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ON THE BEST SOBOLEV INEQUALITY

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ABSTRACT. – We prove that the best constant in the Sobolev inequality ($W^{1,p} \subset L^{p^*}$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and $1 < p < n$) is achieved on compact Riemannian manifolds, or only complete under some hypotheses. We also establish stronger inequalities where the norms are to some exponent which seems optimal. © Elsevier, Paris

1. Introduction

It is well-known that sharp Sobolev inequalities are important in the study of partial differential equations, especially in the study of those arising from geometry and physics. There has been much work on such inequalities and their applications. See, for example, Trudinger [36], Moser [32], Aubin [5,6], Talenti [34], Lieb [30,31], Brezis and Nirenberg [10], Cherrier [14], Brezis and Lieb [9], Carleson and Chang [12], Escobar [17], Carlen and Loss [13], Beckner [8], Adimurthi and Yadava [1], Hebey and Vaugon [25,26], Hebey [23,24], Li and Zhu [28,29], Zhu [37,38], Druet [16], Aubin, Druet and Hebey [7], and the references therein.

For $n \geqslant 2$, it was shown by Aubin [5] and Talenti [34] that, for $1 \leqslant p < n$ and $p^* = np/(n-p)$,

$$\frac{1}{K(n,p)} = \inf\left\{ \frac{\|\nabla u\|_{L^p(\mathbb{R}^n)}}{\|u\|_{L^{p^*}(\mathbb{R}^n)}} \;\Big|\; u \in L^{p^*}(\mathbb{R}^n)\setminus\{0\},\ \nabla u \in L^p(\mathbb{R}^n) \right\}$$

is achieved and the extremal functions are found. In particular,

$$K(n,p) = \frac{p-1}{n-p}\left[\frac{n-p}{n(p-1)}\right]^{1/p}\left[\frac{\Gamma(n+1)}{\Gamma(n/p)\Gamma(n+1-n/p)\omega_{n-1}}\right]^{1/n},$$

for $1 < p < n$, and

$$K(n,1) = \frac{1}{n}\left[\frac{n}{\omega_{n-1}}\right]^{1/n},$$

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where Γ is the gamma function and ω_{n-1} denotes the volume of the standard $(n-1)$-sphere. All the extremal functions for $1 < p < n$ are given by

$$u(x) = c\left(\frac{1}{\mu + |x - \bar{x}|^{p/(p-1)}}\right)^{\frac{n-p}{p}},$$

where $c, \mu > 0$ and $\bar{x} \in \mathbb{R}^n$. It is easy to see that for some $\bar{c}, \bar{\mu} > 0$ the corresponding

$$v(x) = \bar{c}\left(\frac{1}{\bar{\mu} + |x|^{p/(p-1)}}\right)^{\frac{n-p}{p}} \tag{1}$$

is the unique minimizer which satisfies:

$$v(0) = 1, \quad \nabla v(0) = 0, \quad \text{and} \quad \int_{\mathbb{R}^n} v(x)^{p^*}\,\mathrm{d}x = 1.$$

On a compact Riemannian manifold (M_n, g), the Sobolev embedding theorem holds: the inclusion $W^{1,p} \subset L^{p^*}$ is continuous for $1 \leqslant p < n$. Thus there exists a constant C_0 such that any $\varphi \in W^{1,p}$ satisfies $\|\varphi\|_{L^{p^*}} \leqslant C_0\|\varphi\|_{W^{1,p}}$. Recall that $\|\varphi\|_{W^{1,p}} = \|\nabla\varphi\|_{L^p} + \|\varphi\|_{L^p}$. The first author proved in [5] that the Sobolev theorem holds for complete manifolds with positive injectivity radius if the curvature is bounded. It appears now that the result holds if the bound on the curvature is only the Ricci curvature is bounded from below. Moreover on a compact manifold, the inclusion $W^{1,p} \subset L^{p^*}$ is continuous but not compact and $W^{1,p} \subset L^p$ is compact by the Kondrakov theorem. When we are in this situation, there is a best constant associated to the Banach spaces. Namely there are constants C and A such that any $\varphi \in W^{1,p}$ satisfies

$$\|\varphi\|_{L^{p^*}} \leqslant C\|\varphi\|_{W^{1,p}} + A\|\varphi\|_{L^p}.$$

Define $K = \inf C$ such that some A exists. Then $K > 0$. A priori K depends on the three Banach spaces, but the first author proved in [5] that K only depends on n and p. So $K = K(n,p)$ is the norm of the inclusion $W^{1,p} \subset L^{p^*}$ on $\mathbb{R}^n$. Thus for any $\varepsilon > 0$ there exists a constant $A_p(\varepsilon)$ such that every $\varphi \in W^{1,p}(M_n)$ satisfies

$$\|\varphi\|_{L^{p^*}} \leqslant \left[K(n,p) + \varepsilon\right]\|\nabla\varphi\|_{L^p} + A_p(\varepsilon)\|\varphi\|_{L^p}, \tag{2}$$

and $K(n,p)$ is the smallest constant having this property.

A natural question arises: Is the best constant achieved? i.e., does there exist $A_p(0)$? We can expect a positive answer. The first author made a conjecture in [5] concerning the following inequalities:

CONJECTURE. – *There exist constants $A(p)$ such that any $\varphi \in W^{1,p}(M_n)$ satisfies*

$$\|\varphi\|^p_{L^{p^*}} \leqslant K(n,p)^p\|\nabla\varphi\|^p_{L^p} + A(p)\|\varphi\|^p_{L^p} \quad \textit{if } 1 \leqslant p \leqslant 2, \tag{3}$$

and

$$\|\varphi\|^{\frac{p}{p-1}}_{L^{p^*}} \leqslant K(n,p)^{\frac{p}{p-1}}\|\nabla\varphi\|^{\frac{p}{p-1}}_{L^p} + A(p)\|\varphi\|^{\frac{p}{p-1}}_{L^p} \quad \textit{if } 2 < p < n. \tag{4}$$

A stronger form of (4) *is*

$$\|\varphi\|^2_{L^{p^*}} \leqslant K(n,p)^2\|\nabla\varphi\|^2_{L^p} + A(p)\|\varphi\|^2_{L^p} \quad \textit{if } 2 < p < n. \tag{5}$$

From now on we will always use K to denote $K(n,p)$. The above conjecture was made because he proved these inequalities when the manifold is the standard n-sphere $\mathbb{S}^n$. He also proved that the best constant is achieved for manifolds of dimension two, and for manifolds of constant sectional curvature. Related problems on domains of $\mathbb{R}^n$ were studied by Brezis and Nirenberg [10], Brezis and Lieb [9], and Adimurthi and Yadava [1]. Hebey and Vaugon, using techniques of blow up at a point of concentration and the Pohozaev identity, proved in [25] and [26] inequality (3) for $p=2$ under the following condition:

$$\text{(H)} \qquad \begin{cases} (M_n, g) \text{ has a positive injectivity radius } d > 0, \\ |R_{ijkl}| \text{ and } |\nabla_m R_{ijkl}| \text{are bounded by } k. \end{cases}$$

Results on compact manifolds with boundaries, also for $p=2$, were obtained by Li and Zhu in [28] and [29]. Further results were given by Zhu in [37] and [38]. Recently Druet has shown in [16] that inequality (3) is false for $4 < p^2 < n$ if the scalar curvature is positive somewhere. Then Aubin, Druet, and Hebey proved in [7] that inequality (3) holds for all $p \in (1,n)$ on compact manifolds of dimension 2, 3 or 4 with non-positive sectional curvature. In view of our results in Section 6 and the Appendix, this result holds also for complete manifolds of dimension 2, 3 or 4 with non-positive sectional curvature and satisfying (H).

In this paper we establish inequality (3) for $1 < p \leqslant 2$ and inequality (5) for $2 \leqslant p < n$ for Riemannian manifolds satisfying (H). For a complete Riemannian manifold, the larger the exponent of the norms is, the stronger is the inequality, so the conjecture is proved for $1 < p < n$.

THEOREM 1.1. – *Let (M_n, g) be a C^∞ complete Riemannian manifold satisfying* (H). *Then there exist constants $A(p)$, depending also on n, d and k, such that for all $\varphi \in W^{1,p}(M_n, g)$, inequality* (3) *holds for all $1 < p \leqslant 2$, and inequality* (5) *holds for all $2 < p < n$.*

Remark 1.1. – As mentioned earlier, Theorem 1.1 in the special case $p=2$ was established in [25] and [26].

Remark 1.2. – By simple modification of our proof, one can show that $A(p)$ can be chosen as a continuous function in $(1,n)$, i.e., $A(p)$ can be chosen so that it remains bounded on compact subsets of $(1,n)$.

In fact, we establish results stronger than Theorem 1.1. For $n \geqslant 4$, let

$$r^*(n,p) = \frac{np}{n+2-p}, \qquad 1 < p < n,$$

and, for $n = 2, 3$, let

$$r^*(n,p) = \begin{cases} \dfrac{np}{n+2-p}, & p \in \left(1, \dfrac{n+2}{3}\right] \cup (2,n), \\ \dfrac{n(p-1)}{n-p}, & p \in \left(\dfrac{n+2}{3}, \sqrt{n}\right), \\ p, & p \in [\sqrt{n}, 2]. \end{cases}$$

THEOREM 1.2. – *Let (M_n, g) be a C^∞ complete Riemannian manifold satisfying* (H). *For $n \geqslant 4$, let $p \in (1,n)$ and $r > r^*(n,p)$; For $n = 2, 3$, let $p \in (1, \sqrt{n}) \cup (2,n)$ and $r > r^*(n,p)$, or $p \in [\sqrt{n}, 2]$ and $r \geqslant r^*(n,p)$, there exist some constants $A(p,r)$, depending also on n, d and* *k, such that*

$$\|\varphi\|^p_{L^{p^*}} \leqslant K^p\|\nabla\varphi\|^p_{L^p} + A(p,r)\|\varphi\|^p_{L^r} \quad \forall\, \varphi \in W^{1,p}(M_n, g).$$

Remark 1.3. – For $n \geqslant 4$, $1 < p < 2$, we have $r^*(n,p) < p$, so Theorem 1.2 is stronger than Theorem 1.1 in this situation. On the other hand, if the scalar curvature of M_n is positive somewhere, then for $1 \leqslant p \leqslant (n+2)/3$, there does not exist such $A(p,r)$ for any $r < np/(n+2-p)$. This shows, to some extent, the sharpness of $r^*(n,p)$ when $n \geqslant 4$ and $1 < p < n$.

For $n \geqslant 3$ and $2 < p < n$, the exponent 2 in inequality (5) can be improved. Indeed we have

THEOREM 1.3. – *Let (M_n, g) be a complete C^∞ Riemannian manifold satisfying* (H). *Assume p and a satisfy one of the following: For $n = 3, 4$, $2 < p < n$, and $0 < a < p$; For $n > 4$, $2 < p \leqslant \sqrt{n}$ and $0 < a \leqslant 2$; $\sqrt{n} < p < (n+2)/3$ and $0 < a < 2p(n-p)/(-3p^2 + np + 2n)$; $(n+2)/3 \leqslant p < n$ and $0 < a < p$. Then there exist some constant $A(p,a)$, depending also on n, d and k, such that*

$$\|\varphi\|^a_{L^{p^*}} \leqslant K^a\|\nabla\varphi\|^a_{L^p} + A(p,a)\|\varphi\|^a_{L^p} \quad \forall\, \varphi \in W^{1,p}(M_n, g).$$

The proofs of Theorems 1.1–1.3 consist of two parts. The first part is to establish such results on (B, g) for $\varphi \in W_0^{1,p}(B)$ where

$$B = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; \sum_{j=1}^n x_j^2 < 1 \right\}$$

is convex with respect to g, when the curvature tensor and its first covariant derivatives are bounded by sufficiently small number $\delta^* > 0$. The second part is to establish the global results from local results.

The first part is the main part and we briefly describe the proof of this part for Theorem 1.1 in the case $1 < p \leqslant 2$. We want to show that there exists some constant $A(p)$, depending on n, p, and δ^*, such that

$$\|\varphi\|^p_{L^{p^*}(B,g)} \leqslant K(n,p)^p\|\nabla\varphi\|^p_{L^p(B,g)} + A(p)\|\varphi\|^p_{L^p(B,g)} \quad \text{for all } \varphi \in W_0^{1,p}(B).$$

We prove it by contradiction argument. Suppose the contrary, then for any $\alpha > 0$,

$$\lambda_\alpha := \inf_{\varphi \in W_0^{1,p}(B)} I_\alpha(\varphi) < K^{-p}, \quad \text{where } I_\alpha(\varphi) = \frac{\|\nabla\varphi\|^p_{L^p} + \alpha\|\varphi\|^p_{L^p}}{\|\varphi\|^p_{L^{p^*}}}.$$

Due to some results and arguments given in the Appendix, there exists some nonnegative minimizer $\varphi_\alpha \in W_0^{1,p}(B) \cap C^0(\overline{B})$, with $\|\varphi_\alpha\|_{L^{p^*}} = 1$. The Euler–Lagrange equation satisfied by φ_α is

$$-L_g\varphi_\alpha + \alpha\varphi_\alpha^{p-1} = \lambda_\alpha\varphi_\alpha^{p^*-1} \quad \text{in } B, \tag{6}$$

where $L_g\varphi_\alpha = \nabla_g(|\nabla_g\varphi_\alpha|^{p-2}\nabla_g\varphi_\alpha)$ is the p-Laplacian with the metric g.

Let $x_\alpha \in B$ denote a maximum point of φ_α, we show that, after passing to a subsequence, $\varphi_\alpha(x_\alpha) \to \infty$ and φ_α has precisely one point concentration. It is fairly standard to show, by using the previously mentioned classification results on extremal functions for the best Sobolev inequality in $\mathbb{R}^n$, that

$$\varphi_\alpha(x_\alpha)^{p/(n-p)} \operatorname{dist}_g(x_\alpha, \partial B) \to \infty \quad \text{as } \alpha \to \infty.$$

Moreover

$$\lim_{\alpha\to\infty} \left\| \varphi_\alpha(x) - \varphi_\alpha(x_\alpha) v\left(\varphi_\alpha(x_\alpha)^{p/(n-p)} x\right) \right\|_{L^{p^*}(B)} = 0,$$

and

$$\lim_{\alpha\to\infty} \left\| \nabla_g\left(\varphi_\alpha(x) - \varphi_\alpha(x_\alpha) v\left(\varphi_\alpha(x_\alpha)^{p/(n-p)} x\right)\right) \right\|_{L^p(B)} = 0,$$

where v is the function given in (1).

Using the minimality of φ_α, we establish the following Pohozaev type inequality: For some constant $C = C(n, p, \delta^*)$,

$$\alpha \int_B \varphi_\alpha^p \, dv_g \leqslant C \int_B \left(\operatorname{dist}_g(x, \bar{x})^{2-p} \varphi_\alpha^p + \operatorname{dist}_g(x, \bar{x})^2 \varphi_\alpha(x)^{p^*}\right) dv_g$$

holds for large α. Since $p \leqslant 2$, we deduce from the above, with a larger C, that

$$\alpha \int_B \varphi_\alpha^p \, dv_g \leqslant C \int_B \operatorname{dist}_g(x, \bar{x})^2 \varphi_\alpha(x)^{p^*} \, dv_g,$$

namely,

$$\alpha \int_{\Omega_\alpha} v_\alpha^p \, dv_{g_\alpha} \leqslant C \mu_\alpha^{2-p} \int_{\Omega_\alpha} |y|^2 v_\alpha^{p^*} \, dv_{g_\alpha}, \tag{7}$$

where $v_\alpha(y) = \varphi_\alpha(x_\alpha)^{-1} \varphi_\alpha(\psi_\alpha(y))$, $\psi_\alpha(y) = \exp_{x_\alpha}(\varphi_\alpha(x_\alpha)^{p/(p-n)} y)$ is an exponential map (the coordinates are normal at x_α), $y \in \Omega_\alpha := \psi_\alpha^{-1}(B)$, $g_\alpha = \varphi_\alpha(x_\alpha)^{2p/(n-p)} \psi_\alpha^* g$. The left hand side of (7) is bounded below by $C^{-1}\alpha$ since we show that v_α converges uniformly to v on any fixed compact subset of Ω_α. We will show that the right hand side of (7) tends to zero as α tends to infinity. For this, we need the following crucial pointwise estimate of v_α on Ω_α: For some constants $C = C(n, p, \delta^*)$ and $D = D(n, p)$,

$$v_\alpha(y) \leqslant C v(y)^{1 - D\delta^*}, \quad y \in \Omega_\alpha, \tag{8}$$

holds for sufficiently large α.

For $p = 2$, pointwise estimates of this type for radially symmetric solutions of (6) in balls of $\mathbb{R}^n$ were obtained by Atkinson and Peletier [2], and Brezis and Peletier [11]. The estimates were extended by Han [21] to general domains of $\mathbb{R}^n$. Hebey and Vaugon [25] further extended such estimates to general Riemannian manifolds, which play a crucial role in their proof of (3) for $p = 2$. Such estimates on Riemannian manifolds with boundaries, also for $p = 2$, were established by Li and Zhu in [28,29]. The proofs of these pointwise estimates for $p = 2$ rely on the conformal invariance of the conformal Laplacian of the metric g, which is not present when $p \neq 2$. In Section 3 we establish such pointwise estimates by a different method, which works for all $1 < p < n$.

From (8), the right hand side of (7) can be estimated by

$$\mu_\alpha^{2-p} \int\limits_{\Omega_\alpha} |y|^2 v_\alpha^{p^*} \, dv_{g_\alpha} \leqslant C\mu_\alpha^{2-p} \int\limits_{|y| \leqslant C\mu_\alpha^{-1}} |y|^2 v(y)^{(1-D\delta^*)p^*} \, dy \leqslant C,$$

which leads to contradiction. This establishes the first part of the proof of Theorem 1.1. For Theorems 1.2, 1.3, this part is more delicate. In particular we need, in addition to an upper bound like (8), an appropriate lower bound of v_α in certain parts of Ω_α.

The second part of the proof of Theorem 1.1 can be done by a partition of unity argument (see Section 6). It is reasonable to believe that the second part of the proofs of Theorems 1.2, 1.3 could also be done in such a way, though we do not see how to do it at this point. Instead, a general result which establishes global results from local results, which in particular provides the second part of the proofs of Theorems 1.1–1.3, is given in Section 7. The proof relies on heavier machinery (though well known) which include the Moser iteration technique and regularity results on p-harmonic type equations.

The results in this paper were announced by the second author in early September of 1998 at the International Conference on Partial Differential Equations and Related Topics in Mission Beach, Australia. We were informed in late October that Theorem 1.1 was independently obtained by O. Druet.

2. The local version of Theorem 1.1 in the case $1 < p \leqslant 2$

In this section we start to discuss the following local version of Theorem 1.1 in the case $1 < p \leqslant 2$. The proof will be completed in the next section. Throughout the paper we use the following notation:

$$B_\sigma = \{x \in \mathbb{R}^n \mid |x| < \sigma\} \quad \text{and} \quad B = B_1.$$

PROPOSITION 2.1. – *For $n \geqslant 2$ and $1 < p \leqslant 2$, there exist some constants δ^* and A, depending only on n and p, such that for any C^∞ Riemannian metric g in B_2 with the property that B_2 is convex, and the curvature tensor and its first covariant derivatives are bounded by δ^* in B_2, estimate* (3) *holds for all $\varphi \in W_0^{1,p}(B)$.*

We prove Proposition 2.1 by contradiction argument. Suppose the contrary, then for some $1 < p \leqslant 2$ and for any $\alpha > 0$, there exists $u_\alpha \in W_0^{1,p}(B)$ such that

$$\|u_\alpha\|_{L^{p^*}}^p > K^p\big(\|\nabla u_\alpha\|_{L^p}^p + \alpha\|u_\alpha\|_{L^p}^p\big). \tag{9}$$

This implies

$$\lambda_\alpha := \inf_{u \in W_0^{1,p}(B)} I_\alpha(u) < K^{-p}, \tag{10}$$

where

$$I_\alpha(u) = \frac{\|\nabla u\|_{L^p}^p + \alpha\|u\|_{L^p}^p}{\|u\|_{L^{p^*}}^p}.$$

It follows from Proposition 8.1 that there exists some non-negative function $\varphi_\alpha \in W_0^{1,p}(B) \cap C^0(\overline{B})$ satisfying $\|\varphi_\alpha\|_{L^{p^*}} = 1$ and $I_\alpha(\varphi_\alpha) = \lambda_\alpha$. The Euler–Lagrange equation of φ_α is

$$-L_g\varphi_\alpha + \alpha\varphi_\alpha^{p-1} = \lambda_\alpha\varphi_\alpha^{p^*-1}, \quad \varphi_\alpha \geqslant 0, \quad \text{in } B.$$

Here and throughout the paper, L_g denotes the p-Laplacian with the metric g, $L_g\varphi = \nabla_g(|\nabla_g\varphi|^{p-2}\nabla_g\varphi)$. It is well known that $\varphi_\alpha \in C(\overline{B})$. The function φ_α satisfies:

$$\|\nabla\varphi_\alpha\|_{L^p}^p \leqslant \lambda_\alpha < K^{-p}, \quad \alpha\|\varphi_\alpha\|_{L^p}^p \leqslant \lambda_\alpha < K^{-p}, \quad \text{and} \quad \|\varphi_\alpha\|_{L^{p^*}} = 1.$$

Consequently, $\|\varphi_\alpha\|_{L^p} \to 0$, $\max_{\overline{B}}\varphi_\alpha \to \infty$, $\|\varphi_\alpha\|_{L^s} \to 0$ for all $1 \leqslant s < p^*$, and, in view of the Sobolev embedding theorem, $\liminf_{\alpha\to\infty}\lambda_\alpha > 0$. After passing to a subsequence (still using α to denote the subsequence), we also have $\varphi_\alpha \to 0$ almost everywhere.

One of the ingredients in the study of the best constants in Sobolev inequalities on manifolds in [23,25,26] and [28,29] is the use of some Pohozaev type identity. The usual way to derive Pohozaev type identities involves differentiation twice of the solution. In our case, φ_α is not known to be twice differentiable. To avoid addressing this technical difficulty, we obtain instead, as in [22], a Pohozaev type inequality for φ_α by using its minimality. More precisely, we have

LEMMA 2.1. – *There exists some constant C, depending only on n, p and δ^*, such that, for all $\bar{x} \in B$,*

$$\alpha \int_B \varphi_\alpha^p \, dv_g \leqslant C \int_B \left(\mathrm{dist}_g(x,\bar{x})^{2-p}\varphi_\alpha^p + \mathrm{dist}_g(x,\bar{x})^2\varphi_\alpha(x)^{p^*}\right) dv_g.$$

Remark 2.1. – Results and references on Pohozaev type identities for solutions of p-harmonic type equations can be found in [33] and [20].

Proof. – Let (ρ,ω) be some geodesic polar coordinates centered at $\bar{x}$. In this coordinate system, the metric g takes the form

$$g = d\rho^2 + \rho^2 h_{ij}(\rho,\omega)\, d\omega_i\omega_j,$$

where $\{\omega_i\}$ is a coordinate system on $\mathbb{S}^{n-1}$ and h_{ij} satisfies $h_{ij}(\rho,\omega) = \delta_{ij} + O(\rho^2)$. Let $R(\omega) > 0$ be determined by $(R(\omega),\omega) \in \partial B$. In the proof, we drop the subscript α from φ_α. For $t \geqslant 1$, we introduce, using the convexity of B with respect to g,

$$\varphi_t(\rho,\omega) = \begin{cases} \varphi(t\rho,\omega), & 0 \leqslant \rho \leqslant R(\omega)/t, \\ 0, & R(\omega)/t \leqslant \rho \leqslant R(\omega). \end{cases}$$

We will show that $I_\alpha(\varphi_t)$ is differentiable with respect to t and will calculate its derivative at $t = 1$. The desired Pohozaev type inequality will be derived from

$$\frac{d}{dt} I_\alpha(\varphi_t)|_{t=1} \geqslant 0,$$

guaranteed by the minimality of φ.

Making a change of variable, we have

$$\int_B |\nabla\varphi_t|^p \, dv_g = \int_0^{R(\omega)/t} \int_{\mathbb{S}^{n-1}} \left\{|\partial_\rho\varphi_t|^2 + \rho^{-2}h^{ij}\partial_{\omega_i}\varphi_t\partial_{\omega_j}\varphi_t\right\}^{p/2} \rho^{n-1}\sqrt{\det(h_{ij})}\, d\rho\, d\omega$$

$$= t^{p-n} \int_0^{R(\omega)} \int_{\mathbb{S}^{n-1}} \left\{\left|\partial_\sigma\varphi(\sigma,\omega)\right|^2 + \sigma^{-2}h^{ij}(t^{-1}\sigma,\omega)\partial_{\omega_i}\varphi(\sigma,\omega)\partial_{\omega_j}\varphi(\sigma,\omega)\right\}^{p/2}$$

$$\times \sigma^{n-1}\sqrt{\det(h_{ij}(t^{-1}\sigma,\omega))}\,\mathrm{d}\sigma\,\mathrm{d}\omega.$$

So,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_B |\nabla\varphi_t|^p\,\mathrm{d}v_g\bigg|_{t=1} &= (p-n)\int_0^{R(\omega)}\int_{\mathbb{S}^{n-1}} |\nabla\varphi|^{p-2}\big\{|\nabla\varphi|^2 \\ &\quad + \mathrm{O}(\sigma^2)\sigma^{-2}|\partial_\omega\varphi|^2\big\}\sigma^{n-1}\sqrt{\det(h_{ij})}\,\mathrm{d}\sigma\,\mathrm{d}\omega \\ &\quad + \int_0^{R(\omega)}\int_{\mathbb{S}^{n-1}} |\nabla\varphi|^p\mathrm{O}(\sigma^2)\sigma^{n-1}\,\mathrm{d}\sigma\,\mathrm{d}\omega \\ &= (p-n)\int_B |\nabla\varphi|^p\,\mathrm{d}v_g + \int_B \mathrm{O}(\rho^2)|\nabla\varphi|^p\,\mathrm{d}v_g,\end{aligned}$$

where $\rho = \mathrm{dist}_g(x,\bar{x})$. Recall that $\int_B \varphi^{p^*}\,\mathrm{d}v_g = 1$. Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \varphi_t^p\,\mathrm{d}v_g\bigg|_{t=1} = -n\int_B \varphi^p\,\mathrm{d}v_g + \int_B \mathrm{O}(\rho^2)\varphi^p\,\mathrm{d}v_g,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg(\int_B \varphi_t^{p^*}\,\mathrm{d}v_g\bigg)^{p/p^*}\bigg|_{t=1} = (p-n) + \int_B \mathrm{O}(\rho^2)\varphi^{p^*}\,\mathrm{d}v_g.$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\alpha(\varphi_t)\bigg|_{t=1} = -p\alpha\int_B \big[1+\mathrm{O}(\rho^2)\big]\varphi^p\,\mathrm{d}v_g + \int_B \mathrm{O}(\rho^2)\big(|\nabla\varphi|^p + \varphi^{p^*}\big)\,\mathrm{d}v_g.$$

Due to the minimality of φ, we know

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\alpha(\varphi_t)\bigg|_{t=1} \geqslant 0,$$

from which we deduce

$$\alpha\int_B \big[1+\mathrm{O}(\rho^2)\big]\varphi^p\,\mathrm{d}v_g \leqslant C\int_B \rho^2\big(|\nabla\varphi|^p + \varphi^{p^*}\big)\,\mathrm{d}v_g.$$

Multiplying the equation of φ by $\rho^2\varphi$ and integrating by parts yield

$$\begin{aligned}\lambda_\alpha\int_B \rho^2\varphi^{p^*} &= \int_B |\nabla\varphi|^{p-2}\nabla\varphi\nabla\big(\rho^2\varphi\big) + \alpha\int_B \rho^2\varphi^p \\ &= \int_B \rho^2|\nabla\varphi|^p + 2\int_B \rho\varphi|\nabla\varphi|^{p-2}\nabla\varphi\nabla\rho + \alpha\int_B \rho^2\varphi^p \\ &\geqslant \frac{1}{2}\int_B \rho^2|\nabla\varphi|^p + \alpha\int_B \rho^2\varphi^p - C\int_B \rho^{2-p}\varphi^p.\end{aligned}$$

Lemma 2.1 follows immediately. □

Next we provide some asymptotic estimate of φ_α in $W^{1,p}$ norm by a blow up analysis. Let $x_\alpha \in B$ be a maximum point of φ_α, $\varphi_\alpha(x_\alpha) = \max_{\overline{B}} \varphi_\alpha$, then $\mu_\alpha := \varphi_\alpha(x_\alpha)^{1-p^*/p} \to 0$. Set

$$v_\alpha(y) = \mu_\alpha^{\frac{n-p}{p}} \varphi_\alpha\big(\psi_\alpha(y)\big) = \frac{\varphi_\alpha(\psi_\alpha(y))}{\varphi_\alpha(x_\alpha)},$$

where $\psi_\alpha(y) = \exp_{x_\alpha}(\mu_\alpha y)$ is an exponential map, $y \in \Omega_\alpha := \psi_\alpha^{-1}(B)$, we suppose that the coordinates are normal at x_α. It is clear that $0 \leqslant v_\alpha \leqslant 1$, $v_\alpha(0) = 1$, $v_\alpha = 0$ on $\partial\Omega_\alpha$, and v_α satisfies

$$-L_{g_\alpha} v_\alpha + \alpha\mu_\alpha^p v_\alpha^{p-1} = \lambda_\alpha v_\alpha^{p^*-1} \quad \text{in } \Omega_\alpha, \tag{11}$$

where $g_\alpha = \mu_\alpha^{-2}\psi_\alpha^* g$. Multiplying (11) by v_α and integrating by parts lead to

$$\alpha\mu_\alpha^p\Big(\int_{\Omega_\alpha} v_\alpha^p \,\mathrm{d}v_{g_\alpha}\Big) \leqslant \lambda_\alpha \int_{\Omega_\alpha} v_\alpha^{p^*} \,\mathrm{d}v_{g_\alpha} \leqslant \lambda_\alpha \int_{\Omega_\alpha} v_\alpha^p \,\mathrm{d}v_{g_\alpha}$$

which implies

$$\bar\varepsilon_\alpha := \alpha\mu_\alpha^p \leqslant \lambda_\alpha < K^{-p}.$$

We also know that

$$\int_{\Omega_\alpha} v_\alpha^{p^*} \,\mathrm{d}v_{g_\alpha} = 1 \quad \text{and} \quad \int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^p \,\mathrm{d}v_{g_\alpha} + \bar\varepsilon_\alpha \int_{\Omega_\alpha} v_\alpha^p \,\mathrm{d}v_{g_\alpha} = \lambda_\alpha. \tag{12}$$

Since the coefficients of (11) are bounded, it is well known (see [18,35,15], and the references therein) that for some $\beta \in (0,1)$ independent of α, $\|v_\alpha\|_{C^\beta(\overline{\Omega}_\alpha)}$ and $\|v_\alpha\|_{C^{1,\beta}(\mathrm{dist}_{g_\alpha}(y,\partial\Omega_\alpha)>1)}$ are bounded by some constant independent of α. So, after passing to a subsequence,

$$-\sum_{i=1}^n \partial_i\big(|\partial v|^{p-2}\partial_i v\big) + \bar\varepsilon v^{p-1} = \lambda v^{p^*-1} \quad \text{in } O \subset \mathbb{R}^n,$$

where $v \in W^{1,p}(O)$,

$$\bar\varepsilon = \lim_{\alpha\to\infty} \bar\varepsilon_\alpha \in \big[0, K^{-p}\big], \quad \lambda = \lim_{\alpha\to\infty} \lambda_\alpha \in \big(0, K^{-p}\big],$$

$$v_\alpha \to v \quad \text{in } C^{1,\beta'} \text{ norm on any compact set}, \ 0 < \beta' < \beta, \tag{13}$$

and, after a rotation of y-coordinates,

$$O = \left\{ y \in \mathbb{R}^n \;\middle|\; y^n > -\lim_{\alpha\to\infty} \frac{\mathrm{dist}_g(x_\alpha, \partial B)}{\mu_\alpha} \right\},$$

with $v = 0$ on ∂O when $\lim_{\alpha\to\infty} \mathrm{dist}_g(x_\alpha, \partial B)/\mu_\alpha < \infty$.

It follows from Proposition 8.2 that $\lim_{\alpha\to\infty} \mathrm{dist}_g(x_\alpha, \partial B)/\mu_\alpha = \infty$ (so, $O = \mathbb{R}^n$), v is the function given by (1), $\bar\varepsilon = 0$, $\lambda = K^{-p}$, and

$$\lim_{\alpha\to\infty} \int_{\Omega_\alpha} \big(\big|\nabla_{g_\alpha}(v_\alpha - v)\big|^p + |v_\alpha - v|^{p^*}\big) \,\mathrm{d}v_{g_\alpha} = 0. \tag{14}$$

3. The crucial pointwise estimate for blow-up solutions and the proof of Proposition 2.1

Let $\Omega \subset B \subset \mathbb{R}^n$ be a C^∞ domain, g be a C^∞ Riemannian metric on B_2, such that B_2 is convex for g, and the curvature tensor and its first covariant derivatives are bounded by $\delta^* > 0$. Let $1 < p < n$, $1 \leqslant s < p^*$, $x_i \in \Omega$, $\varepsilon_i \geqslant 0$, $0 \leqslant \lambda_i \leqslant \Lambda$, $\mu_i \to 0^+$, $\psi_i(y) = \exp_{x_i}(\mu_i y)$ for $y \in \Omega_i := \psi_i^{-1}(\Omega)$, $g_i := \mu_i^{-2}\psi_i^* g$, $v_i \in W_0^{1,p}(\Omega_i)$ be a solution of

$$-L_{g_i} v_i + \varepsilon_i \|v_i\|_{L^s(\Omega_i, g_i)}^{p-s} v_i^{s-1} = \lambda_i v_i^{p^*-1} \quad \text{in } \Omega_i, \tag{15}$$

with

$$v_i(0) = 1, \qquad 0 \leqslant v_i(y) \leqslant 1 \quad \forall\, y \in \Omega_i. \tag{16}$$

We assume

$$\frac{\operatorname{dist}_g(x_i, \partial\Omega)}{\mu_i} \equiv \operatorname{dist}_{g_i}(0, \Omega_i) \to \infty, \tag{17}$$

and

$$\int_{\Omega_i} \left(\left| \nabla_{g_i}(v_i - v) \right|^p + |v_i - v|^{p^*} \right) \mathrm{d}v_{g_i} \to 0, \tag{18}$$

where v is the function given in (1). Then we have the following crucial pointwise estimate of v_i in Ω_i.

PROPOSITION 3.1. – *For $n \geqslant 2$, $1 < p < n$, $1 \leqslant s < p^*$, let v_i be a sequence of solutions of* (15) *satisfying* (16), (17) *and* (18). *Then there exist some constant $C = C(n, p, s, \Lambda, \delta^*)$ and $D = D(n, p, s, \Lambda)$ such that, for large i,*

$$v_i(y) \leqslant C v(y)^{1 - D\delta^*}, \quad y \in \Omega_i.$$

First we have

LEMMA 3.1. – *Let $h = h_{ij}(y)\,\mathrm{d}y^i\,\mathrm{d}y^j$ be a C^∞ Riemannian metric on $\overline{B}$ such that $|R_{ijkl}|$ and $|\nabla_m R_{ijkl}|$ are bounded by 1. Assume $1 < p < n$ and f is some measurable function with $\|f^+\|_{L^{(n/p)+\delta_0}(B)} \leqslant C_0$ for some $\delta_0 > 0$. Then there exists some constant C, depending only on n, p, C_0 and δ_0, such that for any $u \in W^{1,p}(B) \cap L^\infty(B)$ satisfying*

$$-L_h u \leqslant f|u|^{p-2}u \quad \text{in } B,$$

we have

$$\|u^+\|_{L^\infty(B_{1/2})} \leqslant C\|u^+\|_{L^1(B)}.$$

Proof. – The proof is standard and we only give a sketch. An application of the Moser iteration technique (see, for example, [19] for $p = 2$ case) leads to $\|u^+\|_{L^\infty(B_{1/2})} \leqslant C\|u^+\|_{L^{p_0}(B)}$ for some $p_0 > p$. For $0 < t < s < 1$ and $\bar{x} \in B_t$, an application of the above estimate to $u(\bar{x} + (s-t)x)$ leads to

$$\begin{aligned} \|u^+\|_{L^\infty(B_t)} &\leqslant C(s-t)^{-\frac{n}{p_0}} \|u^+\|_{L^{p_0}(B_s)} \leqslant C(s-t)^{-\frac{n}{p_0}} \|u^+\|_{L^\infty(B_s)}^{\frac{p_0-1}{p_0}} \|u^+\|_{L^1(B_s)}^{\frac{1}{p_0}} \\ &\leqslant \frac{1}{2}\|u^+\|_{L^\infty(B_s)} + C(s-t)^{-n}\|u^+\|_{L^1(B_s)}. \end{aligned}$$

The desired estimate then follows from some elementary calculus lemma (see, for example, Lemma 1.7 in [22]).

Remark 3.1. – In Lemma 3.1, $L^p(B)$ can either be $L^p(B,h)$ or $L^p(B,\mathrm{d}y^2)$. In the rest of this paper we will not specify the metric when there is no need to do so, like here.

We also need the following lemma.

LEMMA 3.2. – *Let $h = h_{ij}(y)\,\mathrm{d}y^i\,\mathrm{d}y^j$ be a C^∞ Riemannian metric on $\overline{B}$ such that B is convex, $|R_{ijkl}|$ and $|\nabla_m R_{ijkl}|$ are bounded by* 1. *Assume $1 < p < n$. Then there exist some positive constants ε_0 and C, depending only on n and p, such that any $u \in W^{1,p}(B) \cap L^\infty(B)$ with*

$$-L_h u \leqslant |u|^{p^*-2}u, \quad \text{in } B, \quad \text{and} \quad \|u^+\|_{L^{p^*}(B)} \leqslant \varepsilon_0$$

satisfies

$$\|u^+\|_{L^\infty(B_{1/2})} \leqslant C\|u^+\|_{L^1(B)}.$$

Proof. – This lemma is deduced from Lemma 3.1. The reduction is standard, though we include it for reader's convenience. We will use ∇ to denote ∇_h and C to denote various constants depending only on n and p. For non-negative $\eta \in C_c^\infty(B)$, multiplying the equation of u by $\eta^p(u^+)^\beta$ for $1 < \beta \leqslant p^* - p$ and integrating by parts lead to

$$\int_B |\nabla u^+|^{p-2}\nabla u^+ \nabla\big(\eta^p(u^+)^\beta\big) \leqslant \int_B (u^+)^{p^*+\beta-1}\eta^p.$$

Let $w = (u^+)^{(p+\beta-1)/p}$, then

$$|\nabla w|^p = \left(\frac{p+\beta-1}{p}\right)^p (u^+)^{\beta-1}|\nabla u^+|^p.$$

A simple calculation yields

$$\begin{aligned}
&\int_B |\nabla u^+|^{p-2}\nabla u^+ \nabla\big(\eta^p(u^+)^\beta\big)\\
&\quad = \beta\left(\frac{p+\beta-1}{p}\right)^{-p}\int_B |\eta\nabla w|^p + \int_B |\nabla u^+|^{p-2}(u^+)^\beta \nabla u^+ \nabla\big(\eta^p\big)\\
&\quad \geqslant \beta\left(\frac{p+\beta-1}{p}\right)^{-p}\int_B |\eta\nabla w|^p - p\left(\frac{p+\beta-1}{p}\right)^{1-p}\int_B \big(\eta|\nabla w|\big)^{p-1}\big(w|\nabla\eta|\big)\\
&\quad \geqslant \frac{\beta}{2}\left(\frac{p+\beta-1}{p}\right)^{-p}\int_B |\eta\nabla w|^p - C\int_B |w\nabla\eta|^p.
\end{aligned}$$

So,

$$\int_B |\eta\nabla w|^p \leqslant C\int_B (u^+)^{p^*+\beta-1}\eta^p + C\int_B |w\nabla\eta|^p,$$

which yields, in view of

$$\int_B |\nabla(\eta w)|^p \leqslant 2^p \int_B |\eta \nabla w|^p + 2^p \int_B |w \nabla \eta|^p$$

and the Sobolev embedding theorem that

$$\left(\int_B (\eta w)^{p^*}\right)^{p/p^*} \leqslant C \int_B (u^+)^{p^*-p} (\eta w)^p + C \int_B |w \nabla \eta|^p.$$

Consequently, using Hölder inequality and the hypothesis $\|u^+\|_{L^{p^*}(B)} \leqslant \varepsilon_0$,

$$\left(\int_B (\eta w)^{p^*}\right)^{p/p^*} \leqslant C\varepsilon_0^{p^*-p} \left(\int_B (\eta w)^{p^*}\right)^{p/p^*} + C \int_B |w \nabla \eta|^p,$$

which implies, once ε_0 is chosen to be small enough from the beginning, that

$$\|u^+\|_{L^{p^*+(p^*/p)(\beta-1)}(B_{3/4})} \leqslant C.$$

Lemma 3.2 then follows from Lemma 3.1 with $f = |u|^{p^*-p}$ and $\delta_0 = p^*(\beta - 1)/[p(p^* - p)]$. $\square$

Using Lemma 3.2, we establish the following initial pointwise estimate of v_i.

LEMMA 3.3. – *There exists some constant C, depending only on n, p, s, Λ, and δ^*, such that, for large i,*

$$v_i(\bar{y}) \leqslant C v(\bar{y})^{\frac{p-1}{p}} \|v_i\|_{L^{p^*}(B_{|\bar{y}|/2}(\bar{y}))} \quad \forall\, \bar{y} \in \Omega_i,\ |\bar{y}| \geqslant 1.$$

Proof. – Multiplying the equation of v_i by v_i and integrating by parts lead to $\varepsilon_i \|v_i\|_{L^s}^{p-s} \leqslant C$. Then by the regularity results for p-harmonic type equations, $v_i \to v$ in $C^{1,\beta}_{\mathrm{loc}}$ and C^β for some $\beta \in (0,1)$. So we only need to find some $\bar{R} > 1$ and to show the estimate for $|\bar{y}| \geqslant 2\bar{R}$. Let $|\bar{y}| = 2R \geqslant 2\bar{R}$ and we will determine the value of $\bar{R}$ in the proof. Consider

$$\tilde{v}_i(y) = R^{\frac{n-p}{p}} v_i(\bar{y} + Ry), \quad |y| \leqslant 1.$$

Then

$$\|\tilde{v}_i\|_{L^{p^*}(B_1, \tilde{g}_i)} = \|v_i\|_{L^{p^*}(B_R(\bar{x}), g_i)} \leqslant \|v_i\|_{L^{p^*}(\Omega_i \setminus B_{\bar{R}}, g_i)}, \tag{19}$$

and

$$-L_{\tilde{g}_i} \tilde{v}_i \leqslant -L_{\tilde{g}_i} \tilde{v}_i + \varepsilon_i R^{p+\frac{n(p-s)}{s}} \|\tilde{v}_i\|_{L^s}^{p-s} \tilde{v}_i^{s-1} = \lambda_i \tilde{v}_i^{p^*-1},$$

where $\tilde{g}_i = (\tilde{g}_i)_{lm}(y)\, \mathrm{d}y^l\, \mathrm{d}y^m$ with $(\tilde{g}_i)_{lm}(y) = (g_i)_{lm}(\bar{y} + Ry)$, $y \in B$.

Because of (18), we can find sufficiently large $\bar{R}$ so that, for large i,

$$\left\|\lambda_i^{1/(p^*-p)} \tilde{v}_i\right\|_{L^{p^*}(B)} \leqslant \varepsilon_0,$$

where ε_0 is the number in Lemma 3.2. Applying Lemma 3.2 with $u = \lambda_i^{1/(p^*-p)}\tilde{v}_i$, we have

$$\tilde{v}_i(0) \leqslant \|\tilde{v}_i\|_{L^1(B)} \leqslant C\|\tilde{v}_i\|_{L^{p^*}(B)},$$

which yields the desired estimate. □

We also need the following comparison lemma.

LEMMA 3.4. – *Let $n \geqslant 1$, D be a bounded domain of $\mathbb{R}^n$, h be a C^∞ Riemannian metric in a neighborhood of $\overline{D}$, $1 < p < \infty$, $a(y) \in L^\infty(\Omega)$. Suppose that $u \in W^{1,p}(D) \cap C^1(\overline{D})$ and $w \in C^2(D) \cap C^0(\overline{D})$ satisfy*

$$-L_h u + a(x)|u|^{p-2}u \leqslant 0, \quad \text{in } D,$$

and

$$-L_h w + a(x)|w|^{p-2}w \geqslant 0, \quad \text{in } D.$$

In addition we assume

$$w(x) > 0 \quad \forall x \in \overline{D}, \qquad |\nabla_h w(x)| \neq 0 \quad \forall x \in D,$$

and

$$u \text{ is } C^{1,1} \text{ in any open set where } \nabla_h u \neq 0.$$

Then if $u \leqslant w$ on ∂D, we have $u \leqslant w$ in $\overline{D}$.

Remark 3.2. – Results and references on the maximum principle for p-harmonic type equations can be found in [35] and [20].

Proof. – For $0 < \varepsilon \leqslant 1$, set $u_\varepsilon = \varepsilon u$, and

$$\bar{\varepsilon} = \sup\{\varepsilon \mid 0 < \varepsilon \leqslant 1,\ u_\varepsilon(x) \leqslant w(x),\ \forall x \in \overline{D}\}.$$

Clearly $\bar{\varepsilon} > 0$. We need to prove $\bar{\varepsilon} = 1$. Suppose the contrary, $\bar{\varepsilon} < 1$; then

$$u_{\bar{\varepsilon}}(\bar{x}) = w(\bar{x}) > 0 \quad \text{for some } \bar{x} \in D,$$

and, by the definition of $\bar{\varepsilon}$,

$$u_{\bar{\varepsilon}}(x) \leqslant w(x), \quad \forall x \in \overline{D}.$$

Let $\bar{x}$ be any such point, it follows from the hypothesis that

$$\nabla u_{\bar{\varepsilon}}(\bar{x}) = \nabla w(\bar{x}) \neq 0.$$

Since both u and w are C^1, we can find a small neighborhood O of $\bar{x}$ so that

$$|\nabla u_{\bar{\varepsilon}}(x) - \nabla w(\bar{x})| \leqslant \frac{1}{2}|\nabla w(\bar{x})|, \qquad |\nabla w(x) - \nabla w(\bar{x})| \leqslant \frac{1}{2}|\nabla w(\bar{x})|,$$

and

$$|u_{\bar{\varepsilon}}(x) - w(\bar{x})| \leqslant \frac{1}{2}w(\bar{x}) \quad \forall x \in O,$$

which ensure that the equation is not degenerate in O and, due to our hypothesis, $u \in C^{1,1}(O)$. Let

$$\xi = w - u_{\bar{\varepsilon}}.$$

Then ξ satisfies

$$-\nabla_h\big(A(x)\nabla_h\xi\big) + b(x)\xi \geqslant 0 \quad \text{in } O,$$

where

$$A(x) = \big(A_{ij}(x)\big) = \left(\frac{1}{p}\int_0^1 \frac{\partial^2(|X|^p)}{\partial X_i X_j}\Big|_{X=t\nabla_h w(x)+(1-t)\nabla_h u_{\bar{\varepsilon}}(x)}\,\mathrm{d}t\right)$$

is positive definite and is Lipschitz in O, and

$$b(x) = a(x)\big[w(x)^{p-1} - u_{\bar{\varepsilon}}(x)^{p-1}\big]/\big(w(x) - u_{\bar{\varepsilon}}(x)\big)$$

is in $L^\infty(O)$. Furthermore

$$\xi(\bar{x}) = 0, \quad \text{and} \quad \xi(x) \geqslant 0 \quad \forall\, x \in O.$$

Since $A_{ij}(x)$ is Lipschitz, we have

$$-A_{ij}(x)\nabla_h^{ij}\xi - \nabla_h^i(A_{ij}(x))\nabla_h^j\xi + b(x)\xi \geqslant 0.$$

It follows from the strong maximum principle [19] that

$$\xi(x) \equiv 0 \quad \forall\, x \in O.$$

This implies that $u_{\bar{\varepsilon}} \equiv w$ in $\overline{D}$, violating $u \leqslant w$ on ∂D. Lemma 3.4 is established. □

Proof of Proposition 3.1. – It follows from Lemma 3.3 that for $|y| \geqslant 4\hat{R} \geqslant 2\bar{R}$, we have

$$v_i(y)^{p^*-p} \leqslant \frac{C\|v_i\|^{p^*-p}_{L^{p^*}(\Omega_i\setminus B_{\hat{R}})}}{|y|^p}.$$

Thus, in view of (15) and (18), for any given $C\delta^* < \delta < n-p$, there exists $\hat{R} \geqslant \bar{R}/2$ such that, for large i,

$$-L_{g_i}v_i - \frac{\varepsilon_2}{|y|^p}v_i^{p-1} \leqslant 0, \quad \Omega_i \setminus B_{\hat{R}}, \tag{20}$$

where

$$\varepsilon_2 = \frac{\delta}{2}\left[\frac{(n-p-\delta)}{p-1}\right]^{p-1} > 0. \tag{21}$$

Next we select a positive test function

$$w(y) = |y|^{-\frac{(n-p-\delta)}{p-1}}.$$

A calculation yields, for large i,

$$-L_{g_i} w(y) \geqslant \left(\delta\left[\frac{(n-p-\delta)}{p-1}\right]^{p-1} - D\delta^*\right)|y|^{-n+\delta}.$$

Here and in the following, C, C', C'' denote constants independent of α, while D, D', D'' denote constants independent of α and δ^*. So, for $\delta \geqslant D\delta^*$ (for a larger D) and for large i,

$$-L_{g_i} w(y) - \frac{\varepsilon_2}{|y|^p} w^{p-1} \geqslant 0, \quad \Omega_i \setminus B_{\hat{R}}. \tag{22}$$

Multiplying w by some constant $\hat{C} = \hat{C}(\hat{R})$, we have

$$v_i \leqslant \hat{C} w \quad \text{on } \partial B_{\hat{R}}.$$

Let $u = v_i/\hat{C}$, since v_i satisfies (15), u is in $C^{1,\beta}$ for some $\beta > 0$ due to the previously mentioned regularity results. Furthermore, in view of the classical results of [27], u is $C^{2,\beta}$ in any open set where ∇u is nonzero. It then follows from Lemma 3.4 that

$$v_i \leqslant \hat{C} w \quad \text{on } \Omega_i \setminus B_{\hat{R}}.$$

Proposition 3.1 is established. □

Proof of Proposition 2.1. – Rewriting the estimate in Lemma 2.1 in terms of v_α, we have, using also $p \leqslant 2$,

$$\alpha \int_{\Omega_\alpha} v_\alpha^p \, dv_{g_\alpha} \leqslant C\mu_\alpha^{2-p} \int_{\Omega_\alpha} |y|^2 v_\alpha^{p^*} \, dv_{g_\alpha}. \tag{23}$$

It follows from Proposition 3.1 that

$$v_\alpha(y) \leqslant C v(y)^{1-D\delta^*}.$$

When δ^* is small, we have $0 < D\delta^* < (n+2-2p)/(np)$, so

$$\mu_\alpha^{2-p} \int_{\Omega_\alpha} |y|^2 v_\alpha^{p^*} \, dv_{g_\alpha} \leqslant C\mu_\alpha^{2-p} \int_{|y| \leqslant C\mu_\alpha^{-1}} |y|^2 v^{(1-D\delta^*)p^*} \, dy \leqslant C,$$

and, in view of (13), that

$$\liminf_{\alpha\to\infty} \alpha \int_{\Omega_\alpha} v_\alpha^p \, dv_{g_\alpha} \geqslant \lim_{\alpha\to\infty} \alpha \int_{|y|\leqslant 1} v^p \, dy = \infty.$$

The above two estimates contradict to (23) for large α. Proposition 2.1 is established. □

4. The local version of Theorem 1.2

In this section we establish the following local version of Theorem 1.2.

PROPOSITION 4.1. – *For $n \geqslant 4$, let $p \in (1,n)$ and $r > r^*(n,p)$; for $n = 2,3$, let $p \in (1,\sqrt{n}) \cup (2,n)$ and $r > r^*(n,p)$, or $p \in [\sqrt{n}, 2]$ and $r \geqslant r^*(n,p)$, then there exist some constants δ^* and A, depending only on n, p and r, such that for any C^∞ Riemannian metric g in B_2 with the property that B_2 is convex, and the curvature tensor and its first covariant derivatives are bounded by δ^* in B_2, we have*

$$\|\varphi\|^p_{L^{p^*}(B,g)} \leqslant K^p \|\nabla\varphi\|^p_{L^p(B,g)} + A\|\varphi\|^p_{L^r(B,g)} \quad \forall\, \varphi \in W_0^{1,p}(B).$$

In view of the Hölder inequality, if the desired inequality holds for some r, then it also holds for any $r' > r$. So we can assume that r is very close to r^* (or equal to r^*, when $n = 2, 3$, and $p \in [\sqrt{n}, 2]$), we assume that $r < p^*$. We establish Proposition 4.1 by contradiction argument. Suppose it were false for some p and r. Let

$$I_\alpha(u) = \frac{\|\nabla u\|^p_{L^p} + \alpha\|u\|^p_{L^r}}{\|u\|^p_{L^{p^*}}}.$$

As in Section 2, there exists some non-negative function $\varphi_\alpha \in W_0^{1,p}(B) \cap C^0(\overline{B})$ with $\|\varphi_\alpha\|_{L^{p^*}} = 1$ and

$$I_\alpha(\varphi_\alpha) = \lambda_\alpha := \inf_{u \in W_0^{1,p}(B)} I_\alpha(u) < K^{-p}.$$

The Euler–Lagrange equation of φ_α takes the form

$$-L_g\varphi_\alpha + \alpha\|\varphi_\alpha\|^{p-r}_{L^r}\varphi_\alpha^{r-1} = \lambda_\alpha\varphi_\alpha^{p^*-1}, \quad \varphi_\alpha \geqslant 0, \quad \text{in } B.$$

We also need a Pohozaev type inequality for φ_α.

LEMMA 4.1. – *There exists some constant C, depending only on n and p such that, for all $\bar{x} \in B$,*

$$\left[\frac{1}{r} - \frac{1}{p^*}\right]\alpha\|\varphi_\alpha\|^p_{L^r} \leqslant C\int_B \left(\operatorname{dist}_g(x,\bar{x})^{2-p}\varphi_\alpha^p + \operatorname{dist}_g(x,\bar{x})^2\varphi_\alpha(x)^{p^*}\right)\mathrm{d}v_g. \tag{24}$$

Proof. – Let (ρ, ω) be some geodesic polar coordinates centered at $\bar{x}$ and we use the same notation as in the proof of Lemma 2.1. We only need to calculate the derivative of $\|\varphi_t\|^p_{L^r}$ at $t = 1$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi_t\|^p_{L^r}\bigg|_{t=1} = -\frac{np}{r}\|\varphi\|^p_{L^r} + \|\varphi\|^{p-r}_{L^r}\int_B \mathrm{O}(d^2)\varphi^r\,\mathrm{d}v_g.$$

The rest of the proof of Lemma 4.1 is essentially the same as that of Lemma 2.1. □

Let $x_\alpha \in B$ be a maximum point of φ_α, $\mu_\alpha = \varphi_\alpha(x_\alpha)^{1-p^*/p}$. It is easy to see, as in Section 2, $\mu_\alpha \to 0$, $\liminf_{\alpha\to\infty}\lambda_\alpha > 0$, $\|\varphi_\alpha\|_{L^s} \to 0$ for all $1 \leqslant s < p^*$, and, after passing to a subsequence, $\varphi_\alpha \to 0$ almost everywhere. Define v_α as in Section 2 and its equation now takes the form

$$-L_{g_\alpha}v_\alpha + \varepsilon_\alpha v_\alpha^{r-1} = \lambda_\alpha v_\alpha^{p^*-1} \quad \text{in } \Omega_\alpha,$$

where

$$\varepsilon_\alpha = \alpha\mu_\alpha^\beta\|\varphi_\alpha\|^{p-r}_{L^r}, \qquad \beta = \left[\frac{n}{r} - \frac{n-p}{p}\right]r.$$

Applying Proposition 8.2, we have $\lambda_\alpha \to K^{-p}$, $\varepsilon_\alpha \to 0$,

$$\frac{\operatorname{dist}_g(x_\alpha, \partial B)}{\mu_\alpha} \to \infty,$$

and

$$\int_{\Omega_\alpha} \left(\left|\nabla_{g_\alpha}(v_\alpha - v)\right|^p + |v_\alpha - v|^{p^*} \right) \mathrm{d}v_{g_\alpha} \to 0. \tag{25}$$

It follows from Proposition 3.1 that

$$v_\alpha(y) \leqslant C v(y)^{1-D\delta^*}. \tag{26}$$

Here and in the following, C, C', C'' denote constants independent of α, while D, D', D'' denote constants independent of α and δ^*.

Rewriting the estimate in Lemma 4.1 in terms of v_α, we have

$$\alpha \|v_\alpha\|^p_{L^r(\Omega_\alpha, g_\alpha)} \leqslant C \mu_\alpha^{2-p-\frac{n(p-r)}{r}} \int_{\Omega_\alpha} \left(|y|^{2-p} v_\alpha^p + |y|^2 v_\alpha^{p^*} \right).$$

When $\delta^* > 0$ in (26) is chosen to be small enough from the beginning, we have

$$|y|^2 v_\alpha^{p^*} = \left(|y|^p v_\alpha^{p^*-p}\right)|y|^{2-p} v_\alpha^p \leqslant C\left(|y|^p v(y)^{(1-D\delta^*)(p^*-p)}\right)|y|^{2-p} v_\alpha^p \leqslant C|y|^{2-p} v_\alpha^p.$$

Since $r > np/(n+2-p)$, we have $2 - p - n(p-r)/r > 0$. It follows that

$$\alpha \|v_\alpha\|^p_{L^r(\Omega_\alpha, g_\alpha)} \leqslant C \mu_\alpha^{2-p-\frac{n(p-r)}{r}} \int_{\Omega_\alpha} |y|^{2-p} v_\alpha^p. \tag{27}$$

Case 1. $1 < p \leqslant (n+2)/3$ and $r > np/(n+2-p)$. Since $1 < p \leqslant (n+2)/3$, we have $2 - p - p(n-p)/(p-1) \leqslant -n$, so for $\bar\delta \in (0, 2-p-n(p-r)/r)$,

$$\mu_\alpha^{2-p-\frac{n(p-r)}{r}} \int_{|y| \leqslant C\mu_\alpha^{-1}} \left(1+|y|\right)^{2-p-\frac{p(n-p)}{p-1}+\bar\delta} \mathrm{d}y \leqslant C \mu_\alpha^{2-p-\frac{n(p-r)}{r}} \mu_\alpha^{-\bar\delta} \leqslant C.$$

Using (27) and (26) (recall that δ^* is small), we obtain:

$$\begin{aligned} \alpha \|v_\alpha\|^p_{L^r(\Omega_\alpha, g_\alpha)} &\leqslant C \mu_\alpha^{2-p-\frac{n(p-r)}{r}} \int_{\Omega_\alpha} |y|^{2-p} v(y)^{(1-D\delta^*)p} \\ &\leqslant C \mu_\alpha^{2-p-\frac{n(p-r)}{r}} \int_{|y| \leqslant C\mu_\alpha^{-1}} |y|^{2-p}\left(1+|y|\right)^{-\frac{p(n-p)}{p-1}+D\delta^*} \mathrm{d}y \leqslant C. \end{aligned}$$

Sending α to ∞ leads to contradiction as in Section 2.

Case 2. $2 \leqslant p < n$ and $r > np/(n+2-p)$. For $p = 2$, the result follows from Theorem 1.1. So we only treat $2 < p < n$ here. Since $r > np/(n+2-p)$, we have $r > p$ and $(p-2)r/(r-$

$p) < n$. So, by using the Hölder inequality,

$$\int_B \operatorname{dist}_g(x,\bar{x})^{2-p}\varphi_\alpha^p \leqslant C\big\|\operatorname{dist}_g(x,\bar{x})^{2-p}\big\|_{L^{r/(r-p)}}\big\|\varphi_\alpha^p\big\|_{L^{r/p}} \leqslant C\|\varphi_\alpha\|_{L^r}^p,$$

i.e.,

$$\mu_\alpha^{2-p-\frac{n(p-r)}{r}}\int_{\Omega_\alpha}|y|^{2-p}v_\alpha^p \leqslant C\|v_\alpha\|_{L^r}^p,$$

which contradicts (27) for $\alpha > C$.

Case 3. $n = 2$, $1 < p < 2$; or $n = 3$, $1 < p < 3$. We only need to consider $(n+2)/3 < p < \sqrt{n}$ since the remaining cases follow from Case 1, Case 2, and Theorem 1.1 in the case $1 < p \leqslant 2$.

We derive from (26) that

$$\int_{\Omega_\alpha}|y|^{2-p}v_\alpha^p \leqslant \int_{\Omega_\alpha}|y|^{2-p}v(y)^{(1-D\delta^*)p} \leqslant C\mu_\alpha^{-[n+2-p-\frac{(n-p)p}{p-1}]+O(\delta^*)}.$$

Using the above to estimate the right hand side of (27), we have

$$\alpha \leqslant C\mu_\alpha^{p[\frac{n-p}{p-1}-\frac{n}{r}]+O(\delta^*)}$$

When δ^* is chosen to be small enough from the beginning, the exponent $p\big[\frac{n-p}{p-1}-\frac{n}{r}\big]+O(\delta^*)$ is positive. This leads to contradiction for large α. Proposition 4.1 is therefore established. □

5. Local version of a result related to Theorem 1.3

In this section we establish the local version of Theorem 1.3. In fact, for the local version, the restriction on a is less than that stated in Theorem 1.3. For this reason we define, for $n = 2, 3$,

$$a^*(n,p) = \begin{cases} 2, & 1 < p \leqslant (n+2)/3, \\ (n-p)/(p-1), & (n+2)/3 < p \leqslant \sqrt{n}, \\ p, & \sqrt{n} < p < n, \end{cases}$$

and, for $n \geqslant 4$,

$$a^*(n,p) = \begin{cases} 2, & 1 < p \leqslant \sqrt{n}, \\ 2p(n-p)/(-3p^2+np+2n), & \sqrt{n} < p \leqslant (n+2)/3, \\ p, & (n+2)/3 < p < n. \end{cases}$$

It is easy to see that $a^*(n,p)$ is a continuous function of p in $(1,n)$ and satisfies $p < a^*(n,p) < 2$ for $n = 2, 3$ and $(n+2)/3 < p < \sqrt{n}$, and $2 < a^*(n,p) < p$ for $n \geqslant 4$ and $\sqrt{n} < p < (n+2)/3$.

PROPOSITION 5.1. – *For $n \geqslant 4$, let $1 < p \leqslant \sqrt{n}$ and $0 < a \leqslant a^*(n,p)$, or $\sqrt{n} < p < n$ and $0 < a < a^*(n,p)$; for $n = 2, 3$, let $1 < p < \sqrt{n}$ and $0 < a \leqslant a^*(n,p)$, or $\sqrt{n} \leqslant p < n$ and $0 < a < a^*(n,p)$, then there exist some positive constants δ^* and A, depending only on n, p and*

a, such that for any C^∞ Riemannian metric g in B_2 with the property that B_2 is convex, and the curvature tensor and its first covariant derivatives are bounded by δ^* in B_2, we have

$$\|\varphi\|^a_{L^{p^*}(B,g)} \leqslant K^a \|\nabla\varphi\|^a_{L^p(B,g)} + A\|\varphi\|^a_{L^p(B,g)}, \quad \forall\, \varphi \in W_0^{1,p}(B).$$

The proof is again by contradiction. We suppose, for some p and a, that for any $\alpha > 0$, there exists $u_\alpha \in W_0^{1,p}(B,g)$ such that

$$\|u_\alpha\|^a_{L^{p^*}} > K^a\big(\|\nabla u_\alpha\|^a_{L^p} + \alpha\|u_\alpha\|^a_{L^p}\big).$$

This implies

$$\lambda_\alpha := \inf_{u\in W_0^{1,p}(B)\setminus\{0\}} I_\alpha(u) < K^{-a},$$

where

$$I_\alpha(u) = \frac{\|\nabla u\|^a_{L^p} + \alpha\|u\|^a_{L^p}}{\|u\|^a_{L^{p^*}}}.$$

It follows from Proposition 8.1 that there exists some non-negative minimizer $\varphi_\alpha \in W_0^{1,p}(B) \cap C^0(\overline{B})$, with $\|\varphi_\alpha\|_{L^{p^*}} = 1$ and $I_\alpha(\varphi_\alpha) = \lambda_\alpha$. It is clear that $C^{-1} \leqslant \|\nabla\varphi_\alpha\|_{L^p} \leqslant C$. Here and throughout this section, C denotes various positive constants which are independent of α and δ^*. The Euler–Lagrange equation is

$$-L_g\varphi_\alpha + \alpha\|\varphi_\alpha\|^{a-p}_{L^p}\|\nabla\varphi_\alpha\|^{p-a}_{L^p}\varphi_\alpha^{p-1} = \lambda_\alpha\|\nabla\varphi_\alpha\|^{p-a}_{L^p}\varphi_\alpha^{p^*-1} \quad \text{in } B.$$

Using the minimality of φ_α, we show as in Section 2, for any $\bar{x} \in B$, that

$$\alpha\|\varphi_\alpha\|^a_{L^p} \leqslant C\alpha\|\varphi_\alpha\|^{a-p}_{L^p}\int_B \rho^2\varphi_\alpha^p + C\int_B \rho^2|\nabla\varphi_\alpha|^p + C\int_B \rho^2\varphi_\alpha^{p^*},$$

where $\rho := \mathrm{dist}_g(x,\bar{x})$ and C is some constant independent of α and $\bar{x}$. Multiplying the equation of φ_α by $\rho^2\varphi_\alpha$ and integrating by parts yield, as in Section 2, that

$$\lambda_\alpha\|\nabla\varphi_\alpha\|^{p-a}_{L^p}\int_B \rho^2\varphi_\alpha^{p^*} \geqslant \frac{1}{2}\int_B \rho^2|\nabla\varphi_\alpha|^p + \alpha\|\varphi_\alpha\|^{a-p}_{L^p}\|\nabla\varphi_\alpha\|^{p-a}_{L^p}\int_B \rho^2\varphi_\alpha^p - C\int_B \rho^{2-p}\varphi_\alpha^p.$$

We deduce from the above two estimates

$$\alpha\|\varphi_\alpha\|^a_{L^p} \leqslant C\int_B \big(\rho^{2-p}\varphi_\alpha^p + \rho^2\varphi_\alpha^{p^*}\big). \tag{28}$$

Let $x_\alpha \in B$ be a maximum point of φ_α and then $\mu_\alpha := \varphi_\alpha(x_\alpha)^{1-p^*/p} \to 0$. Set

$$v_\alpha(y) = \mu_\alpha^{\frac{n-p}{p}}\varphi_\alpha\big(\psi_\alpha(y)\big),$$

where $\psi_\alpha(y) = \exp_{x_\alpha}(\mu_\alpha y)$ is an exponential map, $y \in \Omega_\alpha := \psi_\alpha^{-1}(B)$. It is clear that $0 \leqslant v_\alpha \leqslant 1$, $v_\alpha(0) = 1$, $v_\alpha = 0$ on $\partial\Omega_\alpha$, and v_α satisfies

$$-L_{g_\alpha}v_\alpha + \bar{\varepsilon}_\alpha\|\nabla_{g_\alpha}v_\alpha\|^{p-a}_{L^p}v_\alpha^{p-1} = \lambda_\alpha\|\nabla_{g_\alpha}v_\alpha\|^{p-a}_{L^p}v_\alpha^{p^*-1} \quad \text{in } \Omega_\alpha, \tag{29}$$

where $g_\alpha = \mu_\alpha^{-2}\psi_\alpha^* g$ and $\bar{\varepsilon}_\alpha = \alpha\mu_\alpha^a \|v_\alpha\|_{L^p}^{a-p}$.

It follows from Proposition 8.2 $\lambda_\alpha \to K^{-a}$, $\mu_\alpha^{-1}\,\mathrm{dist}_g(x_\alpha, \partial B) \to \infty$, and

$$\lim_{\alpha\to\infty} \int_{\Omega_\alpha} \left(\left|\nabla_{g_\alpha}(v_\alpha - v)\right|^p + |v_\alpha - v|^{p^*}\right) \mathrm{d}v_{g_\alpha} = 0,$$

where v is the extremal function given by (1). Consequently, $\lambda_\alpha \|\nabla_{g_\alpha} v_\alpha\|_{L^p}^{p-a} \to K^{-p}$.

Applying Proposition 3.1 to v_α, we have

$$v_\alpha(y) \leqslant C v(y)^{1-D\delta^*} \quad \forall\, y \in \Omega_\alpha. \tag{30}$$

Here and in the following, C, C', C'' denote constants independent of α, while D, D', D'' denote constants independent of α and δ^*. As in Section 4, we deduce from (30) that, when δ^* is chosen to be small from the beginning, $|y|^2 v_\alpha^{p^*} \leqslant C|y|^{2-p} v_\alpha^p$ and therefore (28), with $\bar{x} = x_\alpha$, is simplified as $\alpha\|\varphi_\alpha\|_{L^p}^a \leqslant C\int_B \mathrm{dist}(x, x_\alpha)^{2-p}\varphi_\alpha^p$, i.e.,

$$\alpha\|v_\alpha\|_{L^p}^a \leqslant C\mu_\alpha^{2-a} \int_{\Omega_\alpha} |y|^{2-p} v_\alpha^p. \tag{31}$$

Case 1. $n \geqslant 4$. We divide this case into four sub-cases.

Sub-case 1.1. $n \geqslant 4$, $1 < p < (n+2)/3$ and $0 < a \leqslant 2$. Since δ^* is small, it follows from (30) that

$$\limsup_{\alpha\to\infty} \mu_\alpha^{2-a} \int_{\Omega_\alpha} |y|^{2-p} v_\alpha^p \leqslant \limsup_{\alpha\to\infty} C\mu_\alpha^{2-a} \int_{\Omega_\alpha} |y|^{2-p} v^{(1-D\delta^*)p} < \infty.$$

On the other hand,

$$\limsup_{\alpha\to\infty} \alpha\|v_\alpha\|_{L^p}^a \geqslant \limsup_{\alpha\to\infty} \alpha\|v\|_{L^p(|y|<1)}^a = \infty.$$

Contradicting (31) for large α.

Remark 5.1. – The proof for Sub-case 1.1 works for all $n \geqslant 2$, $1 < p < (n+2)/3$, and $0 < a \leqslant 2$.

Sub-case 1.2. $n = 4$ and $p = a = 2$. Obviously (31) can not hold for large α.

Sub-case 1.3. $n \geqslant 4$, $(n+2)/3 < p < n$ and $0 < a < p$. We first estimate the right hand side of (31). Since $p > (n+2)/3$, it follows from (30) that

$$\mu_\alpha^{2-a} \int_{\Omega_\alpha} |y|^{2-p} v_\alpha^p \leqslant C\mu_\alpha^{2-a} \int_{\Omega_\alpha} |y|^{2-p} v^{(1-D\delta^*)p} \leqslant C\mu_\alpha^{-[a-\frac{n-p}{p-1}]+\mathrm{O}(\delta^*)}, \tag{32}$$

where $|\mathrm{O}(\delta^*)| \leqslant D\delta^*$. To estimate the left hand side of (31), we need an appropriate lower bound of v_α. As in [28,29], we use the maximum principle to establish such a lower bound of v_α in $|y| \leqslant R_\alpha$ for appropriate $R_\alpha \to \infty$, which gives an appropriate lower bound of $\|v_\alpha\|_{L^p}$. We derive from (29), in view of (30), that

$$-L_{g_\alpha} v_\alpha(y) \geqslant -C\bar{\varepsilon}_\alpha |y|^{-(n-p)+\mathrm{O}(\delta^*)}, \quad y \in \Omega_\alpha.$$

For $\hat{\delta} = D'\delta^*$, let $d = \frac{1}{2}\min_{|y|=1} v(y)$ and

$$w(y) = d|y|^{-\frac{n-p+\hat{\delta}}{p-1}}.$$

Then, when we make $D' \gg D$ (but independent of α and δ^*), we have, for α large,

$$\begin{aligned}-L_{g_\alpha} w(y) &\leqslant \left(-\hat{\delta} d^{p-1}\left[\frac{n-p+\hat{\delta}}{p-1}\right]^{p-1} + D\delta^*\right)|y|^{-(n+\hat{\delta})} \\ &\leqslant -\frac{\hat{\delta}}{2} d^{p-1}\left[\frac{n-p+\hat{\delta}}{p-1}\right]^{p-1}|y|^{-(n+\hat{\delta})}.\end{aligned}$$

Define, for $D'' \gg D'$ (but independent of α and δ^*),

$$R_\alpha = D''^{-1}(\bar{\varepsilon}_\alpha)^{-1/p} \to \infty.$$

Then, for α large,

$$L_{g_\alpha} v_\alpha - L_{g_\alpha}\big(w - w(R_\alpha)\big) \leqslant 0, \quad 1 \leqslant |y| \leqslant R_\alpha. \tag{33}$$

Since v_α converges strongly to v on compact sets, we have, for α large,

$$v_\alpha(y) \geqslant w(y) - w(R_\alpha), \qquad |y| = 1.$$

Multiplying (33) by $(w - w(R_\alpha) - v_\alpha)^+$ and integrating by parts on $1 \leqslant |y| \leqslant R_\alpha$ yield $v_\alpha \geqslant w - w(R_\alpha)$ on $1 \leqslant |y| \leqslant R_\alpha$, which implies

$$v_\alpha(y) \geqslant C''^{-1}\big(1+|y|\big)^{-\frac{n-p+\hat{\delta}}{p-1}} \quad \forall\, |y| \leqslant R_\alpha/2. \tag{34}$$

It follows, since $p > \sqrt{n}$, that

$$\|v_\alpha\|_{L^p(\Omega_\alpha)} \geqslant C''^{-1}\left(\int\limits_{|y|\leqslant R_\alpha/2} \big(1+|y|\big)^{-\frac{(n-p+\hat{\delta})p}{p-1}}\,\mathrm{d}y\right)^{1/p} \geqslant C''^{-1} R_\alpha^{[\frac{n}{p}-\frac{(n-p)}{p-1}]+\mathrm{O}(\hat{\delta})},$$

i.e.,

$$\|v_\alpha\|_{L^p(\Omega_\alpha)} \geqslant C''^{-1}\big(\bar{\varepsilon}_\alpha\big)^{-\gamma+\mathrm{O}(\hat{\delta})+\mathrm{O}(\delta^*)}, \quad \text{where } \gamma = \frac{1}{p}\left[\frac{n}{p} - \frac{(n-p)}{p-1}\right] \in \left(0, \frac{1}{p}\right).$$

Using the definition of $\bar{\varepsilon}_\alpha$, we have

$$\|v_\alpha\|_{L^p(\Omega_\alpha)} \geqslant C''^{-1}\big(\alpha\mu_\alpha^a\big)^{-\frac{\gamma}{1-\gamma(p-a)}+\mathrm{O}(\hat{\delta})+\mathrm{O}(\delta^*)}. \tag{35}$$

Using the above estimate and (32), we derive from (31) that

$$\alpha^{1-\frac{\gamma a}{1-\gamma(p-a)}}\mu_\alpha^{-\frac{\gamma a^2}{1-\gamma(p-a)}} \leqslant C''\mu_\alpha^{-[a-\frac{n-p}{p-1}]+\mathrm{O}(\hat{\delta})+\mathrm{O}(\delta^*)}. \tag{36}$$

The exponent of α is, in view of $\gamma p < 1$, positive. It is elementary to check

$$\frac{\gamma a^2}{1-\gamma(p-a)} > a - \frac{n-p}{p-1} \quad \forall\, 1 < p < n \text{ and } 0 < a < p.$$

Choosing from the beginning δ^* small enough so that

$$\frac{\gamma a^2}{1-\gamma(p-a)} > a - \frac{n-p}{p-1} + \mathrm{O}(\hat{\delta}) + \mathrm{O}(\delta^*),$$

and letting $\alpha \to \infty$ in (36) lead to a contradiction to (31).

Remark 5.2. – The proof for Sub-case 1.3 works for all $n \geqslant 2$, $\max\{\sqrt{n}, (n+2)/3\} < p < n$, and $0 < a < p$.

Sub-case 1.4. $n \geqslant 4$, $\sqrt{n} < p \leqslant (n+2)/3$ and $0 < a < 2p(n-p)/(-3p^2+np+2n)$. We first point out that

$$\frac{2p(n-p)}{-3p^2+np+2n} \begin{cases} = 2 & \text{if } p = \sqrt{n}, \\ = \dfrac{n+2}{3} & \text{if } p = \dfrac{n+2}{3}, \\ > 2 & \text{if } \sqrt{n} < p < \frac{n+2}{3}. \end{cases}$$

Since $p > \sqrt{n}$, we still have (35) for $\hat{\delta} = D'\delta^*$. On the other hand, since $p \leqslant (n+2)/3$, estimate (32) must be replaced by

$$\mu_\alpha^{2-a} \int\limits_{\Omega_\alpha} |y|^{2-p} v_\alpha^p \leqslant C\mu_\alpha^{2-a} \int\limits_{\Omega_\alpha} |y|^{2-p} v^{(1-D\delta^*)p} \leqslant C\mu_\alpha^{-[a-2]+\mathrm{O}(\delta^*)}.$$

So, instead of (36), we have

$$\alpha^{1-\frac{\gamma a}{1-\gamma(p-a)}} \mu_\alpha^{-\frac{\gamma a^2}{1-\gamma(p-a)}} \leqslant C\mu_\alpha^{-[a-2]+\mathrm{O}(\hat{\delta})+\mathrm{O}(\delta^*)}.$$

It is easy to see that

$$\frac{\gamma a^2}{1-\gamma(p-a)} > a-2,$$

and we reach a contradiction the same way as in Sub-case 1.3.

Case 2. $n = 2, 3$. We divide this case into three sub-cases.

Sub-case 2.1. $n = 2, 3$, $1 < p < (n+2)/3$ and $0 < a \leqslant 2$. This follows from Remark 5.1.

Sub-case 2.2. $n = 2, 3$, $(n+2)/3 \leqslant p \leqslant \sqrt{n}$ and $0 < a < (n-p)/(p-1)$. Since $p \geqslant (n+2)/3$, $p(n-p)/(p-1) + p - 2 \leqslant n$, so when δ^* is sufficiently small, it follows from (30) that

$$\mu_\alpha^{2-a} \int\limits_{\Omega_\alpha} |y|^{2-p} v_\alpha^p \leqslant C\mu_\alpha^{2-a} \int\limits_{\Omega_\alpha} |y|^{2-p} v^{(1-D\delta^*)p} \leqslant C\mu_\alpha^{-[a-\frac{n-p}{p-1}+\mathrm{O}(\delta^*)]} \to 0.$$

Contradicting (31) for large α.

Sub-case 2.3. $n = 2, 3$, $\sqrt{n} < p < n$ and $0 < a < p$. This follows from Remark 5.2.

6. The proof of Theorem 1.1

In this section we establish Theorem 1.1 by using Proposition 2.1, Proposition 5.1, and partition of unity arguments.

For $n \geqslant 2$, let (M, g) be a complete n-dimensional Riemannian manifold (without boundary) satisfying (H). Let $1 < p \leqslant 2$ and $1 < a \leqslant p$, we assume that there exists some $\bar{\varepsilon} > 0$ such that

$$(37) \quad \|\varphi\|^a_{L^{p^*}(B_{\bar{\varepsilon}}(\bar{x}))} \leqslant K^a \|\nabla\varphi\|^a_{L^p(B_{\bar{\varepsilon}}(\bar{x}))} + \bar{A}\|\varphi\|^a_{L^p(B_{\bar{\varepsilon}}(\bar{x}))} \quad \forall\, \bar{x} \in M,\ \varphi \in W_0^{1,p}(B_{\bar{\varepsilon}}(\bar{x})),$$

where $\bar{A}$ is independent of $\bar{x}$ and φ and $B_{\bar{\varepsilon}}(\bar{x})$ denotes the geodesic ball of radius $\bar{\varepsilon}$ centered at $\bar{x}$. Then we have the following theorem which, together with Proposition 2.1 and Proposition 5.1, imply Theorem 1.1.

THEOREM 6.1. – *For $n \geqslant 2$, let (M, g) be a C^∞ complete n-dimensional Riemannian manifold (without boundary) satisfying* (H). *For $1 < a = p \leqslant 2$ or $1 < a \leqslant 2 \leqslant p < n$, we assume* (37). *Then we have, for some A depending only on $p, a, \bar{\varepsilon}, \bar{A}$, and (M, g), that*

$$\|\varphi\|^a_{L^{p^*}(M,g)} \leqslant K^a \|\nabla\varphi\|^a_{L^p(M,g)} + A\|\varphi\|^a_{L^p(M,g)}, \quad \forall\, \varphi \in W^{1,p}(M, g).$$

Proof. – We consider geodesic normal coordinates at x, $|\partial g_{ij}(y)| \leqslant C\rho$ with $\rho = d(x, y)$ and C a constant which depends on the bound of the sectional curvature (see [5, p. 152]). Let us consider a covering of the manifold by balls of radius δ, δ smaller than the injectivity radius and small enough so that the balls are convex (there exists a constant C_0 such that if $C\delta^2 < C_0$ the ball is convex). We know that we can choose the covering uniformly locally finite (each point has a neighborhood whose intersections with the balls are empty except at most $\tilde{k}$ of them), see [5, p. 151]. Let $\{h_i\}$ be a partition of unity subordinated to this covering such that $\{h_i^{1/p}\}$ are bounded in C^2 uniformly in i. For instance we start with a C^∞ radial function $\gamma(\rho)$ which is equal to $\mathrm{e}^{-(\rho-\delta)^{-2}}$ for $\delta/2 < \rho < \delta$ and which is positive inside the ball. We choose $h_i = \gamma^p(\rho_i)/[\sum_j \gamma^p(\rho_j)]$ with $\rho_j = d(x_j, y)$, x_j being the center of the jth ball.

Case 1. $1 < a = p \leqslant 2$. We would like to prove for any $\varphi \in W^{1,p}$ positive

$$(38) \qquad \|\varphi\|^p_{p^*} \leqslant K^p \|\nabla\varphi\|^p_p + A\|\varphi\|^p_p.$$

When $\delta < \bar{\varepsilon}$, we know from (37) that

$$\big\|\varphi h_i^{1/p}\big\|^p_{p^*} \leqslant K^p \big\|\nabla\big(\varphi h_i^{1/p}\big)\big\|^p_p + \bar{A}\big\|\varphi h_i^{1/p}\big\|^p_p.$$

So we can write for such φ

$$\begin{aligned}\|\varphi\|^p_{p^*} = \big\|\varphi^p\big\|_{p^*/p} &= \Big\|\sum_i \varphi^p h_i\Big\|_{p^*/p} \leqslant \sum_i \big\|\varphi^p h_i\big\|_{p^*/p} \\ &\leqslant \sum_i \big\|\varphi h_i^{1/p}\big\|^p_{p^*} \leqslant K^p \sum_i \big\|\nabla\big(\varphi h_i^{1/p}\big)\big\|^p_p + \bar{A}\sum_i \big\|\varphi h_i^{1/p}\big\|^p_p.\end{aligned}$$

It follows that

$$(39) \qquad \|\varphi\|^p_{p^*} \leqslant K^p \sum_i \big\|\nabla\big(\varphi h_i^{1/p}\big)\big\|^p_p + C\|\varphi\|^p_p.$$

The main thing is then to estimate

$$\sum_i \big\|\nabla\big(\varphi h_i^{1/p}\big)\big\|^p_p.$$

Write

$$\|\nabla(\varphi h_i^{1/p})\|_p^p = \int [h_i^{2/p}|\nabla\varphi|^2 + \varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2]^{p/2}.$$

Let

$$V = \{x \in M \mid |\nabla\varphi(x)| \leqslant \varphi(x)\} \quad \text{and} \quad \Theta = \{x \in M \mid |\nabla\varphi(x)| \geqslant \varphi(x)\}.$$

The contribution of the integral on V is easy:

$$\sum_i \int_{\Omega_i \cap V} [h_i^{2/p}|\nabla\varphi|^2 + \varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2]^{p/2}$$

$$\leqslant 2^{p/2}\sum_i \int_{\Omega_i \cap V} \varphi^p[h_i^{2/p} + |\nabla(h_i^{1/p})|^2]^{p/2} \leqslant C\sum_i \int_{\Omega_i} \varphi^p \leqslant C\tilde{k}\int \varphi^p. \tag{40}$$

To estimate the contribution on Θ, we need the following elementary lemma.

LEMMA 6.1. – *Let f and g be two functions with compact support, $f \geqslant 0$, $f + g \geqslant 0$. Then, for $1 < p \leqslant 2$,*

$$\int (f+g)^{p/2} \leqslant \int f^{p/2} + \frac{p}{2}\int \frac{g}{f^{1-p/2}}.$$

Applying the above lemma with $f = h_i^{2/p}|\nabla\varphi|^2$ and $g = \varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2$, we have

$$\sum_i \int_{\Omega_i \cap \Theta} [h_i^{2/p}|\nabla\varphi|^2 + \varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2]^{p/2}$$

$$\leqslant \sum_i \int_{\Omega_i \cap \Theta} h_i|\nabla\varphi|^p + \frac{p}{2}\sum_i \int_{\Omega_i \cap \Theta} \frac{\varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2}{[h_i^{2/p}|\nabla\varphi|^2]^{1-p/2}}. \tag{41}$$

We estimate the right hand side term by term:

$$\sum_i \int_{\Omega_i \cap \Theta} \frac{\varphi^2|\nabla(h_i^{1/p})|^2}{[h_i^{2/p}|\nabla\varphi|^2]^{1-p/2}} \leqslant \left(\frac{1}{p}\right)^2 \sum_i \int_{\Omega_i \cap \Theta} \varphi^p \frac{|\nabla h_i|^2}{h_i} \leqslant C\int \varphi^p,$$

where we have used the fact $|\nabla h_i|^2 \leqslant C h_i$, with C not depending on i. Next,

$$\frac{p}{2}\sum_i \int_{\Omega_i \cap \Theta} \frac{\varphi\nabla^j\varphi\nabla_j(h_i^{2/p})}{[h_i^{2/p}|\nabla\varphi|^2]^{1-p/2}} = \sum_i \int_{\Omega_i \cap \Theta} \frac{\varphi\nabla^j\varphi\nabla_j h_i}{|\nabla\varphi|^{2-p}} = 0,$$

and

$$\sum_i \int_{\Omega_i \cap \Theta} h_i|\nabla\varphi|^p \leqslant \int |\nabla\varphi|^p.$$

Putting the above three estimates into (41), we have

$$\sum_i \int_{\Omega_i \cap \Theta} [h_i^{2/p}|\nabla\varphi|^2 + \varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2]^{p/2} \leqslant \int |\nabla\varphi|^p + C\int \varphi^p. \tag{42}$$

The desired estimate (38) follows easily. Theorem 6.1 in this case is established.

Case 2. $1 < a \leqslant 2 \leqslant p < n$. We would like to prove for any $\varphi \in W^{1,p}$ positive

$$\|\varphi\|_{p^*}^a \leqslant K^a \|\nabla\varphi\|_p^a + A\|\varphi\|_p^a. \tag{43}$$

We write

$$\|\varphi\|_{p^*}^p = \|\varphi^p\|_{p^*/p} = \Big\|\sum_i \varphi^p h_i\Big\|_{p^*/p} \leqslant \sum_i \|\varphi^p h_i\|_{p^*/p} = \sum_i \|\varphi h_i^{1/p}\|_{p^*}^{a(p/a)}.$$

As in Case 1, when $\delta < \bar\varepsilon$, we have

$$\|\varphi h_i^{1/p}\|_{p^*}^a \leqslant K^a \|\nabla(\varphi h_i^{1/p})\|_p^a + \bar A\|\varphi h_i^{1/p}\|_p^a.$$

So, using $p \geqslant a$,

$$\begin{aligned}\|\varphi\|_{p^*}^p &\leqslant \sum_i \big[K^a\|\nabla(\varphi h_i^{1/p})\|_p^a + \bar A\|\varphi h_i^{1/p}\|_p^a\big]^{p/a}\\ &\leqslant \sum_i K^p\|\nabla(\varphi h_i^{1/p})\|_p^p + \sum_i C\|\nabla(\varphi h_i^{1/p})\|_p^{p-a}\|\varphi h_i^{1/p}\|_p^a + D\|\varphi\|_p^p,\end{aligned}$$

for some C and D. In the following we divide into two sub-cases.

Sub-case 1.1. $1 < a \leqslant 2 \leqslant p < n$ and $p \geqslant 4$. We make use of the following elementary inequality, which holds for $t > -1$,

$$(1+t)^k - 1 - kt < \begin{cases} b|t|^k, & \text{when } 1 \leqslant k \leqslant 2,\\ bt^2 + c|t|^k, & \text{when } k > 2,\end{cases}$$

where b and c are some constants depending on k but independent of t. Thus, for $p \geqslant 4$,

$$\begin{aligned}A &:= \sum_i \int \big[h_i^{2/p}|\nabla\varphi|^2 + \varphi\nabla^j\varphi\nabla_j\big(h_i^{2/p}\big) + \varphi^2\big|\nabla\big(h_i^{1/p}\big)\big|^2\big]^{p/2}\\ &\leqslant \int |\nabla\varphi|^p + \frac{p}{2}\sum_i \int h_i^{1-2/p}|\nabla\varphi|^{p-2}\big(\varphi\nabla^j\varphi\nabla_j\big(h_i^{2/p}\big) + \varphi^2\big|\nabla\big(h_i^{1/p}\big)\big|^2\big)\\ &\quad + C\sum_i \int \big(\varphi\nabla^j\varphi\nabla_j\big(h_i^{2/p}\big) + \varphi^2\big|\nabla\big(h_i^{1/p}\big)\big|^2\big)^{p/2}\\ &\quad + b\sum_i \int h_i^{1-4/p}|\nabla\varphi|^{p-4}\big(\varphi\nabla^j\varphi\nabla_j\big(h_i^{2/p}\big) + \varphi^2\big|\nabla\big(h_i^{1/p}\big)\big|^2\big).\end{aligned}$$

But as

$$\sum_i h_i^{1-2/p}\nabla_j\big(h_i^{2/p}\big) = \frac{2}{p}\sum_i \nabla_j h_i = 0,$$

we obtain

$$A \leqslant \|\nabla\varphi\|_p^p + E\|\nabla\varphi\|_p^{p-2}\|\varphi\|_p^2 + F\|\varphi\|_p^p$$

for some constants E and F, where we have used $p-2 \geqslant p/2$, i.e., $p \geqslant 4$.

So

$$\|\varphi\|_{p^*}^p \leqslant K^p\|\nabla\varphi\|_p^p + G\|\nabla\varphi\|_p^{p-a}\|\varphi\|_p^a + H\|\varphi\|_p^p$$
$$\leqslant \left(K^a\|\nabla\varphi\|_p^a + k\|\varphi\|_p^a\right)^{p/a} \tag{44}$$

with G, H, and k some constants. We have obtained the desired inequality (43).

Sub-case 1.2. $1 < a \leqslant 2 \leqslant p < n$ and $2 < p < 4$. We have the inequality for $\varphi \in W^{1,p}$, $\varphi \geqslant 0$:

$$\|\varphi\|_{p^*}^p \leqslant K^p \sum_i \left\|\nabla\left(\varphi h_i^{1/p}\right)\right\|_p^p + C\|\nabla\varphi\|_p^{p-a}\|\varphi\|_p^a + D\|\varphi\|_p^p.$$

Let $V = \{x \in M \mid |\nabla\varphi(x)| < \varphi(x)\}$ and $\Theta = \{x \in M \mid |\nabla\varphi(x)| \geqslant \varphi(x)\}$. We have

$$\sum_i \int_{\Omega_i \cap V} \left[h_i^{2/p}|\nabla\varphi|^2 + \varphi\nabla^j\varphi\nabla_j\left(h_i^{2/p}\right) + \varphi^2\left|\nabla\left(h_i^{1/p}\right)\right|^2\right]^{p/2} \leqslant C\int \varphi^p.$$

For the integral on $\Omega_i \cap \Theta$ we will use the equality

$$\int_{\Omega_i\cap\Theta} (f+g)^{p/2} = \int_{\Omega_i\cap\Theta} f^{p/2} + \frac{p}{2}\int_{\Omega_i\cap\Theta} gf^{p/2-1} + \frac{p}{2}\left(\frac{p}{2}-1\right)\int_0^1 (1-t)\int_{\Omega_i\cap\Theta} \frac{g^2}{(f+tg)^{2-p/2}},$$

here $f \geqslant 0$, $f+g \geqslant 0$, with $f = h_i^{2/p}|\nabla\varphi|^2$ and $g = \varphi\nabla^j\varphi\nabla_j(h_i^{2/p}) + \varphi^2|\nabla(h_i^{1/p})|^2$.

$$\sum_i \int_{\Omega_i\cap\Theta} f^{p/2} = \sum_i \int_{\Omega_i\cap\Theta} h_i|\nabla\varphi|^p = \int_\Theta |\nabla\varphi|^p \leqslant \|\nabla\varphi\|_{L^p}^p,$$

$$\sum_i \int_{\Omega_i\cap\Theta} gf^{p/2-1} = \sum_i \int_{\Omega_i\cap\Theta} h_i^{1-2/p}|\nabla\varphi|^{p-2}\left[\frac{2}{p}h_i^{2/p-1}\varphi\nabla^j\varphi\nabla_j h_i + \varphi^2\left|\nabla\left(h_i^{1/p}\right)\right|^2\right]$$
$$= \sum_i \int_{\Omega_i\cap\Theta} h_i^{1-2/p}|\nabla\varphi|^{p-2}\varphi^2\left|\nabla\left(h_i^{1/p}\right)\right|^2 \leqslant C\|\nabla\varphi\|_{L^p}^{p-2}\|\varphi\|_{L^p}^2.$$

Let Ω_i^+ be the set where $g \geqslant 0$ and Ω_i^- the set where $g < 0$, then:

$$\int_{\Omega_i^+\cap\Theta} \frac{g^2}{(f+tg)^{2-p/2}} \leqslant \int_{\Omega_i^+\cap\Theta} \frac{\left[\frac{2}{p}\varphi h_i^{2/p-1}\nabla^j\varphi\nabla_j h_i + \frac{1}{p^2}\varphi^2 h_i^{2/p-2}|\nabla h_i|^2\right]^2}{h_i^{4/p-1}|\nabla\varphi|^{4-p}}$$
$$\leqslant 2\int_{\Omega_i^+\cap\Theta} \left(\frac{4}{p^2}\frac{\varphi^2(\nabla^j\varphi\nabla_j h_i)^2}{h_i|\nabla\varphi|^{4-p}} + \frac{1}{p^4}\frac{\varphi^4|\nabla h_i|^4}{h_i^3|\nabla\varphi|^{4-p}}\right)$$
$$\leqslant C\int_{\Omega_i^+\cap\Theta} \left(\varphi^2|\nabla\varphi|^{p-2} + \varphi^p\right).$$

Thus

$$\int\limits_{\Omega_i^+\cap\Theta} \frac{g^2}{(f+tg)^{2-p/2}} \leqslant C\left[\|\nabla\varphi\|_{L^p}^{p-2}\|\varphi\|_{L^p}^2 + \|\varphi\|_{L^p}^p\right].$$

On the other hand,

$$A_i := \int_0^1 (1-t) \int\limits_{\Omega_i^-\cap\Theta} \frac{g^2}{(f+tg)^{2-p/2}} \leqslant \int_0^1 (1-t) \int\limits_{\Omega_i^-\cap\Theta} \frac{g^2}{(1-t)^{2-p/2} f^{2-p/2}}$$

$$= \int_0^1 (1-t)^{p/2-1} \int\limits_{\Omega_i^-\cap\Theta} \frac{g^2}{f^{2-p/2}}$$

since $g \geqslant -f$, $f+tg \geqslant (1-t)f$. It follows that

$$\sum_i A_i \leqslant \sum_i \int\limits_{\Omega_i^-\cap\Theta} \frac{g^2}{f^{2-p/2}} \leqslant C\left[\|\nabla\varphi\|_{L^p}^{p-2}\|\varphi\|_{L^p}^2 + \|\varphi\|_{L^p}^p\right].$$

Thus

$$\begin{aligned}\|\varphi\|_{p^*}^p &\leqslant K^p\|\nabla\varphi\|_{L^p}^p + G\|\nabla\varphi\|_{L^p}^{p-a}\|\varphi\|_{L^p}^a + H\|\varphi\|_{L^p}^p \\ &\leqslant \left(K^a\|\nabla\varphi\|_{L^p}^a + k\|\varphi\|_{L^p}^a\right)^{p/a}\end{aligned}$$

with k some constant. Theorem 6.1 in Case 2 is established.

Proof of Theorem 1.1. – Obviously we can find $\bar\varepsilon \in (0,d)$ so that for any $\bar x \in M$ we can dilate the metric g on $B_{\bar\varepsilon}(\bar x)$ to $\bar g$ on B_2 such that, with respect to $\bar g$, B_2 is convex, and the curvature tensor and its first covariant derivatives are bounded by δ^* in B_2. The $\bar\varepsilon$ depends on δ^*. Since the first constant in the Sobolev inequality is invariant under dilation, Theorem 1.1 follows from Theorem 6.1 and the local results Proposition 2.1 and Proposition 5.1.

7. From local to global, and the proof of Theorems 1.2 and 1.3

Though it is more natural to derive global results from local results by partition of unity, we do not see at this point how to do that for the more general situations in Theorems 1.2 and 1.3. In this section we provide a different argument which allows us to establish global results as stated in Theorem 1.2 and Theorem 1.3 from local results established in previous sections.

For $n \geqslant 2$, let (M,g) be a complete n-dimensional Riemannian manifold (without boundary) satisfying (H). Let $1<p<n$, $1<r<p^*$, and $1<a\leqslant p$, and we assume that there exists some $\bar\varepsilon>0$ such that

$$\|\varphi\|_{L^{p^*}(B_{\bar\varepsilon}(\bar x))}^a \leqslant K^a\|\nabla\varphi\|_{L^p(B_{\bar\varepsilon}(\bar x))}^a + \bar A\|\varphi\|_{L^r(B_{\bar\varepsilon}(\bar x))}^a \quad \forall \bar x \in M,\ \varphi \in W_0^{1,p}(B_{\bar\varepsilon}(\bar x)), \tag{45}$$

where $\bar A$ is independent of $\bar x$ and φ and $B_{\bar\varepsilon}(\bar x)$ denotes the geodesic ball of radius $\bar\varepsilon < d$ centered at $\bar x$. Then we have the following global inequality.

THEOREM 7.1. – *For $n \geqslant 2$, let (M, g) be a C^∞ complete n-dimensional Riemannian manifold (without boundary) satisfying* (H). *For $1 < p < n$, $1 < r < p^*$, and $1 < a \leqslant p$, we assume* (45). *Then we have, for some A depending only on $p, r, a, \bar{\varepsilon}, \bar{A}, n, d$, and k, that*

$$\|\varphi\|^a_{L^{p^*}(M,g)} \leqslant K^a \|\nabla\varphi\|^a_{L^p(M,g)} + A\|\varphi\|^a_{L^r(M,g)} \quad \forall \varphi \in W^{1,p}(M,g).$$

Proof. – Suppose the contrary, for all $\alpha > 0$,

$$\inf_{\varphi \in W^{1,p}(M)} I_\alpha(\varphi) < K^{-a}, \quad \text{where } I_\alpha(\varphi) = \frac{\|\nabla\varphi\|^a_{L^p(M)} + \alpha\|\varphi\|^a_{L^r(M)}}{\|\varphi\|^a_{L^{p^*}(M)}}.$$

We can find, for a sequence of $\alpha \to \infty$, compact exhaustion M_α of M, with the second fundamental form of ∂M_α bounded by some constant independent of α (we need the hypothesis on ∂M_α for Proposition 8.2 to hold), such that

$$\lambda_\alpha := \inf_{\varphi \in W_0^{1,p}(M_\alpha)} I_\alpha(\varphi) < K^{-a}.$$

We note that when M is a compact manifold (without boundary), we take $M_\alpha = M$ and $W_0^{1,p}(M_\alpha)$ below simply means $W^{1,p}(M)$. It follows from Propositions 8.1 and 8.2 that there exists non-negative minimizer φ_α of I_α in $W_0^{1,p}(M_\alpha)$ such that $\|\varphi_\alpha\|_{L^{p^*}} = 1$ and, for some $x_\alpha \in M_\alpha$,

$$\lim_{\alpha\to\infty} \left(\|\varphi_\alpha\|_{W^{1,p}(M_\alpha \setminus B_\varepsilon(x_\alpha))} + \|\varphi_\alpha\|_{L^\infty(M_\alpha \setminus B_\varepsilon(x_\alpha))}\right) = 0, \quad \forall \varepsilon > 0. \tag{46}$$

The Euler–Lagrange equation of φ_α is

$$-L_g\varphi_\alpha + \alpha\|\nabla\varphi_\alpha\|^{p-a}_{L^p}\|\varphi_\alpha\|^{a-r}_{L^r}\varphi_\alpha^{r-1} = \lambda_\alpha\|\nabla\varphi_\alpha\|^{p-a}_{L^p}\varphi_\alpha^{p^*-1} \quad \text{on } M_\alpha. \tag{47}$$

For $0 < \varepsilon < \bar{\varepsilon}/9$, let $\eta \equiv \eta_\alpha$ be a smooth cutoff function satisfying $\eta = 1$ in $B_{2\varepsilon}(x_\alpha)$, $\eta = 0$ in $M \setminus B_{4\varepsilon}(x_\alpha)$, $0 \leqslant \eta \leqslant 1$ and $|\nabla\eta| \leqslant (4\varepsilon)^{-1}$ in M. By our assumption (45),

$$\|\eta\varphi_\alpha\|^a_{L^{p^*}} \leqslant K^a \left\|\nabla(\eta\varphi_\alpha)\right\|^a_{L^p} + \bar{A}\|\eta\varphi_\alpha\|^a_{L^r}. \tag{48}$$

It follows, using $a \leqslant p$, that:

$$\begin{aligned}\|\varphi_\alpha\|^a_{L^{p^*}(B_{2\varepsilon}(x_\alpha))} &\leqslant K^a\left\{\|\nabla\varphi_\alpha\|^p_{L^p(B_{2\varepsilon}(x_\alpha))} + C\|\varphi_\alpha\|^p_{W^{1,p}(B_{4\varepsilon}(x_\alpha)\setminus B_{2\varepsilon}(x_\alpha))}\right\}^{a/p} + C\|\varphi_\alpha\|^a_{L^r(B_{4\varepsilon}(x_\alpha))}\\ &\leqslant K^a\|\nabla\varphi_\alpha\|^a_{L^p} + C\|\varphi_\alpha\|^a_{W^{1,p}(B_{4\varepsilon}(x_\alpha)\setminus B_{2\varepsilon}(x_\alpha))} + C\|\varphi_\alpha\|^a_{L^r}.\end{aligned}$$

Since $\|\nabla\varphi_\alpha\|^a_{L^p} + \alpha\|\varphi_\alpha\|^a_{L^r} = \lambda_\alpha$, we have

$$\|\varphi_\alpha\|^a_{L^{p^*}(B_{2\varepsilon}(x_\alpha))} \leqslant \lambda_\alpha K^a - (\alpha K^a - C)\|\varphi_\alpha\|^a_{L^r} + C\|\varphi_\alpha\|^a_{W^{1,p}(B_{4\varepsilon}(x_\alpha)\setminus B_{2\varepsilon}(x_\alpha))}. \tag{49}$$

We easily see, in view of (46) and $\|\varphi_\alpha\|_{L^{p^*}} = 1$,

$$\|\varphi_\alpha\|^a_{L^{p^*}(B_{2\varepsilon}(x_\alpha))} = 1 - \left[a/p^* + o(1)\right]\|\varphi_\alpha\|^{p^*}_{L^{p^*}(M_\alpha\setminus B_{2\varepsilon}(x_\alpha))}.$$

Recall that $\lambda_\alpha K^a < 1$. So we can simplify (49) as

$$\alpha\|\varphi_\alpha\|^a_{L^r} \leqslant C\|\varphi_\alpha\|^{p^*}_{L^{p^*}(M_\alpha\setminus B_{2\varepsilon}(x_\alpha))} + C\|\varphi_\alpha\|^a_{W^{1,p}(B_{4\varepsilon}(x_\alpha)\setminus B_{2\varepsilon}(x_\alpha))}. \tag{50}$$

Now let η be any cutoff function with support in $M \setminus B_\varepsilon(x_\alpha)$ and we multiply equation (47) by $\eta^p \varphi_\alpha$ and integrate by part to obtain

$$\int_M |\nabla \varphi_\alpha|^{p-2} \nabla \varphi_\alpha \nabla(\eta^p \varphi_\alpha) \leqslant C \int_M \eta^p \varphi_\alpha^{p^*},$$

from which we easily deduce

$$\int_M \eta^p |\nabla \varphi_\alpha|^p \leqslant C \int_{\operatorname{supp}\eta} \varphi_\alpha^p + C \int_{\operatorname{supp}\eta} \varphi_\alpha^{p^*}.$$

We point out that in the above derivation we have used the obvious fact that $\{\|\nabla \varphi_\alpha\|_{L^p}^{p-a}\}$ is bounded above by positive constants which are independent of α ($a \leqslant p$ is used here). Selecting η appropriately we have

$$\|\nabla \varphi_\alpha\|_{L^p(B_{4\varepsilon}(x_\alpha)\setminus B_{2\varepsilon}(x_\alpha))} \leqslant C\|\varphi_\alpha\|_{L^p(B_{5\varepsilon}(x_\alpha)\setminus B_\varepsilon(x_\alpha))} + C\|\varphi_\alpha\|_{L^{p^*}(M_\alpha \setminus B_\varepsilon(x_\alpha))}^{p^*/p}.$$

Using the above and (46), we deduce from (50) that:

$$\begin{aligned}
\alpha\|\varphi_\alpha\|_{L^r}^a &\leqslant C\|\varphi_\alpha\|_{L^{p^*}(M_\alpha\setminus B_{2\varepsilon}(x_\alpha))}^{p^*} + C\|\varphi_\alpha\|_{L^{p^*}(M_\alpha\setminus B_\varepsilon(x_\alpha))}^{ap^*/p} + C\|\varphi_\alpha\|_{L^p(B_{5\varepsilon}(x_\alpha)\setminus B_\varepsilon(x_\alpha))}^a \\
&\leqslant C\|\varphi_\alpha\|_{L^{p^*}(M_\alpha\setminus B_\varepsilon(x_\alpha))}^{ap^*/p} + C\|\varphi_\alpha\|_{L^p(B_{5\varepsilon}(x_\alpha)\setminus B_\varepsilon(x_\alpha))}^a.
\end{aligned} \tag{51}$$

We know from (47) that

$$-L_g \varphi_\alpha \leqslant C\varphi_\alpha^{p^*-1} \quad \text{on } M_\alpha.$$

Because of (46), we can apply Lemma 3.1 (Moser iteration) to obtain

$$\|\varphi_\alpha\|_{L^{p^*}(B_{\delta_1}(x))} + \|\varphi_\alpha\|_{L^p(B_{\delta_1}(x))} \leqslant C(\delta_1, \delta_2, \varepsilon)\|\varphi_\alpha\|_{L^r(B_{\delta_2}(x))}$$

for all $0 < \delta_1 < \delta_2$ and $B_{\delta_2}(x) \subset M_\alpha \setminus B_{\varepsilon/2}(x_\alpha)$. It follows immediately that

$$\|\varphi_\alpha\|_{L^p(B_{5\varepsilon}(x_\alpha)\setminus B_\varepsilon(x_\alpha))} \leqslant C\|\varphi_\alpha\|_{L^r},$$

and, by a suitable partition of unity with finite overlapping (using also $r < p^*$),

$$\|\varphi_\alpha\|_{L^{p^*}(M_\alpha\setminus B_\varepsilon(x_\alpha))} \leqslant C\|\varphi_\alpha\|_{L^r}.$$

The above two estimates and (51) lead to contradiction for large α. We have established Theorem 7.1. $\square$

Proof of Theorems 1.2 and 1.3. – The proof is the same as that of Theorem 1.1, only use Theorem 7.1 instead of Theorem 6.1, and also use in addition the local result Proposition 4.1.

8. Appendix

In this appendix, we present some results and arguments used in this paper. Let (M, g) be a C^∞ compact Riemannian manifold with or without boundary. For $p \geqslant 1$, let $W_0^{1,p}(M)$ denote the usual Sobolev space with zero boundary value when M has boundary, and $W_0^{1,p}(M) = W^{1,p}(M)$ when M does not have boundary. For $0 < a < p^*$, $1 \leqslant r < p^*$, $\alpha \geqslant \alpha_0 > 0$, let

$$I_\alpha(\varphi) = \frac{\|\nabla\varphi\|_{L^p}^a + \alpha\|\varphi\|_{L^r}^a}{\|\varphi\|_{L^{p^*}}^a}.$$

We assume

$$\lambda_\alpha := \inf_{\varphi\in W_0^{1,p}(M)} I_\alpha(\varphi) < K^{-a}. \tag{52}$$

PROPOSITION 8.1. – *For $p > 1$, $0 < a < p^*$, $1 \leqslant r < p^*$, and $\alpha > K^{-a} V^{a(\frac{1}{p^*} - \frac{1}{r})}$, $V = \int_M \mathrm{d}v$, we assume (52). Then there exists some non-negative function $\varphi_\alpha \in W_0^{1,p}(M) \cap C^0(M)$ satisfying $\|\varphi_\alpha\|_{L^{p^*}} = 1$, $I_\alpha(\varphi_\alpha) = \lambda_\alpha$, and*

$$-L_g\varphi_\alpha + \alpha\|\varphi_\alpha\|_{L^r}^{a-r}\|\nabla\varphi_\alpha\|_{L^p}^{p-a}\varphi_\alpha^{r-1} = \lambda_\alpha\|\nabla\varphi_\alpha\|_{L^p}^{p-a}\varphi_\alpha^{p^*-1}. \tag{53}$$

LEMMA 8.1. – *Let (M, g) be a C^∞ compact Riemannian manifold with or without boundary, and $1 \leqslant p < n$. Then any non-negative function $\varphi \in W_0^{1,p}(M, g)$ satisfying*

$$-L_g\varphi \leqslant \mu\varphi^{p^*-1} \quad \text{and} \quad \|\varphi\|_{L^{p^*}} = 1 \tag{54}$$

is uniformly bounded if $\mu < K^{-p}$, where K is the best constant $K(n, p)$ in the Sobolev embedding theorem and L_g is the p-Laplacian.

Proof. – Multiplying (54) by φ^{1+kp} and integrating by parts lead to

$$\frac{1+kp}{(1+k)^p}\int\left|\nabla\varphi^{1+k}\right|^p \mathrm{d}v \leqslant \mu\int\varphi^{p^*+kp}\,\mathrm{d}v \leqslant \mu\left\{\int\varphi^{p^*(1+k)}\,\mathrm{d}v\right\}^{p/p^*}. \tag{55}$$

The Sobolev inequality yields

$$\begin{aligned}\left\|\varphi^{1+k}\right\|_{L^{p^*}}^p &\leqslant K^p(1+\varepsilon)\left\|\nabla\varphi^{1+k}\right\|_{L^p}^p + A_\varepsilon\left\|\varphi^{1+k}\right\|_{L^p}^p \\ &\leqslant \frac{(1+k)^p}{1+kp}K^p(1+\varepsilon)\mu\left\|\varphi^{1+k}\right\|_{L^{p^*}}^p + A_\varepsilon\left\|\varphi^{1+k}\right\|_{L^p}^p.\end{aligned}$$

Choose $\varepsilon > 0$ so that $K^p(1+\varepsilon)\mu < 1$, and then pick $k_0 > 0$ such that

$$\frac{(1+k_0)^p}{1+k_0p}K^p(1+\varepsilon)\mu < 1 \quad \text{with } (1+k_0)p \leqslant p^*.$$

We obtain $\|\varphi\|_{L^{(1+k_0)p^*}} \leqslant C$. Now we return to (55) with $k = k_1 = p^*k_0/p$. We have $p^* + k_1p = p^*(1+k_0)$. Thus $\|\varphi\|_{L^{(1+k_1)p^*}} \leqslant C$. So we prove that φ is bounded in any L^r. The Moser iteration technique yields $\sup\varphi \leqslant C$.

COROLLARY 8.1. – *The same result holds for non-negative function $\varphi \in W_0^{1,p}(M, g)$ satisfying*

$$-L_g\varphi \leqslant \mu\varphi^{q-1} \quad \text{and} \quad \|\varphi\|_{L^q} = 1$$

with $1 \leqslant q < p^*$.

Indeed if $\|\varphi\|_{L^{(1+k_0)p^*}} \leqslant 1$ we have nothing to do. Otherwise we write

$$\int \varphi^{q+k_0 p}\,\mathrm{d}v \leqslant \left\{\int \varphi^{p^*(1+k_0)}\,\mathrm{d}v\right\}^{p/p^*},$$

which holds because

$$\frac{kp}{p^*k+p^*-q} < \frac{p}{p^*}.$$

Proof of Proposition 8.1. – It follows from the Sobolev embedding theorem that $\lambda_\alpha > 0$. Since $\alpha > K^{-a}V^{a(\frac{1}{p^*}-\frac{1}{r})}$, λ_α can not be achieved by a constant function. We use the Yamabe method. For $\max(p,r) < q < p^*$, let

$$\lambda_q = \inf_{u\in\mathcal{A}_q}\left(\|\nabla u\|_{L^p}^a + \alpha\|u\|_{L^r}^a\right), \quad \text{where } \mathcal{A}_q = \left\{u \in W_0^{1,p}(M) \mid \|u\|_{L^q} = 1\right\}.$$

A minimizing sequence is bounded in $W_0^{1,p}(M,g)$. Using the Banach theorem and the Kondrakov theorem ($W^{1,p} \subset L^q$ is compact) yields a subsequence which converges weakly in $W_0^{1,p}$, strongly in L^q and a.e. to a non-negative function u_q which satisfies

$$-L_g u_q + \alpha\|u_q\|_{L^r}^{a-r}\|\nabla u_q\|_{L^p}^{p-a}u_q^{r-1} = \lambda_q\|\nabla u_q\|_{L^p}^{p-a}u_q^{q-1} \quad \text{in } M. \tag{56}$$

As in the Yamabe problem, we can prove [5, p. 152] that $\lambda_q \to \lambda_\alpha$ as $q \to p^*$. So for $q_0 \leqslant q < p^*$, $\lambda_q \leqslant \bar{\mu} < K^{-a}$, and $\|\nabla u_q\|_{L^p} > 0$ (recall that $\alpha > K^{-a}V^{a(\frac{1}{p^*}-\frac{1}{r})}$). This implies $-L_g u_q \leqslant \mu u_q^{q-1}$ with $\mu < K^{-p}$. According to Corollary 8.1, $\sup u_q \leqslant C$. The function $\{u_q\}$ ($q_0 \leqslant q < p^*$) are bounded in $W_0^{1,p}$. The Kondrakov theorem then implies that there exists a sequence of $q_i \to p^*$ such that $u_{q_i} \to \varphi_\alpha$ in L^β for any β, and $u_{q_i} \to \varphi_\alpha$ a.e. Thus

$$\|\varphi_\alpha\|_{L^{p^*}} = \lim_{i\to\infty}\|u_{q_i}\|_{L^{q_i}} = 1.$$

Applying the Banach theorem, u_{q_i} converges to φ_α weakly in $W^{1,p}$. It follows that $I_\alpha(\varphi_\alpha) \leqslant \lambda_\alpha$. Since $\varphi_\alpha \in \mathcal{A}$, $I_\alpha(\varphi_\alpha) = \lambda_\alpha$. As pointed out earlier, φ_α is not a constant, so $\|\nabla\varphi_\alpha\|_{L^p} > 0$. Equation (53) is the Euler–Lagrange equation of the minimizer φ_α of I_α. Since φ_α is in L^∞, we know from the regularity results for p-harmonic type equations that $\varphi_\alpha \in C^0(M)$. Proposition 8.1 is established. □

In the rest of this appendix, (M,g) denotes a C^∞ complete connected Riemannian manifold (without boundary) satisfying (H). Let $\alpha \to \infty$ be a sequence of real numbers and let $M_\alpha = M$ when M is compact, and, when M is not compact, let $\{M_\alpha\}$ be a compact sequence of connected submanifolds such that the injectivity radius of M_α is bounded from below by some positive constant independent of α and the second fundamental form of ∂M_α is bounded in absolute value by some constant independent of α. We assume

$$\bar{\lambda}_\alpha := \inf_{\varphi\in W_0^{1,p}(M_\alpha)} I_\alpha(\varphi) \leqslant K^{-a}, \tag{57}$$

and there exist non-negative functions $\varphi_\alpha \in W_0^{1,p}(M_\alpha)\cap C^0(M_\alpha)$ such that

$$\|\varphi_\alpha\|_{L^{p^*}} = 1 \quad \text{and} \quad I_\alpha(\varphi_\alpha) = \bar{\lambda}_\alpha. \tag{58}$$

The Euler–Lagrange equation of φ_α is

$$(59)\qquad -L_g\varphi_\alpha + \alpha\|\varphi_\alpha\|_{L^r}^{a-r}\|\nabla\varphi_\alpha\|_{L^p}^{p-a}\varphi_\alpha^{r-1} = \bar\lambda_\alpha\|\nabla\varphi_\alpha\|_{L^p}^{p-a}\varphi_\alpha^{p^*-1} \quad \text{in } M_\alpha.$$

Since $\alpha\|\varphi_\alpha\|_{L^r}^a \leqslant K^{-a}$, $\|\varphi_\alpha\|_{L^r} \to 0$. After passing to a subsequence, $\varphi_\alpha \to 0$ a.e., $\|\nabla\varphi_\alpha\|_{L^p}^a \to \xi \leqslant K^{-a}$, and $\alpha\|\varphi_\alpha\|_{L^r}^a \to \eta \leqslant K^{-a}$. The Sobolev inequality holds for any $\varepsilon > 0$:

$$\|\varphi\|_{L^{p^*}}^a \leqslant K^a(1+\varepsilon)\|\nabla\varphi\|_{L^p}^a + A_\varepsilon\|\varphi\|_{L^r}^a,$$

take $\varphi = \varphi_\alpha$, and let α tend to infinity, we have $1 \leqslant K^a(1+\varepsilon)\xi$. Since $\varepsilon > 0$ is arbitrary, $\xi = K^{-a}$ and $\eta = 0$.

Since $r < p^*$, we have

$$1 = \int_M \varphi_\alpha^{p^*} \leqslant \Big(\max_{M_\alpha}\varphi_\alpha\Big)^{p^*-r}\|\varphi_\alpha\|_{L^r}^r \leqslant \left(\frac{\bar\lambda_\alpha}{\alpha}\right)^{r/a}\Big(\max_{M_\alpha}\varphi_\alpha\Big)^{p^*-r}.$$

So $\max_{M_\alpha}\varphi_\alpha \to \infty$. Let $x_\alpha \in M_\alpha$ be some maximum point of φ_α, i.e., $\varphi_\alpha(x_\alpha) = \max_{M_\alpha}\varphi_\alpha$. Let $\varepsilon > 0$ be some positive number, independent of α, such that $B_\varepsilon(x)$, the ε-geodesic ball centered at x, is convex with respect to g for all $x \in M$. Define

$$v_\alpha(y) = \mu_\alpha^{\frac{n-p}{p}}\varphi_\alpha\big(\psi_\alpha(y)\big),$$

where $\mu_\alpha := \varphi_\alpha(x_\alpha)^{1-p^*/p} \to 0$, $\psi_\alpha(y) = \exp_{x_\alpha}(\mu_\alpha y)$, $y \in \Omega_\alpha := \psi_\alpha^{-1}(B_\varepsilon(x_\alpha))$. It is clear that $0 \leqslant v_\alpha \leqslant 1$, $v_\alpha(0) = 1$, $v_\alpha = 0$ on $\partial\Omega_\alpha$, and v_α satisfies

$$(60)\qquad -L_{g_\alpha}v_\alpha + \varepsilon_\alpha v_\alpha^{r-1} = \bar\lambda_\alpha\|\nabla\varphi_\alpha\|_{L^p}^{p-a}v_\alpha^{p^*-1} \quad \text{in } \Omega_\alpha,$$

where

$$\varepsilon_\alpha = \alpha\mu_\alpha^\beta\|\varphi_\alpha\|_{L^r}^{a-r}\|\nabla\varphi_\alpha\|_{L^p}^{p-a}, \quad \beta = \left[\frac{n}{r} - \frac{n-p}{p}\right]r, \quad \text{and} \quad g_\alpha = \mu_\alpha^{-2}\psi_\alpha^* g.$$

PROPOSITION 8.2. – *Let $p > 1$, $0 < a < p^*$, $1 \leqslant r < p^*$, and $\{M_\alpha\}$ be as above for a sequence of $\alpha \to \infty$ such that (57) is satisfied. Assume that $\varphi_\alpha \in W_0^{1,p}(M_\alpha) \cap C^0(M_\alpha)$ are non-negative functions with $\|\varphi_\alpha\|_{L^{p^*}} = 1$ and $I_\alpha(\varphi_\alpha) = \bar\lambda_\alpha$, and let x_α be a maximum point of φ_α. Then,*

$$\bar\lambda_\alpha \to K^{-a}, \qquad \lim_{\alpha\to\infty}\frac{\operatorname{dist}_g(x_\alpha, \partial M_\alpha)}{\mu_\alpha} = \infty,$$

$$\|\nabla\varphi_\alpha\|_{L^p} \to K^{-1}, \quad \varepsilon_\alpha \to 0,$$

$$\lim_{\alpha\to\infty}\int_{\Omega_\alpha}\big(\big|\nabla_{g_\alpha}(v_\alpha - v)\big|^p + |v_\alpha - v|^{p^*}\big)\,dv_{g_\alpha} = 0, \quad \textit{and}$$

$$\lim_{\alpha\to\infty}\int_{M_\alpha\setminus B_\varepsilon(x_\alpha)}\big(|\nabla_g\varphi_\alpha|^p + \varphi_\alpha^{p^*}\big) = 0 \quad \forall\,\varepsilon > 0,$$

where v is the extremal function given in (1).

Proof. – Multiplying (59) by φ_α and integrating by parts over M_α lead to

$$\begin{aligned}
&\alpha \|\varphi_\alpha\|_{L^r}^{a-r} \|\nabla\varphi_\alpha\|_{L^p}^{p-a} \int \varphi_\alpha^r \, dv \\
&\leqslant \bar{\lambda}_\alpha \|\nabla\varphi_\alpha\|_{L^p}^{p-a} \int \varphi_\alpha^{p^*} \, dv \leqslant \bar{\lambda}_\alpha \|\nabla\varphi_\alpha\|_{L^p}^{p-a} \Big(\max_M \varphi_\alpha\Big)^{p^*-r} \int \varphi_\alpha^r \, dv \\
&= \bar{\lambda}_\alpha \|\nabla\varphi_\alpha\|_{L^p}^{p-a} \mu_\alpha^{-\beta} \int \varphi_\alpha^r \, dv.
\end{aligned}$$

It follows that

$$\varepsilon_\alpha \leqslant \bar{\lambda}_\alpha \|\nabla\varphi_\alpha\|_{L^p}^{p-a} \leqslant \bar{\lambda}_\alpha^{p/a} \leqslant K^{-p}. \tag{61}$$

The coefficients of (60) are bounded by K^{-p}, so by the regularity results for p-harmonic type equations we have, after passing to a subsequence, $v_\alpha \to w$ in $C^{\beta'}(\overline{\Omega}_\alpha)$ for some $0 < \beta' < 1$ and $w \in C^{\beta'}(\overline{O}) \cap W^{1,p}(\overline{O})$, where (after a possible rotation of the y-coordinates)

$$O = \Big\{ y \in \mathbb{R}^n \mid y^n > -\lim_{\alpha\to\infty} \mu_\alpha^{-1} \operatorname{dist}_g(x_\alpha, \partial M_\alpha) \Big\}$$

and w satisfies $w(0) = 1$ and $w = 0$ on ∂O if $\partial O \neq \phi$.

Next we show $\varepsilon_\alpha \to 0$. Since $\mu_\alpha^\beta \|v_\alpha\|_{L^r}^r = \|\varphi_\alpha\|_{L^r}^r$,

$$\varepsilon_\alpha = \|v_\alpha\|_{L^r}^{-r} \|\nabla\varphi_\alpha\|_{L^p}^{p-a} \alpha \|\varphi_\alpha\|_{L^r}^a.$$

Due to the uniform convergence of v_α to w on compact set and $w(0) = 1$, $\{\|v_\alpha\|_{L^r}\}$ is bounded from below by some positive constant. We have shown $\alpha\|\varphi_\alpha\|_{L^r}^a \to 0$ and $\|\nabla\varphi_\alpha\|_{L^p}^{p-a} \leqslant K^{a-p}$, so $\varepsilon_\alpha \to 0$. Let $\alpha \to \infty$ in (60), w satisfies

$$-L_{g_0} w = K^{-p} w^{p^*-1}, \quad \text{in } O,$$

where g_0 is the Euclidean metric. Multiplying the above equation by w and integrating by parts over O lead to $\|\nabla w\|_{L^p}^p = K^{-p}\|w\|_{L^{p^*}}^{p^*}$. Since v_α weakly converges to w both in $W^{1,p}$ and L^{p^*}, we infer $\|\nabla w\|_{L^p} \leqslant K^{-1}$ and $\|w\|_{L^{p^*}} \leqslant 1$. Thus w is an extremal function to the best Sobolev constant in $\mathbb{R}^n$. Indeed,

$$\|\nabla w\|_{L^p}^p \|w\|_{L^{p^*}}^{-p} = K^{-p}\|w\|_{L^{p^*}}^{p^*-p} \leqslant K^{-p}$$

implies

$$\|\nabla w\|_{L^p}^p \|w\|_{L^{p^*}}^{-p} = K^{-p}, \quad \|w\|_{L^{p^*}} = 1, \quad \text{and} \quad \|\nabla w\|_{L^p} = K^{-1}.$$

So $O = \mathbb{R}^n$ and (recall that $w(0) = 1$ and $\nabla w(0) = 0$) $w = v$, the function given in (1). Thus v_α converges strongly in $W^{1,p}$. The rest of the statements in Proposition 8.2 follow easily.

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