{46}\]

After passing to a subsequence, we have shown that either \( u_i \) goes to zero uniformly on \( 
abla_1 \), or \( u_i(0) \to \infty \). If \( u_i(0) \to \infty \), we see from the equation of \( u_i \) that \( \Delta u_i \leq 0 \) near the origin which, together with the radial symmetry of \( u_i \), implies that 0 is a local maximum point of \( u_i \). Therefore, in view of Step 1, \( \{0\} \) is an isolated blowup point of \( \{u_i\} \). Applying Lemma 3.2, we know, after passing to a subsequence, that
\[
C^{-1}u_i(0) \leq u_i\left(u_i(0)^{-\frac{2}{n-2}}\right) \leq Cu_i(0).
\]

It is easy to see that
\[
\left(-\Delta + \frac{1}{i}\right)(e^{-|x|/|x|^{2-n}}) \leq 0, \quad \forall |x| \leq 1.
\]

Applying the maximum principle in \( B_1 \setminus B_{u_i(0)^{-1/1-(n-2)}}(0) \), we have, for some \( C \) independent of \( i \), that
\[
\nu_i(x) \geq C^{-1}u_i(0)^{-1}[e^{-|x|/|x|^{2-n}} - e^{-1}], \quad \forall u_i(0)^{-\frac{2}{n-2}} \leq |x| \leq 1.
\]

Consequently, for all \( 0 < |x| < 1 \),
\[
\liminf_{i \to \infty} u_i(0)u_i(x) \geq C^{-1}[e^{-|x|/|x|^{2-n}} - e^{-22-n}]. \tag{47}
\]

Now we verify (46) by contradiction argument. Suppose the contrary, \( u_i(0)u_i(1) \leq C \) along a subsequence. It follows from Step 2 that \( \{u_i(0)u_i\} \) is locally bounded in \( 
abla_1 \setminus \{0\} \). After passing to a subsequence,
\[
u_i(0)u_i(x) \to G_0(x) \text{ in } C^2_{\text{loc}}(\nabla_1 \setminus \{0\})
\]

where \( G_0 \) satisfies
\[
\begin{align*}
-\Delta G_0 &= 0, G_0 > 0, \quad \text{in } B_1(0) \setminus \{0\}, \\
\frac{\partial G_0}{\partial v} &= 0 \quad \text{on } \partial B_1(0).
\end{align*}
\]

We see from (47) that \( G_0 \) is singular near the origin, therefore, for some \( a_0 > 0 \),
\[
G_0(x) = \frac{a_0}{|x|^{n-2}} + H_0(x), \quad \text{in } B_2(0) \setminus \{0\},
\]

where \( H_0 \) is a regular harmonic function in \( B_1 \). Apparently the harmonic function \( H_0 \) is in \( C^1(B_1) \) and satisfies \( \frac{\partial H_0}{\partial v} = (n-2)a_0 > 0 \) on \( \partial B_1 \). This violates the maximum principle and the Hopf lemma. Lemma 3.8 follows from Step 1 to Step 4.
We conclude from Lemma 3.8 and the result of Adimurthi and Yadava that the analogue of (23) does not hold in dimension $4 \leq n \leq 6$.

4. An Isolated Blowup Point is in Fact an Isolated Simple Blowup Point

In this section, we prove that an isolated blowup point is in fact an isolated simple blowup point.

Proposition 4.1. Let $u_i$ satisfy (9) and $y_i \to \bar{y}$ be an isolated blowup point. Then $\bar{y}$ must be an isolated simple blowup point.

Proof. Due to our estimates in Sec. 3, the proof is basically the same as that of proposition 3.1 of [42]. For readers' convenience, we include the proof here. We prove it by contradiction argument. Suppose it is not an isolated simple blowup point, then after passing to a subsequence $\{u_{i_j}\}$ (still denoted as $\{u_i\}$, $\{y_i\}$, etc.), $r^{\frac{n-2}{2}} u_{i}\big|_{\bar{y}_i}$ has at least two critical points in $(0,\bar{\mu}_i)$ with $\bar{\mu}_i \to 0$. By passing to another subsequence, we can assume that estimates (18) and (19) hold for some $R_i \to \infty$ and $0 < \varepsilon_i < e^{-\bar{r}_i}$. It follows from (18) that $r^{\frac{n-2}{2}} u_{i}\big|_{\bar{y}_i}$ has precisely one critical point in the interval $0 < \bar{r}_i < R_i u_i(y_i) - \frac{\bar{\varepsilon}_i}{2}$. Let $\mu_i$ be the second critical point of $r^{\frac{n-2}{2}} u_{i}\big|_{\bar{y}_i}$. We have $\mu_i \geq \bar{r}_i$, $\lim_{i \to \infty} \mu_i = 0$. As always we let $x = (x^1, x^2, x^3)$ be some geodesic normal coordinate system centered at $y_i$ given by $\text{exp}_{y_i}(x)$, $h = h_{\alpha\beta}(x) dx^\alpha dx^\beta = g_{\alpha\beta}(\mu_i x) dx^\alpha dx^\beta$ be the scaled metric, and

$$
\xi_i(x) = \mu_i^{\frac{n-2}{2}} u_i(\text{exp}_{y_i}(\mu_i x)), \quad \text{for } |x| < 1/\mu_i.
$$

Then $\xi_i$ satisfies

$$
\begin{cases}
-\Delta_h \xi_i(x) + \tilde{k}_i(x) \xi_i(x) = \xi_i(x)^{p_i}, & |x| < 1/\mu_i, \\
|x|^{\frac{n-2}{n-1}} \xi_i(x) \leq C, & |x| < 1/\mu_i, \\
\lim_{i \to \infty} \xi_i(0) = \infty, \\
r^{\frac{n-2}{n-1}} \xi_i(r) \text{ has precisely one critical point in } 0 < r < 1, \\
\frac{d}{dr} \{r^{\frac{n-2}{n-1}} \tilde{\xi}_i(r)\} \bigg|_{r=1} = 0,
\end{cases}
$$

(48)

where $\tilde{k}_i(x) = \mu_i^2 k_i(\text{exp}_{y_i}(\mu_i x))$ and $\tilde{\xi}_i(r)$ is the spherical average of $\xi_i$ defined in the usual way.

It follows that $\{0\}$ is an isolated simple blowup point of $\{\xi_i\}$. We know from Proposition 3.1 and the Harnack inequality that for all compact set $K \subset \mathbb{R}^3 \setminus \{0\}$,

$$
C(K)^{-1} \leq \xi_i(0) \xi_i \leq C(K) \quad \text{on } K,
$$
and, for some constant $a > 0$, that
\[
\xi_i(0)\xi_i(x) \rightarrow h(x) := a|x|^{-1} + b(x) \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^3 \setminus \{0\}), \tag{49}
\]
where $b(x)$ satisfies $\Delta b = 0$ in $\mathbb{R}^3$. Since $h(x)$ is positive, we have $\liminf_{|x| \to \infty} b(x) \geq 0$. It follows from the maximum principle that $b(x)$ is nonnegative, hence a constant, denoted as $b(x) \equiv b$. We deduce from (48) and (49) that
\[
\frac{d}{dr}\{r^{\frac{1}{2}}h(r)\}_{r=1} = 0.
\]
It follows that
\[
b = a > 0.
\]
This violates Proposition 3.2. Proposition 4.1 is established. $\square$

5. Ruling Out Bubble Accumulations

For $k(x) \in C^1(M)$, we start to analyze compactness properties of $\bigcup_{2 \leq p < 5} M_{k,p}$. The following proposition gives preliminary description of large functions $u \in \bigcup_{2 \leq p < 5} M_{k,p}$, which relies on two Liouville type theorems. One is due to Gidas and Spruck [32] which asserts that, for $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, there is no solution to the following subcritical exponent equation
\[
-\Delta u = u^p, \quad u > 0, \quad \text{in } \mathbb{R}^n.
\]
The other is due to Caffarelli, Gidas and Spruck [17] which asserts that, for $n \geq 3$, any solution of the following critical exponent equation
\[
-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0, \quad \text{in } \mathbb{R}^n
\]
is of the form
\[
u(x) = \left( \frac{\lambda}{1 + \frac{1}{n(n-2)} \lambda^2 |x - \tilde{x}|^2} \right)^{\frac{n-2}{2}}
\]
for some $\lambda > 0$ and $\tilde{x} \in \mathbb{R}^n$. Under some additional decay hypothesis of $u(x)$ at infinity, the above Liouville type theorems are obtained for the subcritical exponent equation in Gidas, Ni and Nirenberg [31], while for the critical exponent equation in Obata [49] and Gidas, Ni and Nirenberg [31].

Roughly speaking, the following proposition shows that for large $u \in \bigcup_{2 \leq p < 5} M_{k,p}$, $p$ has to be very close to 5 and one can find a finite collection of disjoint balls $B_{r_1}(y_1), \ldots, B_{r_N}(y_N)$ ($N$ may depend on $u$) inside which $u$ is very well approximated in strong norms by standard bubbles. Furthermore $u$ satisfies $u(y) \leq C_1 d(y, \{y_1, \ldots, y_N\})^{-\frac{2}{n-2}}$ for all $y$ in $M$, where $C_1$ is some positive constant independent of $u$. 

Proposition 5.1. Given any small $\epsilon > 0$ and large $R$, there exist some positive constants $C_0$ and $C_1$ depending only on $M$, $g$, $\|k\|_{C^1(M)}$, $\epsilon$ and $R$ such that for all $u$ in $\bigcup_{2 \leq p \leq 5} M_{k,p}$ with
\[
\max_M u > C_0,
\]
there exists some integer $N = N(u) \geq 1$ and $N$ local maximum points of $u$, denoted as $y_1, \ldots, y_N$, such that

(i) $5 - \epsilon < p \leq 5$.

(ii) $B_{\delta_j}(y_i) \cap B_{\delta_j}(y_j) = \emptyset$, for $i \neq j$.

and, for each $j$,
\[
\left\| u(y_j)^{-1} u \left( \exp_{y_j} \left( u(y_j)^{-\frac{1}{p-1}} x \right) \right) - \left( 1 + \frac{1}{3} |x|^2 \right)^{-\frac{1}{2}} \right\|_{C^2(|x| < 2R)} < \epsilon, \tag{50}
\]
where $\delta_j = R u(y_j)^{- (p-1)/2}$, $B_{\delta_j}(y_j) \subset M$ denotes the geodesic ball of radius $\delta_j$ and centered at $y_j$, $x = (x^1, x^2, x^3)$ denotes some geodesic normal coordinates centered at $y_j$, and $|x| = (x^1)^2 + (x^2)^2 + (x^3)^2$.

(iii) $d(y_i, y_j)^{2/(p-1)} u(y_j) \geq C_0$ for $j > i$, while $d(y, \{y_1, \ldots, y_N\})^{2/(p-1)} u(y) \leq C_1$ for all $y$ in $M$.

Proposition 5.1 can be derived from the following lemma.

Lemma 5.1. Given any small $\epsilon > 0$ and large $R > 1$. There exists some large positive constants $C_0$, depending only on $M$, $g$, $\epsilon$, $R$, and $\|k\|_{C^1(M)}$ such that for any compact $S \subset M$ and any $u$ in $\bigcup_{2 \leq p \leq 5} M_{k,p}$ with
\[
\max_{y \in M \setminus S} d(y, S)^{2/(p-1)} u(y) \geq C_0,
\]
we have $p > 5 - \epsilon$ and for some local maximum point of $u$ in $M \setminus S$, denoted as $y_0$,
\[
\left\| u(y_0)^{-1} u \left( \exp_{y_0} \left( u(y_0)^{-\frac{1}{p-1}} x \right) \right) - \left( 1 + \frac{1}{3} |x|^2 \right)^{-\frac{1}{2}} \right\|_{C^2(|x| < 2R)} < \epsilon, \tag{51}
\]
where $d(y, S)$ denotes the distance of $y$ to $S$, and $d(y, S) = 1$ if $S = \emptyset$.

Proof. Suppose that for some $\epsilon$ and $R$ such $C_0$ does not exist. Then there exists compact $S_i \subset M$, $2 \leq p_i \leq 5$, $\{k_i\}_{C^1(M)}$ uniformly bounded, and $u_i \in M_{k_i,p_i}$ such that $\max_{y \in M \setminus S_i} d(y, S_i)^{2/(p_i-1)} u_i(y) \geq i$, but no such $y_0$ exists. Pick a point $\tilde{y}_i$ in $M \setminus S_i$ which satisfies $d(\tilde{y}_i, S_i)^{2/(p_i-1)} u_i(\tilde{y}_i) = \max_{y \in M \setminus S_i} d(y, S_i)^{2/(p_i-1)} u_i(y)$. Consider the following rescaled function
\[
u_i(x) = u_i(\tilde{y}_i)^{-1} u_i \left( \exp_{\tilde{y}_i} \left( u_i(\tilde{y}_i)^{- (p_i-1)/2} x \right) \right),
\]
for all $|x| < R_i := \frac{1}{4} u_i(\tilde{y}_i)^{(p_i-1)/2} d(\tilde{y}_i, S_i)$. 


We know that \( R_i \geq i/4 \to \infty \) and, for \( i \) large, \( d(y, S_i) \geq \frac{1}{2}d(\tilde{y}_i, S_i) \) for all \( |x| \leq R_i \) and \( y = \exp_{\tilde{y}_i}(u_i(\tilde{y}_i))^{(p-1)/2}x \). It follows that

\[
\left( \frac{1}{2}d(\tilde{y}_i, S_i) \right)^{- \frac{2}{p-1}} u_i(y) \leq d(y, S_i)^{- \frac{2}{p-1}} u_i(y) \leq d(\tilde{y}_i, S_i)^{- \frac{2}{p-1}} u_i(\tilde{y}_i) \quad \forall |x| \leq R_i,
\]

which implies

\[
w_i(x) \leq 2^{\frac{2}{p-1}} \leq 4, \quad \forall |x| \leq R_i.
\]

It then follows from standard elliptic estimates that after passing to a subsequence, still denoted as \( \{w_i\} \), we have \( p_i \to p \in [2, 5] \),

\[
w_i \to w \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{R}^3),
\]

where \( w \) satisfies

\[-\Delta w = w^p, \quad w > 0, \quad \text{in} \quad \mathbb{R}^3.\]

By the Liouville type theorem of Gidas and Spruck, we know that \( p = 5 \) and in particular \( p_i > 5 - \epsilon \) for large \( i \). In turn, it follows from the Liouville type theorem of Caffarelli, Gidas and Spruck that \( w(x) = \left( \frac{\lambda}{1 + \frac{1}{4} \lambda |x|} \right)^{1/2} \) for some \( \lambda \in \mathbb{R}^3 \). Since \( w(0) = \left( \frac{\lambda}{1 + \frac{1}{4} \lambda |x|} \right)^{1/2} \), and \( w(x) \leq 4 \) for all \( x \in \mathbb{R}^3 \), we see that \( 1 \leq \lambda \leq 2 \). We see from the explicit form of \( w \) that we can find \( x_i \to \tilde{x} \) which are local maximum points of \( w_i(x) \). Clearly \( w_i(x_i) \to \lambda^{1/2} = \max w \). Defining \( y_i = \exp_{\tilde{y}_i}(u_i(\tilde{y}_i))^{(p-1)/2}x_i \), then \( y_i \in M \setminus S_i \) is a local maximum point of \( u_i \). Repeating the scaling with \( \tilde{y}_i \) replaced by \( y_i \), we will obtain a new limit \( w(x) = \left( \frac{\lambda}{1 + \frac{1}{4} \lambda |x|} \right)^{1/2} \). Consequently, for large \( i \), we have

\[
\left\| u_i(y_i)^{-1}u_i \left( \exp_{y_i}(u(y_i)^{1-\epsilon}x) \right) - \left( 1 + \frac{1}{3} |x|^2 \right)^{-\frac{1}{2}} \right\|_{C^2(|x|<2\tau_i)} < \epsilon.
\]

This shows that, for large \( i \), \( u_i \) actually satisfies (51) which violates the contradiction hypothesis we start with. Lemma 5.1 is established. \( \square \)

**Proof of Proposition 5.1.** We first apply Lemma 5.1 with \( S = \phi \) and \( d(y, S) \equiv 1 \). We have \( p > 5 - \epsilon \), \( y_1 = y_0 \) is a maximum point of \( u \), and (51) holds. Next we take

\[
S = B_{\tau_i}(y_1), \quad \tau_i = \frac{2\tau_i}{\pi}(y_1).
\]

If \( \max_{y \in S} d(y, S)^{2/(p-1)}u(y) \leq C_0 \), we stop. Otherwise we obtain \( y_2 = y_0 \) given by Lemma 5.1. It is clear from the lemma that \( B_{\tau_i}(y_1) \cap B_{\tau_i}(y_2) = \emptyset \) since \( \epsilon > 0 \) can be made very small from the begin. We continue this process. The process has to stop at some positive universal constant. Thus we obtain \( \{y_1, \ldots, y_N\} \subset M \) satisfying (ii), and \( d(y, \bigcup_{i=1}^N B_{\tau_i}(y_i))^{2/(p-1)}u(y) \leq C_0 \), for all \( y \in M \setminus \bigcup_{i=1}^N B_{\tau_i}(y_i) \). Now for any \( y \in M \), either \( y \in B_{2\tau_i}(y_i) \) for some \( i \), or \( d(y, y_i) > 2\tau_i \), \( \forall 1 \leq i \leq N \). In the first case, \( d(y, \{y_1, \ldots, y_N\}) \leq d(y, y_i) < 2\tau_i \), so (50) implies
\[ u(y) \leq 2u(y_i) = 2R^{2/p}r_i^{-2/(p-1)}, \quad \text{and therefore} \quad d(y, \{y_1, \ldots, y_N\})^{2/(p-1)}u(y) \leq 2(2R)^{2/(p-1)} \leq 8R^2. \quad \text{In the second case,} \quad d(y, \{y_1, \ldots, y_N\}) \leq 2d(y, \bigcup_{i=1}^N B_{r_i}(y_i)), \quad \text{therefore} \quad d(y, \{y_1, \ldots, y_N\})^{2/(p-1)}u(y) \leq 2^{2/2(p-1)}C_0 \leq 4C_0. \quad \text{Taking} \quad C_1 = 8R^2 + 4C_0, \quad \text{we have established Proposition 5.1.} \]

We are ready to establish our main result in this section.

**Proposition 5.2.** Let \( k \in C^1(M), \epsilon > 0 \) be sufficiently small and \( R > 1 \) be sufficiently large. Then there exists some positive constant \( \delta^* \) depending only on \( M, g, \epsilon, R, \) and \( \|k\|_{C^1(M)} \), such that for all \( u \in \bigcup_{2 \leq p \leq 5} M_{k,p} \) with \( \text{max}_M u \geq C_0 \) we have

\[ d(y_j, y_i) \geq \delta^*, \quad \text{for all} \quad 1 \leq j \neq i \leq N, \]

where \( y_j = y_j(u), y_i = y_i(u), N = N(u) \) and \( C_0 \) are the ones defined in Proposition 5.1.

**Proof.** The proof is similar to that of proposition 4.2 of [42]. Suppose the contrary: for some small \( \epsilon > 0 \), large \( R, k \rightarrow k \) in \( C^1(M), p_i \in [2, 5], p_i \rightarrow p \in [2, 5] \), and \( u_i \in M_{k_i, p_i} \) with \( \text{max}_M u_i \geq C_0 \) and

\[ \lim_{i \to \infty} \min_{j \neq i} d(y_j(u_i), y_i(u_i)) = 0. \]

Without loss of generality, we can assume that

\[ \sigma_i := d(y_1(u_i), y_2(u_i)) = \min_{j \neq i} d(y_j(u_i), y_i(u_i)) \to 0. \]

Since \( B_{R\sigma_i(y_1)} - \epsilon \) \( \cap B_{R\sigma_i(y_2)} - \epsilon \) \( = \emptyset \), we have, in view of Proposition 5.1 (i), that \( u_i(y_1), u_i(y_2) \to \infty \).

Let \( x = (x^1, x^2, x^3) \) be some geodesic geodesic normal coordinate system centered at \( y_1(u_i), \exp_{y_1}(x) \) denote the exponential map, \( g = g_{\alpha\beta}(x)dx^\alpha dx^\beta \) denote the metric in local coordinates. Let \( h_{\alpha\beta}(x) = g_{\alpha\beta}(x) \) and \( h = h_{\alpha\beta}(x)dx^\alpha dx^\beta \) denote the scaled metric. Define the rescaled function

\[ w_i(x) = \sigma_i^{2/(p-1)}u_i(\exp_{y_1}(\sigma_i x)). \]

It follows that \( w_i \) satisfies

\[ \begin{cases} \tilde{\Delta} h_i w_i(x) + \tilde{k}_i(x)w_i(x) = \sigma_i^{2/(p-1)}w_i(x) & \text{in} \quad B_{\frac{1}{\sigma_i}}, \\ w_i(x) > 0 & \text{in} \quad B_{\frac{1}{\sigma_i}}, \end{cases} \quad (52) \]

where \( \tilde{k}_i(x) = \sigma_i^{2/2(\epsilon)}\tilde{k}_i(\exp_{y_1}(\sigma_i x)). \)

Let \( x_j = x_j(u_i) \) be the point so that \( \exp_{y_1}(\sigma_i x_j) = y_j \in B_{\sqrt{\sigma_i}(y_1)}, j \in \{1, \ldots, N\}. \) It is clear that

\[ x_1 = 0, \quad \min_{j \neq 1} |x_j - x_1| \geq 1 + o(1), \quad |x_2| = 1 + o(1). \]
After passing to a subsequence, we have
\[ \bar{x} = \lim_{i \to \infty} x_2(u_i), \quad |\bar{x}| = 1. \]
It is also clear that
\[ \sigma_i \geq \frac{1}{C} \max \left\{ R_{u_i}(y_1)^{-\frac{n-1}{2}}, R_{u_i}(y_2)^{-\frac{n-1}{2}} \right\}. \]
We derive from the above that
\[ \begin{align*}
& w_i(0), w_i(x_2) \geq C_0, \\
& \text{Each } x_j \text{ is a local maximum point of } w_i, \\
& \min_{1 \leq j \leq N} |x - x_j|^{2/(m-1)} w_i(x) \leq C_1 \quad \forall |x| \leq 1/\sigma_i.
\end{align*} \]
We first show that
\[ w_i(0) \to \infty, \quad \text{and } w_i(x_2) \to \infty. \tag{53} \]
If one of them tends to infinity along a subsequence, say \( w_i(0) \to \infty \). We know that \( \{w_i\} \) is an isolated blowup point, and therefore is an isolated simple blowup point. Then \( w_i(x_2) \) has to tend to infinity along the same sequence since otherwise, by an argument used in the proof of Lemma 3.2, \( \{w_i\} \) would be uniformly bounded near \( \{x_2\} \) along a further subsequence. In turn, using Proposition 3.1 and the Harnack inequality, we have \( \{w_i\} \) uniformly goes to zero near \( \{x_2\} \), which violates \( w_i(x_2) \geq C_0 \). On the other hand, if both of \( \{w_i(0)\} \) and \( \{w_i(x_2)\} \) stay bounded, we know from a similar argument as above that \( \{w_i\} \) is locally bounded. It then follows from standard elliptic estimates that after passing to a subsequence, \( w_i \to w \) in \( C^3_{\text{loc}}(\mathbb{R}^3) \) where \( w \) satisfies \( -\Delta w = w^p, w > 0 \) in \( \mathbb{R}^3 \) and \( \nabla w(0) = \nabla w(\bar{x}) = 0 \). This is a contradiction since there is no positive solution to the equation with two distinct critical points according to the Liouville type theorem of Caffarelli, Gidas and Spruck. We have thus established (53).

Apparently \( 0 \) and \( x_2 \to \bar{x} \) are two isolated blowup points of \( \{w_i\} \), and therefore according to Proposition 3.1 are isolated simple blowup points.

After passing to a subsequence, let \( \tilde{S} \) denote the set of blowup points of \( \{w_i\} \). Clearly \( 0 \in \tilde{S} \) and \( d(\tilde{x}, \bar{x}) \geq 1 \) for any two distinct points \( \tilde{x}, \bar{x} \in \tilde{S} \). In view of Proposition 3.1, we know that \( \{w_i(0)w_i\} \) is locally bounded in \( \mathbb{R}^3 \setminus \tilde{S} \).

Multiplying (52) by \( w_i(0) \) and send \( i \) to infinity, we have, after passing to a subsequence, that
\[ \lim_{i \to \infty} w_i(0)w_i = h^* \text{ in } C^2_{\text{loc}}(\mathbb{R}^3 \setminus \tilde{S}), \]
where \( h^* \) is a regular harmonic function in \( \mathbb{R}^3 \setminus \tilde{S} \). Since all the blowup points of \( \{w_i\} \) are isolated simple blowup points, we know from Proposition 3.1 and Böcher's theorem that
\[ h^*(x) = a_1|x|^{-1} + a_2|x - \bar{x}|^{-1} + b^*(x), \quad \forall x \in \mathbb{R}^3 \setminus \tilde{S}, \]
where \(a_1, a_2 > 0\) are some positive constants, and \(b^*\) is some regular harmonic function in \(\mathbb{R}^3 \setminus \{S \setminus \{0, \bar{x}\}\}\). We then derive from the maximum principle that \(b^*(x) \geq 0\) for all \(x \in \mathbb{R}^3 \setminus \{S \setminus \{0, \bar{x}\}\}\). It follows that for some positive constant \(A > 0\)
\[
h^*(x) = a_1|x|^{-1} + A + O(|x|) \quad \text{for } |x| \text{ close to 0.}
\]
This violates Proposition 3.2. Proposition 5.2 is established.

**Proof of Theorem 0.2.** Suppose the contrary, then there exist \(p_i \to p \in (1, 5]\), \(k_i \to k\) in \(C^1(M)\), \(u_i \in M_{k_i, p_i}\) such that \(\|u_i\|_{H^1(M)} \to \infty\), which, in view of standard elliptic estimates, implies that \(\max_M u_i \to \infty\). In turn, it follows from the Liouville type theorem of Gidas and Spruck and some standard blowup argument as in the proof of Lemma 5.1 that \(p_i \to 5\). Applying Proposition 5.2, we know that for some small \(\varepsilon > 0\), large \(R > 0\), and some positive integer \(N\), all independent of \(i\), and \(y_i^{(1)} = y^{(1)}(u_i), \ldots, y_i^{(N)} = y^{(N)}(u_i) \in M\) such that (i)-(iii) in Proposition 5.1 hold. Arguing as in the proof of Proposition 5.2, we know that \(\{y_i^{(1)}\}, \ldots, \{y_i^{(N)}\}\) are isolated blowup points of \(\{u_i\}\), and therefore, in view of Proposition 4.1, that they are isolated simple blowup points of \(\{u_i\}\). We conclude from (18) and Proposition 3.1 that \(\|u_i\|_{H^1(M)}\) is bounded. This is a contradiction. Theorem 0.2 is thus established.

6. Compactness Properties of Solutions

In this section, we study compactness properties of solutions of (5) and establish Theorem 0.3.

**Proof of Theorem 0.3.** Due to standard elliptic estimates, we only need to establish the \(L^\infty\) bound: \(u \leq C\). Suppose the contrary to the \(L^\infty\) bound, then there exist \(p_i \to p \in (1, 5]\), \(k_i \to k\) in \(C^1(M)\) with

\[
\text{the first eigenvalue of } (-\Delta_g + k_i) \geq c_0 > 0, \quad \min_{y \in M} A_{k_i, p_i}(y) \geq c_0 > 0,
\]

for some positive constant \(c_0\) independent of \(i\), and \(u_i \in M_{k_i, p_i}\) such that \(\max_M u_i \to \infty\). As in the proof of Theorem 0.2, we know, after passing to a subsequence, that \(p_i \to 5\) and \(\{u_i\}\) has \(N\) isolated simple blowup points \(y_i^{(1)} \to y^{(1)}, \ldots, y_i^{(N)} \to y^{(N)}\), where \(N \geq 1\) is some positive integer independent of \(i\). It follows from Proposition 3.1 that, after passing to a subsequence,

\[
u_i(y^{(1)} \to y_i(y) \to h(y) := \sum_{j=1}^{N} a_j G_{y^{(j)}}(y) + b(y) \quad \text{in } C^2_{loc}(M \setminus \{y^{(1)}, \ldots, y^{(N)}\}),
\]

where \(a_1, \ldots, a_N\) are some positive constants and \(b \in C^2(M)\) satisfies \((-\Delta_g + k) b = 0\) in \(M\). Since the first eigenvalue of \(k\) is positive, \(b\) is identically zero.
Let \( x = (x^1, x^2, x^3) \) be some geodesic normal coordinate system centered at \( y_i \). It follows from the above discussion that for some positive constant \( A > 0 \) independent of \( i \), we have

\[
h(x) := h \left( \exp_{y_i} (x) \right) = \frac{a_1}{3\omega_3} |x|^{-1} + A_i + O(|x|^\alpha) \quad \text{for } |x| \text{ close to 0},
\]

where \( A_i \geq A > 0 \) for all \( i \), \( \omega_3 \) is the volume of the unit ball in \( \mathbb{R}^3 \) and \( 0 < \alpha < 1 \). This violates Proposition 3.2. Theorem 0.3 is established.

\[ \square \]

7. More General Equations

In this section, we treat more general equations (8), still on a three dimensional smooth compact Riemannian manifold \((M, g)\). Throughout this section we assume, unless otherwise stated, that \( k \in C^1(M) \), \( K \in C^2(M) \) and \( K > 0 \) on \( M \). For \( 1 < p \leq 5 \), we use \( M_{K,k,p} \) to denote the set of solutions of (8) in \( C^2(M) \). We first state the following generalization of Theorem 0.2 which gives a priori estimates in \( H^1(M) \) norm to solutions of (8).

**Theorem 7.1.** Let \((M, g)\) be a three dimensional smooth compact Riemannian manifold. Then for all \( \epsilon_0 > 0 \),

\[
\|u\|_{H^1(M)} \leq C \quad \forall u \in \bigcup_{1+\epsilon_0 \leq p \leq 5} M_{K,k,p},
\]

where \( C \) depends only on \( M, g, \epsilon_0, \|k\|_{C^1(M)}, \|K\|_{C^2(M)} \), and the positive lower bound of \( K \) on \( M \).

We still use \( \lambda_1 \) to denote the first eigenvalue of \(-\Delta_g + k\). Next theorem extends Theorem 0.3.

**Theorem 7.2.** Let \((M, g)\) be a three dimensional smooth compact Riemannian manifold, \( k(y) \in C^1(M) \) with positive \( \lambda_1 \), and \( K(y) \in C^2(M) \) be a positive function. Assume that

\[
\min_{y \in M} A_{k,g}(y) > 0.
\]

Then for all \( \epsilon_0 > 0 \),

\[
1/C \leq u \leq C \quad \text{in } M \quad \text{and} \quad \|u\|_{C^2(M)} \leq C, \quad \forall u \in \bigcup_{1+\epsilon_0 \leq p \leq 5} M_{K,k,p},
\]

where \( C \) is some positive constant depending only on \( M, g, \epsilon_0, \|k\|_{C^1(M)}, \|K\|_{C^2(M)} \), and the positive lower bound of \( \lambda_1, A_{k,g}, \) and \( K \).

We can derive existence results and multiplicity results (counting multiplicity) from the compactness results given in Theorem 7.2 as in the introduction.
Consider the following minimization problem
\[
\min \left\{ E(v) : v \in H^1(M), \quad \int_M K v^5 = 1 \right\},
\] (54)
where \( E(v) \) is the same as in (7). The following theorem extends Theorem 0.4.

**Theorem 7.3.** Under the hypothesis in Theorem 7.2, (54) is attained at some positive function \( v \in C^2(M) \).

We can derive Theorem 7.3 from Theorem 7.2 in the same way as we derive Theorem 0.4 from Theorem 0.3.

We introduce, for \( 0 < \alpha < 1 \), an operator \( \bar{F} : C^{2,\alpha}(M) \to C^{2,\alpha}(M) \) by
\[
\bar{F}(v) = v - (-\Delta_g + k(y))^{-1}(E(v)Kv^5).
\]

Our next theorem extends Theorem 0.5.

**Theorem 7.4.** Under the hypothesis in Theorem 7.2 we have, for \( \Lambda \) large enough, that \( 0 \) does not belong to \( \bar{F}(\partial \Omega_\Lambda) \) and
\[
\deg(\bar{F}, \Omega_\Lambda, 0) = -1.
\]

In particular, \( \mathcal{M}_{K,k,5} \neq \emptyset \).

Using Theorem 7.2, we can prove Theorem 7.4 in the same way as in Schoen [51]. We omit the details. In the rest of this section, we establish Theorems 7.1 and 7.2. The proof is a combination of arguments in earlier sections together with some arguments in [42].

**Remark 7.1.** Let \( (M, g) = (S^3, g_0) \), the standard three dimensional sphere, \( k \in C^1(S^3) \) satisfy \( k(y) < \frac{3}{4} \equiv \frac{1}{8}R_{g_0} \), and \( K \in C^2(S^3) \) be a positive function. Then the hypothesis in Theorem 7.2 holds. Consequently the conclusions given in Theorems 7.2–7.4 all hold.

**Remark 7.2.** Let \( (M, g) \) be a positive three dimensional smooth compact Riemannian manifold which is not conformally equivalent to the standard three dimensional sphere. Then, in view of the positive mass theorem of Schoen and Yau, there exists \( \epsilon = \epsilon(M, g) > 0 \) such that for all \( k \in C^1(M) \) satisfying \( k \leq \frac{1}{8}R_g + \epsilon \) on \( M \) and all positive \( K \in C^2(M) \), the hypothesis in Theorem 7.2 holds. Consequently the conclusions given in Theorems 7.2–7.4 all hold.
Let $\Omega \subset M$ be an open set, $\{k_i\}$ be a sequence of functions converging to $k$ in $C^3(M)$, $\{K_i\}$ be a sequence of functions converging to some positive function $K$ in $C^2(M)$, $\{p_i\}$ be a sequence of numbers satisfying $2 \leq p_i \leq 5$, $p_i \rightarrow 5$, and $\{u_i\}$ be a sequence of $C^2$ functions satisfying
\[-\Delta_g u_i + k_i u_i = K_i u_i^{p_i}, \quad u_i > 0, \quad \text{in } \Omega \subset M. \tag{55}\]

The definitions of isolated blowup points and isolated simple blowup points are the same as in Definitions 1.1 and 1.2.

The Pohozaev type identity for a solution of
\[-\Delta_g u + ku = Ku^p, \quad u > 0, \quad \text{in } \Omega\]
takes the form
\[-\int_{B_\varepsilon} \left( k(x) + \frac{x \cdot \nabla k(x)}{2} \right) u^2 dv + \left( \frac{3}{p+1} - \frac{1}{2} \right) \int_{B_\varepsilon} K u^{p+1} dv \]
\[= \int_{\partial B_\varepsilon} B(\sigma, x, u, \nabla u) ds - \int_{\partial B_\varepsilon} \sigma \left( \frac{ku^2}{2} - \frac{K u^{p+1}}{p+1} \right) ds, \tag{56}\]
where all the notations are the same as in (13).

Now we present properties of isolated and isolated simple blowup points which are parallel to that in Sec. 3. Similar to Lemmas 3.1 and 3.2, we have

**Lemma 7.1.** Let $u_i$ satisfy (55) and $y_i \rightarrow \bar{y} \in \Omega$ be an isolated blowup point. Then for any $0 < r < \bar{r}/3$, we have
\[
\max_{z \in B_2(y_i) \setminus B_\varepsilon(y_i)} u_i(z) \leq C_1 \min_{z \in B_2(y_i) \setminus B_\varepsilon(y_i)} u_i(z),
\]
where $B_s(y_i)$ denotes the geodesic ball of radius $s$ centered at $y_i$, and $C_1$ is some positive constant independent of $i$ and $r$.

**Lemma 7.2.** Let $u_i$ satisfy (55) and $y_i \rightarrow \bar{y} \in \Omega$ be an isolated blowup point. Then for any $R_i \rightarrow +\infty$, $\varepsilon_i \rightarrow 0^+$, we have, after passing to a subsequence $\{u_{i_j}\}$ (still denoted as $\{u_i\}$, $\{y_i\}$, etc.), that
\[
\left\| u_i(y_i)^{-1} u_i \left( \exp_{y_i} \left( u_i(y_i)^{-\frac{p_i-1}{2}} x \right) \right) - \left( 1 + \frac{1}{3} K(y_i) |x|^2 \right)^{-\frac{1}{2}} \right\|_{L^2(B_{2R_i})} \leq \varepsilon_i,
\]
\[+ \left\| u_i(y_i)^{-1} u_i \left( \exp_{y_i} \left( u_i(y_i)^{-\frac{p_i-1}{2}} x \right) \right) - \left( 1 + \frac{1}{3} K(y_i) |x|^2 \right)^{-\frac{1}{2}} \right\|_{L^1(B_{2R_i})} \leq \varepsilon_i, \tag{57}\]
and
\[ \frac{R_i}{\log u_i(y_i)} \to 0 \quad \text{as } i \to \infty, \]
(58)
where \( x = (x^1, x^2, x^3) \) denotes some geodesic normal coordinates given by \( \exp_{y_i}(x) \).

The proof of the above two lemmas is very similar to that of their counterparts, we omit the details.

**Proposition 7.1.** Let \( \{u_i\} \) satisfy (55) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point, with (10) and (11) for all \( i \). Then for some constant \( C \) depending only on \( \rho, \tilde{C} \), \( \|k_i\|_{C^1(\Omega)} \), \( \|K_i\|_{C^2(\Omega)} \), and the positive lower bound of \( \inf_i \inf_{y \in \Omega} K_i(y) \), we have
\[ u_i(y) \leq C u_i(y_i)^{-1} d(y, y_i)^{-1}, \quad \text{for all } d(y, y_i) \leq \rho_i/2, \]
where \( \rho \), \( \tilde{C} \) are given in Definitions 1.1-1.2.

Furthermore, after passing to a subsequence, for some positive constant \( a > 0 \) and \( k(x) = \lim_{i \to \infty} k_i(x) \),
\[ u_i(y_i) u_i \to a \tilde{G}(\cdot, \bar{y}) + b \quad \text{in } C^2_{\text{loc}}(B_{\tilde{\rho}}(\bar{y}) \setminus \{\bar{y}\}), \]
where \( \tilde{\rho} = \min\{\delta_0, \rho_i/2\} \), \( \tilde{G} \) is given by (22) and \( b \in C^2(B_{\tilde{\rho}}(\bar{y})) \) satisfies \( -\Delta_y b + k(y)b = 0 \) in \( B_{\tilde{\rho}}(\bar{y}) \).

Similarly, the above proposition will be established through a series of lemmas.

**Lemma 7.3.** Let \( u_i \) satisfy (55) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \to \infty \) and \( 0 < \epsilon_i < e^{-R_i} \) are sequences with which (57) and (58) hold. Then for any given \( 0 < \delta < 1/100 \), there exists \( \rho_1 \in (0, \rho) \) which is independent of \( i \) (but depending on \( \delta \)) such that
\[ u_i(y) \leq C_2 u_i(y_i)^{-\lambda_i} d(y, y_i)^{-1+\delta} \quad \text{for all } R_i u_i(y_i)^{\frac{\rho_i - 1}{2}} \leq d(y, y_i) \leq \rho_1, \]
\[ |\nabla_y u_i(y)| \leq C_2 u_i(y_i)^{-\lambda_i} d(y, y_i)^{-2+\delta} \quad \text{for all } R_i u_i(y_i)^{\frac{\rho_i - 1}{2}} \leq d(y, y_i) \leq \rho_1, \]
and
\[ |\nabla_y^2 u_i(y)| \leq C_2 u_i(y_i)^{-\lambda_i} d(y, y_i)^{-3+\delta} \quad \text{for all } R_i u_i(y_i)^{\frac{\rho_i - 1}{2}} \leq d(y, y_i) \leq \rho_1, \]
where \( \lambda_i = (1 - \delta) \frac{\rho_i - 1}{2} - 1 \) and \( C_2 \) is some positive constant independent of \( i \).

**Proof.** The proof is basically the same as that of Lemma 3.3. One difference is to replace the operator \( \ell_i \) there by \( \ell_i \varphi = \Delta_y \varphi + K_i(y) u_i(y)^{\rho_i - 1} \varphi - k_i(y) \varphi \). Other modifications are obvious, we omit the details. \( \square \)

Parallel to Lemma 3.4 we have
Lemma 7.4. Let $u_i$ satisfy (55) and $y_i \to \bar{y} \in \Omega$ be an isolated simple blowup point. Assume $R_i \to \infty$ and $0 < \epsilon_i < e^{-R_i}$ are sequences with which (57) and (58) hold. Fixing $0 < \delta < 1/100$. Then for all fixed $0 < \sigma < \rho_1$, we have for large $i$ that
\[
\int_{d(y_i, y) \leq \sigma} d(y, y_i)^2 u_i^{p_i+1} = O(u_i(y_i)^{-4+o(1)}),
\]
\[
\int_{d(y_i, y) \leq \sigma} u_i^2 \leq C_3 \sigma u_i(y_i)^{-2\lambda_i},
\]
\[
\int_{d(y_i, y) \leq \sigma} d(y, y_i)^2 |\nabla u_i|^2 \leq O(u_i(y_i)^{-4+o(1)}) + C_3 \sigma u_i(y_i)^{-2\lambda_i},
\]
\[
\int_{d(y_i, y) \leq \sigma} d(y, y_i)^2 u_i^{p_i+1} = O(u_i(y_i)^{-2\lambda_i}),
\]
where $C_3$ is some constant independent of $i$ and $\sigma$.

Proof. The proof is basically the same as that of Lemma 3.4 with obvious modification, we omit the details. 

In the following, let $(x^1, x^2, x^3)$ be some geodesic normal coordinate system centered at $y_i$. We use the notation explained at the beginning of Sec. 2.

Lemma 7.5. Let $u_i$ satisfy (55) and $y_i \to \bar{y} \in \Omega$ be an isolated simple blowup point. Assume $R_i \to \infty$ and $0 < \epsilon_i < e^{-R_i}$ are sequences with which (57) and (58) hold. Then for $0 < \sigma < \rho_1$, we have
\[
|A(g, u_i)| \leq C_3 \sigma u_i(y_i)^{-2\lambda_i},
\]
where $C_3$ is some constant independent of $i$ and $\sigma$.

Proof. The proof is the same as that of Lemma 3.5 with obvious modification, we omit the details.

Parallel to Lemma 3.6, we have

Lemma 7.6. Let $u_i$ satisfy (55) and $y_i \to \bar{y} \in \Omega$ be an isolated simple blowup point. Assume $R_i \to \infty$ and $0 < \epsilon_i < e^{-R_i}$ are sequences with which (57) and (58) hold. Then
\[
\tau_i = O(u_i(y_i)^{-2\lambda_i}).
\]
Consequently,
\[
u_i(y_i)\tau_i \to 1.
\]
Proof. The proof is a modification of that of Lemma 3.6. For instance, we use (57) instead of (18), and (56) instead of Lemma 2.1.

Lemma 7.7. Let $u_i$ satisfy (55) and $y_i \to \tilde{y} \in \Omega$ be an isolated simple blowup point. Assume $R_i \to \infty$ and $0 \leq \epsilon_i < e^{-R_i}$ are sequences with which (57) and (58) hold. Then for all $0 < \sigma < \epsilon_i/2$, we have

$$
\limsup_{i \to \infty} \max_{y \in \partial B_{\epsilon_i}(y_i)} u_i(y)/u_i(y_i) \leq C(\sigma).
$$

Proof. The proof is basically the same as that of Lemma 3.7 with obvious modification, we omit the details.

Proof of Proposition 7.1. The proof is basically the same as that of Proposition 3.1 with obvious modification, we omit the details.

As in Sec. 3 we have the following three corollaries and one proposition parallel to Corollaries 3.1–3.4.

Corollary 7.1. Let $u_i$ satisfy (55) and $y_i \to \tilde{y} \in \Omega$ be an isolated simple blowup point. Assume $R_i \to \infty$ and $0 \leq \epsilon_i < e^{-R_i}$ are sequences with which (57) and (58) hold. Then there exists $\rho_1 \in (0, \rho)$ which is independent of $i$ such that

$$
|\nabla u_i(y)| \leq C_2 u_i(y_i)^{-1} d(y, y_i)^{-2} \text{ for all } R_i u_i(y_i)^{-\frac{R_i-1}{2}} \leq d(y, y_i) \leq \rho_1,
$$

and

$$
|\nabla^2 u_i(y)| \leq C_2 u_i(y_i)^{-1} d(y, y_i)^{-3} \text{ for all } R_i u_i(y_i)^{-\frac{R_i-1}{2}} \leq d(y, y_i) \leq \rho_1,
$$

where $C_2$ is some positive constant independent of $i$.

Corollary 7.2. Let $u_i$ satisfy (55) and $y_i \to \tilde{y} \in \Omega$ be an isolated simple blowup point. Assume $R_i \to \infty$ and $0 \leq \epsilon_i < e^{-R_i}$ are sequences with which (57) and (58) hold. Then for all fixed $0 < \sigma < \rho_1$, we have for large $i$ that

$$
\int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 u_i^{n+1} = O(u_i(y_i)^{-4+\epsilon(1)}),
$$

$$
\int_{d(y, y_i) \leq \sigma} u_i^2 \leq C_3 \sigma u_i(y_i)^{-2},
$$

$$
\int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 |\nabla u_i|^2 \leq O(u_i(y_i)^{-4+\epsilon(1)}) + C_3 \sigma u_i(y_i)^{-2},
$$

$$
\int_{d(y, y_i) \leq \sigma} d(y, y_i) u_i^{n+1} = O(u_i(y_i)^{-2}),
$$

$$
\int_{d(y, y_i) \leq \sigma} d(y, y_i) |\nabla u_i|^2 = O(u_i(y_i)^{-2} \log u_i(y_i)),
$$

where $C_3$ is some positive constant independent of $i$. 

and
\[ \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2|\nabla^2 u_i|^2 = O(u_i(y_i)^{-2} \log u_i(y_i)), \]
where \( C_3 \) is some constant independent of \( i \) and \( \sigma \).

**Corollary 7.3.** Let \( u_i \) satisfy (55) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. Assume \( R_i \to \infty \) and \( 0 < \epsilon_i < e^{-R_i} \) are sequences with which (57) and (58) hold. Then
\[ |A(g, u_i)| \leq C_3 \sigma u_i(y_i)^{-2}, \]
where \( A(g, u_i) \) is defined as in Lemma 2.1 and \( C_3 \) is some constant independent of \( i \) and \( \sigma \).

**Corollary 7.4.** Let \( u_i \) satisfy (55) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. After passing to some subsequence of \( \{u_i\} \), the following estimates hold:
\[ \lim_{i \to \infty} u_i(y_i)^2 \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 u_i^{p_i+1} = 0, \]
\[ \lim_{\sigma \to 0^+} \lim_{i \to \infty} \sup_{y_i} u_i(y_i)^2 \int_{d(y, y_i) \leq \sigma} d(y, y_i)^2 |\nabla u_i|^2 = 0, \]
\[ \lim_{\sigma \to 0^+} \lim_{i \to \infty} \sup_{y_i} u_i(y_i)^2 \int_{d(y, y_i) \leq \sigma} u_i^2 = 0, \]
and
\[ \lim_{\sigma \to 0^+} \lim_{i \to \infty} \sup_{y_i} u_i(y_i)^2 |A(g, u_i)| = 0, \]
where \( A(g, u_i) \) is given in Lemma 2.1.

Similar to lemmas 2.5 and 2.6 in [42] we have

**Lemma 7.8.** Let \( u_i \) satisfy (55) and \( y_i \to \bar{y} \in \Omega \) be an isolated simple blowup point. Then
\[ \tau_i = 5 - p_i = O(u_i(y_i)^{-2}), \quad |\nabla K_i(y_i)| = O(u_i(y_i)^{-2} \log u_i(y_i)). \]

**Proof.** The estimate of \( \tau_i \) follows immediately from the Pohozaev identity of \( u_i \) given by (56) and the estimates we have obtained for \( u_i \) near isolated simple blowup points. Let \( x = (x^1, x^2, x^3) \) be some geodesic normal coordinates centered at \( y_i \), and \( \eta \) be some smooth cutoff function with \( \eta(x) = 1 \) for \( |x| \leq \sigma/2 \) and \( \eta(x) = 0 \) for \( |x| \geq \sigma \). Multiplying (55) by \( \eta \partial u_i / \partial x_j \) and integrating by parts on \( |x| \leq \sigma \), we have, by using our estimates for \( u_i \) near isolated simple blowup points, that
\[ \left| \int_{|x| \leq \sigma} \frac{\partial K_i}{\partial x_j}(x) u_i(x)^{p_i+1} dx \right| = O(u_i(0)^{-2} \log u_i(0)). \]
Since \( K_i \) is bounded in \( C^2 \) norm, it follows that
\[
|\nabla K_i(0)| \leq O(u_i(0)^{-2} \log u_i(0)) + C \int_{|x| \leq \sigma} |x| u_i(x)^{p_i+1} dx = O(u_i(0)^{-2} \log u_i(0)).
\]

Lemma 7.8 is established.

Now we can state the analogue of Proposition 3.2.

**Proposition 7.2.** Let \( u_i \) satisfy (55) and \( y_i \to \bar{y} \) be an isolated simple blowup point with, for some \( \bar{\rho} > 0, \)
\[
u_i(y_i) u_i \to h \quad \text{in} \ C^2_{\text{loc}}(B_{\bar{\rho}}(\bar{y}) \setminus \{\bar{y}\}).
\]

Assume, for some \( a > 0, \) in some geodesic normal coordinate system \( x = (x^1, x^2, x^3), \) that
\[
h(x) = \frac{a}{|x|} + A + o(1) \quad \text{as} \ |x| \to 0.
\]

Then \( A \leq 0. \)

The proof of Proposition 7.2 is very similar to that of Proposition 3.2. The main difference is, due to the difference in the Pohozaev identities, that we need to establish an extra estimate:
\[
\lim_{i \to \infty} u_i(0) \left| \int_{B_r} (x \cdot \nabla K_i(x)) u_i(x)^{p_i+1} dv \right| = 0. \tag{59}
\]

This can be obtained by writing
\[
\int_{B_r} (x \cdot \nabla K_i(x)) u_i(x)^{p_i+1} dv = \nabla K_i(0) \cdot \int_{B_r} x u_i(x)^{p_i+1} dv + O \left( \int_{B_r} |x|^2 u_i(x)^{p_i+1} dv \right).
\]

Estimate (59) can then be deduced from Lemma 7.8 and Corollary 7.4.

Next we assert, as in Sec. 4, that an isolated blowup point is in fact an isolated simple blowup point.

**Proposition 7.3.** Let \( u_i \) satisfy (55) and \( y_i \to \bar{y} \) be an isolated blowup point. Then \( \bar{y} \) must be an isolated simple blowup point.

With Proposition 7.2, the above proposition as well as Theorems 7.1 and 7.2 can be established with minor modification of the arguments in earlier sections. We omit the details. In view of the positive mass theorem, Theorem 0.1 follows from results in this section.
Proof of Theorem 0.1. The \textit{a priori} estimates follow from Theorem 7.2 and the positive mass theorem of Schoen and Yau. The degree counting of all solutions follows from the \textit{a priori} estimates and arguments in [51].

8. Existence of Minimizers

Let \((M, g)\) be a \(n\) dimensional \((n \geq 3)\) smooth compact Riemannian manifold, \(k \in C^1(M)\) with positive \(\lambda_1\), and \(K \in C^2(M)\) be a function which is positive somewhere.

Consider, for \(1 < p \leq (n + 2)/(n - 2)\), the following minimization problem:

\[ S_p = \inf \left\{ \int_M (|\nabla u|^2 + ku^2) \mid u \in H^1(M), \quad \int_M K|u|^{p+1} = 1 \right\}. \]

It is easy to see that for \(1 < p < (n + 2)/(n - 2)\), \(S_p\) is achieved at some positive function, denoted as \(u_p\), and \(S_p \to S_{(n+2)/(n-2)} \in (0, \infty)\) as \(p \to (n + 2)/(n - 2)\) from the left.

The Euler-Lagrange equation of \(u_p\) is

\[ -\Delta_g u_p + ku_p = S_p K u_p^p, \quad u_p > 0, \quad \text{on } M. \tag{60} \]

Lemma 8.1. Let \(k_i \to k\) in \(C^0(M)\), \(K_i \to K\) in \(C^0(M)\), \(p_i \leq (n + 2)/(n - 2)\), \(p_i \to (n + 2)/(n - 2)\), and \(u_i\) satisfy

\[ -\Delta_g u_i + k_i u_i = K_i u_i^{p_i}, \quad u_i > 0, \quad \text{on } M, \tag{61} \]

and

\[ \{\|u_i\|_{H^1(M)}\} \text{ be uniformly bounded}. \]

Then for some positive numbers \(\epsilon_0\) and \(C\) independent of \(i\),

\[ u_i \leq C \quad \text{on } \{y \in M|K(y) \leq \epsilon_0\}. \]

\textbf{Proof}. Suppose the contrary, then after passing to a subsequence, there exists \(\tilde{y}_i \to \tilde{y} \in M\) such that \(u_i(\tilde{y}_i) \to \infty\) and \(K(\tilde{y}) \leq 0\). Pick \(\epsilon_i \to 0^+\) so that the set

\[ O_i := \{y \in M|K_i(y) < \epsilon_i\} \]

has the property that

\[ d(\tilde{y}_i, \partial O_i)^{2/(n-1)} u_i(\tilde{y}_i) \to \infty. \tag{62} \]

Consider the following function on \(O_i\):

\[ f_i(y) = d(y, \partial O_i)^{2/(n-1)} u_i(y), \]

and let \(y_i \in O_i\) denote a maximum point of \(f_i\) on \(O_i\), namely,

\[ f_i(y_i) = \max_{O_i} f_i. \]
We know from (62) that \( f_i(y_i) \to \infty \).

Let \( \sigma_i = \frac{1}{2}d(y_i, \partial \Omega_i) > 0 \), clearly we have

\[
\sigma_i^{\frac{2}{p_i - 1}} \max_{B_{\sigma_i}(y_i)} u_i \geq 2^{- \frac{2}{p_i - 1}} f_i(y_i) \to \infty,
\]

\[
(2\sigma_i)^{\frac{2}{p_i - 1}} u_i(y_i) \geq \max_{B_{\sigma_i}(y_i)} f_i \geq \sigma_i^{\frac{2}{p_i - 1}} \max_{B_{\sigma_i}(y_i)} u_i.
\]

Consequently,

\[
\sigma_i^{\frac{2}{p_i - 1}} u_i(y_i) \to \infty,
\]

and

\[
u_i(y_i) \geq 2^{- \frac{2}{p_i - 1}} \max_{B_{\sigma_i}(y_i)} u_i. \quad (63)
\]

Let \( x = (x^1, \ldots, x^n) \) denote some geodesic normal coordinates centered at \( y_i \) and we set

\[
w_i(x) = u_i(y_i)^{-1} u_i \left( \exp_{y_i} \left( u_i(y_i)^{- \frac{p_i - 1}{2}} x \right) \right), \quad |x| < u_i(y_i)^{\frac{p_i - 1}{2}} \sigma_i \to \infty.
\]

Clearly, \( w_i \) satisfies, in view of (63),

\[
\begin{aligned}
\{ & -\Delta_h w_i(x) = \tilde{K}_i(x) w_i(x) - u_i(y_i) u_i(y_i)^{-1} k_i(x) w_i(x), \quad |x| < \sigma_i u_i(y_i)^{\frac{p_i - 1}{2}}, \\
& 2^{- \frac{2}{p_i - 1}} w_i(x) \leq w_i(0) = 1, \quad |x| < \sigma_i u_i(y_i)^{\frac{p_i - 1}{2}}.
\end{aligned} \quad (64)
\]

where \( \tilde{K}_i(x) = K_i \left( \exp_{y_i} \left( u_i(y_i)^{- \frac{p_i - 1}{2}} x \right) \right), \) \( \tilde{k}_i(x) = k_i \left( \exp_{y_i} \left( u_i(y_i)^{- \frac{p_i - 1}{2}} x \right) \right) \) and \( h = h_{\alpha \beta}(x) dx^\alpha dx^\beta = g_{\alpha \beta}(u_i(y_i)^{- \frac{p_i - 1}{2}} x) dx^\alpha dx^\beta \) "the scaled metric." "the scaled metric."

On the other hand, we know from Sobolev embedding theorems that \( \int_M u_i^{p_i + 1} \leq C \| u_i \|_{L^{p_i + 1} (M)} \leq C. \) It follows that

\[
\int_{|x| \leq \sigma_i u_i(y_i)^{\frac{p_i - 1}{2}}} w_i^{p_i + 1} \leq C u_i(y_i)^{\frac{p_i - 1}{2} (p_i + 1)} \int_M u_i^{p_i + 1} \leq C. \quad (65)
\]

Sending \( i \) to \( \infty \), we derive from (64), (65), and standard elliptic theorems that, after passing to a subsequence,

\[
w_i \to w \quad \text{in } C^2_{\text{loc}}(\mathbb{R}^n),
\]

where \( w \in L^{2n/(n-2)}(\mathbb{R}^n) \) satisfies, for some constant \( \bar{K} \leq 0 \),

\[
-\Delta w = \bar{K} w^{\frac{n+2}{n-2}}, \quad w > 0, \quad \text{in } \mathbb{R}^n.
\]

It is well known that the above equation has no solution for \( \bar{K} < 0 \); while for \( \bar{K} = 0 \), all solutions are positive constants and therefore can not be in \( L^{2n/(n-2)}(\mathbb{R}^n) \). This leads to a contradiction. Lemma 8.1 is established.
Next we state our main result in this section concerning the compactness of finite energy solutions on three dimensional manifolds when $K$ is allowed to change signs.

**Theorem 8.1.** Let $(M, g)$ be a three dimensional smooth compact Riemannian manifold, $k$ in $C^1(M)$ with positive $\lambda_1$, $K \in C^2(M)$ be positive somewhere. Assume that

$$\min_{y \in M} A_{k, g}(y) > 0.$$  

Then for any $p_i \leq 5$ with $p_i \rightarrow 5$, $k_i \rightarrow k$ in $C^1(M)$, $K_i \rightarrow K$ in $C^2(M)$, and any solutions $u_i$ of (61) with bounded energy $\{\|u_i\|_{H^1(M)}\}$, we have

$$1/C \leq u_i \leq C \quad \text{and} \quad \|u_i\|_{C^0(M)} \leq C, \quad \forall i,$$

where $C$ is some constant independent of $i$.

**Corollary 8.1.** Let $(M, g)$ be a three dimensional smooth compact Riemannian manifold, $k \in C^1(M)$ with positive $\lambda_1$. Assume that

$$\min_{y \in M} A_{k, g}(y) > 0.$$  

Then for any $K \in C^2(M)$ which is positive somewhere, $S_0$ is attained at some positive function. In particular, $M_{K, K, 0} = \emptyset$.

**Remark 8.1.** When $(M, g)$ is a positive three dimensional smooth compact Riemannian manifold which is not conformally equivalent to the standard three dimensional sphere, and $k = \frac{1}{4} R_g$, it follows from the positive mass theorem that $\min_M A_{1/4, g} > 0$, and Corollary 8.1 was already proved by Escobar and Schoen in [29].

**Proof of Theorem 8.1.** It follows from Lemma 8.1 that for some $\epsilon_0 > 0$, $\{u_i\}$ is uniformly bounded in the region where $K \leq \epsilon_0$. Due to our results in Sec. 7, $\{u_i\}$ is either uniformly bounded on $M$, or, after passing to a subsequence, it has finitely many isolated simple blowup points in the region where $K \geq \epsilon_0$. If $\{u_i\}$ is uniformly bounded, then higher derivative estimates follow from standard elliptic theories. Otherwise, let $y_i^{(1)} \rightarrow \tilde{y}^{(1)}, \ldots, y_i^{(m)} \rightarrow \tilde{y}^{(m)}$ be all the isolated simple blowup points. We know from Proposition 7.1 that $\{u_i(y_i^{(1)})u_i\}$ is uniformly bounded on any compact subset of $\{y \in M| K(y) \geq \epsilon_0/2\} \setminus \{\tilde{y}^{(1)}, \ldots, \tilde{y}^{(m)}\}$. Furthermore, using the fact $\{u_i\}$ is uniformly bounded on $\{y \in M| K(y) \leq \epsilon_0\}$ and the Harnack inequality, we know that $\{u_i(y_i^{(1)})u_i\}$ is uniformly bounded on any compact subset of $M \setminus \{\tilde{y}^{(1)}, \ldots, \tilde{y}^{(m)}\}$.

Now, as usual, we can write down the Pohozaev type identity of $u_i$ in $B_{\delta}(y_i^{(1)})$, and reach a contradiction. Here the global hypothesis $\min_{y \in M} A_{k, g}(y) > 0$ is being
used. This shows that the minimizing sequence \(\{u_i\}\) is in fact uniformly bounded. Theorem 8.1 is thus established.

9. Appendix: Positive Solutions in Punctured Balls

In this appendix, we provide some well known descriptions on singular behaviors of positive solutions to some linear elliptic equations in punctured balls. For \(n \geq 3\), let \(B_r\) denote the ball in \(\mathbb{R}^n\) of radius \(r\) and centered at the origin. Throughout this appendix, \(g = g_{ij}dx^idx^j\) denotes some smooth Riemannian metric in \(B_1\) and \(k(x) \in C^1(B_1)\).

Lemma 9.1. Suppose \(u \in C^2(B_1 \setminus \{0\})\) is a solution of

\[-\Delta_g u + k(x)u = 0, \quad \text{in } B_1 \setminus \{0\},\]

and \(u(x) = o(|x|^{2-n})\) as \(|x| \to 0\), then \(u \in C^{2,\alpha}(B_{1/2})\) for any \(0 < \alpha < 1\).

Proof. We first show that \(-\Delta_g u(x) + k(x)u(x) = 0\) in \(B_1\) in the distribution sense.
For any \(\epsilon > 0\), let \(\xi_\epsilon\) be some cutoff function:

\[
\xi_\epsilon(x) = \begin{cases}
1 & \text{for } |x| \leq \epsilon, \\
0 & \text{for } |x| \geq 2\epsilon, \\
|\nabla g \xi_\epsilon| < \frac{C}{\epsilon}, & |\nabla_\nu \xi_\epsilon| < \frac{C}{\epsilon^2}.
\end{cases}
\]

Then for any \(\phi \in C_c^\infty(B_1)\) we have

\[-\int_{B_1} \Delta_g (\phi (1 - \xi_\epsilon)) u + \int_{B_1} k u \phi (1 - \xi_\epsilon) = 0.\]

It follows that as \(\epsilon\) tends to zero,

\[-\int_{B_1} \Delta_g \phi u + \int_{B_1} k(x) u \phi \leq C \epsilon^{-2} \int_{B_{2\epsilon} \setminus B_{\epsilon}} |u| + C \int_{B_{\epsilon}} |u| = o(1),
\]

where we have used \(u(x) = o(|x|^{2-n})\) in the last step.

We know from \(u(x) = o(|x|^{2-n})\) that \(u \in L^s_{loc}(B_1)\) for \(s < \frac{n}{n-2}\). By \(W^{2,s}\) estimates we have \(u \in W^{2,s}_{loc}(B_1)\). The lemma then follows from standard bootstrap methods and elliptic estimates. \(\square\)

Lemma 9.2. There exists some constant \(\delta_0 > 0\) depending on \(n\), \(\|g_{ij}\|_{C^2(B_1)}\) and \(\|k(x)\|_{L^\infty}\), such that the maximum principle holds for \(-\Delta_g + k(x)\) on \(B_{\delta_0}\); and there exists a unique \(G(x) \in C^2(B_{\delta_0} \setminus \{0\})\) satisfying

\[
\begin{cases}
-\Delta_g G + k(x)G = 0 & \text{in } B_{\delta_0} \setminus \{0\} \\
G = 0, & \text{on } \partial B_{\delta_0},
\end{cases}
\]

\[
\lim_{|x| \to 0} |x|^{n-2} G(x) = 1.
\]
Furthermore, \( G(x) = |x|^{2-n} + R(x) \) where \( R(x) \) satisfies for all \( 0 < \epsilon < 1 \) that
\[
|x|^{n-4+\epsilon} |R(x)| + |x|^{n-3+\epsilon} |\nabla R(x)| \leq C(\epsilon), \quad \forall x \in B_0, \ n = 4,
\]
and
\[
|x|^{\epsilon} |R(x) - R(0)| + |x|^\epsilon |\nabla R(x)| \leq C(\epsilon), \quad \forall x \in B_0, \ n = 3,
\]
where \( C(\epsilon) \) is some constant depending only on \( \epsilon, n, \|g_{ij}\|_{C^2(B_1)}, \) and \( \|k\|_{L^\infty(B_1)}. \)

Proof. See, e.g., Appendix B in [45]. □

Lemma 9.3. Assume \( u(x) \in C^2(B_1 \setminus \{0\}) \) satisfies
\[
-\Delta_x u + k(x)u = 0, \quad u > 0, \quad \text{in } B_1 \setminus \{0\}, \tag{66}
\]
then
\[
a = \lim_{r \to 0} \max_{|x| = r} u(x)|x|^{n-2} < +\infty.
\]

Proof. It follows from the Harnack inequality that there exists \( C > 0 \) such that for \( 0 < r < 1, \)
\[
\max_{|x| = r} u(x) \leq C \min_{|x| = r} u(x).
\]

Suppose Lemma 9.3 were false, then \( a = +\infty. \) Therefore for all \( A > 0, \) there exists \( r_i \to 0^+ \) such that
\[
u(x) > A|x|^{2-n}, \quad \forall |x| = r_i.
\]

Set \( v_A(x) = \frac{A}{2} G(x), \) where \( G(x) \) was defined in Lemma 9.2. It follows from the maximum principle that for large \( i \)
\[
u(x) \geq v_A(x), \quad \forall r_i \leq |x| \leq \delta_0.
\]
Sending \( i \) to \( \infty, \) we have
\[
u(x) \geq v_A(x) = \frac{A}{2} G(x), \quad \forall 0 < |x| < \delta_0.
\]
Sending \( A \) to \( \infty, \) we reach a contradiction. □

Proposition 9.1. Suppose \( u \in C^2(B_1 \setminus \{0\}) \) satisfies
\[
-\Delta_x u(x) + k(x)u(x) = 0, \quad u(x) > 0, \quad \text{in } B_1 \setminus \{0\}.
\]

Then there exists some constant \( b \geq 0 \) such that
\[
u(x) = bG(x) + E(x), \quad \text{in } B_0 \setminus \{0\},
\]
where \( G(x), \delta_0 \) are defined in Lemma 9.2, and \( E(x) \in C^2(B_1) \) satisfies
\[
-\Delta_x E(x) + k(x)E(x) = 0, \quad \text{in } B_1.
\]
Proof. Set
\[ b = b(u) = \sup \{ \lambda \geq 0 | \lambda G \leq u \text{ in } B_{\delta_0} \setminus \{0\} \}. \] (67)
We know from Lemma 9.3 that \( 0 \leq b \leq a < \infty \).

Case 1: \( b = 0 \).

In this case we claim: for any \( \varepsilon > 0 \), there exists \( r_\varepsilon \in (0, \delta_0) \) such that
\[ \min_{|x|=r} \{ u(x) - \varepsilon G(x) \} \leq 0 \quad \forall 0 < r < r_\varepsilon. \]

If the above claim were false, then there would exist some \( \varepsilon_0 > 0 \) and \( r_j \to 0^+ \) such that
\[ \min_{|x|=r_j} \{ u(x) - \varepsilon_0 G(x) \} > 0. \]
Notice that \( u(x) - \varepsilon_0 G(x) \geq 0 \) for \( |x| = \delta_0 \). We derive from the maximum principle that \( u(x) - \varepsilon_0 G(x) \geq 0 \) on \( B_{\delta_0} \setminus B_{r_j} \). It follows that \( u(x) - \varepsilon_0 G(x) \geq 0 \) in \( B_{\delta_0} \setminus \{0\} \) which implies that \( b \geq \varepsilon_0 > 0 \), a contradiction.

Therefore, for any \( \varepsilon > 0 \), and \( 0 < r < r_\varepsilon \), there exists \( x_r \) with \( |x_r| = r \), such that \( u(x_r) \leq \varepsilon G(x_r) \). By the Harnack inequality, we have
\[ \max_{|x|=r} u(x) \leq C u(x_r) \leq C \varepsilon G(x_r). \]
It follows that
\[ u(x) \sim o(|x|^{2-n}) \quad \text{as } |x| \to 0. \]
Setting \( E(x) = u(x) \), our result in this case follows from Lemma 9.1.

Case 2: \( b > 0 \).

We consider \( v(x) = u(x) - bG(x) \). From the definition of \( b(u) \), we know that \( v(x) \geq 0 \). By the maximum principle, we know that either \( v(x) = 0 \) or \( v(x) > 0 \) in \( B_{\delta_0} \setminus \{0\} \). In the former case we are done by choosing \( E(x) = 0 \). In the latter case, \( v(x) \) satisfies (66). Set
\[ b(v) = \sup \{ \mu \geq 0 | \mu G \leq v \text{ in } B_{\delta_0} \setminus \{0\} \}. \]
It is easy to see that \( b(v) = 0 \). As in Case 1, we know \( v(x) = o(|x|^{2-n}) \). Letting \( E(x) = v(x) \), our result in this case follows from Lemma 9.1. \( \square \)

The following corollary, used in the proof of Lemma 3.7, is a simple consequence of Proposition 9.1 and Lemma 9.2.

Corollary 9.1. For \( n \geq 3 \), let \( u \) be a solution of (66) which is singular near the origin. Then
\[ \lim_{r \to 0} \int_{\partial B_r} \frac{\partial u}{\partial \nu} = b \cdot \lim_{r \to 0} \int_{\partial B_r} \frac{\partial G}{\partial \nu} = -(n-2)S^{n-1} |b|, \]
where \( b > 0 \) was defined in (67) and \( \nu \) denotes the unit outer normal.
References


