Ausgewählte Kapitel aus der Gruppentheorie: Σ-Theorie

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Lecture 1

1 Notation and terminology

1.1 Groups and their modules:

Definition 1. Let G be a goup. A G-module A is an additive group with a G-action.

A G-module clearly defines a $\mathbb{Z}G$-module structure on A, and a $\mathbb{Z}G$-module structure on A clearly defines a G-module structure on A. The distinction will be blurred in this regard, and one often just says that A is a G-module.

The free G-modules are up to isomorphism $\oplus\mathbb{Z}G$ with basis $\mathfrak{X}$.

For an arbitrary G-module A, there exists a free G-module F and a surjective map $\epsilon : F \twoheadrightarrow A$, such that $G \cong F/\ker\epsilon$. This will be called a *free presentation of the G-module A.*

1.2 Group actions on metric spaces:

Let (M, d) denote a metric space M with metric d.

Definition 2. A group G acts on the metric space (M, d) if G acts by isometries. That is, there is a map $\rho : G \to \operatorname{Isom}(M)$. We will often adopt the notation for $g \in G$, $m \in M$, $gm = \rho(g)\, m = \rho_g(m)$.

Important examples for our consideration are $\mathbb{E}^n$ and $\mathbb{H}^2$.

Example 1. We recall that $\operatorname{Isom}(\mathbb{E}^n) = \operatorname{Transl}(\mathbb{E}^n) \rtimes O(n)$ where we identify $\operatorname{Transl}(\mathbb{E}^n) = \mathbb{R}^n$ and the action of $O(n)$ on $\operatorname{Transl}(\mathbb{E}^n)$ is induced via matrix multiplication on $\mathbb{R}^n$. That is, $(a, A) * (b, B) = (a + Ab, AB)$.

However, for what follows we will be interested mainly in group actions $\rho : G \to \operatorname{Transl}(\mathbb{E}^n)$. In this case, we see that $G' \subseteq \ker\rho$, hence we have the induced homomorphism $\bar{\rho} : G_{\text{ab}} = G/G' \to \operatorname{Transl}(\mathbb{E}^n)$. If we consider $\sqrt{G'} := \left\{g \in G \,|\, \exists k \in \mathbb{N}, g^k \in G'\right\}$, we note that $\sqrt{G'} \subseteq \ker\rho$ since $\operatorname{Transl}(\mathbb{E}^n)$ is torsion free. Thus we have the induced homomorphism $\bar{\bar{\rho}} : G/\sqrt{G'} \to \operatorname{Transl}(\mathbb{E}^n)$. Note that $G/\sqrt{G'}$ is torsion free and Abelian. Thus if we are concerned with finitely generated groups G, it suffices those group actions where $Q = G/\ker\rho \cong \mathbb{Z}^k$.

For $Q = \mathbb{Z}^n$, there is a canonical action of Q on $\mathbb{E}^n$, namely, left translation on the identification $Q \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$, i.e. $g \cdot (g' \otimes r) := (gg') \otimes r$. This action satisfies the following two helpful properties:

1. The orbit of the origin $0 \in \mathbb{R}^n$ under this action is the standard lattice $\mathbb{Z}^k$ which inherits the discrete topology.
2. Let W be the closed cube of unity in $\mathbb{R}^n$. Then the orbit of W under the induced action covers $\mathbb{R}^n$. Such an action is called co-compact.

Not all actions satisfy these two conditions as we now demonstrate.

Example 2. Let $\rho : \mathbb{Z}^n \to \mathrm{Transl}\left(\mathbb{E}^n\right)$ be the canonical action. Define $\rho' : \mathbb{Z}^n \to \mathrm{Transl}\left(\mathbb{E} \oplus \mathbb{E}^n\right)$ by $g\left(x_1, ..., x_{n+1}\right) = \left(x_1, x_2 + g_1, ..., x_{n+1} + g_n\right)$. This action is not co-compact, but condition one is satisfied in the sense that the orbit of 0 is discrete.

Example 3. Define $\rho : \mathbb{Z}^2 \to \mathrm{Transl}\left(\mathbb{E}\right)$ via $gr = r + g_1 + \sqrt{2}g_2$. This is easily seen to be an action which is co-compact, and one can verify that the orbit of 0 is indeed dense in $\mathbb{R}$.

1.3 Boundary of (M, d) and horoballs:

In this paper, we make the convention that (M, d) is always a CAT(0) space. We provide the basic definitions and important properties of CAT(0) spaces, and refer the reader to the book by Bridson & Haefliger for a more precise treatment.

1.3.1 CAT(0) Spaces

Definition 3. Let $\Delta = (\gamma_1, \gamma_2, \gamma_3)$ be a geodesic triangle in (M, d) with corresponding endpoints a_i, where the γ_i are the geodesic edges. Consider a comparison triangle $\Delta' = (\gamma_1', \gamma_2', \gamma_3') \subseteq \mathbb{E}^2$ with corresponding endpoints a_i' such that $d\left(a_i, a_j\right) = d\left(a_i', a_j'\right)$ for all $i, j \in \{1, 2, 3\}$. The metric space (M, d) is then said to be of type CAT(0) if $d\left(\gamma_i\left(t\right), \gamma_j\left(s\right)\right) \leq d\left(\gamma_i'\left(t\right), \gamma_j'\left(s\right)\right)$. for all choices of i, j and all $t \in [0, d\left(a_i, a_{i+1 \mod 3}\right)]$ and $s \in [0, d\left(a_j, a_{j+1 \mod 3}\right)]$.

One can generalize the definition of CAT(0) to CAT(k) by simply replacing the comparison space $\mathbb{E}^2$ with the model surface M_k^2 of constant curvature k. Then a metric space (M, d) is CAT(k) if triangles in M are not fatter than those in M_k^2.

In particular, the Euclidean and hyperbolic spaces are CAT(0) and we will soon drop the generality and concentrate on the Euclidean case.

1.3.2 Horoballs

Definition 4. A geodesic ray is an isometric embedding of the metric space $[0, \infty)$, $\gamma : [0, \infty) \hookrightarrow M$.

We now define an equivalence relation on the geodesic rays of a metric space which generalizes the notion of parallel rays in $\mathbb{E}^n$.

Definition 5. Two geodesic rays γ_1, γ_2 are parallel (or $\gamma_1 \sim \gamma_2$) if and only if $d\left(\gamma_1\left(t\right), \gamma_2\left(t\right)\right)$ is bounded.

Definition 6. We now define $\partial M := \{\gamma \mid \gamma \text{ a geodesic ray}\} / \sim$.

One can also define a topology on ∂M, (which I will hopefully explain shortly). With this, we have the following computations of ∂M.

Example 4. Let $M = \mathbb{E}^n$. Then $\partial \mathbb{E}^n = \mathbb{S}^{n-1}$.

Example 5. Let $M = \mathbb{H}^2$. Then $\partial \mathbb{H}^2 = \mathbb{S}^1$, as in the circle model, equivalence classes of parallel lines are determine by the points along the boundary of the disc.

Before we define half spaces in (M, d), we note that for a point $A \in M$, there is a unique geodesic ray in each equivalence class (one can also say in the direction e), written $\gamma_{e,A}$ which satisfies $\gamma_{e,A}(0) = A$. If we consider the case $M = \mathbb{E}^2$, when given $\gamma_{e,A}$ we can consider for a point $t \in \operatorname{im} \gamma_{e,A}$, the open ball $B(t; d(t, A))$ centered at t with radius $d(t, A)$. The union $\bigcup_{t \in \operatorname{im} \gamma_{e,A}} B(t; d(t, A))$ then gives us a subspace whose closure is the closed half space in $\mathbb{E}^2$. This idea is formalized in the following definition.

Definition 7. In a CAT(0) metric space (M, d), we define the open horoball with respect to $\gamma_{e,A} = \gamma$, a geodesic ray with base point A in the direction e to be

$$\mathcal{H}_{e,A} := \bigcup_{t \in \operatorname{im} \gamma} B(t; d(t, A)).$$

For the closed horoball, we simply take the closure of the open horoball $\overline{\mathcal{H}}_{e,A}$.

Example 6. In $M = \mathbb{H}^2$ seen as the circle model, we can compute $\overline{\mathcal{H}}_{e,A}$ to be the intersection of the Euclidean disc which is tangent to ∂M at e and whose boundary passes through A.

Example 7. In $\mathbb{S}^2$, there do not exist any geodesic rays, and hence, we cannot define a boundary or horoballs in this manner.

1.3.3 Properties of CAT(0) Spaces

$\texttt{CAT}(0)$-1. For all $B \in \partial \mathcal{H}_{e,A}$, we have $\mathcal{H}_{e,A} = \mathcal{H}_{e,B}$.

$\texttt{CAT}(0)$-2. The set of horoballs based at $e \in \partial M$ is linearly ordered by $\subseteq$.

$\texttt{CAT}(0)$-3. For horoballs $\mathcal{H}_{e,A'} \subseteq \mathcal{H}_{e,A}$ and all $B, C \in \partial \mathcal{H}_{e,A}$, we set B' (resp. C') equal to $\operatorname{im} \gamma_{e,B} \cap \partial \mathcal{H}_{e,A'}$. We then have $d(B, B') = d(C, C')$.

2 The Invariant $\Sigma^0(G; A)$

2.1 Control functions:

We begin with a few examples of control functions, and then give an explicit definition.

Example 8. Consider the map $h : G \to M$ defined by choosing an origin $b \in M$ and setting $h(g) = gb$. This then satisfies $h(gg') = gh(g')$, i.e. h is a G-map.

Definition 8. $\mathfrak{f}M = \{A \in \mathcal{P}(M) \mid \operatorname{card} A < \infty\}$ where $\mathcal{P}(M)$ denotes the power set of M.

Definition 9. We can consider an element $\lambda \in RG$ of the group ring over R to be a function $\lambda : G \to R$ in the following way. When $\lambda = \sum_{g \in G} n_g g$, we set $\lambda(g) = n_g$.

Definition 10. Define the support of an element $\lambda \in RG$ to be $\operatorname{supp}(\lambda) := \{g \in G \mid \lambda(g) \neq 0\}$.

Hence with this language, we can consider RG to be all functions $\lambda : G \to RG$ with finite support. One can define the operations in the obvious manner and show that the notions are equivalent.

Example 9. Consider $h : \mathbb{Z}G \to \mathfrak{f}M$ defined by $h(\lambda) = \{gb \mid g \in \operatorname{supp}(\lambda)\}$. This once again is a G-transformation when we define $g\{a_0, ..., a_k\} = \{ga_0, ..., ga_k\}$ for $\{a_0, ..., a_k\} \in \mathfrak{f}M$.

Definition 11. Let F be a finitely generated free G-module with basis $\mathfrak{X}$. We define $Y := G\mathfrak{X} = \{gx \mid g \in G, x \in \mathfrak{X}\}$. Thus Y is a $\mathbb{Z}$-basis of F on which G operates freely, hence $F = \mathbb{Z}Y = \oplus_{y \in Y} \mathbb{Z}y$. We also have for any $w \in F$ a unique expression in terms of the basis $w = \sum_{y \in Y} n_y y$.

We now come to our main definition of a control function.

Definition 12. To construct a control function $h : F \to \mathfrak{f}M$, do the following:

1. choose for each $x \in \mathfrak{X}$ an arbitrary $h(x) \in \mathfrak{f}M \setminus \{\emptyset\}$
2. for $y = gx \in Y$ define $h(y) := gh(x)$;
3. for $w \in F$, define $h(w) = \bigcup_{y \in \operatorname{supp}(w)} h(y)$.

We note that such functions fulfill the following properties:

i. $h(0) = \emptyset$;

ii. $h(gw) = gh(w)$ for all $g \in G$ and all $w \in F$;

iii. $h(mw) = h(w)$ for $m \in \mathbb{Z} \setminus \{0\}$;

iv. $h(w + w') \subseteq h(w) \cup h(w')$.

Control functions don't work nicely with addition. In some circumstances, we do however have $h(w + w') = h(w) \cup h(w')$. A few examples are:

1. $\operatorname{supp}(w) \cap \operatorname{supp}(w') = \emptyset$;
2. $w(y) \geq 0$ for all $y \in Y$.

Remark 1. Note that a control function h depends on the choice of basis $\mathfrak{X}$ and the assignment $\mathfrak{X} \to \mathfrak{f}M$.

We will find it convenient to define generalized control functions.

Definition 13. A generalized control function is a map $h : F \to \mathfrak{f}M$ which satisfies conditions i.–iv. above.

A control function satisfies $h(w) = \emptyset$ iff $w = 0$, whereas a generalized control function may not. One reason why the study of generalized control functions will be useful is because for a control function $h : F \to \mathfrak{f}M$ and a homomorphism $\phi : F' \to F$, the composition $h \circ \phi$ is a generalized control function, but will not be a control function in general.

Definition 14. We define an ϵ-neighborhood (Umgebung) of a set $S \subseteq M$ to be $U_\epsilon(S) := U(S;\epsilon) := \{a \in M \mid \exists s \in S, d(s,a) < \epsilon\}$.

Proposition 1. *Let F be a finitely generated free G-module with respect to the bases $\mathfrak{X}$. Then for a control function $h : F \to \mathfrak{f}M$ defined with respect to $\mathfrak{X}$ and a generalized control function $h' : F \to \mathfrak{f}M$, there exists $\delta > 0$ such that for any $w \in F$, we have $h'(w) \subseteq U_\delta(h(w))$.*

Proof. Choose $\delta := \max\left\{d(a,b) \mid a \in \bigcup_{x\in\mathfrak{X}} h(x), b \in \bigcup_{x\in\mathfrak{X}} h'(x)\right\}$. The maxima are defined as the sets $\mathfrak{X}$, $h(x)$ and $h'(x)$ are all finite sets. From this choice, it is clear that $h'(x) \subseteq U_\delta(h(x))$ for all $x \in \mathfrak{X}$.

Note that for all $g \in G$, we have for $A, B \in \mathfrak{f}M$, $gU_\delta(A) = U_\delta(gA)$ as G acts by isometries.

Now let $w = \sum_{y\in\mathrm{supp}(w)} n_y y$ with $y \in G\mathfrak{X}$. By definition we have $h(w) = \bigcup_{y\in\mathrm{supp}(w)} h(y)$ and $h'(w) \subseteq \bigcup_{y\in\mathrm{supp}(w)} h'(y)$. As $h'(x) \subseteq U_\delta(h(x))$, we have $h'(gx) \subseteq U_\delta(h(gx))$ for all $g \in G$. Hence

$$h'(w) \subseteq \bigcup_{y\in\mathrm{supp}(w)} h'(y) \subseteq \bigcup_{y\in\mathrm{supp}(w)} U_\delta(h(y)) = U_\delta(h(w))$$

as desired. □

Corollary 1. *Let $h, h' : F \to \mathfrak{f}M$ be control functions defined with respect to $\mathfrak{X}$ and $\mathfrak{X}'$ respectively. There then exists $\delta > 0$ such that for any $w \in F$, we have $h'(w) \subseteq U_\delta(h(w))$ and $h(w) \subseteq U_\delta(h'(w))$.*

Lecture 2

2.2 Limit points in ∂M

Definition 15. Let F be a finitely generated free G-module, and $S \subseteq F$. We say that S has an accumulation point at $e \in \partial M$ if for every horoball $\mathcal{H}_e$ there is $s \in S$ such that $h(s) \subseteq \mathcal{H}_e$.

This definition is actually independent from the control function h, which we formulate in the following proposition.

Proposition 2. *If $h, h' : F \to \mathfrak{f}M$ are two control functions, then $h(s) \subseteq \mathcal{H}_e$ implies $h'(s) \subseteq \mathcal{H}_e$.*

Proof. □

Definition 16. Let A be a finitely generated G-module, and $\epsilon : F \twoheadrightarrow A$ a G-homomorphism. For $a \in A$ define

$$L_A^\epsilon(a) := \left\{e \in \partial M \mid e \text{ is acc. point of } \epsilon^{-1}(a) \subseteq F\right\}.$$

This definition is independent of the choice of ϵ. To prove this, we require a couple of simple observations, namely:

i. $L_A^\epsilon(a) \cap L_A^\epsilon(b) \subseteq L_A^\epsilon(a+b)$;

ii. $L_A^\epsilon(ga) = gL_A^\epsilon(a)$.

Proposition 3. *If $f : A \to A'$ is a G-homomorphism, with A and A' finitely generated G-modules, and there exist finite presentations $\epsilon : F \twoheadrightarrow A$ and $\epsilon' : F' \twoheadrightarrow A'$, then $L_A^\epsilon(a) \subseteq L_{A'}^{\epsilon'}(f(a))$.*

Proof. □

Corollary 2. *$L_A^\epsilon(a)$ is independent of the choice of ϵ.*

Proof. □

Because of this result, we write $L_A(a) := L_A^\epsilon(a)$.

Definition 17. $\Sigma^0(\rho; A) = \bigcap_{a \in A} L_A(a)$.

Recall that for this definition, G is a group and A is a finitely generated G-module given by the action ρ of G on A.

2.3 Explicit Interpretation of $e \in \Sigma^0(\rho, A)$

2.3.1 Interpretation via a condition on finite generation

As our definition of Σ^0 is independent of the choice of h, we pick a simple one that will be easy to work with. With $x \in \mathfrak{X}$, define $h(x) = \{b\}$ where $b \in M$ is a chosen origin, or base point. Thus we compute quite easily $h(y) = h(gx) = \{gb\}$ for all $y \in Y$. Thus $h(\lambda x) = \operatorname{supp}(\lambda) \cdot b$, and furthermore $h(\sum \lambda_i x_i) = \cup_i \operatorname{supp}(\lambda_i) b$ for all $\sum \lambda_i x_i \in F$.

We now investigate exactly what it means in this case for $e \in L_A(a)$. The condition $e \in L_A(a)$ means that for every horoball $\mathcal{H}_e$ there exists $w \in \epsilon^{-1}(a)$ such that $h(w) \subseteq \mathcal{H}_e$. To make this more concise, define

$$G_{\mathcal{H}_e} = \{g \in G \mid gb \in \mathcal{H}_e\}.$$

Thus $h(w) \subseteq \mathcal{H}_e$ means that $w \in \mathbb{Z}G_{\mathcal{H}_e}\mathfrak{X}$. We thus can conclude:

Proposition 4. *$e \in L_A(a)$ if and only if $a \in \mathbb{Z}G_{\mathcal{H}_e}\epsilon(\mathfrak{X})$ for all horoballs $\mathcal{H}_e$.*

Remark 2.

Theorem 1. $\Sigma^0(\rho, \mathbb{Z}) = \partial M$ if and only if ρ is co-compact.

Proof. No proof given. □

Theorem 2. Suppose the orbit Gb is discrete in M, with $G_e^+ \neq \emptyset$ for all $e \in \partial M$ and set $K := \ker \rho$. Then $\Sigma^0(\rho, A) = \partial M$ if and only if A is a finitely generated as a K-module.

Proof. □

Lecture 3

2.3.2 Dynamic Interpretation

Roughly said, in this section we will come up an equivalence with $e \in \Sigma^0(\rho; A)$ and the existence of a function $f : F \to F$ which satisfies for $y \in Y$

i. $\epsilon(\phi(y)) = \epsilon(y)$ and

ii. $h(f(y))$ is closer to e than $h(y)$.

To make this precise, we need to utilize the properties of CAT(0) spaces mentioned above.

Definition 18. Let $b \in M$ be a chosen origin, and $e \in \partial M$. The Busemann function with respect to b and e is $\beta_{e,b} : M \to \mathbb{R}$ which is defined for $A \in M$ by computing $d(b, A')$ where $A' = \gamma_e \cap \partial\mathcal{H}_{e,A}$ and $b \in \operatorname{im}\gamma_e$. Then by convention, if $\mathcal{H}_{e,A} \subseteq \mathcal{H}_{e,b}$ we set $\beta_e(A) = d(b, A')$ and if $\mathcal{H}_{e,b} \subseteq \mathcal{H}_{e,A}$ we set $\beta_e(A) = -d(b, A')$.

Definition 19. We extend the Busemann functions to be defined on $\mathfrak{f}M$ by setting for $L \in \mathfrak{f}M$

$$\beta_e(L) := \max\{\beta_e(m) \mid m \in L\}.$$

Definition 20. Let F be a finitely generated free $\mathbb{Z}G$-module and $f : F \to F$ a $\mathbb{Z}$-endomorphism. We say that f pushes F towards e if there exists $\delta > 0$ such that

$$\beta_e(h(f(w))) \geq \beta_e(h(w)) + \delta$$

for all $w \in F$. Alternatively, we say that f pushes F in the direction e.

Theorem 3.

$$\Sigma^0(\rho; A) = \{e \mid \exists f : F \twoheadrightarrow F \ni \epsilon \circ f = f \text{ and } f \text{ pushes in direction } e\}$$

2.3.3 Pushing with G-homomorphisms

Observation 1. The action $\rho : G \to \operatorname{Isom}(M)$ induces an action on the geodesic rays, and furthermore, an action on ∂M.

Proposition 5. *Supposing that $e \in \Sigma^0(\rho; A)$, the $\mathbb{Z}$-epimorphisms pushing F towards e are G-homomorphisms if and only if $Ge = e$.*

Proof. □

Example 10. In the case where $M = \mathbb{E}^n$ and G acts by translations, we have $Ge = e$.

Definition 21. Let $b \in M$ be a chosen origin, $a \in M$ and let $e \in \partial M$. We then define $\chi_e^a : G \to \mathbb{R}$ by $\chi_e^a(g) := \beta_{e,a}(gb) - \beta_{e,a}(b)$.

Proposition 6. *The map χ_e^a is a group homomorphism into the additive group $\mathbb{R}$ and independent of the choice a. We thus write $\chi_e := \chi_e^a$.*

Proof.

$$\begin{aligned}
\chi_e^a(g_1g_2) &= \beta_{e,a}(g_1g_2b) - \beta_{e,a}(b) \\
&= \beta_{e,a}(g_1g_2b) + (-\beta_{e,a}(g_2b) + \beta_{e,a}(g_2b)) - \beta_{e,a}(b) \\
&= (\beta_{e,a}(g_1g_2b) - \beta_{e,a}(g_2b)) + (\beta_{e,a}(g_2b) - \beta_{e,a}(b)) \\
&= \chi_e^a(g_1) + \chi_e^a(g_2)
\end{aligned}$$

From $\mathtt{CAT}(0)$-3, we obtain for $c \in M$ the equation $\beta_{e,a}(A) = \beta_{e,c}(A) - \beta_{e,c}(a)$. We now compute

$$\begin{aligned}
\chi_e^a(g) &= \beta_{e,a}(gb) - \beta_{e,a}(b) \\
&= \beta_{e,c}(gb) - \beta_{e,c}(a) - (\beta_{e,c}(b) - \beta_{e,c}(a)) \\
&= \beta_{e,c}(gb) - \beta_{e,c}(b) \\
&= \chi_e^c(g).
\end{aligned}$$

It is helpful to note that we can define an action of G on $\mathbb{R}$ which makes χ_e G-equivariant. We define for $r \in \mathbb{R}$ the action $g \cdot r = \chi_e(g) + r$.

□

Definition 22. We define a useful monoid

$$G_e := \{g \in G \mid \chi_e(g) \geq 0\}$$

and the associated semi-group

$$G_e^+ := \{g \in G \mid \chi_e(g) > 0\}.$$

The geometric interpretation of G_e^+ is that all of its elements push in the direction of e, i.e. gb is closer to e than b. We likewise see that G_e is the set of all elements which satisfy gb is not further away from e than b.

If we assume $Ge = e$ for all $e \in \partial M$, we obtain yet another description of $\Sigma^0(\rho; A)$. Suppose $A = \sum_{i=1}^k \mathbb{Z}Ga_i$ and set $\overline{a} = (a_1, a_2, ..., a_k)$. Then

Proposition 7.

$$\Sigma^0(\rho; A) = \left\{e \mid \exists \Lambda \in \mathcal{M}_n\left(\mathbb{Z}G_e^+\right) \ni \Lambda\overline{a} = \overline{a}\right\}$$

2.4 The Euclidean Case

We now drop the generality and focus on the case where G is finitely generated and acts via left translation on $G/G' \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{E}^n$. That is, the action is given by $\tau : G \to \mathrm{Transl}(G/G' \otimes \mathbb{R})$ where $\tau(g)(h \otimes r) = g \otimes 1 + h \otimes r$. In this case, we make the convention that

$$\Sigma^0(G; A) := \Sigma^0(\tau; A)$$

as is only depends on the choice of G and the G-module A now.

With this added restriction, we gain intuition and a few nice properties which we list here:

1. Via the vector space isomorphism $M = G/G' \otimes \mathbb{R} \cong \mathbb{R}^n$, we get an induced inner product $\langle -, - \rangle$ on $G/G' \otimes \mathbb{R}$.

2. We also have the association $\partial M = \mathbb{S}^{n-1}$, and thus every direction e is given by the unit vector $u_e \in \mathbb{S}^{n-1}$.

3. We take the base point b to be $b = 0$.

4. Furthermore, $Gu_e = u_e$ for all $u_e \in \partial M$, and the additive character χ_e takes the form $\chi_e(g) = \langle u_e, g \cdot b \rangle$.

5. The description of Σ^0 in Proposition 7 is valid in this case. If the group G is Abelian, we can then formulate a more convenient description of Σ^0 by the use of determinants.

   Proposition 8. *If the group G is Abelian, then*

$$\begin{aligned} \Sigma^0(G;A) &= \left\{ e \mid \exists \mu \in \mathbb{Z}G_e^+ \, \forall a \in A \ni \mu a = a \right\} \\ &= \left\{ e \mid \exists \mu \in \mathbb{Z}G_e^+ \ni 1 - \mu \in \operatorname{Ann}_{\mathbb{Z}G}(A) \right\} \end{aligned}$$

   Proof. (adjoint matrix trick and the description of Σ^0 in Prop. 7) □

Lecture 4

In addition to the above listed consequences of restricting our attention to Euclidean metric spaces, we also have a few helpful descriptions of Σ^0 to work with, which we list here:

1. $\Sigma^0(G;A) = \{e \mid A \text{ is fin. gen. over } \mathbb{Z}G_e\}$

2. $\Sigma^0(G;A) = \{e \mid \exists \text{ G-Hom } \phi : F \to F \ni \epsilon\phi = \phi \text{ and pushes } F \text{ in direction } e\}$

3. $\Sigma^0(G;A) = \{e \mid \exists \Lambda \in \mathcal{M}_n(\mathbb{Z}G_e^+) \ni \Lambda\overline{a} = \overline{a} \ni \{a_i\} \text{ generate } A\}$

Theorem 4. $\Sigma^0(G;A) = \partial M$ if and only if A is finitely generated as a module over the kernel $K = \ker\tau$ of the operation $\tau : G \to \operatorname{Transl}(M)$.

Proof. □

For G a finitely generated Abelian group, we have

$$\Sigma^0(G;A) = \left\{ e \in \partial M \mid \exists \lambda \in \mathbb{Z}G_e^+ \ni 1 - \lambda \in \operatorname{Ann}_{\mathbb{Z}G} A \right\}.$$

Thus for finitely generated Abelian groups G, this description shows Σ^0 only depends on the group G and the annihilator of A, hence we conclude $\Sigma^0(G;A) = \Sigma^0(G;\mathbb{Z}G/\operatorname{Ann}_{\mathbb{Z}G}(A))$.

We now make a few computational observations about $\Sigma^0(G;A)$.

1. $\Sigma^0(G;\mathbb{Z}G/IJ) = \Sigma^0(G;\mathbb{Z}G/I) \cap \Sigma^0(G;\mathbb{Z}G/J)$.

2. $\Sigma^0(G;\mathbb{Z}G/IJ) = \Sigma^0(G;\mathbb{Z}G/(I \cap J))$.

3. $\Sigma^0(G;\mathbb{Z}G/I) = \Sigma^0\left(G;\mathbb{Z}G/\sqrt{I}\right)$

4. Thus from observations 1–3 we conclude that when $\mathbb{Z}G$ is Noetherian, computing $\Sigma^0(G;\mathbb{Z}G/I)$ for any ideal I is reduced to computing $\Sigma^0(G;\mathbb{Z}G/\mathfrak{p}_i)$ where the $\mathfrak{p}_i$ are prime ideals in $\mathbb{Z}G$ such that $\sqrt{I} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_l$.

Lecture 5

Example 11 (1-Relator Modules)**.**

Definition 23. A one relator $\mathbb{Z}G$-module is a module of the form $\mathbb{Z}G/I$ where $I = \lambda\mathbb{Z}G = (\lambda)$ for $\lambda \in \mathbb{Z}G$.

Observation 2. Given a specified direction $e \in \partial M$, we then recall $\chi_e : G \to \mathbb{R}$ defined by $\chi_e(g) = \langle u_e, h(g)\rangle$ (where $u_e \in S^{n-1}$ is the corresponding unit vector to the direction e) is a homomorphism. With this homomorphism, there is an $\mathbb{R}$-grading of $\mathbb{Z}G$ given by $\mathbb{Z}G = \oplus_{r\in\mathbb{R}}\mathbb{Z}(G_r \setminus G_r^+)$. That is, for $\lambda \in \mathbb{Z}G$, we can write $\lambda = \sum_{r\in\mathbb{R}} \lambda_r$ where $\chi_e(\lambda_r) = r$. In particular, $\mathrm{supp}(\lambda_r) = \{g \in \mathrm{supp}(\lambda) \mid \chi_e(g) = r\}$.

From the $\mathbb{R}$-grading on $\mathbb{Z}G$, we can define a valuation $v_e : \mathbb{Z}G \to \mathbb{R}_\infty$ as follows: set $v_e(\lambda) := \min\{r \mid \lambda_r \neq 0\}$ for $\lambda \neq 0$ and $v_e(0) = \infty$. We also define $\lambda_e := \lambda_{v_e(\lambda)}$, which we also call the initial term of λ. We observe that v_e satisfies the following properties which almost makes it into a valuation in the normal sense:

1. $v_e(\lambda + \mu) \geq \min\{v_e(\lambda), v_e(\mu)\}$;
2. $v_e(g\mu) = \chi_e(g) + v_e(\mu)$;
3. $v_e(0) = \infty$;
4. $v_e(-\lambda) = v_e(\lambda)$.

Lemma 1. *If $\mathbb{Z}G$ does not have any non-trivial zero divisors, then $v_e(\lambda\mu) = v_e(\lambda) + v_e(\mu)$ and $(\lambda\mu)_e = \lambda_e\mu_e$.*

Proof. Write $\lambda = \lambda_e + \lambda^+$ and $\mu = \mu_e + \mu^+$. Then $\lambda\mu = \lambda_e\mu_e + \lambda^+\mu_e + \lambda_e\mu^+ + \lambda^+\mu^+$. By assumption, $\lambda_e\mu_e \neq 0$ and $v_e(\lambda\mu - \lambda_e\mu_e) > v_e(\lambda) + v_e(\mu)$, whence the result follows. □

We now restrict our attention to the case where G is a finitely generated Abelian group. In this case, we can utilize the description from Propositon 8 to the following description of $\Sigma^0(G; \mathbb{Z}G/(\lambda))$:

$$\begin{aligned}\Sigma^0(G; \mathbb{Z}G/(\lambda)) &= \left\{e \mid \exists\zeta \in \mathbb{Z}G_e^+ \ni 1 - \zeta \in \mathrm{Ann}_{\mathbb{Z}G}(A)\right\} \\ &= \left\{e \mid \exists\mu \in I \wedge \exists\zeta \in \mathbb{Z}G_e^+ \ni \mu = 1 - \zeta\right\} \\ &= \{e \mid \exists\mu \in I \ni \mu_e = 1\}.\end{aligned}$$

For especially nice group rings $\mathbb{Z}G$, we can make the result even more precise.

Theorem 5. If $\mathbb{Z}G$ contains only trivial zero divisors and trivial units (those of the form $\pm g$ for $g \in G$), then

$$\Sigma^0(G; \mathbb{Z}G/(\lambda)) = \{e \mid \lambda_e \in \pm G\}.$$

Proof. As $(\lambda) = \{\mu\lambda \mid \mu \in \mathbb{Z}G\}$, we need to determine when $(\mu\lambda)_e = 1$ by the above description of $\Sigma^0(G, \mathbb{Z}G/(\lambda))$. By Lemma ? we have $(\mu\lambda)_e = \mu_e\lambda_e$. We have $\mu_e\lambda_e = 1$ if and only if λ_e is a unit in $\mathbb{Z}G$, and by assumption, the only units are the elements of $\pm G$, whence the theorem holds. □

Remark 3. If G contains a nontrivial element of finite order, then $\mathbb{Z}G$ contains non-trivial zero divisors. If $|g| = n$, then $(g-1)\left(1+g+g^2+\cdots+g^{n-1}\right) = 0$. The finitely generated free Abelian groups $Q = \mathbb{Z}^n$ have group rings $\mathbb{Z}Q$ which do not contain non-trivial zero divisors and non-trivial units.

To give a very explicit example, we compute $\Sigma^0\left(\mathbb{Z}^2; \mathbb{Z}\mathbb{Z}^2/(\lambda)\right)$ where $\mathbb{Z}^2 = \langle x, y \mid [x, y]\rangle$ and $\lambda = 2\cdot 1 + x + y + x^2y^2$. The set Σ^0 in this case is illustrated in Figure 1. We make the association $x \to (1,0)$ and $y \to (0,1)$ for the following computations. The actual computation of $\Sigma^0\left(\mathbb{Z}^2; \mathbb{Z}\mathbb{Z}^2/(\lambda)\right)$ lies in computing $\chi_{(a,b)}((0,0))$, $\chi_{(a,b)}((1,0))$, $\chi_{(a,b)}((0,1))$ and $\chi_{(a,b)}((2,2))$ for all directions $e = (a,b) \in \mathbb{S}^1$, and determining the which obtain the minimum value. It is easy to verify the following chart:

$e = (a,b)$	λ_e	in $\pm G$?
$a, b > 0$	$2\cdot 1$	no
$a = 0 \wedge b > 0$	$2\cdot 1 + x$	no
$b = 0 \wedge a > 0$	$2\cdot 1 + y$	no
$b < 0 \wedge -b/2 < a$	y	yes
$b < 0 \wedge -b/2 = a$	$y + x^2y^2$	no
$a < 0 \wedge -a/2 < b$	x	yes
$a < 0 \wedge -a/2 = b$	$x + x^2y^2$	no
$-b/2 > a \wedge -a/2 > b$	x^2y^2	yes

from which Figure 1 follows by Theorem 5.

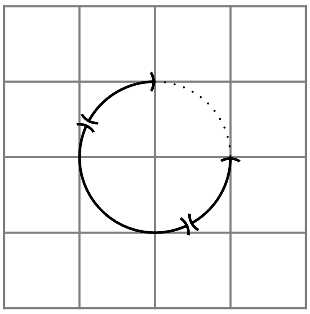

Figure 1: The solid black line represents the points in $\Sigma^0\left(\mathbb{Z}^2; \mathbb{Z}\mathbb{Z}^2/(\lambda)\right)$ while the dotted line represents $\partial\mathbb{R}^2$ which we are identifying with $\mathbb{S}^1$.

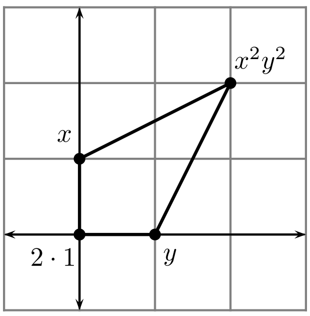

Figure 2: This polyhedron, namely, the convex hull of supp (λ), enables an easy way to compute $\Sigma^0\left(\mathbb{Z}^2; \mathbb{Z}\mathbb{Z}^2/(\lambda)\right)$.

Example 12. We now look at the case when G is Abelian and $A = \mathbb{Z}G/\mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal, or equivalently, $\mathbb{Z}G/\mathfrak{p}$ is an integral domain. Let $L = \kappa(A)$ be the field of fractions of A.

Now given $e \in \partial M$, we have the additive character $\chi_e(-) = \langle u_e, - \cdot b\rangle$ where u_e is the unit vector representing the direction e and b is the origin. We can extend an additive character to a valuation $v_e : \mathbb{Z}G \to \mathbb{R}_\infty$. We generalize the construction in the following proposition.

Proposition 9. *For $v \in \operatorname{Hom}(Q, \mathbb{R})$ an additive character, we extend v to a valuation $v_* : \mathbb{Z}Q \to \mathbb{R}_\infty$ by defining:*

i. $v_*(\lambda) = \min\{v(q) : q \in \operatorname{supp}(\lambda)\}$*;*

ii. $v_*(0) = \infty$.

Proof. We need to show that for $\lambda, \mu \in \mathbb{Z}Q$ the equations

i. $v_*(\lambda\mu) = v_*(\lambda) + v_*(\mu)$ and

ii. $v_*(\lambda + \mu) \geq \inf\{v_*(\lambda), v_*(\mu)\}$

are satisfied.

We write $\lambda = \sum_{q \in \operatorname{supp}(\lambda)} \lambda_q x_q$, likewise for μ and we compute

$$\lambda\mu = \sum_{\substack{q \in \operatorname{supp}(\lambda) \\ r \in \operatorname{supp}(\mu)}} (\lambda_q \mu_r) x_{q+r}.$$

We now see

$$v_*(\lambda\mu) = \min\{v(q + r) : q \in \operatorname{supp}(\lambda),\ r \in \operatorname{supp}(\mu)\}$$

$$v_*(\lambda) + v_*(\mu) = \min\{v(q) : q \in \operatorname{supp}(\lambda)\} + \min\{v(r) : r \in \operatorname{supp}(\mu)\}$$

and it is clear that $v(qr) = v(q) + v(r)$ will be minimal when $v(q)$ and $v(r)$ are minimalwhence the first equation holds.

From the easily verified inclusion $\operatorname{supp}(\lambda+\mu) \subseteq \operatorname{supp}(\lambda) \cup \operatorname{supp}(\mu)$, the second equation follows.

□

With this construction, we can define $A_e := \operatorname{im} \pi|_{\mathbb{Z}G_e}$ and $I_e := \operatorname{im} \pi|_{\mathbb{Z}G_e^+}$. We thus obtain the following diagram:

$$\begin{array}{ccccccc}
\mathbb{Z}G_e^+ & \hookrightarrow & \mathbb{Z}G_e & \hookrightarrow & \mathbb{Z}G & & \\
\downarrow{\scriptstyle \pi|_{\mathbb{Z}G_e^+}} & & \downarrow{\scriptstyle \pi|_{\mathbb{Z}G_e}} & & \downarrow{\scriptstyle \pi} & & \\
I_e & \hookrightarrow & A_e & \hookrightarrow & A & \hookrightarrow & L \\
 & & & & \| & & \\
 & & & & \mathbb{Z}G/\mathfrak{p} & &
\end{array}$$

Proposition 10. *$e \in \Sigma^0(G; A)^c$ if and only if $I_e \neq A_e$, or equivalently, $e \in \Sigma^0(G; A)$ if and only if $I_e = A_e$.*

Proof. We utilize the description of $\Sigma^0(G;A)$ from Proposition 8, namely $\Sigma^0(G;A) = \{e \,|\, \exists \mu \in \mathbb{Z}G_e^+ \ni 1-\mu \in \mathrm{Ann}_{\mathbb{Z}G}(A)\}$.

Suppose $I_e = A_e$. Then $\overline{1} \in I_e$—that is, there exists $\lambda \in \mathbb{Z}G_e^+$ such that $\overline{1} = \overline{\lambda}$. Thus there exists $\gamma \in \mathfrak{p}$ such that $1 = \lambda + \gamma$, and we have $1 - \lambda = \gamma \in \mathfrak{p} = \mathrm{Ann}(A)$. Hence $e \in \Sigma^0(G;A)$.

Now suppose $e \in \Sigma^0(G;A)$. Then there exists $\lambda \in \mathbb{Z}G_e^+$ such that $1 - \lambda \in \mathrm{Ann}(A) = \mathfrak{p}$, hence $1 - \lambda = \gamma$ for some $\gamma \in \mathfrak{p}$. Now let $\overline{\mu} \in A_v$ such that $v_e(\mu) = 0$. Then $\mu = \mu(\lambda + \gamma) = \mu\lambda + \mu\gamma$. We compute $v_e(\mu\lambda) = v_e(\mu) + v_e(\lambda) = v_e(\lambda) > 0$. Thus $\overline{\mu} = \overline{\mu\lambda + \mu\gamma} = \overline{\mu\lambda} \in I_e$. Clearly for those $\overline{\mu} \in A_e$ such that $v_e(\mu) > 0$ we have $\mu \in I_e$, whence $I_e = A_e$. □