Bona Recurrence

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1 Introduction

Definitions...

Following Bona, we define a permutation to be *(in)decomposable* if it can(not) be cut into two (strictly smaller) parts such that everything before the cut is larger than everything after the cut. We further define a permutation to be p-*(in)decomposable* if it can(not) be cut into two parts such that everything before the cut is larger than everything after the cut and at least p elements occur before the cut.

2 Permutations Avoiding 1342

We will develop a system of recurrences to count the permutations avoiding 1342. We begin with the usual prefix scheme:

$$A_{\emptyset}(n) = \sum_{i=1}^{n} A_1(i;n); \tag{1}$$

i.e. the number of permutations avoiding 1342 of length n is given by summing up, over all $1 \leq i \leq n$, the number of such permutations beginning with i. Next, consider the second letter of the permutation and call it j. If $i > j$, then i is reversely deletable; if $i < j$ then neither is reversely deletable, and so we find:

$$A_1(i;n) = \sum_{j=1}^{i} A_1(j;n-1) + \sum_{j=i+2} A_{12}(i,j;n). \tag{2}$$

But, there is a problem. No matter how long we make the prefix $12\ldots l$, none of its elements will ever be reversely deletable. We are saved by the fact that we know a great deal about the structure of permutation with a 12 prefix. The following observation is immediately clear.

Observation 2.1. *If a 1342-avoiding permutation begins with ij where $i < j$, it follows that all elements in the interval (i,j) must precede all elements in the interval $(j,n]$.*

The following lemma shows that even more is true.

Lemma 2.2. *Let π be a permutation avoiding 1342 which begins with ij where $i < j$. Then, there exists $k \leq i$ such that every element in $[k, j]$ precedes in π every other element.*

Proof. We proceed by induction on i. If $i = j - 1$, then the lemma is trivial. Otherwise (i, j) is nonempty. Let k_1 be the smallest element of π which precedes an element of (i, j), and let j_1 be the last-occuring element of (i, j). If $k_1 = i$, then the lemma follows Observation 2.1; otherwise $k_1 < i$. Form π' by adding the prefix $k_1 j_1$ to portion of π following j_1, and reducing the resulting word. Note that π' is a permutation avoiding 1342 which begins with $k_1 j_1$ where $k_1 < j_1$ and $k_1 < i$. By the induction hypothesis, there exists $k \leq k_1$ such that $[k, j_1]$ precedes in π' every other element.

We claim that, in π, every element in $[k, j]$ precedes every other element. Suppose that is not the case, and there is an element $x \in [k, j]$ which comes after $y \notin [k, j]$. First consider the case when $y > j$. By Observation 2.1, y cannot occur before j_1, and so π' contains an element corresponding to y which is larger than j_1. Since x comes after y, it also occurs after j_1, so π' contains an element corresponding to x which is in $[k, j_1]$ and follows the element corresponding to y. This contradicts the fact that each element in $[k, j_1]$ precedes every other element in π'. Next consider the case when $y < k$. By the choice of k_1, y comes after j_1, and so π' contains an element corresponding to y which is less than k. As in the previous case π' also contains an element corresponding to x which is in $[k, j_1]$ and follows the element corresponding to y, and again this contradicts the fact that each element in $[k, j_1]$ precedes every other element in π'. Thus, the lemma holds. □

Lemma 2.2 suggests dividing a permutation π with a 12 prefix into two pieces: π_{pref} consisting of the elements in $[k, j]$ and π_{suff} consisting of all the others. The following lemma establishes conditions on π_{pref} and π_{suff} that guarantee that π will avoid 1342.

Lemma 2.3. *Let $\pi = \pi_{pref}\pi_{suff}$ where π_{pref} consists of the elements of π in $[k, j]$ (assume $k < j$) and π_{suff} consists of those elements in $[1, k-1] \cup [j+1, n]$. Then π avoids 1342 if and only if π_{pref} avoids 1342 and $k\pi_{suff}$ avoids 1342.*

Proof. The only if direction is easy; π_{pref} and $k\pi_{\text{suff}}$ are both subpermutations of π, and so if either contains 1342, π must as well.

To show the if direction, suppose that π contains 1342, write this occurrence as $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}$ where π_i is the i^{th} element of π. We claim that either $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4} \in \pi_{\text{pref}}$, $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4} \in \pi_{\text{suff}}$, or $\pi_{i_2}\pi_{i_3}\pi_{i_4} \in \pi_{\text{suff}}$ and $\pi_{i_2}, \pi_{i_3}, \pi_{i_4} > k$.

Suppose that $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4} \notin \pi_{\text{pref}}$ and $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4} \notin \pi_{\text{suff}}$, it follows that $\pi_{i_1} \in \pi_{\text{pref}}$ and $\pi_{i_4} \in \pi_{\text{suff}}$. Since $\pi_{i_1}\pi_{i_2}\pi_{i_3}\pi_{i_4}$ is an occurrence of 1342, we conclude that $\pi_{i_2}, \pi_{i_3}, \pi_{i_4} > j$, so $\pi_{i_2}\pi_{i_3}\pi_{i_4} \in \pi_{\text{suff}}$, and, since $j > k$, $\pi_{i_2}, \pi_{i_3}, \pi_{i_4} > k$. Thus, $k\pi_{\text{suff}}$ contains 1342. □

Combining Lemmas 2.2 and 2.3 suggests we compute $A_{12}(i,j;n)$ by finding the number of possible π_{pref}s and multiplying by the number of possible of possible π_{suff}, which would give the (incorrect) recurrence

$$A_{12}(i,j;n) = \sum_{k=1}^{i} A_1(i-k+1, j-k+1; j-k) \cdot A_1(k; n-j+k)). \qquad (3)$$

This recurrence is incorrect because the right-hand side counts permutations once for every k such that the permutation can be written as $\pi_{\text{pref}}\pi_{\text{suff}}$ with π_{pref} containing the elements in $[k,j]$ and π_{pref} containing all other elements. This leads to a lot of double-counting!

To fix it, we will count permutations only once, corresponding to the maximal k that allows them to be appropriately broken up. If we write π as $\pi_{\text{pref}}\pi_{\text{suff}}$ and find that π_{pref} is $j-i+1$-decomposable, that would imply that we could have actually ended π_{pref} after a prefix and obtained a different expression $\pi_{\text{pref}}\pi_{\text{suff}}$ with a larger k. Therefore, we define $A_{12}(p;i,j;n)$ to be the number of length-n 1342-avoiding permutations beginning with ij such that there is no proper prefix of length $\geq p$ whose elements are all greater than the following elements. We redefine $A_1(p;i;n)$ similarly.

This leads us to the final and correct system of recurrences.

Theorem 2.4. *The following system of recurrences holds:*

$$\begin{aligned}
A_{\emptyset}(n) &= \sum_{i=1}^{n} A_1(n;i;n) \\
A_1(p;i;n) &= \sum_{j=1}^{n-p-1} A_1(n-j;j;n-1) + \sum_{j=n-p}^{i} A_1(p-1;j;n-1) + \sum_{j=i+2}^{n} A_{12}(p;i,j;n) \\
A_{12}(p;i,j;n) &= \sum_{k=1}^{n-p-1} A_1(j-i-1;i-k+1;j-k) \cdot A_1(n-j+1;k;n-j+k) \\
&\quad + \sum_{k=n-p}^{i} A_1(j-i-1;i-k+1;j-k) \cdot A_1(p-(j-k);k;n-j+k).
\end{aligned}$$

Proof. The first equation follows from Equation 1 and the fact that no permutation can have a proper prefix of length n, so $A_1(n;i;n)$ counts all length-n permutations which avoid 1342.

The second equation looks a great deal like Equation 2 except it keeps track of forbidden prefix lengths. Since j is the new letter being added to the permutation, it is impossible for the final permutation to be p-decomposable with $p < n-j$; otherwise adding a j simply makes the prefix one letter longer. Thus, we replace $A_1(j, n-1)$ with $A_1(\max(n-j, p-1); j; n-1)$ which accounts for the first two summations. The third summation comes directly from the second summation in Equation 2.

Now we turn to the third equation. Each term of the two summations counts the number of permutations of length n beginning with ij where k is the largest value such that all elements in $[k, j]$ precede all other elements. The first factor, $A_1(j-i-1; i-k+1; j-k)$ counts the number of ways to arrange the elements in $[k, j]$ (i.e. the number of potential π_{pref}s); the length of each prefix is $j-k+1$, but we ignore j because we know it occurs as the second element and does not affect where any other element can be placed. This portion of the permutation begins with i which is the $i-k+1^{\text{th}}$ largest element of $[k, j]$, and it must be $j-i-1$-indecomposable to ensure that k is maximal. The second factor, $A_1(n-j+1; k; n-j+k)$ counts the number of ways to arrange the remaining elements in $[1, k-1] \cup [j+1, n]$ (i.e. the number of potential π_{suff}s). The length of each suffix is $n-j+k-1$, but we pretend that there is a k at the very beginning since Lemma 2.3 demands that the elements $[1, k-1] \cup [j+1, n]$ be arranged such that they avoid 1342 even when k is prepended. If $k < n-p$, then we cause the final permutation to be p-decomposable if and only if π_{suff} is $n-j+1$-decomposable, while, if $k \geq n-p$, we cause the final permutation to be p-decomposable if and only if π_{suff} is $p-(j-k)$-decomposable.

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