MOTIVIC STABLE STEMS OVER FINITE FIELDS

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ABSTRACT OF THE DISSERTATION

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Let ℓ be a prime. For any algebraically closed field $\overline{F}$ of positive characteristic $p \neq \ell$, we show that there is an isomorphism $\pi_n^s[\frac{1}{p}] \cong \pi_{n,0}(\overline{F})[\frac{1}{p}]$ for all $n \geq 0$ of the nth stable homotopy group of spheres with the $(n, 0)$ motivic stable homotopy group of spheres over $\overline{F}$ after inverting the characteristic of the field $\overline{F}$. For a finite field $\mathbb{F}_q$ of characteristic p, we calculate the motivic stable homotopy groups $\pi_{n,0}(\mathbb{F}_q)[\frac{1}{p}]$ for $n \leq 18$ with partial results when $n = 19$ and $n = 20$. This is achieved by studying the properties of the motivic Adams spectral sequence under base change and computer calculations of Ext groups.

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Dedication

This dissertation is dedicated to my parents, my brother, Mr. Stover, and Ms. Besante.

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Chapter 1

Introduction

For any field F, Morel and Voevodsky construct a triangulated category $\mathcal{SH}_F$ in which one can use techniques of stable homotopy theory to study schemes over F [MV99]. Just as for the stable homotopy category $\mathcal{SH}$ of topology, it is an interesting problem to compute the stable motivic homotopy groups of spheres $\pi_{s,w}(F) = \mathcal{SH}_F(\Sigma^{s,w}\mathbb{1}, \mathbb{1})$ over F, where $\mathbb{1}$ denotes the motivic sphere spectrum. In this dissertation, we use the motivic Adams spectral sequence (MASS) to determine the structure of the motivic stable homotopy groups of spheres $\pi_{n,0}(\mathbb{F}_q)$ over finite fields $\mathbb{F}_q$ when $n \leq 18$ with the assistance of computer calculations.

For a field F of characteristic different from ℓ, write $\mathcal{A}^{**}(F)$ for the bigraded mod ℓ motivic Steenrod algebra over F and $H^{**}(F)$ for the mod ℓ motivic cohomology ring of F, which are discussed in chapter 3. The mod ℓ motivic Adams spectral sequence of the sphere spectrum $\mathbb{1}$ over F is defined in chapter 4 and has second page

$$E_2^{f,(s,w)} = \mathrm{Ext}^{f,(s+f,w)}_{\mathcal{A}^{**}(F)}(H^{**}(F), H^{**}(F)).$$

The motivic Adams spectral sequence of $\mathbb{1}$ over F converges to the homotopy groups of the H-nilpotent completion of the sphere spectrum $\pi_{s,w}(\mathbb{1}^\wedge_H(F)) = \mathcal{SH}_F(\Sigma^{s,w}\mathbb{1}, \mathbb{1}^\wedge_H)$ for fields F of finite mod ℓ cohomological dimension by proposition 4.17. We show in proposition 4.21 that the motivic Adams spectral sequence over finite fields and algebraically closed fields converges to the ℓ-primary part of $\pi_{s,w}(F)$ for $s > w \geq 0$. Our argument relies on the fact that the groups $\pi_{s,w}(F)$ are torsion for $s > w \geq 0$ [ALP15].

Dugger and Isaksen have calculated the 2-complete stable motivic homotopy groups of spheres up to the 34 stem over the complex numbers [DI10] by using the motivic Adams spectral sequence. Isaksen has extended this work largely up to the 70 stem [Isa14a, Isa14b]. We are led to wonder, how do the motivic stable homotopy groups vary for different base fields?

Morel determined a complete description of the 0-line $\pi_{n,n}(F)$ in terms of Milnor-Witt K-theory [Mor12]. In particular, $\pi_{0,0}(F)$ is isomorphic to the Grothendieck-Witt group of F and for all $n > 0$ there is an isomorphism $\pi_{n,n}(F) \cong W(F)$ where $W(F)$ is the Witt group of quadratic forms of F. See [Wei13, II.5.6] for a definition of $GW(F)$ and $W(F)$.

For the 1-line $\pi_{n+1,n}(F)$, partial results have been obtained in [OØ14]. Ormsby has investigated the case of related invariants over p-adic fields [Orm11] and the rationals [OØ13], and Dugger and Isaksen have analyzed the case over the real numbers [DI15]. It is now possible to perform similar calculations over fields of positive characteristic, thanks to work on the motivic Steenrod algebra in positive characteristic [HKØ13].

Denote the nth topological stable stem by π_n^s. Over the complex numbers, Levine showed there is an isomorphism $\pi_n^s \cong \pi_{n,0}(\mathbb{C})$ [Lev14, Cor. 2]. We obtain a similar result in theorem 5.6 for an algebraically closed field $\overline{F}$ of positive characteristic after ℓ-completion away from the characteristic of $\overline{F}$. Our argument uses the motivic Adams spectral sequence and properties of the motivic Adams spectral sequence under base change. In particular, we must work with the motivic stable homotopy category over the ring of Witt vectors of a field of positive characteristic. We use the construction of a spectrum which represents motivic cohomology by Spitzweck [Spi13] to construct the motivic Adams spectral sequence over Dedekind domains.

Theorem 1.1. Let $\overline{F}$ be an algebraically closed field of positive characteristic p. For all $s \geq w \geq 0$, there are isomorphisms $\pi_{s,w}(\overline{F})[\frac{1}{p}] \cong \pi_{s,w}(\mathbb{C})[\frac{1}{p}]$.

Proof. When $s > w \geq 0$, the groups $\pi_{s,w}(\overline{F})$ and $\pi_{s,w}(\mathbb{C})$ are torsion by proposition 4.21. The isomorphism $\pi_{s,w}(\overline{F})[\frac{1}{p}] \cong \pi_{s,w}(\mathbb{C})[\frac{1}{p}]$ follows when $s > w \geq 0$ from theorem 5.6 by summing up the ℓ-primary parts. When $s = w \geq 0$ the result follows by Morel's identification of the 0-line in [Mor12]. $\square$

Corollary 1.2. Let $\overline{F}$ be an algebraically closed field of positive characteristic p. For all $n \geq 0$ the homomorphism $\mathbb{L}c : \pi_n^s[\frac{1}{p}] \to \pi_{n,0}(\overline{F})[\frac{1}{p}]$ is an isomorphism.

For a finite field $\mathbb{F}_q$ with an algebraic closure $\overline{\mathbb{F}}_p$, theorem 1.1 helps us analyze the mod ℓ motivic Adams spectral sequence over $\mathbb{F}_q$ by comparing the spectral sequences over

$\mathbb{F}_q$ and $\overline{\mathbb{F}}_p$. In particular, we obtain the following calculation of the motivic stable stems over $\mathbb{F}_q$.

Theorem 1.3. Let $\mathbb{F}_q$ be a finite field of characteristic p. For all $0 \leq n \leq 18$, there is an isomorphism $\pi_{n,0}(\mathbb{F}_q)[\frac{1}{p}] \cong (\pi_n^s \oplus \pi_{n+1}^s)[\frac{1}{p}]$.

Proof. Propositions 6.8, 6.11, 6.12 calculate the ℓ-completion of $\pi_{n,0}(\mathbb{F}_q)$ for primes $\ell \neq p$ when the Bockstein acts trivially on $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$. Propositions 7.8, 7.10, and 7.11 calculate the ℓ-completion of $\pi_{n,0}(\mathbb{F}_q)$ for primes $\ell \neq p$ when the action of the Bockstein on $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$ is non-trivial. The ℓ-completions of $\pi_{n,0}(\mathbb{F}_q)$ are shown to agree with the ℓ-primary part of $\pi_{n,0}(\mathbb{F}_q)$ for $n > 0$ in proposition 4.21. When $n = 0$, the result follows by Morel's identification of $\pi_{0,0}(\mathbb{F}_q)$ with the Grothendieck-Witt ring of $\mathbb{F}_q$, since $GW(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ [Sch85, Ch. 2, 3.3]. $\square$

In the case of a finite field $\mathbb{F}_q$ where the Bockstein acts non-trivially on the motivic cohomology of $\mathbb{F}_q$ with $\mathbb{Z}/2$ coefficients, i.e., when $q \equiv 3 \bmod 4$, we use computer calculations to identify the E_2 page of the mod 2 motivic Adams spectral sequence. We discuss the methods of calculation in chapter 8.

It is interesting to note that at the prime $\ell = 2$, the pattern $\pi_{n,0}(\mathbb{F}_q)^\wedge_2 \cong (\pi_n^s \oplus \pi_{n+1}^s)^\wedge_2$ obtained in theorem 1.3 does not hold in general. Recall that $(\pi_{19}^s)^\wedge_2 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$, $(\pi_{20}^s)^\wedge_2 \cong \mathbb{Z}/8$, and $(\pi_{21}^s)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. We show that if $q \equiv 5 \bmod 8$, then

$$\pi_{19,0}(\mathbb{F}_q)^\wedge_2 \cong (\pi_{19}^s)^\wedge_2 \oplus \mathbb{Z}/4 \text{ and } \pi_{20,0}(\mathbb{F}_q)^\wedge_2 \cong (\pi_{20}^s)^\wedge_2 \oplus \mathbb{Z}/2.$$

In the mod 2 Adams spectral sequence of topology, the class $\bar{\kappa} \in \pi_{20}^s$ is detected by the class g which is in Adams filtration 4. The calculation in proposition 6.9 implies that the class $\bar{\kappa} \in \pi_{20,0}(\mathbb{F}_q)$ is in Adams filtration 3 when $q \equiv 5 \bmod 8$ but in Adams filtration 4 if $q \equiv 1 \bmod 8$. See 6.10 for more details and references. It is still an open question whether or not $\pi_{n,0}(\mathbb{F}_q)^\wedge_2 \cong (\pi_n^s \oplus \pi_{n+1}^s)^\wedge_2$ holds when $q \equiv 3 \bmod 4$ and $n = 19$ or $n = 20$.

The 2-primary calculations presented in this dissertation have been submitted for publication [WØ16]. The odd primary calculations have not yet appeared elsewhere.

Chapter 2

The stable motivic homotopy category

We first sketch a construction of the stable motivic homotopy category that will be convenient for our purposes and set our notation. Treatments of stable motivic homotopy theory can be found in [Ayo07, DRØ03, DLØ$^+$07, Jar00, Hu03].

2.1 Base schemes

A *base scheme* S is a Noetherian separated scheme of finite Krull dimension. We write Sm/S for the category of smooth schemes of finite type over S. Denote the category of presheaves of sets on Sm/S by $\mathrm{Pshv}(\mathrm{Sm}/S)$. A *space over* S is a simplicial presheaf on Sm/S. The collection of spaces over S forms the category $\mathrm{Spc}(S) = \Delta^{op}\mathrm{Pshv}(\mathrm{Sm}/S)$, where morphisms are natural transformations of functors. We write $\mathrm{Spc}_*(S)$ for the category of pointed spaces. Note that $\mathrm{Spc}(S)$ is naturally equivalent to the category of presheaves on Sm/S with values in the category of simplicial sets $\underline{\mathrm{sSet}}$. We will occasionally switch between the two perspectives.

We will be focused on the special cases where S is the Zariski spectrum of a Hensel local ring in which ℓ is invertible or a field of positive characteristic different from ℓ. For a field F of positive characteristic, the *ring of Witt vectors of* F is a complete Hensel local ring $W(F)$ with residue field F. A thorough analysis of the ring of Witt vectors is given in [Ser79, II §6].

2.2 The projective model structure

We are able to perform familiar constructions from homotopy theory with schemes thanks to the construction of the motivic model category due to Morel and Voevodsky [MV99].

We will assume the reader is familiar with the basic properties of model categories, which can be found in [Hir03, Hov99].

The first model category structure we endow $\mathrm{Spc}(S)$ with is the projective model structure, see [Bla01, 1.4], [DRØ03, 2.7], [Hir03, 11.6.1].

Definition 2.1. A map $f : X \to Y$ in $\mathrm{Spc}(S)$ is a *(global) weak equivalence* if for any $U \in \mathrm{Sm}/S$ the map $f(U) : X(U) \to Y(U)$ of simplicial sets is a weak equivalence. The *projective fibrations* are those maps $f : X \to Y$ for which $f(U) : X(U) \to Y(U)$ is a Kan fibration for any $U \in \mathrm{Sm}/S$. The *projective cofibrations* are those maps in $\mathrm{Spc}(S)$ which satisfy the left lifting property for trivial projective fibrations. The projective model structure on $\mathrm{Spc}(S)$ consists of the global weak equivalences, the projective fibrations, and the projective cofibrations.

The category $\mathrm{Spc}(S)$ equipped with the projective model structure is cellular, proper, and simplicial [Bla01, 1.4]. Furthermore, $\mathrm{Spc}(S)$ has the structure of a simplicial monoidal model category, with product $\times$ and internal hom $\underline{\mathrm{Hom}}$. We write $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ for the simplicial mapping space for spaces $\mathcal{X}$ and $\mathcal{Y}$.

Definition 2.2. For a smooth scheme X over S, we write h_X for the *representable presheaf* of simplicial sets. For $U \in \mathrm{Sm}/S$, the simplicial set $h_X(U)$ is given by $h_X(U)_n = \mathrm{Sm}/S(U, X)$ for all $n \in \Delta$ where the face and degeneracy maps are the identity map. We will frequently abuse notation and write X for h_X.

The *constant presheaf functor* $c : \underline{\mathrm{sSet}} \to \mathrm{Spc}(S)$ associates to a simplicial set A the presheaf cA defined by $cA(U) = A$ for any $U \in \mathrm{Sm}/S$.

The functor c is a left Quillen functor when $\mathrm{Spc}(S)$ is equipped with the projective model structure. Its right adjoint $\mathrm{Ev}_S : \mathrm{Spc}(S) \to \underline{\mathrm{sSet}}$ satisfies $\mathrm{Ev}_S(X) = X(S)$. One can show that representable presheaves and constant presheaves in $\mathrm{Spc}(S)$ are cofibrant in the projective model structure.

2.3 The Nisnevich local model structure

Although the representable presheaf functor embeds Sm/S into $\mathrm{Spc}(S)$, colimits which exist in Sm/S are not necessarily preserved in $\mathrm{Spc}(S)$. That is, if $X = \mathrm{colim}\, X_i$ in Sm/S, it need not be true that $h_X = \operatorname{colim} h_{X_i}$, e.g., $\operatorname{colim}(h_{\mathbb{A}^1} \leftarrow h_{\mathbb{G}_m} \to h_{\mathbb{A}^1}) \neq h_{\mathbb{P}^1}$. To fix this, one introduces the Nisnevich topology on Sm/S.

Definition 2.3. Let S be a base scheme. For any $X \in \mathrm{Sm}/S$, let $\mathcal{U} = \{U_i \to X\}$ be a finite family of étale maps in Sm/S. We say $\mathcal{U}$ is a *Nisnevich covering* of X if for any $x \in X$ there exists a map $U_i \to X$ in $\mathcal{U}$ and a point $u \in U_i$ for which the induced map of residue fields $k(x) \to k(u)$ is an isomorphism. The Nisnevich covers determine a Grothendieck topology on Sm/S, which is called the *Nisnevich topology.*

Definition 2.4. An elementary distinguished square is a pull-back square in Sm/S

$$\begin{tikzcd} V' \arrow[r] \arrow[d] & X' \arrow[d, "f"] \\ V \arrow[r, "j"] & X \end{tikzcd}$$

for which f is an étale map, j is an open embedding, and $f^{-1}(X - V) \to X - V$ is an isomorphism, where these subschemes are given the reduced structure.

Morel and Voevodsky proved in [MV99, 3.1.4] that the Nisnevich topology is generated by covers coming from the elementary distinguished squares. That is, a presheaf of sets F on Sm/S is a Nisnevich sheaf if and only if for any elementary distinguished square, as in definition 2.4, the resulting square

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(V) \arrow[d] \\ F(X') \arrow[r] & F(V') \end{tikzcd}$$

is a pull-back square.

We now set out to modify the projective model structure on the category of spaces over a base scheme S. In particular, we would like to declare a collection of maps $\mathcal{C}$ to be weak equivalences which may not already be weak equivalences in the projective model structure. The general procedure for this is Bousfield localization, which is defined by Hirschhorn in [Hir03, 3.3.1] and proven to exist in good circumstances in [Hir03, 4.1.1], such as when $\mathcal{C}$ is a set. To be brief, for a model category $\mathcal{M}$ and a class of morphisms $\mathcal{C}$ in $\mathcal{M}$, the *left Bousfield localization of* $\mathcal{M}$ *at* $\mathcal{C}$—if it exists—is a model category $L_{\mathcal{C}}\mathcal{M}$ with the same underlying category as $\mathcal{M}$, but the weak equivalences are the $\mathcal{C}$-local weak equivalences, the cofibrations are the same as in $\mathcal{M}$, and the fibrations are determined by the right lifting property. The $\mathcal{C}$-local weak equivalences include the weak equivalences of $\mathcal{M}$ and all maps in $\mathcal{C}$.

Definition 2.5. For a pointed space $\mathcal{X}$ and $n \geq 0$, the nth *simplicial homotopy sheaf* $\pi_n\mathcal{X}$ *of* $\mathcal{X}$ is the Nisnevich sheafification of the presheaf $U \mapsto \pi_n(\mathcal{X}(U))$.

Write W_{Nis} for the class of maps $f : \mathcal{X} \to \mathcal{Y}$ which satisfy $f_* : \pi_n\mathcal{X} \to \pi_n\mathcal{Y}$ is an isomorphism of Nisnevich sheaves for all $n \geq 0$. The *Nisnevich local model structure* on $\mathrm{Spc}_*(S)$ is the left Bousfield localization of the projective model structure with respect to W_{Nis}.

Definition 2.6. Let $W_{\mathbb{A}^1}$ be the class of maps $\pi_X : (X \times \mathbb{A}^1)_+ \to X_+$ for $X \in \mathrm{Sm}/S$. The *motivic model structure* on $\mathrm{Spc}_*(S)$ is the left Bousfield localization of the projective model structure with respect to $W_{Nis} \cup W_{\mathbb{A}^1}$. We write $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ for the category of pointed spaces equipped with the motivic model structure. The pointed motivic homotopy category $\mathcal{H}_*^{\mathbb{A}^1}(S)$ is the homotopy category of $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$.

For pointed spaces $\mathcal{X}$ and $\mathcal{Y}$, write $[\mathcal{X}, \mathcal{Y}]$ for the set of maps $\mathcal{H}_*^{\mathbb{A}^1}(S)(\mathcal{X}, \mathcal{Y})$. The nth *motivic homotopy sheaf* of a pointed space $\mathcal{X}$ over S is the sheaf $\pi_n\mathcal{X}$ associated to the presheaf $U \mapsto [S^n \wedge U_+, \mathcal{X}]$.

There are two circles in the category of pointed spaces: the constant simplicial presheaf S^1 pointed at its 0-simplex and the representable presheaf $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$ pointed at 1. These determine a bigraded family of spheres $S^{i,j} = (S^1)^{\wedge i-j} \wedge \mathbb{G}_m^{\wedge j}$.

Definition 2.7. For a pointed space $\mathcal{X}$ over S and natural numbers i and j, write $\pi_{i,j}\mathcal{X}$ for the set of maps $[S^{i,j}, X]$.

The category of pointed spaces $\mathrm{Spc}_*(S)$ equipped with the induced motivic model category structure has many good properties which make it amenable to Bousfield localization. In particular, $\mathrm{Spc}_*(S)$ is closed symmetric monoidal, pointed simplicial, left proper, and cellular.

2.4 The stable Nisnevich local model structure

With the unstable motivic model category in hand, we now construct the stable motivic model category using the general framework laid out in [Hov01].

Let T be a cofibrant replacement of $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$. One can show that T is weak equivalent to $S^{2,1}$ in $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ [MV99, 3.2.15]. The functor $T \wedge -$ on $\mathrm{Spc}_*^{\mathbb{A}^1}(S)$ is a left Quillen functor, which we may invert by creating a category of T-spectra.

Definition 2.8. A *T-spectrum* is a sequence of spaces $X_n \in \mathrm{Spc}_*^{\mathbb{A}^1}(S)$ equipped with structure maps $\sigma_n : T \wedge X_n \to X_{n+1}$. A map of T-spectra $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ which is compatible with the structure maps. We write $\mathrm{Spt}_T(S)$ for the category of T-spectra of spaces.

The *level model structure* on $\mathrm{Spt}_T(S)$ is given by declaring a map $f : X \to Y$ to be a weak equivalence (respectively fibration) if every map $f_n : X_n \to Y_n$ is a weak equivalence (respectively fibration) in the motivic model structure on $\mathrm{Spc}_*(S)$. The cofibrations for the level model structure are determined by the left lifting property for trivial level fibrations.

Definition 2.9. Let X be a T-spectrum. For integers i and j, the (i,j) *stable homotopy sheaf* of X is the sheaf $\pi_{i,j}X = \mathrm{colim}_n\, \pi_{i+2n,j+n}X_n$. A map $f : X \to Y$ is a *stable weak equivalence* if for all integers i and j, the induced maps $f_* : \pi_{i,j}X \to \pi_{i,j}Y$ are isomorphisms.

Definition 2.10. The *stable model structure* on $\mathrm{Spt}_T(S)$ is the model category where the weak equivalences are the stable weak equivalences and the cofibrations are the cofibrations in the level model structure. The fibrations are those maps with the right lifting property with respect to trivial cofibrations. We write $\mathcal{SH}_S$ for the homotopy category of $\mathrm{Spt}_T(S)$ equipped with the stable model structure.

The stable model structure on $\mathrm{Spt}_T(S)$ can be realized as a left Bousfield localization of the level-wise model structure [Hov01, 3.3].

Just as for the category Spt_{S^1} of simplicial S^1-spectra, there is not a symmetric monoidal category structure on $\mathrm{Spt}_T(S)$ which lifts the smash product $\wedge$ in $\mathcal{SH}_S$. One remedy is to use a category of symmetric T-spectra $\mathrm{Spt}_T^\Sigma(S)$. The construction of this category is given by Hovey in [Hov01, 7.7]. It is proven in [Hov01, 9.1] that there is a zig-zag of Quillen equivalences from $\mathrm{Spt}_T^\Sigma(S)$ to $\mathrm{Spt}_T(S)$, so $\mathcal{SH}_S$ is equivalent to the homotopy category of $\mathrm{Spt}_T^\Sigma(S)$ as well. Since Quillen equivalences induce equivalences of homotopy categories, the category $\mathcal{SH}_S$ is a symmetric monoidal, triangulated category, where the shift functor is given by $[1] = S^{1,0} \wedge -$.

Definition 2.11. For a T-spectrum E over S, write $\pi_{i,j}E$ for the group $\mathcal{SH}_S(\Sigma^{i,j}\mathbb{1}, E)$. In the case where $E = \mathbb{1}$ and $S = \mathrm{Spec}(D)$, we simply write $\pi_{i,j}(D)$ for $\mathcal{SH}_S(\Sigma^{i,j}\mathbb{1}, \mathbb{1})$.

In addition to the category of T-spectra, we will find it convenient to work with the category of $(\mathbb{G}_m, S^1)$ bispectra, see [Jar00, DLØ+07].

Definition 2.12. Consider the simplicial circle S^1 as a space over S, given by the constant presheaf. An S^1*-spectrum* over S is a sequence of spaces $X_n \in \mathrm{Spc}_*(S)$ equipped with structure maps $\sigma_n : S^1 \wedge X_n \to X_{n+1}$. A map of S^1-spectra over S is a sequence of maps $f_n : X_n \to Y_n$ that are compatible with the structure maps. The collection of S^1-spectra over S with compatible maps between them forms a category $\mathrm{Spt}_{S^1}(S)$.

First equip $\mathrm{Spt}_{S^1}(S)$ with the level model structure with respect to the Nisnevich local model structure on $\mathrm{Spc}_*(S)$. The nth *stable homotopy sheaf* of an S^1-spectrum E over S is the Nisnevich sheaf $\pi_n E = \mathrm{colim}\, \pi_{n+j}E_j$. A map $f : E \to F$ of S^1-spectra over S is a *simplicial stable weak equivalence* if for all $n \in \mathbb{Z}$ the induced map $f_* : \pi_n E \to \pi_n F$ is an isomorphism of sheaves. The *stable Nisnevich local model category structure* on $\mathrm{Spt}_{S^1}(S)$ is obtained by localizing at the class of simplicial stable equivalences, as in definition 2.10.

The *motivic stable model category structure* on $\mathrm{Spt}_{S^1}(S)$ is obtained from the simplicial stable model category structure by left Bousfield localization at the class of maps $W_{\mathbb{A}^1} = \{\Sigma^\infty X_+ \wedge \mathbb{A}^1 \to \Sigma^\infty X_+ \mid X \in \mathrm{Sm}/S\}$. Write $\mathrm{Spt}_{S^1}^{\mathbb{A}^1}(S)$ for the motivic stable model category $L_{W_{\mathbb{A}^1}}\mathrm{Spt}_{S^1}(S)$ and write $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)$ for its homotopy category. For S^1-spectra E and F over S, write $[E, F]$ for the group $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)(E, F)$. The nth *motivic stable homotopy sheaf* of an S^1-spectrum E is the Nisnevich sheaf $\pi_n^{\mathbb{A}^1}E$ associated to the presheaf $U \mapsto [S^n \wedge \Sigma^\infty U_+, E]$.

Definition 2.13. In the projective model structure on $\mathrm{Spc}_*(S)$, the space $\mathbb{G}_m$ pointed at 1 is not cofibrant. We abuse notation, and write $\mathbb{G}_m$ for a cofibrant replacement of $\mathbb{G}_m$. A $(\mathbb{G}_m, S^1)$ *bispectrum* over S is a $\mathbb{G}_m$-spectrum of S^1-spectra. More concretely, a $(\mathbb{G}_m, S^1)$ bispectrum E is a bigraded family of spaces $E_{m,n}$ with bonding maps $\sigma_{m,n} : S^1 \wedge E_{m,n} \to E_{m,n+1}$ and $\gamma_{m,n} : \mathbb{G}_m \wedge E_{m,n} \to E_{m+1,n}$ which are compatible, meaning that the following diagram commutes.

$$\begin{tikzcd}
\mathbb{G}_m \wedge S^1 \wedge E_{m,n} \arrow[rr, "\tau \wedge E_{m,n}"] \arrow[d, "\mathbb{G}_m \wedge \sigma"'] & & S^1 \wedge \mathbb{G}_m \wedge E_{m,n} \arrow[d, "S^1 \wedge \gamma"] \\
\mathbb{G}_m \wedge E_{m,n+1} \arrow[r, "\gamma"] & E_{m+1,n+1} & S^1 \wedge E_{m+1,n} \arrow[l, "\sigma"']
\end{tikzcd}$$

We write $\mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$ for the category of $(\mathbb{G}_m, S^1)$ bispectra over S. Consider $\mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$ as the category of $\mathbb{G}_m$-spectra of S^1-spectra, we first equip $\mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$ with the level model category structure with respect to the motivic stable model category structure on $\mathrm{Spt}_{S^1}(S)$. The *motivic stable model category structure* on $\mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$ is the left Bousfield localization at the class of stable equivalences.

There are left Quillen functors $\Sigma^\infty_{S^1} : \mathrm{Spc}_*(S) \to \mathrm{Spt}_{S^1}(S)$ and $\Sigma^\infty_{\mathbb{G}_m} : \mathrm{Spt}_{S^1}(S) \to \mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$. Additionally, the category $\mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$ equipped with the motivic stable model structure is Quillen equivalent to the stable model category structure on $\mathrm{Spt}_T(S)$; see [DLØ$^+$07, p. 216].

Definition 2.14. To any S^1-spectrum of simplicial sets $E \in \mathrm{Spt}_{S^1}$ we may associate the constant S^1-spectrum cE over S with value E. That is, cE is the sequence of spaces cE_n with the evident bonding maps. For a simplicial spectrum E, we also write cE for the $(\mathbb{G}_m, S^1)$ bispectrum $\Sigma^\infty_{\mathbb{G}_m} cE$. This defines a left Quillen functor $c : \mathrm{Spt}_{S^1} \to \mathrm{Spt}_{\mathbb{G}_m,S^1}(S)$ with right adjoint given by evaluation at S. Compare with [Lev14, 6.5].

2.5 Base change of stable model categories

Definition 2.15. Let $f : R \to S$ be a map of base schemes. Pull-back along f determines a functor $f^{-1} : \mathrm{Sm}/S \to \mathrm{Sm}/R$, which induces Quillen adjunctions $(f^*, f_*) : \mathrm{Spc}^{\mathbb{A}^1}_*(S) \to \mathrm{Spc}^{\mathbb{A}^1}_*(R)$ and $(f^*, f_*) : \mathrm{Spt}_T(S) \to \mathrm{Spt}_T(R)$.

We now discuss some of the properties of base change. A more thorough treatment is given in [Mor05, §5]. The map f_* sends a space $\mathcal{X}$ over R to the space $\mathcal{X} \circ f^{-1}$ over S.

The adjoint f^* is given by the formula $(f^*\mathcal{Y})(U) = \operatorname*{colim}_{U \to f^{-1}V} \mathcal{Y}(V)$, as described in [HW14, §12.1]. For a smooth scheme X over S, a standard calculation shows $f^*X = f^{-1}X$. Additionally, if cA is a constant simplicial presheaf on Sm/S, it follows that $f^*(cA) = cA$.

The Quillen adjunction (f^*, f_*) extends to both the model category of T-spectra and $(\mathbb{G}_m, S^1)$ bispectra by applying the maps f^*, and respectively f_*, term-wise to a given spectrum. In the case of f^* for T-spectra, for instance, the bonding maps for f^*E are given by

$$T \wedge f^*E_n \xrightarrow{\cong} f^*(T \wedge E_n) \to f^*(E_{n+1})$$

as $f^*T = T$. The same reasoning shows that the adjunction (f^*, f_*) extends to $(\mathbb{G}_m, S^1)$ bispectra.

Write Q (respectively R) for the cofibrant (respectively fibrant) replacement functor in $\mathrm{Spt}_T(S)$. The derived functors $\mathbb{L}f^*$ and $\mathbb{R}f_*$ are given by the formulas $\mathbb{L}f^* = f^*Q$ and $\mathbb{R}f_* = f_*R$.

Let $f : C \to B$ be a smooth map. The functor $f_\# : \mathrm{Sm}/C \to \mathrm{Sm}/B$ sends a scheme $X \to C$ to $X \to C \xrightarrow{f} B$, and induces a functor $f_\# : \mathrm{Spc}_*^{\mathbb{A}^1}(B) \to \mathrm{Spc}_*^{\mathbb{A}^1}(C)$ by restricting a presheaf on Sm/B to a presheaf on Sm/C. The functor f^* is canonically equivalent to $f_\#$ on the level of spaces and spectra.

2.6 The connectivity theorem

Morel establishes the connectivity of the sphere spectrum $\mathbb{1}$ over fields F by studying the effect of Bousfield localization at $W_{\mathbb{A}^1}$ of the stable Nisnevich local model category structure on $\mathrm{Spt}_{S^1}(\mathrm{Spec}(F))$ (see definition 2.12).

An S^1-spectrum E over S is said to be *simplicially k-connected* if for any $n \leq k$, the simplicial stable homotopy sheaves $\pi_n E$ of definition 2.12 are trivial. An S^1-spectrum E is *k-connected* if for all $n \leq k$ the motivic stable homotopy sheaves $\pi_n^{\mathbb{A}^1} E$ are trivial.

Theorem 2.16 (Morel's connectivity theorem)**.** If E is an S^1-spectrum over S which is simplicially k-connected, then E is also k-connected.

Proof. When F is an infinite field, the argument given in [Mor12] goes through. When

F is a finite field, Morel's argument relies on a proof of Gabber's presentation lemma in the case of one point for a finite field (see [Mor12, 1.15] for the statement). A letter of Gabber to Morel [Gab15] establishes this case of Gabber's presentation lemma, and so the connectivity theorem holds without qualification on the field F. $\square$

The connectivity theorem along with the work in [Mor04, §5] yield the following. This also follows from [Voe98, 4.14].

Corollary 2.17. Over a field F, the sphere spectrum $\mathbb{1}$ is (-1)-connected. In particular, for all $s - w < 0$ the groups $\pi_{s,w}(F)$ are trivial.

2.7 Comparison to the classical stable homotopy category

The following result of Levine plays a fundamental role in our calculations [Lev14, Thm. 1].

Theorem 2.18. If $S = \operatorname{Spec}(\mathbb{C})$, the map $\mathbb{L}c : \mathcal{SH} \to \mathcal{SH}_S$ is fully faithful.

Proposition 2.19. Let $f : R \to S$ be a map of base schemes. The following diagram of stable homotopy categories commutes.

$$\begin{tikzcd} & \mathcal{SH} \arrow[dl, "\mathbb{L}c"'] \arrow[dr, "\mathbb{L}c"] & \\ \mathcal{SH}_S \arrow[rr, "\mathbb{L}f^*"] & & \mathcal{SH}_R \end{tikzcd}$$

Proof. The result follows by establishing $f^* \circ c = c$ on the level of model categories. For a constant space $cA \in \operatorname{Spc}(S)$, we have $f^*cA = cA$ by the calculation

$$(f^*cA)(U) = \operatorname*{colim}_{U \to f^{-1}V} cA(V) = A$$

given the formula for f^* in section 2.5. As the base change map is extended to T-spectra by applying f^* term-wise, the claim follows. $\square$

Proposition 2.20. Let S be a base scheme equipped with a map $\operatorname{Spec}(\mathbb{C}) \to S$. Then $\mathbb{L}c : \mathcal{SH} \to \mathcal{SH}_S$ is faithful.

Proof. For symmetric spectra X and Y, the map $\mathbb{L}c : \mathcal{SH}(X, Y) \to \mathcal{SH}_{\mathbb{C}}(cX, cY)$ factors through $\mathcal{SH}_S(cX, cY)$ by proposition 2.19. Hence $\mathbb{L}c : \mathcal{SH}(X, Y) \to \mathcal{SH}_S(cX, cY)$ is injective by theorem 2.18. $\square$

Chapter 3

Motivic cohomology

Spitzweck has constructed a spectrum $H\mathbb{Z}$ in $\mathrm{Spt}_T^{\Sigma}(S)$ which represents motivic cohomology $H^{a,b}(X;\mathbb{Z})$ defined using Bloch's cycle complex when S is the Zariski spectrum of a Dedekind domain [Spi13]. Spitzweck establishes enough nice properties of $H\mathbb{Z}$ so that we may construct the motivic Adams spectral sequence over general base schemes and establish comparisons between the motivic Adams spectral sequence over a Hensel local ring in which ℓ is invertible and its residue field.

3.1 Integral motivic cohomology

Definition 3.1. Over the base scheme $\mathrm{Spec}(\mathbb{Z})$, the spectrum $H\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ is defined by Spitzweck in [Spi13, 4.27]. For a general base scheme S, we define $H\mathbb{Z}_S$ to be $f^*H\mathbb{Z}_{\mathrm{Spec}(\mathbb{Z})}$ where $f: S \to \mathrm{Spec}(\mathbb{Z})$ is the unique map.

Let S be the Zariski spectrum of a Dedekind domain D. For $X \in \mathrm{Sm}/S$, there is a canonical isomorphism $\mathcal{SH}_S(\Sigma^\infty X_+, \Sigma^{i,n}H\mathbb{Z}) \cong H^{a,b}(X;\mathbb{Z})$, where $H^{a,b}(-;\mathbb{Z})$ denotes motivic cohomology [Spi13, 7.19]. The isomorphism is functorial with respect to maps in Sm/S. Additionally, if $i: \{s\} \to S$ is the inclusion of a closed point with residue field $k(s)$, there is a commutative diagram for $X \in \mathrm{Sm}/S$.

$$\begin{tikzcd}
\mathcal{SH}_S(\Sigma^\infty X_+, \Sigma^{i,n}H\mathbb{Z}) \arrow[r, "\cong"] \arrow[d] & H^{a,b}(X;\mathbb{Z}) \arrow[d] \\
\mathcal{SH}_{k(s)}(\mathbb{L}i^*\Sigma^\infty X_+, \Sigma^{a,b}H\mathbb{Z}) \arrow[r, "\cong"] & H^{a,b}(\mathbb{L}i^*X, \mathbb{Z})
\end{tikzcd}$$

If the residue field $k(s)$ has positive characteristic, there is a canonical isomorphism of ring spectra $\mathbb{L}i^*H\mathbb{Z}_S \cong H\mathbb{Z}_{k(s)}$ [Spi13, 9.16].

Proposition 3.2. If $f: R \to S$ is a smooth map of base schemes, then $\mathbb{L}f^*H\mathbb{Z}_S \cong H\mathbb{Z}_R$.

Proof. Since f is smooth, $\mathbb{L}f^* = f^*$ by the discussion in [Mor05, p. 44]. The result now follows, as it is straightforward to see that $f^* H\mathbb{Z}_S \cong H\mathbb{Z}_R$. $\square$

3.2 Motivic cohomology with coefficients $\mathbb{Z}/\ell$

For a prime ℓ, write $H\mathbb{Z}/\ell$ for the cofiber of the map $\ell\cdot : H\mathbb{Z} \to H\mathbb{Z}$ in $\mathcal{SH}_S$. The spectrum $H\mathbb{Z}/\ell$ represents motivic cohomology with $\mathbb{Z}/\ell$ coefficients. For a smooth scheme X over S, we write $H^{**}(X;\mathbb{Z}/\ell)$ for the motivic cohomology of X with $\mathbb{Z}/\ell$ coefficients. If S is the Zariski spectrum of a ring R, we will write $H^{**}(R;\mathbb{Z}/\ell)$ for the motivic cohomology of R. In this section, we calculate $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$ when q and ℓ are relatively prime.

The now established Beilinson-Lichtenbaum conjecture gives a ring homomorphism $H^{a,b}(\mathbb{F}_q;\mathbb{Z}/\ell) \to H^a_{et}(\mathbb{F}_q;\mu_\ell^{\otimes b})$ which is an isomorphism when $a \leq b$; when $a > b$, the groups $H^{a,b}(\mathbb{F}_q;\mathbb{Z}/\ell)$ vanish [MVW06, 3.6]. Furthermore, $H^{n,n}(F;\mathbb{Z}/\ell) \cong K^M_n(F)/\ell$ where $K^M_n(F)$ denotes the Milnor K-theory of F. In particular, $K^M_1(F) = F^\times$ is the group of units of F and $K^M_1(F)/\ell = F^\times/F^{\times\,\ell}$ is the group of units of F modulo ℓth powers.

We will also identify the action of the Bockstein homomorphism β on $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$. The Bockstein homomorphism is the connecting homomorphism in the long exact sequence associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z}/\ell \to \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0.$$

Definition 3.3. We set some notation which will be used in our description of the mod ℓ motivic cohomology of finite fields. For a field F with characteristic different from 2, $-1 \in \mu_2(F)$ is a non-trivial second root of unity. We write τ for the class corresponding to -1 via the isomorphism $H^{0,1}(F;\mathbb{Z}/2) \cong \mu_2(F)$ and ρ for the class corresponding to -1 in $H^{1,1}(F;\mathbb{Z}/2) \cong F^\times/F^{\times\,2}$. Note that the class ρ is trivial if and only if $\sqrt{-1} \in F$. For a finite field $\mathbb{F}_q$ of odd order, the group of units $\mathbb{F}_q^\times$ is cyclic, there is an isomorphism, $H^{1,1}(\mathbb{F}_q;\mathbb{Z}/2) \cong \mathbb{Z}/2$, and we write u for the non-trivial class of $H^{1,1}(\mathbb{F}_q;\mathbb{Z}/2)$. We remark that $u = \rho$ if and only if $q \equiv 3 \bmod 4$.

For a prime $\ell > 2$, write ζ for a primitive ℓth root of unity in a field F, should the field F have one. If F has a primitive ℓth root of unity ζ, we write γ for the class of ζ in

$F^\times/F^{\times\,\ell}$, which is trivial if and only if $\sqrt[\ell]{\zeta} \in F$.

A finite field $\mathbb{F}_q$ will contain an ℓth root of unity if and only $q \equiv 1 \bmod \ell$. For a general finite field $\mathbb{F}_q$, let i be the smallest positive integer for which $q^i \equiv 1 \bmod \ell$. Then the field extension $\mathbb{F}_{q^i}$ contains an ℓth root of unity, and we write ζ for the class corresponding to a primitive ℓth root of unity in $H^{0,i}(\mathbb{F}_q;\mathbb{Z}/\ell) \cong \mu_\ell(\mathbb{F}_{q^i})$. We write γ for the class corresponding to the primitive ℓth root of unity ζ in $H^{1,i}(\mathbb{F}_q;\mathbb{Z}/\ell) \cong H^1_{et}(\mathbb{F}_q;\mu_\ell^{\otimes i}) \cong \mathbb{F}_{q^i}^\times/\mathbb{F}_{q^i}^{\times\,\ell}$, and u for a generator of $H^{1,i}(\mathbb{F}_q;\mathbb{Z}/\ell)$. Note that γ is trivial in $H^{1,i}(\mathbb{F}_q;\mathbb{Z}/\ell)$ if and only if $q^i \equiv 1 \bmod \ell^2$. When $q^i \not\equiv 1 \bmod \ell^2$, γ is non-trivial and we take u to be γ.

Proposition 3.4. 1. For any finite field $\mathbb{F}_q$ with q odd, there are isomorphisms

$$H^{**}(\mathbb{F}_q;\mathbb{Z}/2) \cong \begin{cases} \mathbb{F}_2[\tau,u]/(u^2) & \text{if } q \equiv 1 \bmod 4 \\ \mathbb{F}_2[\tau,\rho]/(\rho^2) & \text{if } q \equiv 3 \bmod 4. \end{cases}$$

The action of the Bockstein is determined by $\beta(\tau) = \rho$ which is trivial if and only if $q \equiv 1 \bmod 4$. The bidegree of τ is $(0,1)$ and the bidegree of ρ and u is $(1,1)$.

2. Suppose ℓ is an odd prime, $\mathbb{F}_q$ is a finite field with characteristic different from ℓ, and let i be the smallest positive integer for which $\ell \mid q^i - 1$. The motivic cohomology of $\mathbb{F}_q$ is the associative, graded-commutative $\mathbb{Z}/\ell$-algebra

$$H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell) \cong \begin{cases} \mathbb{F}_2[\zeta,u]/(u^2) & \text{if } q^i \equiv 1 \bmod \ell^2 \\ \mathbb{F}_2[\zeta,\gamma]/(\gamma^2) & \text{if } q^i \not\equiv 1 \bmod \ell^2 \end{cases}$$

The action of the Bockstein is determined by $\beta(\zeta) = \gamma$, which is trivial if and only if $q^i \equiv 1 \bmod \ell^2$. The bidegree of ζ is $(0,i)$ and the bidegree of u and γ is $(1,i)$.

Proof. Because of the Beilinson-Lichtenbaum conjecture, we reduce the problem to calculating $H^p_{et}(\mathbb{F}_q;\mu_\ell^{\otimes q})$ when $p \leq q$. A calculation in Galois cohomology by Soulé in [Sou79, III.1.4] shows that $H^0_{et}(\mathbb{F}_q;\mu_\ell^{\otimes j})$ and $H^1_{et}(\mathbb{F}_q;\mu_\ell^{\otimes j})$ are the cyclic group of order $\gcd(\ell, q^j - 1)$, and all higher cohomology groups vanish. When $\ell \neq 2$, the sheaf $\mu_\ell^{\otimes i}$ is the constant sheaf $\mathbb{Z}/\ell$, and so the products

$$H^0_{et}(\mathbb{F}_q;\mu_\ell^{\otimes b}) \otimes H^a_{et}(\mathbb{F}_q;\mu_\ell^{\otimes c}) \to H^a_{et}(\mathbb{F}_q;\mu_\ell^{\otimes b+c})$$

are isomorphisms.

For $x, y \in H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$, the Bockstein homomorphism is a derivation, i.e.,

$$\beta(xy) = \beta(x)y + (-1)^{|x|}x\beta(y)$$

where $|x|$ denotes the topological degree [Voe03, (8.1)]. It then suffices to identify the action of the Bockstein on τ or ζ.

The Bockstein fits into the following exact sequence in weight i.

$$\begin{tikzcd}
H^0_{\text{et}}(\mathbb{F}_q; \mu_{\ell^2}^{\otimes i}) \arrow[r] \arrow[d, "\cong"] & H^0_{\text{et}}(\mathbb{F}_q; \mu_{\ell}^{\otimes i}) \arrow[r, "\beta"] \arrow[d, "\cong"] & H^1_{\text{et}}(\mathbb{F}_q; \mu_{\ell^2}^{\otimes i}) \arrow[d, "\cong"] \\
\mu_{\ell^2}(\mathbb{F}_{q^i}) \arrow[r, "-^{\ell}"] & \mu_{\ell}(\mathbb{F}_{q^i}) \arrow[r] & \mathbb{F}_{q^i}^{\times}/\mathbb{F}_{q^i}^{\times\,\ell}
\end{tikzcd}$$

Hence the Bockstein then is trivial if and only if $\mathbb{F}_{q^i}$ contains a primitive ℓ^2 root of unity. This occurs if and only if $q^i \equiv 1 \bmod \ell^2$. □

We remark that in the case of a finite field $\mathbb{F}_q$, the element $\zeta \in \mu_\ell(\mathbb{F}_{q^i})$ is the analog of $\tau \in \mu_2(\mathbb{F}_q)$ at an odd prime ℓ. Furthermore, $\gamma \in \mathbb{F}_{q^i}^{\times}/\mathbb{F}_{q^i}^{\times\,\ell}$ is the analog of the class $\rho \in \mathbb{F}_q^{\times}/\mathbb{F}_q^{\times\,2}$ at an odd prime ℓ. We use distinct notation to avoid confusion with the specific role τ and ρ play in Voevodsky's calculation of the motivic Steenrod algebra and its dual in [Voe03].

We will make frequent use of Geisser's rigidity theorem for motivic cohomology [Gei04, 1.2(3)], which we adapt to our needs.

Proposition 3.5. Let D be a Hensel local ring in which ℓ is invertible. Write F for the residue field of D and write $\pi : D \to F$ for the quotient map. Then the map $\pi^* : H^{**}(D; \mathbb{Z}/\ell) \to H^{**}(F; \mathbb{Z}/\ell)$ is an isomorphism of $\mathbb{Z}/\ell$-algebras. Furthermore, the action of the Bockstein is the same in either case.

Proof. The rigidity theorem for motivic cohomology [Gei04, 1.2(3)] gives the isomorphism. The map $\mathbb{L}\pi^*$ gives comparison maps for the long exact sequences which define the Bockstein over D and F. The rigidity theorem shows that the long exact sequences are isomorphic, so the action of the Bockstein is the same in either case. □

3.3 Mod ℓ motivic cohomology operations of finite fields

The *mod ℓ motivic Steenrod algebra* over a base scheme S, which we write as $\mathcal{A}^{**}(S)$, is the algebra of bistable mod ℓ motivic cohomology operations. A *bistable cohomology operation* is a family of operations $\theta_{**} : H^{**}(-;\mathbb{Z}/\ell) \to H^{*+a,*+b}(-;\mathbb{Z}/\ell)$ which are compatible with the suspension isomorphism for both the simplicial circle S^1 and the Tate circle $\mathbb{G}_m$.

In the case where S is the Zariski spectrum of a characteristic 0 field, Voevodsky identified the structure of this algebra in [Voe03, Voe10]. Voevodsky's calculation was extended to hold where the base is the Zariski spectrum of a field of positive characteristic $p \neq \ell$ by Hoyois, Kelly, and Østvær in [HKØ13]. In both cases, the structure of the algebra of mod ℓ motivic cooperations was also identified. We now adapt these calculations to the particular case where the base scheme is $\mathrm{Spec}(\mathbb{F}_q)$.

Proposition 3.6. The mod 2 motivic Steenrod algebra $\mathcal{A}^{**}(\mathbb{F}_q)$ over $\mathbb{F}_q$ with q odd is the associative $\mathbb{Z}/2$-algebra generated by the Steenrod square operations Sq^i for $i \geq 1$ and the cup products $x \cup -$ for $x \in H^{**}(\mathbb{F}_q;\mathbb{Z}/2)$. The Steenrod square operation Sq^{2i} has bidegree $(2i, i)$ and Sq^{2i+1} has bidegree $(2i+1, i)$. The operation Sq^1 agrees with the Bockstein β. The Steenrod square operations Sq^i satisfy modified Adem relations which are listed below. We include these formulas for completeness, as they are explicitly used in the computer calculations discussed in chapter 8. See definition 3.3 for our notation of the elements τ, ρ, and u in $H^{**}(\mathbb{F}_q;\mathbb{Z}/2)$

1. (Cartan formula): For cohomology classes x and y, the following relations hold.

$$\mathrm{Sq}^1(xy) = \mathrm{Sq}^1(x)y + x\mathrm{Sq}^1(y)$$
$$\mathrm{Sq}^{2i}(xy) = \sum_{r=0}^{i} \mathrm{Sq}^{2r}(x)\mathrm{Sq}^{2i-2r}(y) + \tau \sum_{r=0}^{i-1} \mathrm{Sq}^{2r+1}(x)\mathrm{Sq}^{2i-2r-1}(y)$$

$$\mathrm{Sq}^{2i+1}(xy) = \sum_{r=0}^{i} \left(\mathrm{Sq}^{2r+1}(x)\mathrm{Sq}^{2i-2r}(y) + \mathrm{Sq}^{2r}(x)\mathrm{Sq}^{2i-2r+1}(y)\right) + \rho \sum_{r=0}^{i-1} \mathrm{Sq}^{2r+1}(x)\mathrm{Sq}^{2i-2r-1}(y)$$

2. (Adem relations): If $a+b \equiv 0 \bmod 2$, then

$$\mathrm{Sq}^a\mathrm{Sq}^b = \begin{cases} \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j & a, b \text{ odd} \\ \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \tau^j \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j & a, b \text{ even.} \end{cases}$$

If a is odd and b is even, then

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{\substack{j=0 \\ j \text{ even}}}^{[a/2]} \binom{b-1-j}{a-1-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j + \sum_{\substack{j=0 \\ j \text{ odd}}}^{[a/2]} \binom{b-1-j}{a-1-2j} \rho \mathrm{Sq}^{a+b-1-j}\mathrm{Sq}^j.$$

If a is even and b is odd

$$\mathrm{Sq}^a\mathrm{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j}\mathrm{Sq}^j + \sum_{\substack{j=1 \\ j \text{ odd}}}^{[a/2]} \binom{b-1-j}{a-2j} \rho \mathrm{Sq}^{a+b-j-1}\mathrm{Sq}^j.$$

3. (Right multiplication by $H^{*,*}$):

$$\mathrm{Sq}^1\tau = \tau\mathrm{Sq}^1 + \rho$$

For $i > 0$, one can derive the following formulas from the Cartan relation.

$$\mathrm{Sq}^{2i}\tau = \tau\mathrm{Sq}^{2i} + \tau\rho\mathrm{Sq}^{2i-1}$$

$$\mathrm{Sq}^{2i+1}\tau = \tau\mathrm{Sq}^{2i+1} + \rho\mathrm{Sq}^{2i}$$

For all $a > 0$,

$$\mathrm{Sq}^a\rho = \rho\mathrm{Sq}^a \text{ and } \mathrm{Sq}^a u = u\mathrm{Sq}^a.$$

Proposition 3.7. Let ℓ be an odd prime and suppose $\mathbb{F}_q$ is a finite field of characteristic p with $p \neq \ell$. The mod ℓ Steenrod algebra is an associative $\mathbb{Z}/\ell$-algebra generated by the Bockstein β, the reduced power operations P^i for $i \geq 1$, and the cup products $x \cup -$ for $x \in H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$ subject to the usual Adem relations [Voe03, 10.3].

In the case where $\beta(\zeta) = \gamma$, we have the additional relations $\beta\zeta = \zeta\beta + \gamma$, $\beta\gamma = \gamma\beta$, and the reduced power operations commute with cup products. If $\beta(\zeta) = 0$, the reduced power operations and the Bockstein commute with cup products.

3.4 The Hopf algebroid of cooperations for $H\mathbb{Z}/\ell$

The dual Steenrod algebra $\mathcal{A}_{**} = \mathrm{Hom}_{H^{**}}(\mathcal{A}^{**}, H^{**})$ has the structure of a Hopf algebroid which was identified over fields of characteristic 0 by Voevodsky in [Voe03, 12.6] and over fields of positive characteristic by Hoyois, Kelly, and Østvær in [HKØ13, 5.5]. A homogeneous homomorphism $f : \mathcal{A}^{**} \to H^{**}$ is said to have bidegree (a, b) if it decreases bidegree by (a, b).

For $x \in H^{a,b}$, consider x also as the element of $\mathcal{A}_{-a,-b}$ which is the left H^{**}-module homomorphism which kills the Steenrod squares, respectively the reduced power operations and the Bockstein, and maps 1 to x. Given the isomorphism $H^{a,b} \cong H_{-a,-b}$, this construction defines a homomorphism $\eta_L : H_{**} \to \mathcal{A}_{**}$.

For a prime $\ell > 2$, write $\mathcal{A}_*^{top}$ for the topological dual Steenrod algebra. Milnor studied the structure of $\mathcal{A}_*^{top}$ in [Mil58] and found that $\mathcal{A}_*^{top}$ is the graded-commutative $\mathbb{F}_\ell$-algebra $\mathbb{F}_\ell[\tau_i, \xi_j \,|\, i \geq 0, j \geq 1]/(\tau_i^2)$ where the degree of τ_j is $2\ell^j - 1$ and the degree of ξ_j is $2(\ell^j - 1)$. We may also give $\mathcal{A}_*^{top}$ a second grading by declaring the weight of τ_j and ξ_j to be $\ell^j - 1$. Furthermore, Milnor identified the Hopf algebra structure of $(\mathbb{F}_\ell, \mathcal{A}_*^{top})$. We now record the structure of the motivic dual Steenrod algebra at a prime ℓ over a finite field, which is due to [HKØ13], cf. [Voe03].

Proposition 3.8. 1. The mod 2 dual Steenrod algebra $\mathcal{A}_{**}(\mathbb{F}_q)$ for a finite field $\mathbb{F}_q$ of characteristic different from 2 is an associative, commutative algebra of the following form.

$$\mathcal{A}_{**}(\mathbb{F}_q) \cong H_{**}(F)[\tau_i, \xi_j \,|\, i \geq 0, j \geq 1]/(\tau_i^2 - \tau\xi_{i+1} - \rho\tau_{i+1} - \rho\tau_0\xi_{i+1})$$

Here τ_i has bidegree $(2^{i+1} - 1, 2^i - 1)$ and ξ_i has bidegree $(2^{i+1} - 2, 2^i - 1)$.

The structure maps for the Hopf algebroid $(H_{**}(\mathbb{F}_q), \mathcal{A}_{**}(\mathbb{F}_q))$, which we write simply as $(H_{**}, \mathcal{A}_{**})$, are as follows.

(a) The left unit $\eta_L : H_{**} \to \mathcal{A}_{**}$ is given by $\eta_L(x) = x$.

(b) The right unit $\eta_R : H_{**} \to \mathcal{A}_{**}$ is determined as a map of $\mathbb{Z}/2$-algebras by $\eta_R(\rho) = \rho$ and $\eta_R(\tau) = \tau + \rho\tau_0$. In the case where ρ is trivial, i.e., $q \equiv 1 \bmod 4$, the right and left unit agree $\eta_R = \eta_L$.

(c) The augmentation $\epsilon : \mathcal{A}_{**} \to H_{**}$ kills τ_i and ξ_i, and for $x \in H_{**}$, $\epsilon(x) = x$.

(d) The coproduct $\Delta : \mathcal{A}_{**} \to \mathcal{A}_{**} \otimes_{H_{**}} \mathcal{A}_{**}$ is a map of graded $\mathbb{Z}/2$-algebras determined by $\Delta(x) = x \otimes 1$ for $x \in H_{**}$, $\Delta(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i + \sum_{j=0}^{i-1} \xi_{i-j}^{\ell^j} \otimes \tau_j$, $\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i + \sum_{j=1}^{i-1} \xi_{i-j}^{\ell^j} \otimes \xi_j$.

(e) The antipode c is a map of $\mathbb{Z}/2$-algebras determined by $c(\rho) = \rho$, $c(\tau) = \tau + \rho\tau_0$, $c(\tau_i) = -\tau_i - \sum_{j=0}^{i-1} \xi_{i-j}^{\ell^j} c(\tau_j)$, and $c(\xi_i) = -\xi_i - \sum_{j=1}^{i-1} \xi_{i-j}^{\ell^j} c(\xi_j)$.

2. Let ℓ be an odd prime and $\mathbb{F}_q$ a finite field with characteristic different from ℓ. The mod ℓ dual Steenrod algebra over $\mathbb{F}_q$ is the associative, graded-commutative algebra (in the first index) given by $\mathcal{A}_{**}(\mathbb{F}_q) \cong H_{**}(\mathbb{F}_q) \otimes_{\mathbb{F}_\ell} \mathcal{A}_*^{top}$. Furthermore, $(H_{**}, \mathcal{A}_{**})$ is a Hopf algebroid, and the structure maps are as follows.

(a) The left unit $\eta_L : H_{**} \to \mathcal{A}_{**}$ is given by $\eta_L(x) = x$.

(b) The right unit $\eta_R : H_{**} \to \mathcal{A}_{**}$ is determined as a map of $\mathbb{Z}/\ell$-algebras by $\eta_R(\zeta) = \zeta - \gamma\tau_0$ and $\eta_R(\gamma) = \gamma$. In the case where the Bockstein acts trivially on H_{**}, the right and left unit agree $\eta_R = \eta_L$.

(c) The augmentation $\epsilon : \mathcal{A}_{**} \to H_{**}$ kills τ_i and ξ_i, and $\epsilon(x) = x$ for $x \in H_{**}$.

(d) The coproduct $\Delta : \mathcal{A}_{**} \to \mathcal{A}_{**} \otimes_{H_{**}} \mathcal{A}_{**}$ is a map of $\mathbb{Z}/\ell$-algebras determined by $\Delta(x) = x \otimes 1$ for $x \in H_{**}$, $\Delta(\tau_i) = \tau_i \otimes 1 + 1 \otimes \tau_i + \sum_{j=0}^{i-1} \xi_{i-j}^{\ell^j} \otimes \tau_j$, $\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i + \sum_{j=1}^{i-1} \xi_{i-j}^{\ell^j} \otimes \xi_j$. Note that in $\mathcal{A}_{**} \otimes \mathcal{A}_{**}$, the product is given by $(x \otimes x')(y \otimes y') = (-1)^{|x'||y|} xy \otimes x'y'$.

(e) The antipode c is a map of $\mathbb{Z}/\ell$-algebras determined by $c(\gamma) = \gamma$, $c(\zeta) = \zeta + \gamma\tau_0$, $c(\tau_i) = -\tau_i - \sum_{j=0}^{i-1} \xi_{i-j}^{\ell^j} c(\tau_j)$, and $c(\xi_i) = -\xi_i - \sum_{j=1}^{i-1} \xi_{i-j}^{\ell^j} c(\xi_j)$.

3.5 Steenrod algebra over Dedekind domains

Let S be the Zariski spectrum of a Dedekind domain D in which ℓ is invertible. Spitzweck shows in [Spi13, 11.24] that for an odd prime ℓ, the mod ℓ dual Steenrod algebra with $\mathbb{Z}/\ell$ coefficients $\mathcal{A}_{**}(D)$ has the structure of a Hopf algebroid similar to that of proposition 3.8. At the prime 2, Spitzweck's result shows $\mathcal{A}_{**}(D)$ is generated by the elements τ_i and ξ_j, but does not establish the relations for τ_i^2.

Definition 3.9. Let D be a Dedekind domain, and let C denote the set of sequences $(\epsilon_0, r_1, \epsilon_1, r_2, \ldots)$ with $\epsilon_i \in \{0,1\}$, $r_i \geq 0$, and only finitely many non-zero terms. The elements $\tau_i \in \mathcal{A}_{2\ell^i-1,\ell^i-1}(D)$ and $\xi_i \in \mathcal{A}_{2\ell^i-2,\ell^i-1}(D)$ are constructed in [Spi13, 11.23]. For any sequence $I = (\epsilon_0, r_1, \epsilon_1, r_2, \ldots)$ in C, write $\omega(I)$ for the element $\tau_0^{\epsilon_0}\xi_1^{r_1}\cdots$ and $(a(I), b(I))$ for the bidegree of the operation $\omega(I)$.

In [Spi13, 11.24] Spitzweck identifies the structure of the mod ℓ dual Steenrod algebra, which we record here.

Proposition 3.10. Let D be a Dedekind domain. As an $H\mathbb{Z}/\ell$-module, there is a weak equivalence $\bigvee_{I\in C}\Sigma^{a(I),b(I)}H\mathbb{Z}/\ell \to H\mathbb{Z}/\ell \wedge H\mathbb{Z}/\ell$. The map is given by $\omega(I)$ on the factor $\Sigma^{a(I),b(I)}H\mathbb{Z}/\ell$.

For $\ell > 2$, this establishes the isomorphism $\mathcal{A}_{**}(D) \cong H_{**}(D) \otimes_{\mathbb{F}_\ell} \mathcal{A}_*^{top}$ as an associative, graded-commutative $\mathbb{Z}/\ell$-algebra.

In the case $\ell = 2$, one must be careful about the relations for τ_i^2 in $\mathcal{A}_{**}$. In particular, we need the analog of [Voe03, 6.10]. We give an argument when D is a Hensel local ring.

Proposition 3.11. Let D be a Hensel local ring in which 2 is invertible and let F denote the residue field of D. Then the following isomorphism holds.

$$H^{**}(B\mu_2, \mathbb{Z}/2) \cong H^{**}(D, \mathbb{Z}/2)[[u,v]]/(u^2 = \tau v + \rho u)$$

Here $\rho \in H^{1,1}(D;\mathbb{Z}/2) \cong F^\times/F^{\times 2}$ and $\tau \in H^{0,1}(D;\mathbb{Z}/2) \cong \mu_2(F)$ are given in definition 3.3. Further, v is the class $v_2 \in H^{2,1}(B\mu_2)$ defined in [Spi13, p. 81], and $u \in H^{1,1}(B\mu_2;\mathbb{Z}/2)$ is the unique class satisfying $\tilde{\beta}(u) = v$, where $\tilde{\beta}$ is the integral Bockstein determined by the coefficient sequence $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2$.

Proof. The motivic classifying space $B\mu_2$ over D (respectively F) fits into a triangle $B\mu_{2+} \to (\mathcal{O}(-2)_{\mathbb{P}^\infty})_+ \to \mathrm{Th}(\mathcal{O}(-2))$ by [Voe03, (6.2)] and [Spi13, (25)]. From this triangle, we obtain a long exact sequence in mod 2 motivic cohomology [Voe03, (6.3)] and [Spi13, (26)]. The comparison map $\mathbb{L}\pi^* : \mathcal{SH}_D \to \mathcal{SH}_F$ induces a homomorphism of these long exact sequences. The rigidity theorem 3.5 and the 5-lemma then show that the comparison maps are all isomorphisms. As the desired relation holds in the motivic cohomology of $B\mu_2$ over F, and the choices of u and v are compatible with base change, the result follows. □

Proposition 3.12. Let D be a Hensel local ring in which ℓ is invertible. The coproduct Δ for $\mathcal{A}_{**}(D)$ is as given in proposition 3.8(d).

Proof. This follows from the calculation in [Spi13, 11.23]. □

All that remains to identify the structure of $\mathcal{A}_{**}(D)$ is the right unit and the antipode.

Proposition 3.13. Over a Hensel local ring in which ℓ is invertible, the right unit η_R and the antipode c are given by the formulas in proposition 3.8(b,e).

Proof. Naturality of the reduced power operations guarantees that the actions on $H^{**}(D)$ and $H^{**}(F)$ agree, since the cohomology groups are isomorphic. This additional structure determines the right unit and the antipode. □

Remark 3.14. Let D be a Dedekind domain in which ℓ is invertible, and consider the map $f : \mathbb{Z}[1/\ell] \to D$. A key observation in the proof of [Spi13, 11.24] is that $f^* : \mathcal{A}_{**}(\mathbb{Z}[1/\ell]) \to \mathcal{A}_{**}(D)$ satisfies $f^*\tau_i = \tau_i$ and $f^*\xi_i = \xi_i$ for all i. For a map $j : D \to \widetilde{D}$ of Dedekind domains in which ℓ is invertible, it follows that $j^*\tau_i = \tau_i$ and $j^*\xi_i = \xi_i$ for all i.

Proposition 3.15. Let D be a Hensel local ring in which ℓ is invertible and let F denote the residue field of D. Then the comparison map $\pi^* : \mathcal{A}_{**}(D) \to \mathcal{A}_{**}(F)$ is an isomorphism of Hopf algebroids.

Proof. Remark 3.14 shows that the map $\pi^* : \mathcal{A}_{**}(D) \to \mathcal{A}_{**}(F)$ is an isomorphism of left $H_{**}(F)$-modules. The compatibility of the isomorphism with the coproduct, right unit, and antipode follows from propositions 3.12 and 3.13. □

Definition 3.16. A set of bigraded objects $A = \{x_{(a,b)}\}$ is said to be *motivically finite* [DI10, 2.11] if for any bigrading (a, b) there are only finitely many objects $y_{(a',b')} \in X$ for which $a \geq a'$ and $2b - a \geq 2b' - a'$.

We say a bigraded algebra or module is motivically finite if it has a generating set which is motivically finite. One benefit is that a motivically finite $H^{**}(X)$-module is a finite dimensional $\mathbb{Z}/\ell$-vector space in each bidegree.

Proposition 3.17. Let D be a Dedekind domain in which ℓ is invertible. The Steenrod algebra over D is isomorphic to the dual of $\mathcal{A}_{**}(D)$, that is,

$$\mathcal{A}^{**}(D) \cong \mathrm{Hom}_{H_{**}}(\mathcal{A}_{**}(D), H\mathbb{Z}/\ell_{**}(D)).$$

Proof. As the algebra of cooperations $\mathcal{A}_{**}(S)$ is motivically finite, we may identify its dual with the Steenrod algebra. See [HKØ13, 5.2] and [Spi13, 11.25]. □

Corollary 3.18. Let D be a Hensel local ring in which ℓ is invertible with residue field F. The comparison map $\pi^* : \mathcal{A}^{**}(D) \to \mathcal{A}^{**}(F)$ is an isomorphism.

3.6 Base change for finite fields

Proposition 3.19. Consider a prime power $q = p^n$ and let ℓ be a prime different from p. For a field extension $f : \mathbb{F}_q \to \mathbb{F}_{q^j}$ where j is relatively prime to $\ell(\ell-1)$, the induced map $f^* : H^{**}(\mathbb{F}_q) \to H^{**}(\mathbb{F}_{q^j})$ and $f^* : \mathcal{A}^{**}(\mathbb{F}_q) \to \mathcal{A}^{**}(\mathbb{F}_{q^j})$ are isomorphisms.

Proof. Let i be the smallest positive integer for which $q^i \equiv 1 \bmod \ell$. In other words, i is the order of q in $\mathbb{F}_\ell^\times$. Since j is relatively prime to $\ell - 1$, the integers q and q^j have the same order in $\mathbb{F}_\ell^\times$. By proposition 3.4, we see that there are isomorphisms $H^{*,*}(\mathbb{F}_q; \mathbb{Z}/\ell) \cong H^{*,*}(\mathbb{F}_{q^j}; \mathbb{Z}/\ell)$. We first show $f^* : H^{**}(\mathbb{F}_q) \to H^{**}(\mathbb{F}_{q^j})$ is an isomorphism by using the presentation in proposition 3.4. The map on mod ℓ motivic cohomology is determined by its behavior on $H^0_{et}(\mathbb{F}_q; \mu_\ell^{\otimes i}) \cong \mu_\ell(\mathbb{F}_{q^i})$ and $H^1_{et}(\mathbb{F}_q; \mu_\ell^{\otimes i}) \cong \mathbb{F}_{q^i}^\times / \mathbb{F}_{q^i}^{\times\,\ell}$. The map $\mu_\ell(\mathbb{F}_{q^i}) \to \mu_\ell(\mathbb{F}_{q^{ij}})$ is an isomorphism, as an ℓth root of unity in $\mathbb{F}_{q^i}$ is sent to an ℓth root of unity in $\mathbb{F}_{q^{ij}}$.

As long as j is relatively prime to ℓ, the map $\mathbb{F}_{q^i}^\times / \mathbb{F}_{q^i}^{\times\,\ell} \to \mathbb{F}_{q^{ij}}^\times / \mathbb{F}_{q^{ij}}^{\times\,\ell}$ is an isomorphism. For suppose conversely, that for a generator $\alpha \in \mathbb{F}_{q^i}^\times / \mathbb{F}_{q^i}^{\times\,\ell}$ there is an ℓth root of α in $\mathbb{F}_{q^{ij}}$. In this case, the extension $\mathbb{F}_{q^i} \to \mathbb{F}_{q^{ij}}$ would factor through the splitting field F of $x^\ell - \alpha$ over $\mathbb{F}_{q^i}$. The degree of the extension $\mathbb{F}_{q^i} \to F$ is ℓ, since $\mathbb{F}_{q^i}$ has a primitive ℓth root of unity. Hence $\ell \mid j$, contradicting $(j, \ell) = 1$.

Finally, remark 3.14 establishes that $f^* : \mathcal{A}_{**}(\mathbb{F}_q) \to \mathcal{A}_{**}(\mathbb{F}_{q^j})$ is an isomorphism. Hence by proposition 3.17, $f^* : \mathcal{A}^{**}(\mathbb{F}_q) \to \mathcal{A}^{**}(\mathbb{F}_{q^j})$ is an isomorphism as well. □

Proposition 3.20. Let q be a prime power which is relatively prime to ℓ. Write $\widetilde{\mathbb{F}}_q$ for the union of the field extensions $\mathbb{F}_{q^j}$ over $\mathbb{F}_q$ with j relatively prime to $\ell(\ell-1)$. The field extension $f : \mathbb{F}_q \to \widetilde{\mathbb{F}}_q$ induces isomorphisms $f^* : H^{**}(\mathbb{F}_q) \to H^{**}(\widetilde{\mathbb{F}}_q)$ and $f^* : \mathcal{A}^{**}(\mathbb{F}_q) \to \mathcal{A}^{**}(\widetilde{\mathbb{F}}_q)$.

Proof. This follows by a colimit argument, using proposition 3.19. □

Proposition 3.21. Let q be a prime power and suppose ℓ is relatively prime to q. For a field extension $f : \mathbb{F}_q \to \mathbb{F}_{q^j}$ where $\ell \mid j$, the map $f^* : H^{1,*}(\mathbb{F}_q) \to H^{1,*}(\mathbb{F}_{q^j})$ is trivial and $f^* : H^{0,*}(\mathbb{F}_q) \to H^{0,*}(\mathbb{F}_{q^j})$ is injective.

Proof. We follow the argument given for proposition 3.19. Let i be the order of q in $\mathbb{F}_\ell^\times$. The map $\mu_\ell(\mathbb{F}_{q^i}) \to \mu_\ell(\mathbb{F}_{q^{ij}})$ is injective, so all that remains is to identify the map $\mathbb{F}_{q^i}^\times/\mathbb{F}_{q^i}^{\times\,\ell} \to \mathbb{F}_{q^{ij}}^\times/\mathbb{F}_{q^{ij}}^{\times\,\ell}$. Let $\alpha \in \mathbb{F}_{q^i}^\times$ be a generator of $\mathbb{F}_{q^i}^\times/\mathbb{F}_{q^i}^{\times\,\ell}$. Then since $\ell \mid j$, the extension $\mathbb{F}_{q^i} \to \mathbb{F}_{q^{ij}}$ factors through the splitting field of $x^\ell - \alpha$ over $\mathbb{F}_{q^i}$, so that $\sqrt[\ell]{\alpha} \in \mathbb{F}_{q^{ij}}$. It now follows that the map $\mathbb{F}_{q^i}^\times/\mathbb{F}_{q^i}^{\times\,\ell} \to \mathbb{F}_{q^{ij}}^\times/\mathbb{F}_{q^{ij}}^{\times\,\ell}$ is trivial. □

Corollary 3.22. Let $\mathbb{F}_q$ be a finite field with algebraic closure $f : \mathbb{F}_q \to \overline{\mathbb{F}}_p$ and let ℓ be a prime different from p. Then $f^* : H^{1,*}(\mathbb{F}_q) \to H^{1,*}(\overline{\mathbb{F}}_p)$ is trivial and $f^* : H^{0,*}(\mathbb{F}_q) \to H^{0,*}(\overline{\mathbb{F}}_p)$ is injective.

Chapter 4

Motivic Adams spectral sequence

The motivic Adams spectral sequence over a base scheme S may be defined using the appropriate notion of an Adams resolution, see [Ada95, Swi75, Rav86] for treatments in the topological case. We recount the definition for completeness and establish some basic properties of the motivic Adams spectral sequence under base change. We follow Dugger and Isaksen [DI10, §3] for the definition of the motivic Adams spectral sequence. See also [HKO11, §6].

Let p and ℓ be distinct primes and let $q = p^n$ for some integer $n \geq 1$. We will be interested in the specific case of the motivic Adams spectral sequence over a field and over a Hensel discrete valuation ring with residue field of characteristic p. We write H for the spectrum $H\mathbb{Z}/\ell$ over the base scheme S and $H^{**}(S)$ for the motivic cohomology of S with $\mathbb{Z}/\ell$ coefficients. The spectrum H is a ring spectrum and is cellular in the sense of Dugger and Isaksen [DI05] by [Spi13, 11.4].

4.1 Construction of the mod ℓ motivic Adams spectral sequence

Definition 4.1. Consider a spectrum X over the base scheme S and let $\overline{H}$ denote the spectrum in the cofibration sequence $\overline{H} \to \mathbb{1} \to H \to \Sigma\overline{H}$. The *standard H-Adams resolution of* X is the tower of cofibration sequences $X_{f+1} \to X_f \to W_f$ given by $X_f = \overline{H}^{\wedge f} \wedge X$ and $W_f = H \wedge X_f$ [Ada95, §15].

$$\begin{tikzcd}
X_0 = X \arrow[dr, "j_0"'] & & \overline{H} \wedge X \arrow[ll, "i_1"'] \arrow[dr, "j_1"'] & & \overline{H} \wedge \overline{H} \wedge X \arrow[ll, "i_2"'] & \cdots \arrow[l] \\
& H \wedge X \arrow[ur, "\partial_0"', "\bullet"] & & H \wedge \overline{H} \wedge X \arrow[ur, "\partial_1"', "\bullet"] & &
\end{tikzcd}$$

Definition 4.2. Let X be a T-spectrum over S and let $\{X_f, W_f\}$ be the standard H-Adams resolution of X. The *motivic Adams spectral sequence* for X with respect to H is the spectral sequence determined by the following exact couple.

$$\begin{tikzcd} \bigoplus \pi_{**} X_f \arrow[rr, "i_*"] & & \bigoplus \pi_{**} X_f \arrow[dl, "j_*"] \\ & \bigoplus \pi_{**} W_f \arrow[ul, "\partial_*"] & \end{tikzcd}$$

The E_1 term of the motivic Adams spectral sequence is $E_1^{f,(s,w)} = \pi_{s,w} W_f$. The index f is called the Adams filtration, s is the stem, and w is the motivic weight. The *Adams filtration* of $\pi_{**}X$ is given by $F_i \pi_{**} X = \mathrm{im}(\pi_{**} X_i \to \pi_{**} X)$.

Proposition 4.3. Let $\mathfrak{S}$ denote the category of spectral sequences in the category of Abelian groups. The associated spectral sequence to the standard H-Adams resolution defines a functor $\mathfrak{M} : \mathcal{SH}_S \to \mathfrak{S}$. Furthermore, the motivic Adams spectral sequence is natural with respect to base change.

Proof. The construction of the standard H-Adams resolution is functorial because $\mathcal{SH}_S$ is symmetric monoidal. Given $X \to X'$ we get induced maps of standard H-Adams resolutions $\{X_f, W_f\} \to \{X'_f, W'_f\}$. As $\pi_{**}(-)$ is a triangulated functor, we get an induced map of the associated exact couples, and hence of spectral sequences $\mathfrak{M}(X) \to \mathfrak{M}(X')$.

Let $f : R \to S$ be a map of base schemes. The claim is that there is a natural transformation between $\mathfrak{M} : \mathcal{SH}_S \to \mathfrak{S}$ and $\mathfrak{M} \circ \mathbb{L}f^* : \mathcal{SH}_S \to \mathcal{SH}_R \to \mathfrak{S}$. Let $X \in \mathcal{SH}_S$ and let $\{X_f, W_f\}$ be the standard H_S-Adams resolution of X in $\mathcal{SH}_S$. We may as well assume X is cofibrant, so $QX = X$ where Q is the cofibrant replacement functor. Let $\{X'_f, W'_f\}$ denote the standard H_R-Adams resolution of $\mathbb{L}f^* X = f^* X$. Observe that $\{f^* X_f, f^* W_f\} = \{X'_f, W'_f\}$, since $f^* \mathbb{1} = \mathbb{1}$, $f^* H_S = H_R$, and $\mathbb{L}f^*$ is a monoidal functor. We therefore have a map $\{\mathbb{L}f^* X_f, \mathbb{L}f^* W_f\} \to \{X'_f, W'_f\}$. Applying $\mathbb{L}f^* : \mathcal{SH}_S(\Sigma^{s,w}\mathbb{1}, -) \to \mathcal{SH}_R(\Sigma^{s,w}\mathbb{1}, \mathbb{L}f^* -)$ to $\{X_f, W_f\}$ gives a map of exact couples, and therefore a map $\Phi_X : \mathfrak{M}_S(X) \to \mathfrak{M}_R(\mathbb{L}f^* X)$. It is straightforward to verify that Φ determines a natural transformation. $\square$

Corollary 4.4. For a map of base schemes $f : R \to S$, there is a map of motivic Adams spectral sequences $\Phi : \mathfrak{M}_S(\mathbb{1}) \to \mathfrak{M}_R(\mathbb{1})$. The map Φ is furthermore compatible with the induced map $\pi_{**}(S) \to \pi_{**}(R)$.

4.2 The E_2 page of the motivic Adams spectral sequence

We now turn our attention to the identification of the second page of the motivic Adams spectral sequence. The arguments in topology, which can be found in [Ada95, Swi75], largely translate to the motivic setting with some modifications. The arguments given by Hu, Kriz, and Ormsby in [HKO11] and Dugger and Isaksen in [DI10, §7] go through in the case where the base scheme S is the Zariski spectrum of a Dedekind domain by the work of Spitzweck [Spi13]. There are two different approaches to identify the E_2 page of the motivic Adams spectral sequence: homological and cohomological. The homological approach uses the structure of the Hopf algebroid of mod ℓ homology cooperations and the methods turn out to be useful in greater generality. The cohomological approach uses the structure of the mod ℓ Steenrod algebra and is more amenable to machine calculations.

Definition 4.5. A particularly well behaved family of spectra in $\mathcal{SH}_S$ are the cellular spectra in the sense of [DI05, 2.10]. A spectrum $E \in \mathcal{SH}_S$ is *cellular* if it can be constructed out of the spheres $\Sigma^\infty S^{a,b}$ for any integers a and b by homotopy colimits. A cellular spectrum is of *finite type* if for some k it has a cell decomposition with no cells $S^{a,b}$ for $a-b<k$ and at most finitely many cells $S^{a,b}$ for any a and b [HKO11, §2].

Proposition 4.6. Suppose X is a cellular spectrum over the base scheme S. The motivic Adams spectral sequence for X has E_2 page given by

$$E_2^{f,(s,w)} \cong \mathrm{Ext}^{f,(s+f,w)}_{\mathcal{A}_{**}(S)}(H_{**}S, H_{**}X).$$

with differentials $d_r : E_r^{f,(s,w)} \to E_r^{f+r,(s-1,w)}$ for $r \geq 2$. Here Ext is taken in the category of $\mathcal{A}_{**}$-comodules.

See [Ada95, Swi75, Rav86] for details on the homological algebra of comodules.

Proof. The argument given for [DI10, 7.10] goes through given that H is a cellular spectrum, as was proven in [Spi13, 11.4]. The cellularity of X and H is sufficient to ensure that the Künneth theorem holds, which is needed in the argument. $\square$

Corollary 4.7. If X and X' are cellular spectra over S and $X \to X'$ induces an isomorphism $H_{**}X \to H_{**}X'$, then the induced map $\mathfrak{M}(X) \to \mathfrak{M}(X')$ is an isomorphism of spectral sequences from the E_2 page onwards.

Corollary 4.8. Let $f : R \to S$ be a map of base schemes, and consider a cellular spectrum X over S. Suppose $f^* : H_{**}(S) \to H_{**}(R)$, $f^* : \mathcal{A}_{**}(S) \to \mathcal{A}_{**}(R)$, and $f^* : H_{**}X \to H_{**}(\mathbb{L}f^*X)$ are all isomorphisms. Then $\mathfrak{M}_S(X) \to \mathfrak{M}_R(\mathbb{L}f^*X)$ is an isomorphism of spectral sequences from the E_2 page onwards.

Corollary 4.9. Let D be a Hensel local ring in which ℓ is invertible and write F for the residue field of D. Then the comparison map $\mathfrak{M}(D) \to \mathfrak{M}(F)$ is an isomorphism at the E_2 page.

Proof. Propositions 3.15, 3.5, and corollary 4.8 give the result when $X = \mathbb{1}$. $\square$

The argument for proposition 4.6 is based on the construction of the *reduced cobar resolution of* $H_{**}X$. From the standard H-Adams resolution $\{X_f, W_f\}$ of X, we extract the following sequence

$$X \to H \wedge X \to \Sigma H \wedge \overline{H} \wedge X \to \Sigma^{2,0} H \wedge \overline{H}^{\wedge 2} \wedge X \to \cdots . \tag{4.10}$$

Applying $H_{**}(-)$ to this sequence yields the reduced cobar resolution of $H_{**}X$

$$H_{**}X \to \mathcal{A}_{**} \otimes H_{**}X \to \mathcal{A}_{**} \otimes \overline{\mathcal{A}}_{**} \otimes H_{**}X \to \cdots \tag{4.11}$$

where $\overline{\mathcal{A}}_{**} = H_{**}\overline{H}$ is the kernel of the augmentation map $\epsilon : \mathcal{A}_{**} \to H_{**}$, and the tensor products are taken over H_{**}. Here one must take care to distinguish the left and right module actions of H_{**} on $\mathcal{A}_{**}$ and $\overline{\mathcal{A}}_{**}$!

The complex obtained from (4.11) by applying $\mathrm{Hom}_{\mathcal{A}_{**}}(H_{**}, -)$ is called the *reduced cobar complex*, and its homology gives the E_2 page of the Adams spectral sequence by a standard argument, which we outline. Note that applying $\pi_{**}(-)$ to (4.10) yields the E_1 term of the motivic Adams spectral sequence. The Hurewicz map

$$\pi_{**}(H \wedge \overline{H}^{\wedge f} \wedge X) \cong \mathrm{Hom}_{\mathcal{A}_{**}}(H_{**}, H_{**}(H \wedge \overline{H}^{\wedge f} \wedge X)),$$

is an isomorphism since $H_{**}(H \wedge \overline{H}^{\wedge f} \wedge X)$ is an extended $\mathcal{A}_{**}$-comodule, and the map

$$\mathrm{Hom}_{\mathcal{A}_{**}}(H_{**}, H_{**}(H \wedge \overline{H}^{\wedge f} \wedge X)) \to \mathrm{Hom}_{\mathcal{A}_{**}}(H_{**}, H_{**}(H \wedge \overline{H}^{\wedge f+1} \wedge X))$$

agrees with d_1 in the motivic Adams spectral sequence; see [DI10, 7.10], [Ada95, p. 323], [Swi75, p. 469]. When the left and right units of the Hopf algebroid $(H_{**}(S), \mathcal{A}_{**}(S))$ do not agree, the reduced cobar complex is difficult to use for practical calculations. However, the (unreduced) cobar complex takes a simpler form in this case.

Definition 4.12. Let S be a base scheme, and let $(H_{**}, \mathcal{A}_{**})$ denote the Hopf algebroid for mod ℓ motivic homology over S. The *cobar complex* is the chain complex $(\mathcal{C}^*(S), d_{\mathcal{C}})$ with terms $\mathcal{C}^s(S) = H_{**} \otimes_{H_{**}} \mathcal{A}_{**}^{\otimes s}$. The standard notation for an element $\alpha \otimes x_1 \otimes \cdots \otimes x_s$ in $\mathcal{C}^s(S)$ is $\alpha[x_1|\cdots|x_s]$. The map $d^s_{\mathcal{C}}$ sends the element $\alpha[x_1|\cdots|x_s]$ to

$$[\eta_R(\alpha)|x_1|\cdots|x_s] + \sum_{i=1}^{s}(-1)^s\alpha[x_1|\cdots|\Delta(x_i)|\cdots|x_s] + (-1)^{s+1}\alpha[x_1|\cdots|x_s|1].$$

When $s = 0$, the map $d^0_{\mathcal{C}}$, given by $\alpha[\,] \mapsto [\eta_R(\alpha)] - [\alpha]$, can be identified with $\eta_R - \eta_L$. The juxtaposition product of $x_0[x_1|\cdots|x_s] \in \mathcal{C}^s(S)$ and $y_0[y_1|\cdots|x_t] \in \mathcal{C}^t(S)$ is given by

$$x_0[x_1|\cdots|x_s] * y_0[y_1|\cdots|y_t] = x_0[x_1|\cdots|x_s|y_0y_1|\cdots|y_t].$$

We now turn our attention to the cohomological approach.

Definition 4.13. Let X be a T-spectrum over S. An *H^{**}-Adams resolution of X* is a tower of cofibration sequences $X_{f+1} \to X_f \to W_f$ in $\mathcal{SH}_S$ of the following form. Each spectrum W_f has a description $W_f \cong \vee_\alpha \Sigma^{p_\alpha, q_\alpha} H$ where the set of indices $\{(p_\alpha, q_\alpha)\}$ is motivically finite (see definition 3.16), and the induced map $H^{**}W_f \to H^{**}X_f$ is a surjection.

$$\begin{tikzcd}
X_0 = X \arrow[dr, "j_0"'] & & X_1 \arrow[ll, "i_1"'] \arrow[dr, "j_1"'] & & X_2 \arrow[ll, "i_2"'] & \cdots \arrow[l] \\
& W_0 \arrow[ur, "\partial_0"', "\bullet"] & & W_1 \arrow[ur, "\partial_1"', "\bullet"] &
\end{tikzcd}$$

Proposition 4.14. Let X be a cellular spectrum over S, and suppose $H^{**}(X)$ is a motivically finite free H^{**}-module. The standard H-Adams resolution of X (see definition 4.1) is an H^{**}-Adams resolution.

Proof. The conditions on X ensure that for any f the spectrum W_f is a motivically finite wedge of suspensions of H. Furthermore, the map $j_f : X_f \to H \wedge X_f = W_f$ induces a surjection $j_f^* : H^{**}W_f \to H^{**}X_f$, since if $x : X_f \to H$ represents a class in $H^{**}X_f$, the class in $H^{**}W_f$ represented by $H \wedge X_f \xrightarrow{H\wedge x} H \wedge H \xrightarrow{\mu} H$ maps to x under j_f^*. $\square$

Proposition 4.15. Suppose $X \in \mathcal{SH}_S$ satisfies the hypotheses of proposition 4.14. The motivic Adams spectral sequence for X has E_2 page given by

$$E_2^{f,(s,w)} \cong \mathrm{Ext}_{\mathcal{A}^{**}(S)}^{f,(s+f,w)}(H^{**}X, H^{**}S)$$

with differentials $d_r : E_r^{f,(s,w)} \to E_r^{f+r,(s-1,w)}$ for $r \geq 2$. Here Ext is calculated in the category of left modules over the Steenrod algebra $\mathcal{A}^{**}$.

Proof. The usual argument given in the topological situation goes through. The cellularity of X and H, and consequently of W_f and X_f, allow one to use the Künneth theorem to calculate $\pi_{**}(W_f)$ and $H^{**}(W_f)$. As each W_f is a motivically finite cell spectrum, there is an isomorphism $\pi_{**}(W_f) \cong \mathrm{Hom}_{\mathcal{A}^{**}}(H^{**}(W_f), H^{**})$. For further details, consult the treatment in [Rav86, §2.1]. $\square$

4.3 Convergence of the motivic Adams spectral sequence

To simplify the notation, write $\mathrm{Ext}(R)$ for $\mathrm{Ext}_{\mathcal{A}^{**}(R)}(H^{**}(R), H^{**}(R))$ when working over the base scheme $S = \mathrm{Spec}(R)$. For any Abelian group G and any prime ℓ, we write $G_{(\ell)}$ for the ℓ-primary part of G and $G_\ell^\wedge = \varprojlim G/\ell^\nu$ for the ℓ-completion of G.

Definition 4.16. Let ℓ be a prime and X a spectrum over S. The *ℓ-completion* of X is the homotopy limit $X_\ell^\wedge = \mathrm{holim}\, X/\ell^\nu$. For H the mod ℓ motivic cohomology spectrum and $\{X_f, W_f\}$ the standard H-Adams resolution of X, the *H-nilpotent completion* of X is the spectrum $X_H^\wedge = \mathrm{holim}_f\, X/X_f$ [Bou79, §5]. The H-nilpotent completion has a tower given by $C_i = \mathrm{holim}_f(X_i/X_f)$.

Recall that the homotopy limit of an inverse system of spectra (X_ν, g_ν) may be defined as the homotopy fiber of $\prod_\nu X_\nu \xrightarrow{\mathrm{id}-G} \prod X_\nu$ where G_ν is the composition $\prod X_\nu \to X_\nu \xrightarrow{g_\nu} X_{\nu-1}$; see [Bou79, 1.8], [Ada95, §15].

Proposition 4.17. Let S be the Zariski spectrum of a field F with characteristic $p \neq \ell$, and let X be a cellular spectrum X over S of finite type (definition 4.5). If either $\ell > 2$ and F has finite mod ℓ cohomological dimension, or $\ell = 2$ and $F[\sqrt{-1}]$ has finite mod 2 cohomological dimension, the motivic Adams spectral sequence converges to the homotopy groups of the H-nilpotent completion of X

$$E_2^{f,(s,w)} \Rightarrow \pi_{s,w}(X_H^\wedge).$$

Furthermore, there is a weak equivalence $X_H^\wedge \cong X_\ell^\wedge$.

Proof. The argument given in [HKO11] carries over to the positive characteristic case from the work of [HKØ13]. See [OØ14, 3.1] for the analogous argument for the motivic Adams-Novikov spectral sequence. □

We say a line $s = mf + b$ in the (f, s)-plane is a *vanishing line* for a bigraded group $G^{f,s}$ if $G^{f,s}$ is zero whenever $0 < s < mf + b$.

Proposition 4.18. If $\overline{F}$ is an algebraically closed field of characteristic $p \neq \ell$, then a vanishing line for $\mathrm{Ext}^{**}(\overline{F}) \cong \mathrm{Ext}^{**}(W(\overline{F}))$ at the prime ℓ is $s = (2\ell - 3)f$. If $\mathbb{F}_q$ is a finite field of characteristic $p \neq \ell$, then a vanishing line for $\mathrm{Ext}^{**}(\mathbb{F}_q) \cong \mathrm{Ext}^{**}(W(\mathbb{F}_q))$ at the prime ℓ is $s = (2\ell - 3)f - 1$.

Proof. A vanishing line exists for $\mathrm{Ext}(\overline{F}) \cong \mathrm{Ext}(W(\overline{F}))$ when $\overline{F}$ is an algebraically closed fields by comparison with $\mathbb{C}$ and the topological case [DI10]. The vanishing line $s = f(2\ell - 3)$ from topology [Ada61] is therefore a vanishing line for $\mathrm{Ext}(\overline{F}) \cong \mathrm{Ext}(W(\overline{F}))$.

For a finite field $\mathbb{F}_q$, the line $s = f(2\ell - 3) - 1$ is a vanishing line for $\mathrm{Ext}(\mathbb{F}_q) \cong \mathrm{Ext}(W(\mathbb{F}_q))$ by the identification of the E_2 page of the motivic Adams spectral sequence given in sections 7.1 and 6.1 below. □

We now discuss the convergence of the motivic Adams spectral sequence over the ring of Witt vectors associated to a finite field or an algebraically closed field.

Proposition 4.19. Let $W(F)$ be the ring of Witt vectors of a field F that is either a finite field or an algebraically closed field of characteristic p and let ℓ be a prime different from p. The motivic Adams spectral sequence for $\mathbb{1}$ over $W(F)$ converges to $\pi_{**}(\mathbb{1}_H^\wedge)$ filtered by the Adams filtration, where $\mathbb{1}_H^\wedge$ is the H-nilpotent completion of $\mathbb{1}$ (see definition 4.16).

Proof. The convergence $\mathfrak{M}_{W(F)}(\mathbb{1}) \Rightarrow \pi_{**}(\mathbb{1}_H^\wedge)$ follows by the argument given for [DI10, 7.15] given the vanishing line in the motivic Adams spectral sequence by proposition 4.18. □

Proposition 4.20. Let R and S be base schemes for which the motivic Adams spectral sequence for $\mathbb{1}$ converges to $\pi_{**}(\mathbb{1}^\wedge_H)$; see propositions 4.17 and 4.19 for examples. A map of base schemes $f : R \to S$ yields a comparison map $\mathfrak{M}_S(\mathbb{1}^\wedge_H) \to \mathfrak{M}_R(\mathbb{1}^\wedge_H)$ which is compatible with the induced map $\pi_{**}(\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{L}f^*\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{1}^\wedge_H(R))$.

Proof. Let $\{X_f(S), W_f(S)\}$ denote the standard H-Adams resolution of $\mathbb{1}$ over S. We now construct a map $\pi_{**}(\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{1}^\wedge_H(R))$. Recall from proposition 4.3 that $f^*X_f(S) = X_f(R)$. Since $\mathbb{L}f^*$ is a triangulated functor, there are maps $\mathbb{L}f^*(\mathbb{1}/X_f(S)) \to \mathbb{1}/X_f(R)$, and so a map $\mathbb{L}f^*\mathbb{1}^\wedge_H(S) \to \mathbb{1}^\wedge_H(R)$ by the universal property for $\mathbb{1}^\wedge_H(R) = \operatorname{holim} \mathbb{1}/X_f(R)$. Let $C_i(S)$ denote the tower for $\mathbb{1}^\wedge_H(S)$ over S defined in 4.16. Similar considerations give a map of towers $\mathbb{L}f^*C_i(S) \to C_i(R)$. Hence $\mathfrak{M}_S(\mathbb{1}^\wedge_H) \to \mathfrak{M}_R(\mathbb{1}^\wedge_H)$ is compatible with the induced map $\pi_{**}(\mathbb{1}^\wedge_H(S)) \to \pi_{**}(\mathbb{1}^\wedge_H(R))$. □

Proposition 4.21. Let F be a field of characteristic p with finite mod ℓ cohomological dimension for all primes $\ell \neq p$. Suppose the mod ℓ motivic Adams spectral sequence for $\mathbb{1}$ over F has a vanishing line, such as when F is a finite field or an algebraically closed field. Then the ℓ-primary part of $\pi_{s,w}(F)$ is finite whenever $s > w \geq 0$.

Proof. Ananyevsky, Levine, and Panin show that the groups $\pi_{s,w}(F)$ are torsion for $s > w \geq 0$ in [ALP15]. It follows that the group $\pi_{s,w}(F)$ is the sum of its ℓ-primary subgroups $\pi_{s,w}(F)_{(\ell)}$. We set out to show that $\pi_{s,w}(F)_{(\ell)}$ is finite when $\ell \neq p$.

The motivic Adams spectral sequence converges to $\pi_{**}(\mathbb{1}^\wedge_\ell)$ by proposition 4.17. The vanishing line in the motivic Adams spectral sequence shows that the Adams filtration of $\pi_{s,w}(\mathbb{1}^\wedge_\ell)$ has finite length, and as each group $E_2^{f,(s,w)}$ is a finite dimensional $\mathbb{F}_\ell$-vector space we conclude the groups $\pi_{s,w}(\mathbb{1}^\wedge_\ell)$ are finite. From the long exact sequence of homotopy groups associated to the triangle $\mathbb{1}^\wedge_\ell \to \prod \mathbb{1}/\ell^\nu \to \prod \mathbb{1}/\ell^\nu$ defining $\mathbb{1}^\wedge_\ell$, we extract the following short exact sequence of finite groups.

$$0 \to \varprojlim{}^1 \pi_{s+1,w}(\mathbb{1}/\ell^\nu) \to \pi_{s,w}(\mathbb{1}^\wedge_\ell) \to \varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu) \to 0 \tag{4.22}$$

Similarly, from the triangles $\mathbb{1} \xrightarrow{\ell^\nu \cdot} \mathbb{1} \to \mathbb{1}/\ell^\nu$ we extract the short exact sequences

$$0 \to \pi_{s,w}(\mathbb{1})/\ell^\nu \to \pi_{s,w}(\mathbb{1}/\ell^\nu) \to {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1}) \to 0,$$

which form a short exact sequence of towers. The maps in the tower $\{\pi_{s,w}(\mathbb{1})/\ell^\nu\}$ are given by the reduction maps $\pi_{s,w}(\mathbb{1})/\ell^\nu \to \pi_{s,w}(\mathbb{1})/\ell^{\nu-1}$. Since the tower $\{\pi_{s,w}(\mathbb{1})/\ell^\nu\}$ satisfies the Mittag-Leffler condition, we have $\varprojlim^1 \pi_{s,w}(\mathbb{1})/\ell^\nu = 0$. The associated long exact sequence for the inverse limit gives the short exact sequence

$$0 \to \pi_{s,w}(\mathbb{1})^\wedge_\ell \to \varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu) \to \varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1}) \to 0. \tag{4.23}$$

The group $\varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1})$ is the ℓ-adic Tate module of $\pi_{s-1,w}(\mathbb{1})$, which is torsion-free. Since $\varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu)$ is finite by (4.22), the map $\varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu) \to \varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1})$ is trivial. But since the sequence (4.23) is exact, the group $\varprojlim {}_{\ell^\nu}\pi_{s-1,w}(\mathbb{1})$ is trivial, $\pi_{s,w}(\mathbb{1})^\wedge_\ell \cong \varprojlim \pi_{s,w}(\mathbb{1}/\ell^\nu)$, and $\pi_{s,w}(\mathbb{1})^\wedge_\ell$ is finite.

Write $K(i)$ for the kernel of the canonical map $\pi_{s,w}(\mathbb{1})^\wedge_\ell \to \pi_{s,w}(\mathbb{1})/\ell^i$. The tower $\cdots K(i) \subseteq K(i-1) \subseteq \cdots \subseteq K(1)$ consists of finite groups and so it must stabilize. Hence the tower

$$\cdots \to \pi_{s,w}(\mathbb{1})/\ell^\nu \to \pi_{s,w}(\mathbb{1})/\ell^{\nu-1} \to \cdots \to \pi_{s,w}(\mathbb{1})/\ell$$

must also stabilize. There is then some N for which $\ell^N\pi_{s,w}(\mathbb{1}) = \ell^\nu\pi_{s,w}(\mathbb{1})$ for all $\nu \geq N$, and so $\ell^N\pi_{s,w}(\mathbb{1})$ is ℓ-divisible. From the short exact sequence of towers $\ell^\nu\pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1})/\ell^\nu$, taking the inverse limit yields the exact sequence

$$0 \to \ell^N\pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1}) \to \pi_{s,w}(\mathbb{1})^\wedge_\ell \to 0.$$

Since $\pi_{s,w}(\mathbb{1})^\wedge_\ell$ is finite it is ℓ-primary, and there is a short exact sequence

$$0 \to \ell^N\pi_{s,w}(\mathbb{1})_{(\ell)} \to \pi_{s,w}(\mathbb{1})_{(\ell)} \to \pi_{s,w}(\mathbb{1})^\wedge_\ell \to 0.$$

The group $\ell^N\pi_{s,w}(\mathbb{1})_{(\ell)}$ must be zero. Suppose for a contradiction that it is non-zero. Then $\ell^N\pi_{s,w}(\mathbb{1})_{(\ell)}$ must contain $\mathbb{Z}/\ell^\infty$ as a summand, which shows the ℓ-adic Tate module of $\pi_{s,w}(\mathbb{1})$ is non-zero—a contradiction. □

We now identify the groups $\pi_{s,s}(\mathbb{1}^\wedge_\ell)$ for $s \geq 0$.

Proposition 4.24. Let F be a finite field or an algebraically closed field of characteristic $p \neq \ell$. When $s = w \geq 0$, the motivic Adams spectral sequence of $\mathbb{1}$ over F converges to the ℓ-completion of $\pi_{s,w}(F)$.

Proof. From proposition 4.17 it follows that at bidegree $(s, w) = (s, s)$ the motivic Adams spectral sequence converges to the group $\pi_{s,w}(\mathbb{1}^\wedge_\ell)$. Since $\pi_{s-1,s}(\mathbb{1}) = 0$ by Morel's connectivity theorem, the short exact sequence (see [HKO11, (2)])

$$0 \to \mathrm{Ext}(\mathbb{Z}/\ell^\infty, \pi_{s,s}(\mathbb{1})) \to \pi_{s,s}(\mathbb{1}^\wedge_\ell) \to \mathrm{Hom}(\mathbb{Z}/\ell^\infty, \pi_{s-1,s}(\mathbb{1})) \to 0$$

gives an isomorphism $\mathrm{Ext}(\mathbb{Z}/\ell^\infty, \pi_{s,s}(\mathbb{1})) \cong \pi_{s,s}(\mathbb{1}^\wedge_\ell)$. In [Mor12, 1.25], Morel has calculated $\pi_{0,0}(F) \cong GW(F)$ and $\pi_{s,s}(F) \cong W(F)$ for $s > 0$ where $W(F)$ is the Witt group of the field F. For the fields under consideration, $GW(F)$ and $W(F)$ is a finitely generated Abelian group. But for any finitely generated Abelian group A, there is an isomorphism $\mathrm{Ext}(\mathbb{Z}/\ell^\infty, A) \cong A^\wedge_\ell$ [BK72, Ch.VI§2.1], which concludes the proof. $\square$

Chapter 5

Stable stems over algebraically closed fields

Let $\overline{F}$ be an algebraically closed field of positive characteristic p. We write $W = W(\overline{F})$ for the ring of Witt vectors of $\overline{F}$, $K = K(\overline{F})$ for the field of fractions of W, and $\overline{K} = \overline{K}(\overline{F})$ for the algebraic closure of K. Note that K is a field of characteristic 0. The previous sections have set us up with enough machinery to compare the motivic Adams spectral sequences at a prime $\ell \neq p$ over the associated base schemes $\mathrm{Spec}(\overline{F})$, $\mathrm{Spec}(W)$, and $\mathrm{Spec}(\overline{K})$. We will often write the ring instead of the Zariski spectrum of the ring in our notation. For any Dedekind domain R, we write $\mathrm{Ext}(R)$ for the trigraded ring $\mathrm{Ext}_{\mathcal{A}^{**}(R)}(H^{**}(R), H^{**}(R))$.

Proposition 5.1. Let $\overline{F}$ be an algebraically closed field of positive characteristic p, and let ℓ be a prime different from p. The E_2 page of the mod ℓ motivic Adams spectral sequence for $\mathbb{1}$ over W, the ring of Witt vectors of $\overline{F}$, is given by

$$E_2^{f,(s,w)}(W) \cong \mathrm{Ext}^{f,(s+f,w)}(W) \cong \mathrm{Ext}^{f,(s+f,w)}(\overline{F}).$$

Proof. Since W is a Hensel local ring with residue field $\overline{F}$, proposition 4.9 applies. $\square$

Proposition 5.2. Let $\overline{F}$ be an algebraically closed field of characteristic p. The homomorphism $f : W \to \overline{K}$ induces isomorphisms of graded rings $f^* : H_{**}(W) \to H_{**}(\overline{K})$ and $f^* : \mathcal{A}_{**}(W) \to \mathcal{A}_{**}(\overline{K})$.

Proof. It is sufficient to establish isomorphisms for motivic cohomology, as $H^{**}(S) \cong H_{-*,-*}(S)$. Since $H^{**}(W) \cong H^{**}(\overline{\mathbb{F}}_p)$, we have $H^{**}(W) \cong \mathbb{F}_\ell[\zeta]$ where $\zeta \in H^{0,1}(W) \cong \mu_\ell(W)$. We also have $H^{**}(\overline{K}) \cong \mathbb{F}_\ell[\zeta]$. To identify the ring map $f^* : H^{**}(W) \to H^{**}(R)$ it suffices to identify the value of $f^*(\zeta)$. The homomorphism $f^* : H^{0,1}(W) \to H^{0,1}(\overline{K})$ may be identified with $\mu_\ell(W) \to \mu_\ell(\overline{K})$, which is an isomorphism. Hence $f^* : H^{**}(W) \to H^{**}(\overline{K})$ is an isomorphism. The argument given for proposition 3.15 establishes that $j^* : \mathcal{A}_{**}(W) \to \mathcal{A}_{**}(\overline{K})$ is an isomorphism. $\square$

Corollary 5.3. Let $\overline{F}$ be an algebraically closed field of characteristic p. The homomorphisms $W \to \overline{K}$ and $W \to \overline{F}$ induce isomorphisms of motivic Adams spectral sequences for $\mathbb{1}$ from the E_2 page onwards. In particular, $\mathrm{Ext}(\overline{F}) \cong \mathrm{Ext}(W) \cong \mathrm{Ext}(\overline{K})$.

Lemma 5.4. Let $f : \bar{k} \to \overline{K}$ be an extension of algebraically closed fields of characteristic 0. For all $s \geq w \geq 0$, base change induces an isomorphism $\pi_{s,w}(\bar{k}) \to \pi_{s,w}(\overline{K})$.

Proof. Let ℓ be a prime. The maps $f^* : H_{**}(\bar{k}) \to H_{**}(\overline{K})$ and $f^* : \mathcal{A}_{**}(\bar{k}) \to \mathcal{A}_{**}(\overline{K})$ are isomorphisms, hence the induced map of cobar complexes $\mathcal{C}^*(\bar{k}) \xrightarrow{f^*} \mathcal{C}^*(\overline{K})$ is an isomorphism. It follows that the map $\mathfrak{M}_{\bar{k}}(\mathbb{1}) \to \mathfrak{M}_{\overline{K}}(\mathbb{1})$ is an isomorphism from the E_2 page onwards. The homomorphism $\mathbb{L}f^* : \pi_{**}(\mathbb{1}_H^\wedge) \to \pi_{**}(\mathbb{1}_H^\wedge)$ is therefore an isomorphism since it is compatible with the map of spectral sequences. Propositions 4.21 and 4.24 identify $\pi_{s,w}(\mathbb{1}_H^\wedge)$ with $\pi_{s,w}(\mathbb{1})_\ell^\wedge$ for all $s \geq w \geq 0$ over both $\bar{k}$ and $\overline{K}$. By [ALP15], the groups $\pi_{s,w}(\bar{k})$ and $\pi_{s,w}(\overline{K})$ are torsion for $s > w \geq 0$ and so they are the sum of their ℓ-primary parts. This establishes the result for $s > w \geq 0$. When $s = w \geq 0$, the result follows by proposition 4.24 and Morel's identification of the groups $\pi_{n,n}(F)$. $\square$

Corollary 5.5. Let $\overline{K}$ be an algebraically closed field of characteristic 0. For any $n \geq 0$, the map $\mathbb{L}c : \pi_n^s \to \pi_{n,0}(\overline{K})$ is an isomorphism.

Proof. The statement is true when $\overline{K} = \mathbb{C}$. The previous proposition extends the result to an arbitrary algebraically closed field of characteristic 0. $\square$

Theorem 5.6. Let $\overline{F}$ be an algebraically closed field of characteristic p and let ℓ be a prime different from p. Then there is an isomorphism $\pi_{s,w}(\overline{F})_\ell^\wedge \cong \pi_{s,w}(\mathbb{C})_\ell^\wedge$ for all $s \geq w \geq 0$.

Proof. Consider the homomorphisms $\overline{F} \leftarrow W \to \overline{K}$. The induced maps on the motivic Adams spectral sequence are compatible with the maps of homotopy groups

$$\pi_{**}(\mathbb{1}_H^\wedge(\overline{F})) \leftarrow \pi_{**}(\mathbb{1}_H^\wedge(W)) \to \pi_{**}(\mathbb{1}_H^\wedge(\overline{K}))$$

By corollary 5.3, the maps $\mathfrak{M}_{\overline{F}}(\mathbb{1}) \leftarrow \mathfrak{M}_W(\mathbb{1}) \to \mathfrak{M}_{\overline{K}}(\mathbb{1})$ are isomorphisms at the E_2 page, and so there are isomorphisms $\pi_{**}(\mathbb{1}_H^\wedge(\overline{F})) \cong \pi_{**}(\mathbb{1}_H^\wedge(W)) \cong \pi_{**}(\mathbb{1}_H^\wedge(\overline{K}))$. For

$s \geq w \geq 0$, propositions 4.21 and 4.24 give isomorphisms $\pi_{s,w}(\mathbb{1}_H^\wedge(\overline{F})) \cong \pi_{s,w}(\overline{F})_\ell^\wedge$ and $\pi_{s,w}(\mathbb{1}_H^\wedge(\overline{K})) \cong \pi_{s,w}(\overline{K})_\ell^\wedge$. The result now follows from lemma 5.4. $\square$

Corollary 5.7. Let $\overline{F}$ be an algebraically closed field of characteristic p and let ℓ be a prime different from p. The homomorphism $\mathbb{L}c : (\pi_n^s)_\ell^\wedge \to \pi_{n,0}(\overline{F})_\ell^\wedge$ is an isomorphism for all $n \geq 0$.

Proof. The previous theorem yields the following diagram for all $n \geq 0$.

$$\begin{tikzcd}
(\pi_n^s)_\ell^\wedge \arrow[dr, "\mathbb{L}c"] \arrow[drr, bend left, "\mathbb{L}c", "\cong"'] \arrow[ddr, bend right, "\mathbb{L}c"'] & & \\
& \pi_{n,0}(\mathbb{1}_H^\wedge(W)) \arrow[r, "\cong"] \arrow[d, "\cong"] & \pi_{n,0}(\overline{K})_\ell^\wedge \\
& \pi_{n,0}(\overline{F})_\ell^\wedge &
\end{tikzcd}$$

The map $\mathbb{L}c : (\pi_n^s)_\ell^\wedge \to \pi_{n,0}(\overline{K})_\ell^\wedge$ is an isomorphism by corollary 5.5, and so all of the maps in the above diagram are isomorphisms. $\square$

Let $t_{\mathbb{C}} : \mathcal{SH}_{\mathbb{C}} \to \mathcal{SH}$ denote the topological realization functor defined in [MV99, Dug01]. For a prime $\ell > 2$, we consider $\mathbb{F}_\ell$ as a module over the polynomial ring $\mathbb{F}_\ell[\zeta]$ where $\zeta \in H^{0,1}(\overline{F})$ acts as the identity.

When it is clear from context, we will write $\mathfrak{M}(F)$ for the mod ℓ motivic Adams spectral sequence for the sphere spectrum over $\mathrm{Spec}(F)$ instead of $\mathfrak{M}_F(\mathbb{1})$. For a prime p and $\ell \neq p$, we establish an isomorphism $\mathfrak{M}(\overline{\mathbb{F}}_p) \cong \mathfrak{M}(\mathbb{C})$. When $\ell = 2$, this isomorphism shows that the differential calculations for $\mathfrak{M}(\mathbb{C})$ by Isaksen in [Isa14b] also hold for $\mathfrak{M}(\overline{\mathbb{F}}_p)$.

Proposition 5.8. Let ℓ be a prime and suppose p is a prime different from ℓ. There is an isomorphism $\mathfrak{M}(\overline{\mathbb{F}}_p) \cong \mathfrak{M}(\mathbb{C})$ from the E_2 page onwards, hence the E_2 page of the motivic Adams spectral sequence over $\overline{\mathbb{F}}_p$ is given by $E_2^{f,(s,w)}(\overline{\mathbb{F}}_p) \cong \mathrm{Ext}^{f,(s+f,w)}(\mathbb{C})$.

When ℓ is odd, $\mathrm{Ext}(\mathbb{C})$ takes the form

$$\mathrm{Ext}(\mathbb{C}) \cong \mathbb{F}_\ell[\zeta] \otimes_{\mathbb{F}_\ell} \mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_\ell, \mathbb{F}_\ell).$$

Write $\mathfrak{A}$ for the mod ℓ Adams spectral sequence for the sphere spectrum in topology.

Topological realization induces an isomorphism of spectral sequences

$$\mathfrak{M}(\mathbb{C}) \otimes_{\mathbb{F}_\ell[\zeta]} \mathbb{F}_\ell \to \mathfrak{A}.$$

The differentials in $\mathfrak{M}(\mathbb{C})$ are determined by $d_r(\zeta^j) = 0$ for all $r \geq 2$ and all j, and for any $x \in E_r(\mathbb{C})$ the differential $d_r(x)$ is zero if and only if $d_r(t_{\mathbb{C}}(x))$ is zero in $\mathfrak{A}$.

Proof. The proof of proposition 5.6 shows that for distinct primes ℓ and p there are isomorphisms of mod ℓ motivic Adams spectral sequences $\mathfrak{M}(\overline{\mathbb{F}}_p) \cong \mathfrak{M}(\mathbb{C})$. At the prime $\ell = 2$, the differentials in $\mathfrak{M}(\mathbb{C})$ are analyzed in [DI10, Isa14b].

Let ℓ be an odd prime. We calculate $\mathrm{Ext}(\mathbb{C})$ using the cobar complex for $\mathcal{A}_{**}(\mathbb{C})$ and $\mathcal{A}_*^{top}$. Let $\mathcal{C}_{top}^*$ denote the cobar complex for the Hopf algebra $(\mathbb{F}_\ell, \mathcal{A}_*^{top})$ defined in [Rav86, A1.2.11], (see also 4.12). Recall from section 3.4 that $\mathcal{A}_*^{top}$ has a bigrading by assigning the appropriate weights to the generators τ_j and ξ_j. Since $\mathcal{A}_{**}(\mathbb{C})$ is the Hopf algebra $\mathcal{A}_{**}(\mathbb{C}) \cong \mathbb{F}_\ell[\zeta] \otimes \mathcal{A}_*^{top}$, there is an isomorphism of cobar complexes $\mathcal{C}^*(\mathbb{C}) \cong \mathbb{F}_\ell[\zeta] \otimes \mathcal{C}_{top}^*$. The universal coefficient theorem then establishes the isomorphism $\mathrm{Ext}(\mathbb{C}) \cong \mathbb{F}_\ell[\zeta] \otimes \mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_\ell, \mathbb{F}_\ell)$.

Topological realization induces a map from the motivic Adams spectral sequence over $\mathbb{C}$ to the topological Adams spectral sequence as pointed out by Dugger and Isaksen in [DI10, §3.2]. Voevodksy proved in [Voe10, §3.4] that $H^{**}(\mathbb{C}) \xrightarrow{t_{\mathbb{C}}} H_{top}^*$ sends ζ to 1 and induces an isomorphism $H^{**}(\mathbb{C}) \otimes_{\mathbb{F}_\ell[\zeta]} \mathbb{F}_\ell \to H_{top}^*$. Furthermore, the topological realization $\mathcal{A}^{**}(\mathbb{C}) \xrightarrow{t_{\mathbb{C}}} \mathcal{A}_{top}^*$ factors through the isomorphism $\mathcal{A}^{**} \otimes_{\mathbb{F}_\ell[\zeta]} \mathbb{F}_\ell \to \mathcal{A}_{top}^*$. We obtain similar results for the topological realization of $H_{**}(\mathbb{C})$ and $\mathcal{A}_{**}(\mathbb{C})$ by dualizing. The map of cobar complexes $\mathcal{C}^*(\mathbb{C}) \to \mathcal{C}_{top}^*$ induced by topological realization is determined by $\zeta \mapsto 1$, $\tau_j \mapsto \tau_j$ and $\xi_j \mapsto \xi_j$. But we then have a map of spectral sequences $\mathfrak{M}(\mathbb{C}) \otimes_{\mathbb{F}_\ell[\zeta]} \mathbb{F}_\ell \to \mathfrak{A}$ which is an isomorphism at the E_2 page. □

We conclude this section with some remarks about the effect of base change between finite fields on the motivic Adams spectral sequence, and how the results of this section can be used in the analysis of the motivic Adams spectral sequence over finite fields.

Proposition 5.9. Let q be a prime power which is relatively prime to ℓ. If j is relatively prime to $\ell(\ell - 1)$, the induced map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\mathbb{F}_{q^j})$ is an isomorphism of spectral sequences from the E_2 page onwards.

Proof. This follows from proposition 3.19 and corollary 4.8. □

Proposition 5.10. Let $\mathbb{F}_q$ be a finite field and let ℓ be a prime different from the characteristic of $\mathbb{F}_q$. Write $\widetilde{\mathbb{F}}_q$ for the union of the field extensions $\mathbb{F}_{q^j}$ over $\mathbb{F}_q$ with j relatively prime to $\ell(\ell-1)$. The map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\widetilde{\mathbb{F}}_q)$ is an isomorphism of spectral sequences from the E_2 page onwards.

Proof. This follows from proposition 3.20 and corollary 4.8. □

Corollary 5.11. For a finite field $\mathbb{F}_q$ of characteristic p, the comparison map $\mathbb{L}f^*$: $\pi_{i,j}(\mathbb{F}_q)[\frac{1}{p}] \to \pi_{i,j}(\widetilde{\mathbb{F}}_q)[\frac{1}{p}]$ is an isomorphism.

The next proposition enables differentials in $\mathfrak{M}(\mathbb{F}_q)$ to be inferred from the known differential calculations in $\mathfrak{M}(\overline{\mathbb{F}}_p) \cong \mathfrak{M}(\mathbb{C})$. See 3.3 for the definition of u, ρ, and γ. We remark that over any finite field the class u is defined to always be non-trivial.

Proposition 5.12. Let $\mathbb{F}_q$ be a finite field with algebraic closure $\overline{\mathbb{F}}_p$, and consider a prime $\ell \neq p$. The class $u \in \mathrm{Ext}(\mathbb{F}_q)$ maps to 0 in $\mathrm{Ext}(\overline{\mathbb{F}}_p)$, whereas any class $x \in \mathrm{Ext}(\mathbb{F}_q)$ which is not divisible by u maps to a non-zero class in $\mathrm{Ext}(\overline{\mathbb{F}}_p)$.

Proof. The induced map on the cobar complex $\mathcal{C}(\mathbb{F}_q) \to \mathcal{C}(\overline{\mathbb{F}}_p)$ kills u, and induces an injection $\mathcal{C}(\mathbb{F}_q)/u\mathcal{C}(\mathbb{F}_q) \to \mathcal{C}(\overline{\mathbb{F}}_p)$ by proposition 3.21. The result now follows from the calculation of $\mathrm{Ext}(\mathbb{F}_q)$ given in sections 6.1 and 7.1 below. □

Proposition 5.13. Let ℓ be a prime different from the characteristic of $\mathbb{F}_q$. For $n > 0$, the map $\mathbb{L}c : (\pi_n^s)^\wedge_\ell \to \pi_{n,0}(\mathbb{F}_q)^\wedge_\ell$ injects as a direct summand.

Proof. The map $\mathbb{L}c : (\pi_n^s)^\wedge_\ell \to \pi_{n,0}(\overline{\mathbb{F}}_p)^\wedge_\ell$ is an isomorphism by proposition 5.6 and factors through $\mathbb{L}c : (\pi_n^s)^\wedge_\ell \to \pi_{n,0}(\mathbb{F}_q)^\wedge_\ell$. Hence $(\pi_n^s)^\wedge_\ell \to \pi_{n,0}(\mathbb{F}_q)^\wedge_\ell$ must be injective and the comparison $\pi_{n,0}(\mathbb{F}_q)^\wedge_\ell \to \pi_{n,0}(\overline{\mathbb{F}}_p)^\wedge_\ell$ gives a splitting. □

Chapter 6

The motivic Adams spectral sequence for finite fields $\mathbb{F}_q$ with trivial Bockstein action

Throughout this chapter, we assume the Bockstein acts trivially on $H^{**}(\mathbb{F}_q)$. That is, at the prime $\ell = 2$ we assume $q \equiv 1 \bmod 4$. At the prime $\ell > 2$, consider a prime power q which is relatively prime to ℓ and write i for the order of q in $\mathbb{F}_\ell$. Then the action of the Bockstein on $H^{**}(\mathbb{F}_q)$ is trivial if and only if $q^i \equiv 1 \bmod \ell^2$.

6.1 The E_2 page of the mod ℓ motivic Adams spectral sequence

We will make frequent use of the structure of $H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$ which was determined in proposition 3.4. For a field F, we write $\mathrm{Ext}(F)$ for $\mathrm{Ext}_{\mathcal{A}^{**}(F)}(H^{**}(F), H^{**}(F))$. Let $\mathcal{A}_*^{top}$ denote the mod ℓ dual Steenrod algebra of topology. We declare the weight of the elements τ_j and ξ_j in $\mathcal{A}_*^{top}$ to be $\ell^j - 1$, so that $\mathcal{A}_*^{top}$ is bigraded.

Proposition 6.1. The E_2 page of the mod 2 motivic Adams spectral sequence for the sphere spectrum over $\mathbb{F}_q$ with $q \equiv 1 \bmod 4$ is the trigraded algebra

$$E_2 \cong \mathrm{Ext}(\mathbb{F}_q) \cong \mathbb{F}_2[\tau, u]/(u^2) \otimes_{\mathbb{F}_2[\tau]} \mathrm{Ext}(\overline{\mathbb{F}}_p).$$

For $\ell > 2$ the E_2 page of the mod ℓ motivic Adams spectral sequence for $\mathbb{F}_q$ when $q^i \equiv 1 \bmod \ell^2$ is the trigraded algebra

$$E_2 \cong \mathrm{Ext}(\mathbb{F}_q) \cong \mathbb{F}_\ell[\zeta, u]/(u^2) \otimes_{\mathbb{F}_\ell} \mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_\ell, \mathbb{F}_\ell).$$

Proof. We prove the proposition in the case $\ell = 2$, since the proof for $\ell > 2$ is similar. Consult [DI10, 3.5] for a similar argument. Recall from proposition 3.8 that $\mathcal{A}^{**}(\mathbb{F}_q) \cong \mathcal{A}^{**}(\overline{\mathbb{F}}_p) \otimes_{\mathbb{F}_2[\tau]} \mathbb{F}_2[\tau, u]/(u^2)$ and $H^{**}(\mathbb{F}_q) \cong H^{**}(\overline{\mathbb{F}}_p) \otimes \mathbb{F}_2[\tau, u]/(u^2)$. Since $\mathbb{F}_2[\tau, u]/(u^2)$ is flat as a module over $\mathbb{F}_2[\tau]$, a free resolution $H^{**}(\overline{\mathbb{F}}_p) \leftarrow P^\bullet$ determines a free resolution

$H^{**}(\mathbb{F}_q) \leftarrow P^{\bullet} \otimes \mathbb{F}_2[\tau, u]/(u^2)$.

For a field F, consider the functor $\mathrm{Hom}_{\mathcal{A}^{**}}(-, H^{**})$ of motivically finitely generated bigraded modules over $\mathcal{A}^{**}(F)$. The canonical map

$$\mathrm{Hom}_{\mathcal{A}^{**}(\overline{\mathbb{F}}_p)}(-, H^{**}(\overline{\mathbb{F}}_p)) \otimes \mathbb{F}_2[\tau, u]/(u^2) \to \mathrm{Hom}_{\mathcal{A}^{**}(\mathbb{F}_q)}(- \otimes \mathbb{F}_2[\tau, u]/(u^2), H^{**}(\mathbb{F}_q)).$$

is a natural isomorphism, since a generating set for a module M over $\mathcal{A}^{**}(\overline{\mathbb{F}}_p)$ is also a generating set for $M \otimes \mathbb{F}_2[\tau, u]/(u^2)$ over $\mathcal{A}^{**}(\mathbb{F}_q)$ by proposition 3.8. We conclude that $\mathrm{Ext}(\overline{\mathbb{F}}_p) \otimes \mathbb{F}_2[\tau, u]/(u^2) \cong \mathrm{Ext}(\mathbb{F}_q)$. □

Remark 6.2. In proposition 6.1 we must treat the case $\ell = 2$ separately from the case $\ell > 2$ for two reasons. First, when $\ell = 2$ there is no isomorphism $\mathcal{A}^{top}_* \otimes \mathbb{F}_2[\tau] \cong \mathcal{A}_{**}(\mathbb{C})$ because the relations for τ_i^2 in $\mathcal{A}_{**}(\mathbb{C})$ differ from those in $\mathcal{A}^{top}_*$ by Voevodsky's calculation in [Voe03, 12.6] (compare with proposition 3.8). Second, when $\ell > 2$ the generators ζ and γ of $H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$ have weight i where i is the smallest positive integer for which $q^i \equiv 1 \bmod \ell$. But over $\overline{\mathbb{F}}_p$, the generator $\zeta \in H^{**}(\overline{\mathbb{F}}_p; \mathbb{Z}/\ell)$ has weight 1. Hence if $i > 1$, there is not a bigraded isomorphism between $\mathrm{Ext}(\mathbb{F}_q)$ and $\mathbb{F}_\ell[\zeta, u]/(u^2) \otimes_{\mathbb{F}_\ell[\zeta]} \mathrm{Ext}(\overline{\mathbb{F}}_p)$.

6.2 Differentials in the mod ℓ motivic Adams spectral sequence

We begin with the motivic Adams spectral sequence for $X = H\mathbb{Z}[\frac{1}{p}]$ over a finite field $\mathbb{F}_q$ of characteristic p, as defined in 4.2. In proposition 6.4 we identify the differentials for $\mathfrak{M}_{\mathbb{F}_q}(H\mathbb{Z}[\frac{1}{p}])$ which converges to $\pi_{**}(H\mathbb{Z}[\frac{1}{p}]^\wedge_\ell) \cong H_{**}(\mathbb{F}_q; \mathbb{Z})^\wedge_\ell$. We accomplish this by working backwards from our knowledge of $H^{**}(\mathbb{F}_q; \mathbb{Z})^\wedge_\ell$ by a calculation due to Soulé [Sou79, IV.2].

Lemma 6.3. Let $\mathbb{F}_q$ be a finite field of characteristic p, and let ℓ be a prime different from p. The spectrum $H\mathbb{Z}[\frac{1}{p}]$ is cellular, and for $H = H\mathbb{Z}/\ell$, the H-nilpotent completion of $H\mathbb{Z}[\frac{1}{p}]$ is weak equivalent to $H\mathbb{Z}^\wedge_\ell$.

Proof. The spectrum $H\mathbb{Z}[\frac{1}{p}]$ is cellular by the Hopkins-Morel theorem [Hoy15, §8.1]. We show $H\mathbb{Z}^\wedge_\ell$ is weak equivalent to the H-nilpotent completion of $H\mathbb{Z}[\frac{1}{p}]$ by showing that the tower $H\mathbb{Z}/\ell \leftarrow H\mathbb{Z}/\ell^2 \leftarrow H\mathbb{Z}/\ell^3 \leftarrow \cdots$ under $H\mathbb{Z}[\frac{1}{p}]$ is an H-nilpotent resolution under $H\mathbb{Z}[\frac{1}{p}]$ (defined in [Bou79, 5.6]). It will then follow that the homotopy limit of this tower is weak equivalent to the H-nilpotent completion of $H\mathbb{Z}[\frac{1}{p}]$, that is, $H\mathbb{Z}_\ell^\wedge \cong H\mathbb{Z}[\frac{1}{p}]_H^\wedge$, by the discussion in [DI10, §7.7] which shows Bousfield's result [Bou79, 5.8] holds in the motivic stable homotopy category.

The spectrum $H\mathbb{Z}[\frac{1}{p}]$ is the homotopy colimit of the diagram $H\mathbb{Z} \xrightarrow{p\cdot} H\mathbb{Z} \xrightarrow{p\cdot} \cdots$. From the triangle $H\mathbb{Z} \xrightarrow{\ell^\nu\cdot} H\mathbb{Z} \to H\mathbb{Z}/\ell^\nu$ we obtain a triangle $H\mathbb{Z}[\frac{1}{p}] \xrightarrow{\ell^\nu\cdot} H\mathbb{Z}[\frac{1}{p}] \to H\mathbb{Z}/\ell^\nu$ after inverting p, since $p \neq \ell$ and $H\mathbb{Z}/\ell^\nu \xrightarrow{p\cdot} H\mathbb{Z}/\ell^\nu$ is a homotopy equivalence. Consider the following cofibration sequence of towers.

$$\begin{tikzcd}
H\mathbb{Z}[\frac{1}{p}] \arrow[d, "="'] & H\mathbb{Z}[\frac{1}{p}] \arrow[l, "\ell\cdot"'] \arrow[d, "\ell\cdot"] & H\mathbb{Z}[\frac{1}{p}] \arrow[l, "\ell\cdot"'] \arrow[d, "\ell^2\cdot"] & \cdots \arrow[l] \\
H\mathbb{Z}[\frac{1}{p}] \arrow[d] & H\mathbb{Z}[\frac{1}{p}] \arrow[l, "="'] \arrow[d] & H\mathbb{Z}[\frac{1}{p}] \arrow[l, "="'] \arrow[d] & \cdots \arrow[l] \\
\mathrm{pt} & H\mathbb{Z}/\ell \arrow[l] & H\mathbb{Z}/\ell^2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

It is clear that $H\mathbb{Z}/\ell^\nu$ is H-nilpotent for all $\nu \geq 1$. For any H-nilpotent spectrum N we show that the induced map $\mathrm{colim}_\nu \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}/\ell^\nu, N) \to \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[\frac{1}{p}], N)$ is an isomorphism following the proof of [Bou79, 5.7]. This isomorphism holds if and only if

$$\mathrm{colim}\{\mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[\tfrac{1}{p}], N) \xrightarrow{\ell\cdot} \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[\tfrac{1}{p}], N)\} \cong \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[\tfrac{1}{p}], N)[\tfrac{1}{\ell}]$$

vanishes for all H-nilpotent N. This follows by an inductive proof with the following filtration of the H-nilpotent spectra given in [Bou79, 3.8]. Take C_0 to be the collection of spectra $H \wedge X$ for X any spectrum, and let C_{m+1} be the collection of the spectra N for which either N is a retract of an element of C_m or there is a triangle $X \to N \to Z$ with X and Z in C_m.

If $N = H \wedge X$, it is clear that $\mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[\frac{1}{p}], N) \xrightarrow{\ell} \mathcal{SH}_{\mathbb{F}_q}(H\mathbb{Z}[\frac{1}{p}], N)$ is the zero map, which establishes the base case. If the claim holds for N in filtration C_m, the claim holds for N in filtration C_{m+1} by a standard argument. The claim now follows. $\square$

Proposition 6.4. The mod 2 motivic Adams spectral sequence for $X = H\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{F}_q$ when $q \equiv 1 \bmod 4$ has E_1 page given by

$$E_1 \cong \mathbb{F}_2[\tau, u, h_0]/(u^2)$$

where $h_0 \in E_1^{1,(0,0)}$. Write ν_2 for the 2-adic valuation, and write $\epsilon(q)$ for $\nu_2(q-1)$. For all $r \geq 1$ the differentials d_r vanish on $u\tau^j$ and h_0^j. If $r < \epsilon(q) + \nu_2(j)$ the differentials $d_r\tau^j$

vanish, and

$$d_{\epsilon(q)+\nu_2(j)}\tau^j = u\tau^{j-1}h_0^{\epsilon(q)+\nu_2(j)}.$$

In particular, the differential d_1 is trivial, so $E_2 \cong E_1$.

For $\ell > 2$, the mod ℓ motivic Adams spectral sequence for $H\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{F}_q$ with $q^i \equiv 1 \bmod \ell^2$ has E_1 page

$$E_1 \cong \mathbb{F}_\ell[\zeta, u, a_0]/(u^2)$$

where $a_0 \in E_1^{1,(0,0)}$. Note here that $\zeta \in E_2^{0,(0,-i)}$ and $u \in E_2^{0,(1,-i)}$. In this case, the differentials are controlled by $\epsilon(q) = \nu_\ell(q^i - 1)$. For all $r \geq 1$ the differentials d_r vanish on $u\zeta^j$ and a_0. If $r < \epsilon(q) + \nu_\ell(j)$ the differentials $d_r\zeta^j$ vanish, and

$$d_{\epsilon(q)+\nu_\ell(j)}\zeta^j = u\zeta^{j-1}a_0^{\epsilon(q)+\nu_\ell(j)}.$$

In particular, the differential d_1 is trivial, so $E_2 \cong E_1$.

Proof. We build the following H^{**}-Adams resolution of $H\mathbb{Z}[\frac{1}{p}]$ utilizing the triangles constructed in 6.3.

$$\begin{tikzcd}
H\mathbb{Z}[\frac{1}{p}] \arrow[dr, "j_0"'] & & H\mathbb{Z}[\frac{1}{p}] \arrow[ll, "\ell\cdot"'] \arrow[dr, "j_1"'] & & H\mathbb{Z}[\frac{1}{p}] \arrow[ll, "\ell\cdot"'] & \cdots \arrow[l] \\
& H\mathbb{Z}/\ell \arrow[ur, "\bullet", "\partial_0"'] & & H\mathbb{Z}/\ell \arrow[ur, "\bullet", "\partial_1"'] & &
\end{tikzcd} \tag{6.5}$$

The spectrum $H\mathbb{Z}[\frac{1}{p}]$ is cellular, so the motivic Adams spectral sequence for $X = H\mathbb{Z}[\frac{1}{p}]$ converges to $\pi_{**}(H\mathbb{Z}[\frac{1}{p}]^\wedge_H)$ by proposition 4.17. Lemma 6.3 shows that $\pi_{**}(H\mathbb{Z}[\frac{1}{p}]^\wedge_H) \cong \pi_{**}(H\mathbb{Z}^\wedge_\ell)$, so the spectral sequence converges $E_2^{f,(s,w)} \Rightarrow H^{-s,-w}(\mathbb{F}_q;\mathbb{Z})^\wedge_\ell$.

The groups $H^{**}(\mathbb{F}_q;\mathbb{Z})^\wedge_\ell$ were calculated by Soulé in [Sou79, IV.2].

$$H^{-s,-w}(\mathbb{F}_q;\mathbb{Z})^\wedge_\ell \cong \begin{cases} \mathbb{Z}_\ell & \text{if } s = w = 0 \\ \mathbb{Z}/(q^j-1)^\wedge_\ell & \text{if } s = -1, w \geq 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

Note that $\nu_2(q^j - 1) = \epsilon(q) + \nu_2(j)$ for all natural numbers j. The formulas for the differentials on τ^j and ζ^j are the only choice to give $H^{**}(\mathbb{F}_q;\mathbb{Z})^\wedge_\ell$ as the E_∞ term. $\square$

Corollary 6.7. Let $\mathbb{F}_q$ be a finite field of characteristic p, and let $\ell \neq p$ be a prime. The unit map $\mathbb{1} \to H\mathbb{Z}[\frac{1}{p}]$ induces a map of spectral sequences $\mathfrak{M}(\mathbb{1}) \to \mathfrak{M}(H\mathbb{Z}[\frac{1}{p}])$ over $\mathbb{F}_q$

which is surjective on the E_2 page. The differentials calculated in proposition 6.4 hold in the motivic Adams spectral sequence for $\mathbb{1}$ over $\mathbb{F}_q$.

In addition to the above differential calculations, propositions 5.12 and 5.13 help identify differentials. Let $\mathbb{F}_q \to \overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_q$, and write $\Phi : \mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$ for the induced map of spectral sequences. For $x \in E_r(\mathbb{F}_q)$, we must have $d_r(\Phi(x)) = \Phi(d_r(x))$. By proposition 5.8, we often know something about $d_r(\Phi(x))$, and can use this to determine $d_r(x) \in \Phi^{-1}(d_r(x))$.

6.3 The prime 2

We now analyze the 2-complete stable stems $\hat{\pi}_{**}(\mathbb{F}_q) = \pi_{**}(\mathbb{F}_q)^{\wedge}_2$ when q and 2 are relatively prime and $q \equiv 1 \bmod 4$. The results of the previous sections allow us to identify the nth classical 2-complete stable stem $\hat{\pi}^s_n = (\pi^s_n)^{\wedge}_2$ as a summand of $\hat{\pi}_{n,0}(\mathbb{F}_q)$. Using this, we are able to determine the E_∞ page of the motivic Adams spectral sequence for $\mathbb{F}_q$ for stems $s \leq 20$. For the remainder of this section, let H denote the spectrum representing motivic cohomology with $\mathbb{Z}/2$ coefficients.

Proposition 6.1 shows that the irreducible elements of $\mathrm{Ext}(\overline{\mathbb{F}}_p)$ are also irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ when $q \equiv 1 \bmod 4$. The only additional irreducible element in $\mathrm{Ext}(\mathbb{F}_q)$ is the class u. Table 6.1 gives the list of irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ up to stem $s \leq 21$. In this table, P is an operation of tridegree $4,(8,4)$ defined on elements $x \in \mathrm{Ext}(\mathbb{F}_q)$ which satisfy $h_0^4 x = 0$ given by the Massey product $P(x) = \langle h_3, h_0^4, x\rangle$. This table was obtained by computer calculation and is consistent with [Isa14b, Table 8].

Elt.	Filtr. (f,s,w)	Elt.	Filtr. (f,s,w)	Elt.	Filtr. (f,s,w)
u	$(0,-1,-1)$	c_0	$(3,8,5)$	e_0	$(4,17,10)$
τ	$(0,0,-1)$	Ph_1	$(5,9,5)$	P^2h_1	$(9,17,9)$
h_0	$(1,0,0)$	Ph_2	$(5,11,6)$	f_0	$(4,18,10)$
h_1	$(1,1,1)$	d_0	$(4,14,8)$	P^2h_2	$(9,19,10)$
h_2	$(1,3,2)$	h_4	$(1,15,8)$	c_1	$(3,19,11)$
h_3	$(1,7,4)$	Pc_0	$(7,16,9)$	$[\tau g]$	$(4,20,11)$

Table 6.1: Irreducible elements in $\mathrm{Ext}(\mathbb{F}_q)$ with stem $s \leq 21$ for $q \equiv 1 \bmod 4$

We now begin an analysis of the differentials in the motivic Adams spectral sequence in the range $s \leq 21$ to identify the two-complete stable stems over $\mathbb{F}_q$. To assist the reader with the computations presented below, figure 8.1 displays the E_2 page and figures 8.2 and 8.3 display the E_∞ page of the motivic Adams spectral sequence over $\mathbb{F}_q$ in the range $s \leq 21$ when $q \equiv 1 \bmod 8$ and $q \equiv 5 \bmod 8$.

Morel proved the stem $\pi_{0,0}(\mathbb{F}_q)$ is isomorphic to the Grothendieck-Witt group $GW(\mathbb{F}_q)$ in [Mor04] and Scharlau calculated $GW(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ in [Sch85, Ch. 2, 3.3]. Recall that there are isomorphisms $\pi_0^s \cong \mathbb{Z}$ and $\pi_1^s \cong \mathbb{Z}/2$, so there is an isomorphism $\pi_{0,0}(\mathbb{F}_q) \cong \pi_0^s \oplus \pi_1^s$. We show in proposition 6.8 that the pattern $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$ continues after 2-completion for small values of $n \geq 0$. However, proposition 6.9 shows the pattern fails when $n = 19$ and $q \equiv 5 \bmod 8$.

Proposition 6.8. When $q \equiv 1 \bmod 4$ and $0 \leq n \leq 18$, there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$.

Proof. The irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$ in this range are given in table 6.1. All differentials d_r for $r \geq 2$ vanish on $h_0, h_1, h_3, c_0, Ph_1, d_0, Pc_0, P^2h_1$ for degree reasons. As $\hat{\pi}_{3,0}(\mathbb{F}_q)$ must contain $\hat{\pi}_3^s \cong \mathbb{Z}/8$ as a summand by proposition 5.13, we conclude $d_2(\tau^2 h_2) = \tau^2 d_2(h_2) = 0$. The only possible non-zero value for $d_2(h_2)$ is uh_1^3. If $d_2(h_2) = uh_1^3$, then $d_2(\tau^2 h_2) = u\tau^2 h_1^3$ would be non-zero by the product structure of $\mathrm{Ext}(\mathbb{F}_q)$ in proposition 6.1—a contradiction. Hence $d_2(h_2) = 0$.

The non-zero Massey product $Ph_2 = \langle h_3, h_0^4, h_2 \rangle$ has no indeterminacy, because $h_3 E_2^{4,(3,2)} + E_2^{4,(7,4)} h_2 = 0$. Since $\hat{\pi}_{11}^s \cong \mathbb{Z}/8$ is a summand of $\hat{\pi}_{11,0}$, the differential $d_2 Ph_2$ must vanish. The non-zero Massey product $P^2h_2 = \langle h_3, h_0^4, h_2 \rangle$ has no indeterminacy, because $h_3 E_2^{8,(11,6)} + E_2^{4,(7,4)} Ph_2 = 0$. Since $d_2 Ph_2 = 0$, the topological result of Moss [McC01, 9.42(2)] implies $d_2 P^2 h_2 = 0$.

The comparison map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$ shows that $d_2(h_4)$ and $d_3(h_0h_4)$ must be non-zero, as these differentials are non-zero in $\mathfrak{M}(\overline{\mathbb{F}}_p)$ by proposition 5.12 and [Isa14b, Table 8] over $\mathbb{C}$. The only possible choice for $d_2(h_4)$ is $h_0 h_3^2$, but $d_3(h_0h_4)$ is either $h_0 d_0$ or $h_0 d_0 + u h_1 d_0$. In order to have $\hat{\pi}_{14}^s \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ as a summand of $\hat{\pi}_{14,0}$, we must have $d_3(h_0h_4) = h_0 d_0$. A similar argument establishes $d_2(e_0) = h_1^2 d_0$ and $d_2(f_0) = h_0^2 e_0$. Note that $d_4(h_0^3 h_4) = 0$ for degree reasons.

The elements in weight 0 are all of the form $\tau^j x$ or $u\tau^{j-1}x$ where x is not a multiple of

τ and of weight j. The differentials of the elements in weight 0 are now readily identified by using the Leibniz rule from proposition 6.4. Since $\hat{\pi}^s_n$ is a summand of $\hat{\pi}_{n,0}(\mathbb{F}_q)$ for all $n \geq 0$, we see that there are no hidden extensions for $0 < n \leq 18$. $\square$

Recall that there are isomorphisms $\hat{\pi}^s_{19} \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ and $\hat{\pi}^s_{20} \cong \mathbb{Z}/8$.

Proposition 6.9. When $q \equiv 5 \bmod 8$, $\hat{\pi}_{19,0}(\mathbb{F}_q) \cong \hat{\pi}^s_{19} \oplus \mathbb{Z}/4$ and $\hat{\pi}_{20,0}(\mathbb{F}_q) \cong \hat{\pi}^s_{20} \oplus \mathbb{Z}/2$. When $q \equiv 1 \bmod 8$ and $19 \leq n \leq 20$, there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}^s_n \oplus \hat{\pi}^s_{n+1}$.

Proof. The differential $d_2[\tau g]$ is trivial as it lands in $E_2^{6,(19,11)} = 0$. Since $[\tau g]$ has weight 11, the class $\tau^{11}[\tau g]$ is in $E_\infty^{4,(20,0)}$.

In the case $q \equiv 5 \bmod 8$, we calculate $d_2\tau^{11}[\tau g] = u\tau^{10}h_0^2[\tau g] \neq 0$ by proposition 6.4. This resolves all of the differentials in the 19 and 20 stems, and the calculation of the 19 stem follows. As $\hat{\pi}^s_{20} \cong \mathbb{Z}/8$ must be a summand of $\hat{\pi}_{20,0}(\mathbb{F}_q)$ which has order 16, we conclude $\hat{\pi}_{20,0} \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ and there is a hidden extension from $u\tau^{11}h_2^2h_4 = u\tau^{11}h_3^3$ to $\tau^{12}h_2e_0$.

When $q \equiv 1 \bmod 8$, proposition 6.4 shows $d_2\tau^{11} = 0$, hence $d_2\tau^{11}[\tau g] = 0$. This resolves all of the differentials in the stems 19, 20, and 21. We note that there are no possible hidden extensions in the 19 or 20 stem in this case. The result then follows. $\square$

Remark 6.10. The element $\bar{\kappa} \in \pi^s_{20} \cong \mathbb{Z}/24$ is detected by g in the 2-primary Adams spectral sequence. Toda calculated π^s_n for $n \leq 19$ in [Tod62], Mimura and Toda calculated the 20 stem π^s_{20} in [MT63], and May analyzed the Adams spectral sequence for stems $s \leq 28$ in [May65]. Over a finite field $\mathbb{F}_q$ with $q \equiv 5 \bmod 8$, the class $\mathbb{L}c(\bar{\kappa}) \in \pi_{20,0}(\mathbb{F}_q)$ is detected by $u\tau^{11}h_3^3$ which is in Adams filtration 3, and not 4. But over $\overline{\mathbb{F}}_q$, the class $\mathbb{L}c(\bar{\kappa})$ is detected by $\tau^{11}[\tau g]$ in Adams filtration 4.

6.4 The prime 3

Ravenel gives a description of the 3-primary Adams spectral sequence in [Rav86, §1.2] which may be used to calculate $\mathrm{Ext}(\mathbb{F}_q)$ given proposition 6.1. In this section, $\hat{\pi}$ denotes the 3-completion of the group π and i denotes the order of q in $\mathbb{F}_3^\times$. The finite fields $\mathbb{F}_q$ with trivial Bockstein action are those fields with either $q \equiv 1 \bmod 9$ or $q \equiv 8 \bmod 9$.

The non-zero irreducible elements of the E_2 page in the range $s \leq 20$ are given in table 6.2. Write $R(x)$ for the Massey product $\langle h_0, h_0, x\rangle$, and $Q(x)$ for the Massey product $\langle h_0, a_0, x\rangle$.

Elt.	Filtr. (f, s, w)	Elt.	Filtr. (f, s, w)
a_0	$(1, 0, 0)$	h_1	$(1, 11, 6)$
h_0	$(1, 3, 2)$	$R(h_1)$	$(2, 18, 10)$
$R(a_0)$	$(2, 7, 4)$	$Q^2(R(a_0))$	$(4, 15, 8)$
b_0	$(2, 10, 6)$	$Q^3(R(a_0))$	$(5, 19, 10)$

Table 6.2: Irreducible elements in $\mathrm{Ext}(\mathbb{F}_q)$ at the prime $\ell = 3$

We now calculate the 3-complete stems $\hat{\pi}_{n,0}$ for $n \leq 20$. Note that $\hat{\pi}_n^s$ is non-trivial for the following values of $n \leq 20$: 0, 3, 7, 10, 11, 13, 15, 19, and 20.

Proposition 6.11. At the prime $\ell = 3$ and for a field $\mathbb{F}_q$ with a trivial Bockstein action on $H^{**}(\mathbb{F}_q)$, i.e., $q \equiv \pm 1 \bmod 9$, there are isomorphisms $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$ for $0 \leq n \leq 20$.

Proof. Most of the differentials on irreducible elements in the range $s \leq 20$ vanish for degree reasons, except for possibly $d_2(h_1)$, and $d_2(R(h_1))$. In weight 0, the corresponding differentials to analyze are $d_2(\zeta^{6/i}h_1)$, $d_2(\gamma\zeta^{(6/i)-1}h_1)$, $d_2(\zeta^{10/i}R(h_1))$, $d_2(\gamma\zeta^{(10/i)-1}R(h_1))$, $d_2(\zeta^{12/i}h_0R(h_1))$, and $d_2(\gamma\zeta^{(12/i)-1}h_0R(h_1))$

We use the comparison map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$ to calculate the differentials $d_2(h_1)$ and $d_2(R(h_1))$. Proposition 5.12 shows that the comparison map sends h_1 to h_1 in the cobar complex. Since $d_2(h_1) = a_0b_0$ holds over $\overline{\mathbb{F}}_p$ by [Rav86], we must then have $d_2(h_1) = a_0b_0$ since there is nothing divisible by γ in this position. The same comparison also shows $d_2(R(h_1)) = b_0R(a_0)$ is non-zero. Since $d_2(h_0) = 0$ for degree reasons, the Leibniz rule implies that $d_2(h_0R(h_1)) = h_0d_2(R(h_1))$ is non-zero.

The differentials in weight 0 are now readily calculated using the Leibniz rule. In particular, $d_2(\zeta^{6/i}h_0) = \zeta^{6/i}a_0b_0$ since we have shown $d_2(\zeta^{6/i}) = 0$ in proposition 6.4. $\square$

6.5 The primes $\ell \geq 5$

We continue to write $\hat{\pi}$ for the ℓ-completion of π when the prime ℓ is clear from context. We now identify $\hat{\pi}_{n,0}(\mathbb{F}_q)$ for the remaining odd primes. Let ℓ be a prime, $\mathbb{F}_q$ a finite field with characteristic different from ℓ, and write i for the smallest positive integer satisfying $q^i \equiv 1 \bmod \ell$. The action of the Bockstein on $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$ is trivial if and only if $q^i \equiv 1 \bmod \ell^2$. Since $0 < i < \ell$, every solution of $q^i \equiv 1 \bmod \ell$ lifts uniquely to a solution of $q^i \equiv 1 \bmod \ell^2$ by Hensel's lemma. We list these congruences in table 6.3.

Prime ℓ	Congruence
2	$q \equiv 1 \bmod 4$
3	$q \equiv \pm 1 \bmod 9$
5	$q \equiv \pm 1, \pm 7 \bmod 25$
7	$q \equiv \pm 1, \pm 18, \pm 19 \bmod 49$
11	$q \equiv \pm 1, \pm 3, \pm 9, \pm 27, \pm 40 \bmod 121$

Table 6.3: Congruences for trivial Bockstein action on $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$

Proposition 6.12. Let $\mathbb{F}_q$ be a finite field of characteristic p. For any prime $\ell > 3$ with $\ell \neq p$ for which the action of the Bockstein on $H^{**}(\mathbb{F}_q;\mathbb{Z}/\ell)$ is trivial, we calculate $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}^s_n \oplus \hat{\pi}^s_{n+1}$ for $0 \leq n \leq 20$.

Proof. For any prime $\ell > 2$, the first non-zero group $\hat{\pi}_{n,0}(\mathbb{F}_q)$ with $n > 0$ occurs at $n = 2\ell - 4$, detected by the class $u\zeta^{\ell-2}[\xi_1]$ in the cobar complex. So for any prime $\ell > 11$, the group $\hat{\pi}_{n,0}(\mathbb{F}_q)$ is trivial when $0 < n \leq 20$. We now turn our attention to the remaining primes 5, 7, and 11.

Since we are assuming the action of the Bockstein on $H^{**}(\mathbb{F}_q)$ is trivial, the E_2 page of the mod ℓ motivic Adams spectral sequence is $E_2 \cong H_{**}(\mathbb{F}_q) \otimes \mathrm{Ext}_{\mathcal{A}^{top}_*}(\mathbb{F}_\ell, \mathbb{F}_\ell)$ by proposition 6.1. For the primes 5, 7, and 11, the necessary calculations in the cobar complex for $\mathcal{A}^{top}_*$ can be carried out in stems $s \leq 20$ without much trouble. See figures 8.8, 8.9, and 8.10 for charts of the E_2 page of the motivic Adams spectral sequence at the primes 5, 7, and 11. We find that the irreducible elements in $\mathrm{Ext}_{\mathcal{A}^{top}_*}(\mathbb{F}_\ell, \mathbb{F}_\ell)$ which appear in stem $s \leq 20$ are $a_0 = [\tau_0]$, $h_0 = [\xi_1]$, and the Massey product $R(a_0) = \langle h_0, h_0, a_0 \rangle$.

When $\ell = 5$, there are three cases based on the order of q in $\mathbb{F}_\ell^\times$, i.e., i is 1, 2, or 4. In any case, the class h_0 has weight 4, so $\zeta^{4/i}h_0$ and $\gamma\zeta^{4/i-1}h_0$ are non-zero classes in weight 0, and as the class $R(a_0)$ has weight 8, $\zeta^{8/i}R(a_0)$ and $\gamma\zeta^{8/i-1}R(a_0)$ are in weight 0. All differentials in the range $0 < s \leq 21$ are trivial $\square$

Chapter 7

The motivic Adams spectral sequence for finite fields $\mathbb{F}_q$ with non-trivial Bockstein action

Throughout this chapter, we assume the action of the Bockstein on $H^{**}(\mathbb{F}_q)$ is non-trivial. That is, at the prime $\ell = 2$ we work over a finite field $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$ and for a prime $\ell > 2$, we work over a finite field $\mathbb{F}_q$ with q relatively prime to ℓ which additionally satisfies $q^i \not\equiv 1 \bmod \ell^2$ where i is the order of q in $\mathbb{F}_\ell^\times$.

7.1 The E_2 page of the mod ℓ motivic Adams spectral sequence

We first analyze the E_2 page of the mod 2 motivic Adams spectral sequence, which is isomorphic to $\mathrm{Ext}(\mathbb{F}_q) = \mathrm{Ext}_{\mathcal{A}^{**}(\mathbb{F}_q)}(H^{**}(\mathbb{F}_q), H^{**}(\mathbb{F}_q))$. The action of the Bockstein on $H^{**}(\mathbb{F}_q; \mathbb{Z}/2)$ is non-trivial if and only if $q \equiv 3 \bmod 4$. In this case, the class $\rho \in H^{1,1}(\mathbb{F}_q; \mathbb{Z}/2)$ is non-trivial (see definition 3.3) and the action of the Bockstein on $H^{**}(\mathbb{F}_q)$ is determined by $\beta(\tau) = \rho$. We directly calculate the structure of $\mathrm{Ext}(\mathbb{F}_q)$ up to stem $s = 21$ using computer calculations discussed in chapter 8.

Proposition 7.1. When $q \equiv 3 \bmod 4$, the irreducible elements of $E_2(\mathbb{F}_q) \cong \mathrm{Ext}(\mathbb{F}_q)$ up to stem $s = 21$ are the classes listed in table 7.1.

Proof. This was obtained by computer calculation. See chapter 8 for more details about the program. Note the class τ does not appear in $\mathrm{Ext}^{0,(0,-1)}(\mathbb{F}_q)$ by the following calculation with the cobar complex (see definition 4.12).

$$d_{\mathcal{C}}(\tau[\,]) = [\eta_R(\tau)] + \tau[1] = [\tau] + [\rho\tau_0] + \tau[1] = \rho[\tau_0] \qquad \square$$

The ρ-Bockstein spectral sequence assists in the calculation of $\mathrm{Ext}(\mathbb{F}_q)$ when the action of the Bockstein is non-trivial. We will use proposition 7.1 to identify a non-trivial

Elt.	Filtr. (f,s,w)	Elt.	Filtr. (f,s,w)	Elt.	Filtr. (f,s,w)
ρ	$(0,-1,-1)$	$[\tau c_0]$	$(3,8,4)$	$[\tau Pc_0]$	$(7,16,8)$
$[\rho\tau]$	$(0,-1,-2)$	Ph_1	$(5,9,5)$	e_0	$(4,17,10)$
$[\tau^2]$	$(0,0,-2)$	$[\tau Ph_1]$	$(5,9,4)$	P^2h_1	$(9,17,9)$
h_0	$(1,0,0)$	Ph_2	$(5,11,6)$	$[\tau P^2h_1]$	$(9,17,8)$
h_1	$(1,1,1)$	$[\tau h_0h_3^2]$	$(3,14,7)$	f_0	$(4,18,10)$
$[\tau h_1]$	$(1,1,0)$	d_0	$(4,14,8)$	P^2h_2	$(9,19,10)$
h_2	$(1,3,2)$	$[\tau h_0^2d_0]$	$(6,14,7)$	c_1	$(3,19,11)$
$[\tau h_2^2]$	$(2,6,3)$	h_4	$(1,15,8)$	$[\tau c_1]$	$(3,19,10)$
h_3	$(1,7,4)$	$[\tau h_0^7h_4]$	$(8,15,7)$	$[\rho\tau g]$	$(4,19,10)$
$[\tau h_0^3h_3]$	$(4,7,3)$	Pc_0	$(7,16,9)$	$[\tau^2 g]$	$(4,20,10)$
c_0	$(3,8,5)$				

Table 7.1: Irreducible elements in $\mathrm{Ext}(\mathbb{F}_q)$ with stem $s \leq 21$ for $q \equiv 3 \bmod 4$

differential in the ρ-Bockstein spectral sequence in proposition 7.3. We briefly describe the construction of the ρ-Bockstein spectral sequence and refer the reader to [DI15, OØ13, Orm11] for more details.

Let $\mathcal{C}$ be the cobar complex for $\mathbb{F}_q$ defined in 4.12 at the prime $\ell = 2$. The filtration of $\mathcal{C}$ given by $0 \subseteq \rho\mathcal{C} \subseteq \mathcal{C}$ determines a spectral sequence, which in this case is just the long exact sequence associated to the short exact sequence of complexes

$$0 \to \rho\mathcal{C} \to \mathcal{C} \to \mathcal{C}/\rho\mathcal{C} \to 0.$$

Since the complexes $\rho\mathcal{C}$ and $\mathcal{C}/\rho\mathcal{C}$ are both isomorphic to the cobar complex over $\mathbb{C}$, the ρ-Bockstein spectral sequence is the following long exact sequence.

$$\cdots \rho\,\mathrm{Ext}^i(\mathbb{C}) \longrightarrow \mathrm{Ext}^i(\mathbb{F}_q) \longrightarrow \mathrm{Ext}^i(\mathbb{C}) \xrightarrow{d_1} \rho\,\mathrm{Ext}^{i+1}(\mathbb{C}) \cdots \tag{7.2}$$

Proposition 7.3. In the ρ-Bockstein spectral sequence for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, every irreducible element x of $\mathrm{Ext}(\mathbb{C})$ in stem $s \leq 19$ other than τ has $d_1x = 0$, whereas $d_1\tau = \rho h_0$ and $d_1([\tau g]) = \rho h_2 e_0$. Here, $[\tau g]$ is the irreducible element of $\mathrm{Ext}(\mathbb{C})$ in stem 20, weight 11, and filtration 4.

Proof. The differential d_1 vanishes on the classes h_0, h_1, c_0, Ph_1, d_0, Pc_0, e_0, P^2h_1 for degree reasons. The remaining differentials follow from the structure of $\mathrm{Ext}(\mathbb{F}_q)$ given in proposition 7.1. □

Example 7.4. In the ρ-Bockstein spectral sequence, we calculate $d_1\tau h_1 = \rho h_0 h_1 = 0$, since $d_1 h_1 = 0$ and $h_0 h_1$ vanishes in $\mathrm{Ext}(\mathbb{C})$. There is thus a class $[\tau h_1] \in \mathrm{Ext}^{1,(1,0)}(\mathbb{F}_q)$ which is irreducible.

At the primes $\ell > 2$, we use the analog of the ρ-Bockstein spectral sequence to identify the E_2 page of the mod ℓ motivic Adams spectral sequence. Recall from definition 3.3 that when the action of the Bockstein on $H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$ is non-trivial, the class $\gamma \in H^{1,1}(\mathbb{F}_q)$ is non-zero and $\beta(\zeta) = \gamma$. The γ-Bockstein spectral sequence is the long exact sequence of cohomology groups associated to the short exact sequence of chain complexes

$$0 \to \gamma\mathcal{C} \to \mathcal{C} \to \mathcal{C}/\gamma\mathcal{C} \to 0.$$

The complexes $\gamma\mathcal{C}$ and $\mathcal{C}/\gamma\mathcal{C}$ are isomorphic to the cobar complex for the Hopf algebroid $(\mathbb{F}_\ell[\zeta], \mathbb{F}_\ell[\zeta] \otimes \mathcal{A}_*^{top})$. Note that the Hopf algebroid $(\mathbb{F}_\ell[\zeta], \mathbb{F}_\ell[\zeta] \otimes \mathcal{A}_*^{top})$ is isomorphic as a bigraded Hopf algebroid to $(H_{**}(\mathbb{C}), \mathcal{A}_{**}(\mathbb{C}))$ if and only if $q \equiv 1 \bmod \ell$.

Proposition 7.5. The E_1 page of the γ-Bockstein spectral sequence is given by

$$E_1 \cong \mathrm{Ext}_{\mathbb{F}_\ell[\zeta]\otimes\mathcal{A}_*^{top}}(\mathbb{F}_\ell[\zeta], \mathbb{F}_\ell[\zeta]) \oplus \gamma\, \mathrm{Ext}_{\mathbb{F}_\ell[\zeta]\otimes\mathcal{A}_*^{top}}(\mathbb{F}_\ell[\zeta], \mathbb{F}_\ell[\zeta]).$$

The differential d_1 is determined by $d_1(\zeta^j[\,]) = -j\gamma\zeta^{j-1}[\tau_0]$.

Proof. Note that $\mathrm{Ext}_{\mathbb{F}_\ell[\zeta]\otimes\mathcal{A}_*^{top}}(\mathbb{F}_\ell[\zeta], \mathbb{F}_\ell[\zeta]) \cong \mathbb{F}_\ell[\zeta] \otimes \mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_\ell, \mathbb{F}_\ell)$. The cobar complex $\mathcal{C}(\mathbb{F}_q)$ is a differential graded algebra with respect to the juxtaposition product defined in 4.12. Let α be a cobar complex representative for a homogeneous class in $\mathrm{Ext}_{\mathbb{F}_\ell[\zeta]\otimes\mathcal{A}_*^{top}}(\mathbb{F}_\ell[\zeta], \mathbb{F}_\ell[\zeta])$. Then $\alpha = \zeta^j[\,] * \alpha'$ where α' is an element in the cobar complex for $\mathcal{A}_*^{top}$. We calculate $d_1(\zeta^j[\,] * \alpha') = d_1(\zeta^j[\,])\alpha'$ by our assumption that $d_{\mathcal{C}}(\alpha)$ is zero. But then the class $d_1(\zeta^j[\,])\alpha' = -j\gamma\zeta^{j-1}[\tau_0] * \alpha'$ will vanish if and only if $\ell \mid j$ or $[\tau_0] * \alpha' = 0$ in $\mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_\ell, \mathbb{F}_\ell)$. □

Remark 7.6. Note that this argument does not work at the prime $\ell = 2$ because one cannot pass between the cobar complex over $\mathbb{C}$ and the topological cobar complex since the relations on τ_i^2 over $\mathbb{C}$ involve the class τ.

7.2 Differentials in the mod ℓ motivic Adams spectral sequence

We now establish the analog of proposition 6.4 in the case where the Bockstein acts non-trivially on $H^{**}(\mathbb{F}_q)$, that is, we use the motivic Adams spectral sequence for $X = H\mathbb{Z}[\frac{1}{p}]$ over $\mathbb{F}_q$ to identify the differentials on τ^j and ζ^j in $\mathfrak{M}_{\mathbb{F}_q}(\mathbb{1})$.

Proposition 7.7. When $\ell = 2$ and $q \equiv 3 \bmod 4$, the E_1 page of $\mathfrak{M}(H\mathbb{Z}[\frac{1}{p}])$ is given by

$$E_1 \cong \mathbb{F}_2[\tau, \rho, h_0]/(\rho^2)$$

where $h_0 \in E_1^{1,(0,0)}$. For all $r \geq 1$ the differentials d_r vanish on $\rho\tau^j$ and h_0^j. For odd natural numbers j, we calculate $d_1\tau^j = \rho h_0$. Write $\lambda(q)$ for $\nu_2(q^2-1)$. If $r < \lambda(q) + \nu_2(n)$ the differentials $d_r\tau^{2n}$ vanish, and

$$d_{\lambda(q)+\nu_2(n)}\tau^{2n} = \rho\tau^{2n-1}h_0^{\lambda(q)+\nu_2(n)}.$$

Now let $\ell > 2$ and consider a finite field $\mathbb{F}_q$ with non-trivial Bockstein action on $H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$. Then the E_1 page of $\mathfrak{M}(H\mathbb{Z}[\frac{1}{p}])$ is the graded-commutative $\mathbb{F}_\ell$-algebra

$$E_1 \cong \mathbb{F}_\ell[\zeta, \gamma, a_0]/(\gamma^2)$$

where $a_0 \in E_1^{1,(0,0)}$. The differential d_1 vanishes on a_0^j, $\gamma\zeta^j$, and $\zeta^{\ell j}$ for all $j \geq 0$, but $d_1\zeta^j = \gamma\zeta^{j-1}a_0$ for natural numbers j with $\ell \nmid j$. The E_2 page takes the form

$$E_2 \cong \mathbb{F}_\ell[\gamma, \zeta^\ell, a_0]/(\gamma^2, a_0\gamma) \oplus \bigoplus_{j=1}^{\ell-1} \gamma\zeta^j\mathbb{F}_\ell[\zeta^\ell, a_0]/(a_0\gamma\zeta^j)$$

Let $\lambda(q) = \nu_\ell(q^{\ell i} - 1)$. For all $r \geq 2$ the differentials d_r vanish on a_0 and $\gamma\zeta^j$ for $0 \leq j \leq \ell - 1$. The differentials $d_r(\zeta^{\ell n})$ are trivial for $r < \lambda(q) + \nu_\ell(n)$ and

$$d_{\lambda(q)+\nu_\ell(n)}(\zeta^{\ell n}) = \gamma\zeta^{\ell n-1}a_0^{\lambda(q)+\nu_\ell(n)}$$

up to multiplication by a unit in $\mathbb{F}_\ell$.

Proof. This follows the proof of proposition 6.4. The H^{**}-Adams resolution given in (6.5) gives $E_1^{f,(s,w)} \cong \pi_{s,w}H\mathbb{Z}/\ell$. When $\ell = 2$, the order of $E_\infty^{*,(-1,j)}$ is $\nu_2(q^j - 1)$, so we conclude $d_1\tau = \rho h_0$. As we have $\nu_2(q^{2j} - 1) = \lambda(q) + \nu_2(j)$ for all natural numbers j, the claimed formulas for the differentials on τ^{2n} hold. A similar analysis goes through for $\ell > 2$, since for all natural numbers n we have $\nu_\ell(q^{\ell in} - 1) = \lambda(q) + \nu_\ell(n)$. $\square$

7.3 The prime 2

We work at the prime $\ell = 2$ in this section and consider a finite field $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$. We write $\hat{\pi}_{**}(\mathbb{F}_q)$ for $\pi_{**}(\mathbb{F}_q)^{\wedge}_2$. Multiplication by 2 in $\pi_{**}(\mathbb{F}_q)$ is detected in the mod 2 motivic Adams spectral sequence by the class $h_0 + \rho h_1$ in $\mathrm{Ext}(\mathbb{F}_q)$. A chart of the E_2 page of the motivic Adams spectral sequence is given in figure 8.4 and a chart of the E_∞ page is given in figure 8.5.

The stem $\pi_{0,0}(\mathbb{F}_q)$ is isomorphic to the Grothendieck-Witt group $GW(\mathbb{F}_q)$ by [Mor04]. The isomorphism $GW(\mathbb{F}_q) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ was established by Scharlau in [Sch85, Ch. 2, 3.3]. Recall that $\pi_0^s \cong \mathbb{Z}$ and $\pi_1^s \cong \mathbb{Z}/2$. Hence we conclude $\pi_{0,0}(\mathbb{F}_q) \cong \pi_0^s \oplus \pi_1^s$.

Proposition 7.8. When $q \equiv 3 \bmod 4$ and $0 \leq n \leq 18$, there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$.

Proof. The discussion preceding this proposition establishes the claim when $n = 0$. We now analyze the differentials and the group extension problem for $0 < n \leq 18$. The differentials d_r for $r \geq 2$ vanish on the following generators for degree reasons: $[\rho\tau]$, ρ, h_0, h_1, h_3, $[\tau h_2^2]$, $[\tau c_0]$, $[\tau P h_1]$, d_0, $[\tau P c_0]$, $[\tau P^2 h_1]$. Since $\hat{\pi}_1^s \cong \mathbb{Z}/2$ is a summand of $\hat{\pi}_{1,0}(\mathbb{F}_q)$, we must have $d_r[\tau h_1] = 0$ for all $r \geq 2$. Since $\hat{\pi}_3^s \cong \mathbb{Z}/8$ is a summand of $\hat{\pi}_{3,0}(\mathbb{F}_q)$, we must have $d_2(h_2) = 0$. An argument similar to that given for proposition 6.8, we conclude $d_2(h_4) = h_0 h_3^2$, $d_2(e_0) = h_1^2 d_0$, and $d_2(f_0) = h_0^2 e_0$ by comparison to $\mathfrak{M}(\overline{\mathbb{F}}_q)$. Also, we determine $d_r[\tau c_1] = 0$ for $r \geq 2$ by comparing with $\mathfrak{M}(\overline{\mathbb{F}}_q)$, as the class $[\tau c_1]$ must be a permanent cycle.

The one exceptional case is $d_3(h_0 h_4)$. Here we must have $d_3(h_0 h_4) = h_0 d_0 + \rho h_1 d_0$ in order for $\hat{\pi}_{14}^s = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ to be a summand of $\hat{\pi}_{14,0}(\mathbb{F}_q)$.

The elements in weight 0 are all of the form $[\tau^2]^i x$ or $[\rho\tau][\tau^2]^{i-1} x$ where x is not a multiple of τ^2 and weight $2i$, or of the form $\rho[\tau^2]^i x$ if x is not a multiple of τ^2 and of weight $2i+1$. The differentials of the elements in weight 0 are now determined by using the Leibniz rule. Since $\lambda(q) = \nu_2(q^2 - 1) \geq 3$, we have $d_2(\tau^2) = 0$. This is sufficient to ensure that for elements x in stem $s \leq 19$ there are no non-trivial differentials of the form $d_r[\tau^2]^i x = \rho\tau^{2i-1} h_0^r x$ when $[\tau^2]^i x$ has weight 0. This resolves all differentials in weight 0 for stems $s \leq 19$ and there are no hidden 2-extensions in this range. Hence for $0 < n \leq 18$ there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}^s_n \oplus \hat{\pi}^s_{n+1}$. □

Remark 7.9. It is unclear whether $d_2[\tau^2 g] = [\rho\tau g]$ or $d_2[\tau^2 g] = 0$. This is all that obstructs the identification of the stems $\hat{\pi}_{19,0}(\mathbb{F}_q)$ and $\hat{\pi}_{20,0}(\mathbb{F}_q)$.

7.4 The prime 3

Throughout this section, $\mathbb{F}_q$ is a finite field with characteristic different from 3. Let i be the smallest positive integer for which $q^i \equiv 1 \bmod 3$. The assumption that the Bockstein acts non-trivially on $H^{**}(\mathbb{F}_q; \mathbb{Z}/3)$ is equivalent to $q^i \not\equiv 1 \bmod 9$ which amounts to $q \equiv \pm 2, \pm 3, \pm 4 \bmod 9$. We write $\hat{\pi}$ for the 3-completion of a group π.

Proposition 7.10. Let $\mathbb{F}_q$ be a finite field of characteristic different from 3 for which $q^i \not\equiv 1 \bmod 9$. For any natural number n with $0 \leq n \leq 20$, there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}^s_n \oplus \hat{\pi}^s_{n+1}$.

Proof. We use proposition 7.5 and a calculation of $\mathrm{Ext}_{\mathcal{A}^{top}_*}(\mathbb{F}_3, \mathbb{F}_3)$ (see [Rav86, p. 11] or chapter 8) to identify the E_2 page of the motivic Adams spectral sequence. Every class $x \in \mathrm{Ext}^{f,s,w}_{\mathcal{A}^{top}_*}(\mathbb{F}_3, \mathbb{F}_3)$ with $a_0 x = 0$ contributes classes $\gamma\zeta^{w-1}x$ and $\zeta^w x$ in weight 0. If $a_0 x$ is non-zero, only the classes $\zeta^{\ell j}x$ are non-zero in the E_2 page. In the range $n \leq 21$, only the classes b_0, h_1, and b_0^2 are not killed by multiplication by a_0. Note that the weights of b_0, h_1, and b_0^2 are 6, 6, and 12 respectively. Since all of these weights are divisible by 3, we conclude that $\zeta^{6/i}b_0$, $\zeta^{6/i}h_1$, $\zeta^{12/i}b_0^2$, $\gamma\zeta^{(6/i)-1}b_0$, $\gamma\zeta^{(6/i)-1}b_0$, and $\gamma\zeta^{(12/i)-1}b_0^2$ are all non-zero classes in weight 0. This analysis identifies the E_2 page of the motivic Adams spectral sequence given in figure 8.7.

We deduce the differentials $d_2(h_1) = a_0 b_0$ and $d_2(R(h_1)) = b_0 R(a_0)$ from the comparison map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$ and the calculations in topology in [Rav86]. The Leibniz rule shows $d_2(\zeta^{6/i}h_1) = d_2(\zeta^{6/i})h_1 + \zeta^{6/i}a_0 b_0$. In the case where $\nu_\ell(q^6 - 1) = 2$, proposition 7.7 shows $d_2(\zeta^{6/i})$ is non-zero, and when $\nu_\ell(q^6 - 1) > 2$ the differential $d_2(\zeta^{6/i})$ vanishes. In either case, the group structure is as claimed in the 10 and 11 stem.

Since ζ and ζ^2 do not survive the γ-Bockstein spectral sequence, we must be careful calculating $d_2(\zeta^{10/i}R(h_1))$. If $i = 1$, the class in weight 0 coming from $R(h_1)$ is $[\zeta^3]^3[\zeta R(h_1)]$ and if $i = 2$, the class in weight 0 coming from $R(h_1)$ is $[\zeta^3]^2[\zeta^2 R(h_1)]$. When $i = 1$, the comparison map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$ sends $[\zeta^3]^3[\zeta R(h_1)]$ to $\zeta^{10}R(h_1)$. Since $d_2(R(h_1)) = b_0 R(a_0)$ is non-zero in $\mathfrak{M}(\overline{\mathbb{F}}_p)$ by [Rav86], it follows that $d_2(\zeta^{10}R(h_1)) = \zeta^{10}b_0R(a_0)$ in $\mathfrak{M}(\overline{\mathbb{F}}_p)$. As the differential $d_2([\zeta^3]^3[\zeta R(h_1)])$ must have image $\zeta^{10}b_0R(a_0)$ under the comparison map $\mathfrak{M}(\mathbb{F}_q) \to \mathfrak{M}(\overline{\mathbb{F}}_p)$, $d_2([\zeta^3]^3[\zeta R(h_1)])$ is non-zero. Similarly, when $i = 2$, $[\zeta^3]^2[\zeta^2 R(h_1)]$ maps to $\zeta^{10}R(h_1)$ in $\mathrm{Ext}(\overline{\mathbb{F}}_p)$ and so $d_2([\zeta^3]^2[\zeta^2 R(h_1)])$ is non-zero.

The remaining differential $d_2([\zeta^3]^{4/i}h_0R(h_1))$ is found to be non-zero by using the Leibniz rule. This resolves all differentials up to stem 21. The computed product structure shows there are no possible hidden 3-extensions in this range, hence the result. □

7.5 The primes $\ell \geq 5$

We continue to write $\hat{\pi}$ for the ℓ-completion of π when the prime ℓ is clear from context. We now identify $\hat{\pi}_{n,0}(\mathbb{F}_q)$ for the remaining odd primes. Let ℓ be a prime, $\mathbb{F}_q$ a finite field with characteristic different from ℓ, and write i for the smallest positive integer satisfying $q^i \equiv 1 \bmod \ell$. The action of the Bockstein on $H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$ is non-trivial if and only if $q^i \not\equiv 1 \bmod \ell^2$. At the primes up to 11, the appropriate congruences are those which do not appear in table 6.3.

Proposition 7.11. Let $\mathbb{F}_q$ be a finite field of characteristic p. For any prime $\ell \geq 5$ with $\ell \neq p$ for which the action of the Bockstein on $H^{**}(\mathbb{F}_q; \mathbb{Z}/\ell)$ is non-trivial, we calculate $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}^s_n \oplus \hat{\pi}^s_{n+1}$ for $0 \leq n \leq 20$.

Proof. For any prime $\ell > 2$, the first non-zero group $\hat{\pi}_{n,0}(\mathbb{F}_q)$ with $n > 0$ occurs at $n = 2\ell - 4$, detected by the class $\gamma\zeta^{\ell-2}[\xi_1]$ in the cobar complex. So for any prime $\ell > 11$, the group $\hat{\pi}_{n,0}(\mathbb{F}_q)$ is trivial when $0 < n \leq 20$.

At the primes 5, 7, and 11, simple cobar complex calculations can be used to identify the structure of $\mathrm{Ext}(\mathbb{F}_q)$ in the range $s \leq 21$. The reader may consult chapter 8 for a discussion about a computer program to perform these calculations. Charts of the E_2

page of the motivic Adams spectral sequence for the primes 5, 7, and 11 are given in figures, 8.8, 8.9, and 8.10.

At the prime $\ell = 5$, there are three cases to consider depending on the order of q in $\mathbb{F}_5^\times$, i.e., q has order 1, 2, or 4. In $\mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_5, \mathbb{F}_5)$, the irreducible elements in the range $s \leq 21$ are $a_0 = [\tau_0]$, $h_0 = [\xi_1]$, and $R(a_0) = \langle h_0, h_0, a_0 \rangle$. The products $a_0 h_0$ and $a_0 R(a_0)$ are trivial, so that there are non-trivial classes coming from $\zeta^j h_0$ and $\zeta^j R(a_0)$. For example, when $i = 1$ a cobar representative for the non-zero class in $E_2^{1,(7,0)}$ is

$$\zeta^4[\xi_1] + \gamma\zeta^3(4[\tau_1] + [\tau_0\xi_1]),$$

and a representative for the non-zero class in $E_2^{2,(15,0)}$ is

$$\zeta^8(2[\xi_1^2|\tau_0] - [\xi_1|\tau_1]) + \gamma\zeta^7([\tau_1|\tau_1] + 3[\tau_0\xi_1|\tau_1] - [\xi_1|\tau_0\tau_1] + [\tau_1\xi_1|\tau_0] - [\tau_0\xi_1^2|\tau_0]).$$

In the range $0 < s \leq 21$, all differentials vanish for degree reasons when $\ell = 5$. We conclude that $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$ for $0 \leq n \leq 20$. The non-trivial groups may be read off from figure 8.8.

The E_2 page of the motivic Adams spectral sequence at the prime $\ell = 7$ in the range $s \leq 20$ may be calculated in the same fashion as the prime 5. In $\mathrm{Ext}_{\mathcal{A}_*^{top}}(\mathbb{F}_7, \mathbb{F}_7)$ for $s \leq 21$, the only irreducible elements are ζ, $a_0 = [\tau_0]$ and $h_0 = [\xi_1]$. An analysis of the γ-Bockstein spectral sequence yields the structure of the E_2 page in the same manner as at the prime $\ell = 5$. We conclude that when $\ell = 7$ there is an isomorphism $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$ for $0 \leq n \leq 20$. The non-trivial groups may be read off from figure 8.9.

The E_2 page of the motivic Adams spectral sequence at the prime $\ell = 11$ is entirely analogous to the situation at the prime 7 in the range $s \leq 20$. It follows that $\hat{\pi}_{n,0}(\mathbb{F}_q) \cong \hat{\pi}_n^s \oplus \hat{\pi}_{n+1}^s$ for $0 \leq n \leq 20$. The non-trivial groups may be read off from figure 8.10. $\square$

Chapter 8

Computer assisted Ext calculations

8.1 Minimal resolution

The computer calculations used in this dissertation at the prime 2 were performed with the program written by Fu and Wilson [FW15] at `https://github.com/glenwilson/MassProg`. The program is written in python, and calculates $\mathrm{Ext}(F)$ for the fields $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}_q$ by producing a minimal resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$-modules in a range. The program then applies the functor $\mathrm{Hom}_{\mathcal{A}^{**}(F)}(-, H^{**}(F))$ to the minimal resolution and calculates cohomology in each degree.

To calculate a free resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$-modules, we first need the program to efficiently perform calculations in $\mathcal{A}^{**}(F)$. The mod 2 motivic Steenrod algebra is a free left H^{**}-module with the Steenrod square operations Sq^I corresponding to the admissible sequences I as a basis. Given any class $x \in \mathcal{A}^{**}(F)$, the program applies the relations given in proposition 3.6 to arrive at a canonical form for x in terms of the basis $\{\mathrm{Sq}^I \mid I \text{ is admissible}\}$. This can be very time consuming, so the program uses a database to store the canonical forms of certain elements $x \in \mathcal{A}^{**}(F)$.

With the algebra of $\mathcal{A}^{**}(F)$ available to the program, it then proceeds to calculate a minimal resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$-modules. This is where a great deal of computational effort is spent. To clarify what a minimal resolution is in practice, let $\prec$ denote the order on $\mathbb{Z} \times \mathbb{Z}$ given by $(m_1, n_1) \prec (m_2, n_2)$ if and only if $m_1 + n_1 < m_2 + n_2$, or $m_1 + n_1 = m_2 + n_2$ and $n_1 < n_2$. The reader is encouraged to compare this definition with the definition by McCleary in [McC01, 9.3] and consult Bruner's primer [Bru09] for detailed calculations of a minimal resolution for the Adams spectral sequence of topology.

Definition 8.1. A resolution of $H^{**}(F)$ by $\mathcal{A}^{**}(F)$-modules $H^{**}(F) \leftarrow P^\bullet$ is a minimal resolution if the following conditions are satisfied.

1. Each module P^i has an ordered basis $h_i(j)$ such that if $j \leq k$ then $\deg h_i(j) \preceq \deg h_i(k)$.
2. For any k, $\mathrm{im}(h_i(k)) \notin \mathrm{im}(\langle h_i(j) \,|\, j < k \rangle)$.
3. The element $\deg h_i(j)$ is minimal with respect to degree in the order $\prec$ over all elements in $P^{i-1} \setminus \mathrm{im}(\langle h_i(j) \,|\, j < k \rangle)$.

The computer program calculates the first n maps and modules in a minimal resolution up to bidegree $(2n, n)$. With this, it then calculates the dual of the resolution by applying the functor $\mathrm{Hom}_{\mathcal{A}^{**}(F)}(-, H^{**}(F))$ to the resolution $P^\bullet$. With the cochain complex $\mathrm{Hom}_{\mathcal{A}^{**}(F)}(P^\bullet, H^{**}(F))$ in hand, the program calculates cohomology in each degree, that is, $\mathrm{Ext}^{f,(s+f,w)}(\mathbb{F}_q)$.

As the program calculates an explicit resolution of $H^{**}(F)$, the products of elements in $\mathrm{Ext}(F)$ can be obtained from the composition product (see [McC01, 9.5]).

8.2 Cobar complex

The computer calculations used in this dissertation at the primes $\ell > 2$ were performed with the program available at `https://github.com/glenwilson/CobarComplex`. The program calculates $\mathrm{Ext}(\mathbb{F}_q)$ and $\mathrm{Ext}_{\mathcal{A}^{top}_*}(\mathbb{F}_\ell, \mathbb{F}_\ell)$ with a straightforward implementation of the cobar complex. This is inefficient, but it suffices to identify the structure needed in the low degrees considered in this dissertation.

8.3 Charts

The weight 0 part of the E_2 page of the mod 2 motivic Adams spectral sequence over $\mathbb{F}_q$ is depicted in figures 8.1 and 8.4 according to the case $q \equiv 1 \bmod 4$ or $q \equiv 3 \bmod 4$. The weight 0 part of the E_∞ page of the mod 2 motivic Adams spectral sequence over $\mathbb{F}_q$ can be found in figures 8.2, 8.3, and 8.5.

In each chart, a circular or square dot in grading (s, f) represents a generator of the $\mathbb{F}_2$ vector space in the graded piece of the spectral sequence. The square dots are used to indicate that the given element is divisible by u, ρ, or $\rho\tau$, depending on the case. Circular dots denote elements which are not divisible by u, ρ, or $\rho\tau$. In figure 8.5, there is an oval dot which corresponds to the class with representative $\tau^8\rho h_1 d_0 \equiv \tau^8 h_0 d_0$, as the class $\rho h_1 d_0 + h_0 d_0$ is killed.

The labels in the chart correspond to those elements coming from irreducible elements of $\mathrm{Ext}(\mathbb{F}_q)$. However, most of these elements must be multiplied by the appropriate power of τ or $[\tau^2]$ to land in the weight 0 part of the spectral sequence. We leave this off in the notation, as the weights of these elements are listed in tables 6.1 and 7.1.

We indicate that the product of a given class by h_0 with a solid, vertical line. The arrow in the 0-stem indicates that h_0^j is non-zero for all natural numbers j. In the case $q \equiv 3 \bmod 4$ multiplication by ρh_1 plays an important role, so non-zero products by ρh_1 are indicated by dashed vertical lines. In particular, when $q \equiv 3 \bmod 4$, multiplication by 2 in $\hat{\pi}_{**}(\mathbb{F}_q)$ is detected by multiplication by $h_0 + \rho h_1$. The lines of slope 1 indicate multiplication by τh_1 or $[\tau h_1]$ depending on the case.

Dotted lines are used in two separate instances in these charts. The first use is in figure 8.3, where dotted lines indicate hidden extensions by h_0 and τh_1. The other instance is in figure 8.5 to indicate an unknown d_2 differential.

The chart for $\mathrm{Ext}(\mathbb{F}_q)$ at the prime $\ell = 3$ with trivial Bockstein is given in figure 8.6 and with nontrivial Bockstein in figure 8.7. In figure 8.7, we indicate the names of the classes which appear in weight 0 after multiplying by an appropriate power of $[\zeta^3]$. A label which appears above a dot corresponds to the case where $q \equiv 1 \bmod 3$ and labels which appear below a dot correspond to the case $q \equiv 2 \bmod 3$, but dots with just a single label are valid in either case. The vertical lines indicate multiplication by a_0 which detects multiplication by 3.

The charts for $\mathrm{Ext}(\mathbb{F}_q)$ at the primes 5, 7, and 11 are given in figures 8.8, 8.9, and 8.10. We do not indicate the names of the irreducible elements in the figure, but rather give the weights in which the classes appear. The regions which are skipped are trivial.

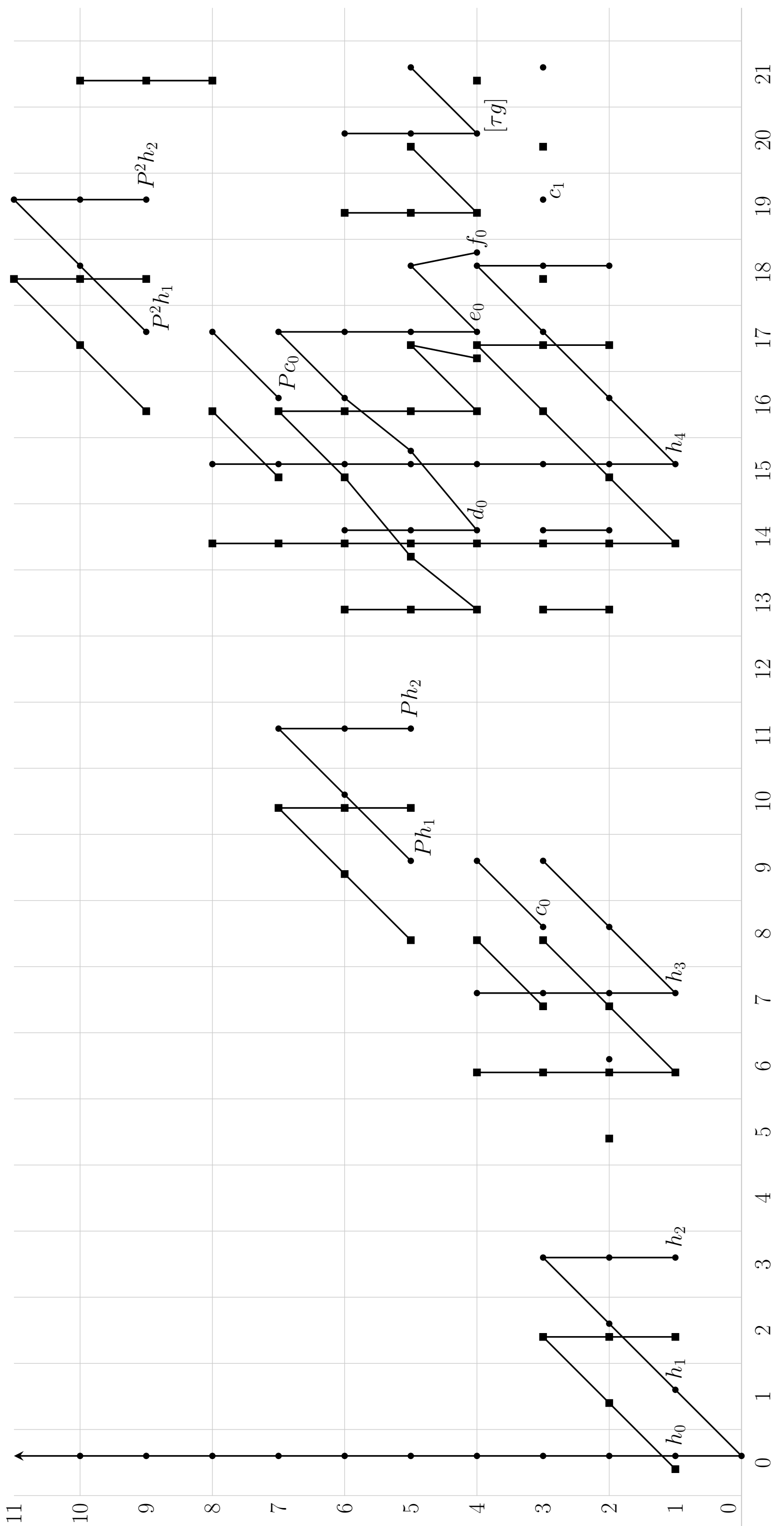

Figure 8.1: E_2 page of the mod 2 MASS for $\mathbb{F}_q$ with $q \equiv 1 \bmod 4$, weight 0

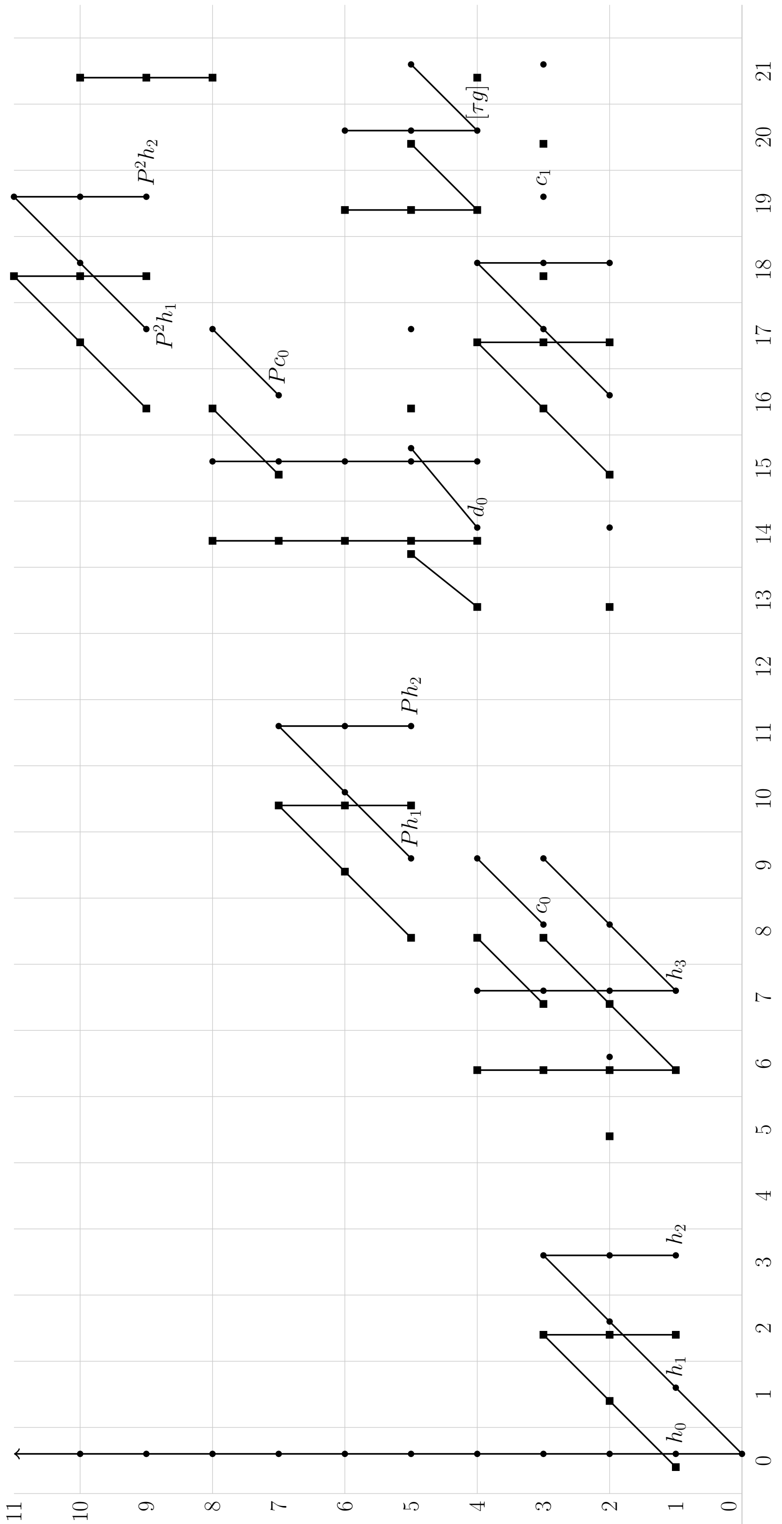

Figure 8.2: E_∞ page of the mod 2 MASS for $\mathbb{F}_q$ with $q \equiv 1 \bmod 8$, weight 0

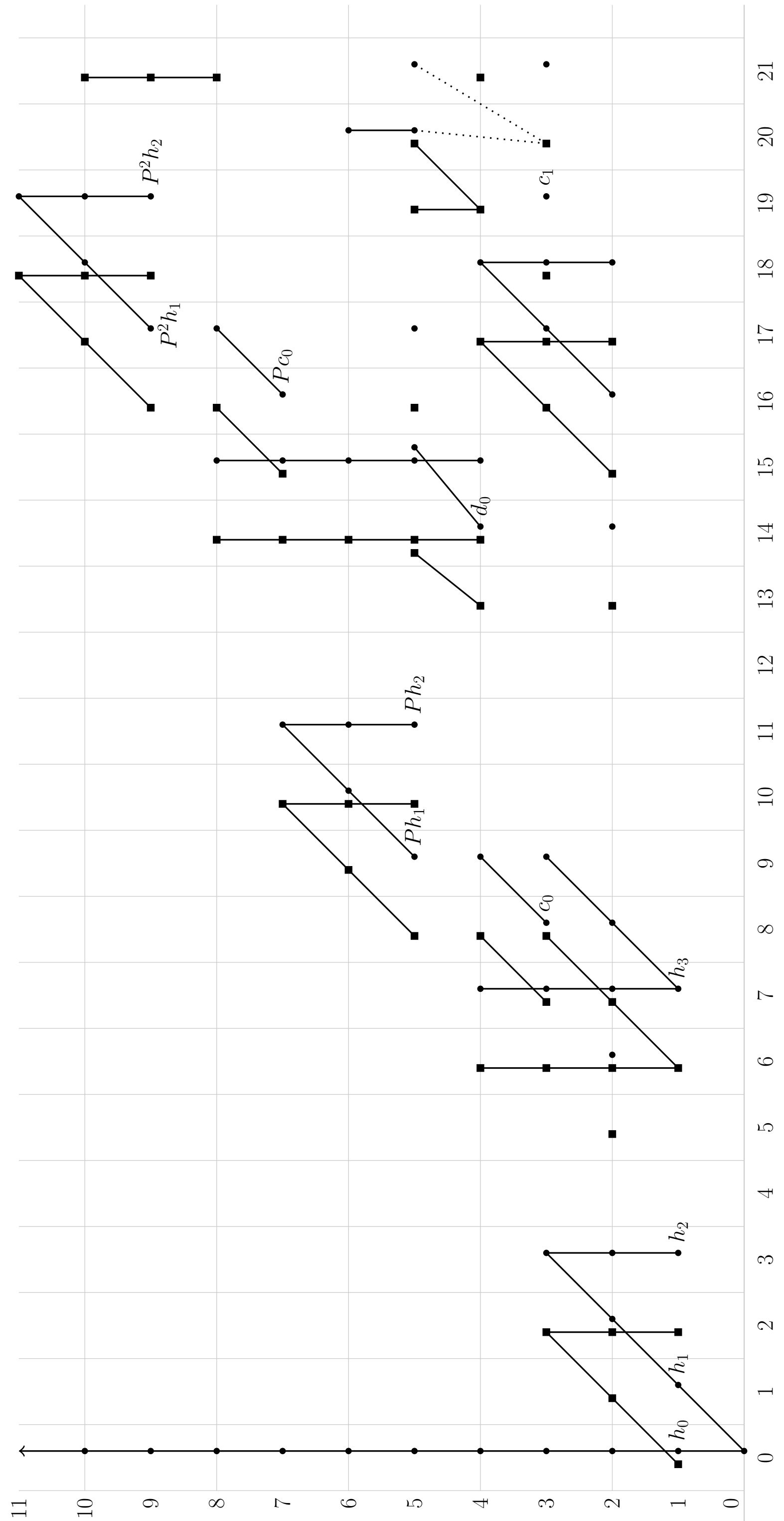

Figure 8.3: E_∞ page of the mod 2 MASS for $\mathbb{F}_q$ with $q \equiv 5 \bmod 8$, weight 0

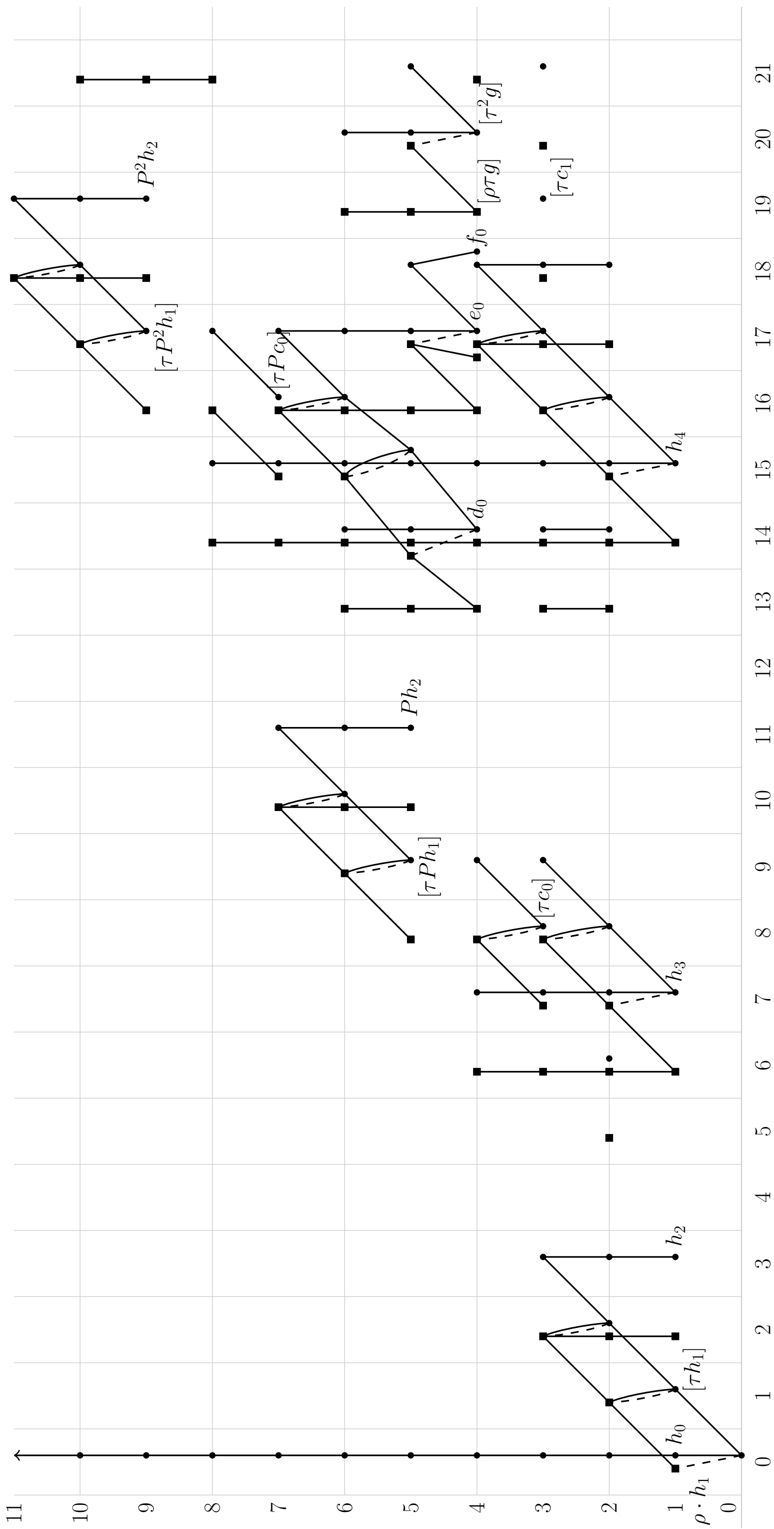

Figure 8.4: E_2 page of the mod 2 MASS for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, weight 0

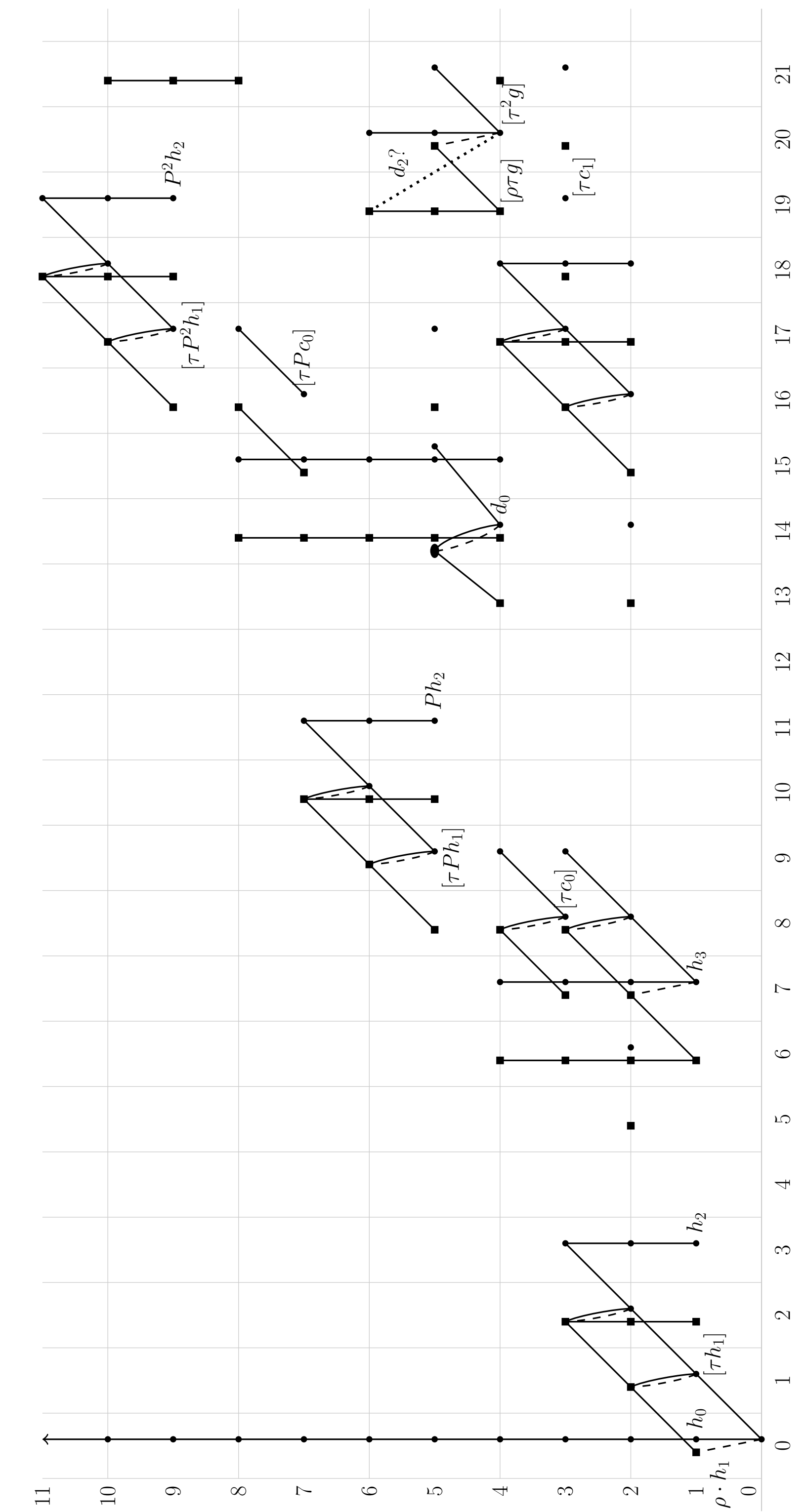

Figure 8.5: E_∞ page of the mod 2 MASS for $\mathbb{F}_q$ with $q \equiv 3 \bmod 4$, weight 0

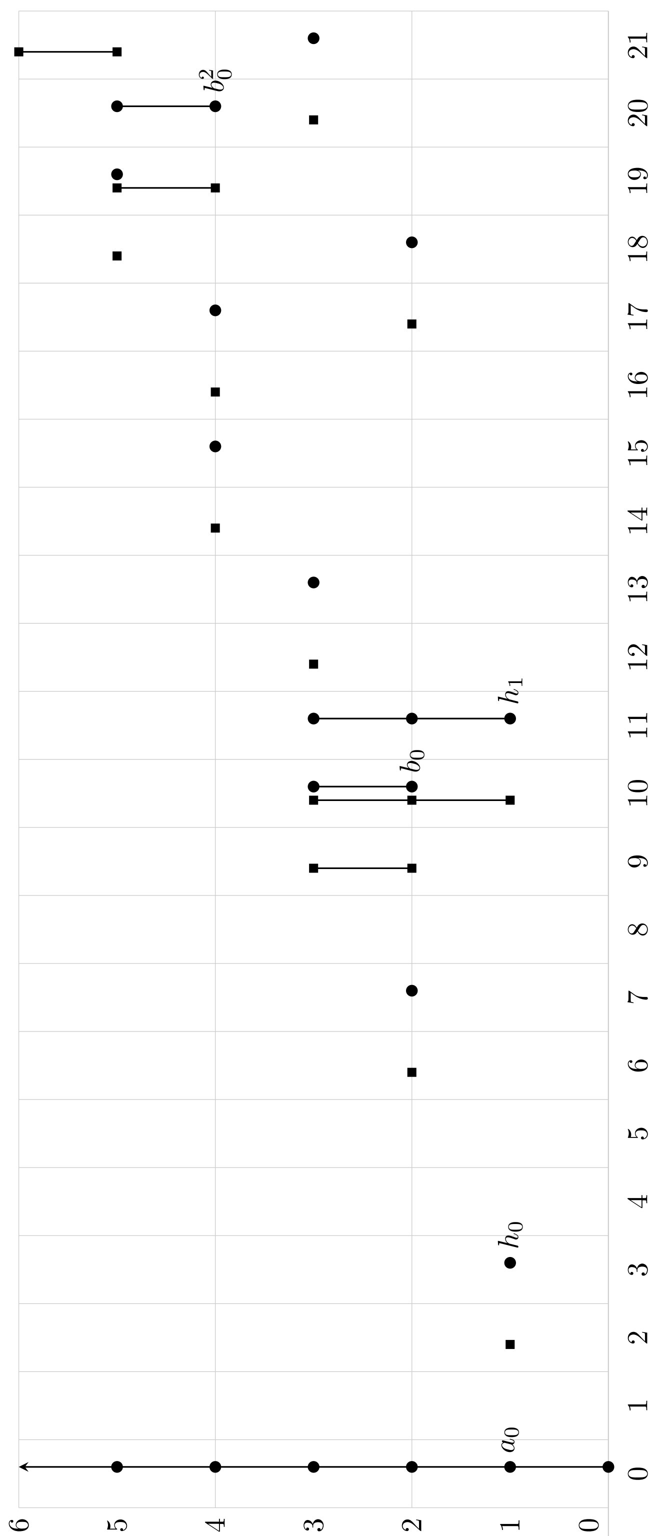

Figure 8.6: E_2 page of the mod 3 MASS for $\mathbb{F}_q$ with $\beta = 0$, weight 0

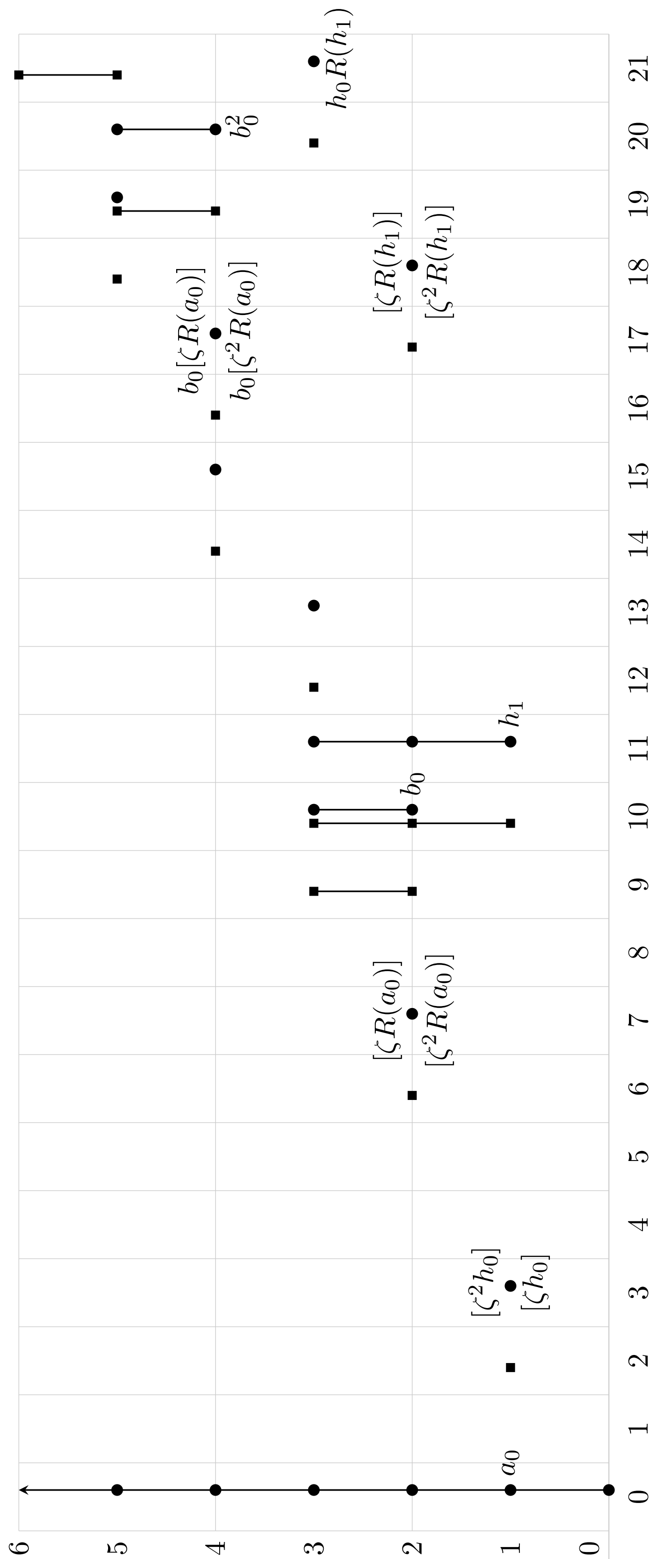

Figure 8.7: E_2 page of the mod 3 MASS for $\mathbb{F}_q$ with $\beta \neq 0$, weight 0

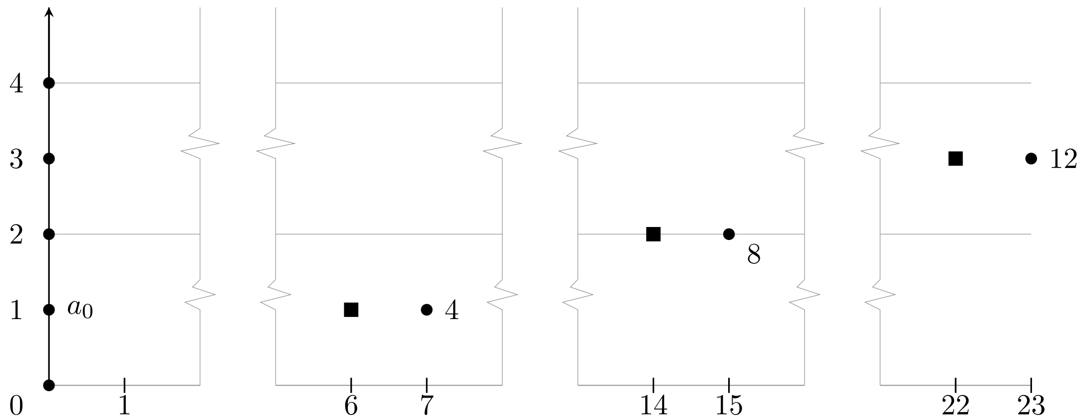

Figure 8.8: E_2 page of the mod 5 MASS for $\mathbb{F}_q$, weight 0

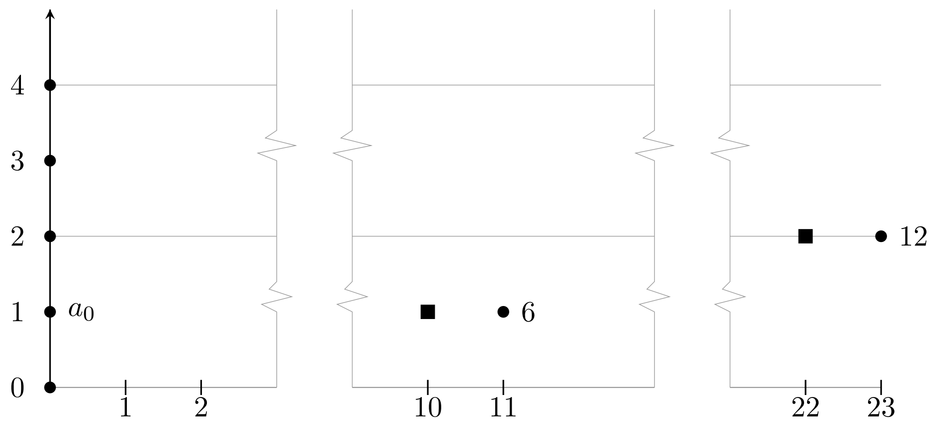

Figure 8.9: E_2 page of the mod 7 MASS for $\mathbb{F}_q$, weight 0

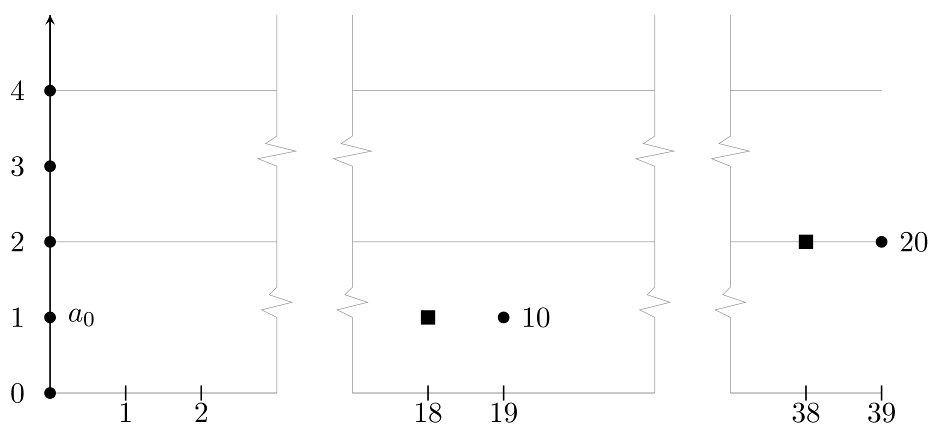

Figure 8.10: E_2 page of the mod 11 MASS for $\mathbb{F}_q$, weight 0

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