Quiz 3
Math 250

Show that any vector in $\mathcal{R}^3$ is in the span of $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

Are the vectors in $S$ linearly independent?

To show that these vectors span $\mathcal{R}^3$, we can show that the rank of a matrix with these vectors as columns is 3. That would mean that the reduced row echelon form has no nonzero rows, so no matter which vector $\vec{b}$ in $\mathcal{R}^3$ you augment it with, you will still get a consistent system, ie, there exist coefficients for the vectors in $S$ such that the linear combination is $\vec{b}$. (Also, see Theorem 1.6(b)(c) on page 70.)

Therefore, we need to form a matrix such that its colums are the vectors in $S$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

We now row reduce $A$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \xleftrightarrow{-r_1+r_2\to r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$
$$\xleftrightarrow{-r_2+r_3\to r_3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

At this point, we have completed the forward pass, and have a matrix in row echelon form (although it is still not reduced). However this is enough to tell us the rank, since we now know what the pivot positions are, and it is impossible to get an additional zero rows after this point. Therefore, we see that the rank of $A$ is 3, so every vector in $\mathcal{R}^3$ is in the span of $S$.

(Note that an easier alternative to doing all the work above is to reorder the vectors in $S$ before putting them into a matrix. We cannot reorder the columns of a matrix, but we can reorder a set of vectors. This means that we can rewrite $S$ as

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

If we now form a matrix $B$ using the vectors in this order as its columns, you get a matrix that is already in row echelon form, although, once again, it is not in reduced row echelon form.)

Finally, to show that the vectors are linearly independent, we again want to show that the rank of a matrix $A$ with these vectors is 3. Here, this means that every column contains a pivot position, so there are no free variables in the solution of $A\vec{x} = \vec{0}$, and so the system has no solution besides $\vec{0}$. (Also, see Theorem 1.8(a)(d) on pages 78-79.) But we have already shown this to be the case, and so the vectors in $S$ must also be linearly independent.