GENERAL FUNCTIONS AND JACOBIANS

ADDITIONAL MULTIVARIABLE CALCULUS MATERIAL: HANDOUT 2

Please review vector-valued functions, functions of several variables, partial derivatives and gradient vectors from the textbook before attempting to read this material. Please also refer to Handout 1 which addresses matrices and their multiplication.

1. Functions from $\mathbb{R}^m \to \mathbb{R}^n$

We have already seen functions from $\mathbb{R}$ to $\mathbb{R}^n$, called *vector valued functions* as well as functions from $\mathbb{R}^n \to \mathbb{R}$ called *functions of several variables*. Here is a typical vector valued function:

$$\vec{h}(t) = \langle \cos(t), \sin(t), t \rangle$$

which parametrizes the *helix*. This is clearly a function from $\mathbb{R}$ to $\mathbb{R}^3$ since it takes a single number as input and outputs a 3-vector. The derivative of this function is $\vec{h'}(t) = \langle -\sin(t), \cos(t), 1 \rangle$ which is again a function from $\mathbb{R}$ to $\mathbb{R}^3$.

On the other hand, an example for a function of several variables is

$$f(x, y) = x^2 - 3y^2$$

This is a function from $\mathbb{R}^2 \to \mathbb{R}$ which takes in 2 variables as input and outputs a single number. The full picture of the derivative of this function is given by its *gradient vector* $\nabla f = \langle 2x, -6y \rangle$ which is just a collection of the partial derivatives of f with respect to the input variables x and y. We have not seen how to differentiate functions where *neither the input nor output dimension equals* 1. First let us see what such a function might look like. Here is an example.

$$F(x, y, z) = (x^2 - yz, xyz)$$

We must learn to quickly identify the input and output dimension of this function: but this is **easy**. The function takes in three variables, so the input dimension is 3. It then outputs a vector of dimension 2, and we can see how the output is defined in terms of the input variables. The first coordinate, call it $F_1(x, y, z)$, equals $x^2 - yz$. This is a standard function of three variables. Similarly, the second coordinate is $F_2(x, y, z) = xyz$. So, a general function from $\mathbb{R}^3$ to $\mathbb{R}^2$ is just a vector of *two* functions, each taking *three variables* as input. Of course, we can *evaluate* this function at any given point P in the input space $\mathbb{R}^3$. So if $P = (1, 3, 2)$ then our function evaluated at P is just $F(1, 3, 2)$ which is a vector in $\mathbb{R}^2$ with first component $F_1(1, 3, 2) = 1^2 - (3)(2) = -5$ and second component $F_2(1, 3, 2) = (1)(3)(2) = 6$. So, $F(1, 3, 2) = (-5, 6)$. Evaluation is easy: just plug in numbers for the input variables!

Similarly, a general function G from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a vector of size n where each component is a function of m variables. That is,

$$G(x_1, \ldots, x_m) = (G_1(x_1, \ldots, x_m), \ldots, G_n(x_1, \ldots, x_m))$$

where each of G_1 through G_n is a function of the m variables x_1 through x_m. To convince yourself that you have understood general functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, you should try the following easy exercises.

1.1. **Exercises.**

1. Consider $H(x, y) = (2xy - y^2, 3y - 7, x + y)$. What are the input and output dimensions of H? Compute $H(1, 1)$, $H(2, 1)$ and $H(0, 0)$.

2. Create a function G from $\mathbb{R}^3$ to $\mathbb{R}^5$. Now evaluate your function at the origin in $\mathbb{R}^3$.

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2. Derivatives as Matrices: The Jacobian

Let's bring back the function $F(x, y)$ from the previous section. Here it is again:

$$F(x, y, z) = (x^2 - yz, xyz)$$

We remember that this is a function with input dimension 3 and output dimension 2. The derivative of F is a *matrix* called the *Jacobian of F*. It is denoted by $JF(x, y, z)$, and defined by

$$JF(x, y, z) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} 2x & -z & -y \\ yz & xz & xy \end{bmatrix}$$

The Jacobian of F is clearly a 2×3 matrix of functions. The columns contain each component function of F (i.e., F_1 and F_2) being differentiated with respect to the variables x, y and z in order. The rows are just the gradients ∇F_1 and ∇F_2 laid one on top of the another in order. So, we learn that for a function F from $\mathbb{R}^3$ to $\mathbb{R}^2$, the derivative is the Jacobian JF, a matrix of size 2×3. This is *not* an accident! The following rule is very general:

If F is a function from $\mathbb{R}^m$ to $\mathbb{R}^n$ then JF is a matrix of size $n \times m$

Anyway, back to the Jacobian $JF(x,y,z) = \left[\begin{smallmatrix} 2x & -z & -y \\ yz & xz & xy \end{smallmatrix}\right]$. Much like the function F itself, the Jacobian can also be evaluated at a point like $P = (1,3,2)$. There are various notations for this, we will use $J_P F$. As you might expect, computing $J_P F$ only requires us to plug the values into the matrix of functions and get a matrix of numbers instead. So, $J_P F = \left[\begin{smallmatrix} 2 & -2 & -3 \\ 6 & 2 & 3 \end{smallmatrix}\right]$.

Now we will see a *general formula* for the derivative of a function from $\mathbb{R}^m$ to $\mathbb{R}^n$. The general function G that we saw in the previous section was

$$G(x_1, \dots, x_m) = (G_1(x_1, \dots, x_m), \dots, G_n(x_1, \dots, x_m))$$

And to compute its Jacobian, we will construct a matrix which has as many columns as G has inputs (i.e., m) and as many rows as G has outputs, (i.e., n). Each of the m columns is indexed by the input variables $x_1, \dots, x_m$ in order from left to right and the rows are indexed by the output functions $G_1, \dots G_m$ from top to bottom. The component of JG in row i and column j is the partial derivative of G_i with respect to x_j. Here is the final picture:

$$JG(x_1, \dots, x_m) = \begin{bmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_n}{\partial x_1} & \cdots & \frac{\partial G_n}{\partial x_m} \end{bmatrix}$$

And as expected, we have a $n \times m$ matrix! We will conclude with some exercises in computing and evaluating Jacobians.

2.1. **Exercises.**

1. Let $F(x,y,z) = x^2y + 2z$. Compute JF and evaluate it at $P = (1,-1,3)$. Confirm that the Jacobian of a function of several variables has only one row.

2. Let $F(x) = (3x, x^2, \sin(2x))$. Compute JF and evaluate it at $P = 2$. Confirm that the Jacobian of a vector valued function variables has only one column.

3. Let $F(x,y) = (xy^2, -3xy, 12x)$. Compute JF and evaluate it at $P = (2,0)$.

4. Let $F(x,y,z) = (y\sin(z), 3xy\ln(z))$. Compute JF and evaluate it at $P = (-1,3,2)$.

5. Let $F(x,y,z) = (xe^y, 3xz\sin(y), -x^2\ln(y+z))$. Compute JF and evaluate it at $P = (1,0,1)$.