Study Guide

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0. Generalized cohomology and homology theories

Definition 0.1: [Eilenberg-Steenrod Homology Theory]. An E-S homology theory is a collection of functors H_n from pairs of CGWH spaces to Ab satisfying the following axioms:

1. (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ such that $f \simeq g$, then $f_* = g_*$.
2. (Boundary map) For any pair (X, A), there is a natural transformation $\partial : H_n(X, A) \to H_{n-1}(A)$. That is, for any map $f : (X, A) \to (Y, B)$ the following diagram commutes:
$$\begin{tikzcd} H_n(X,A) \arrow[r, "f_*"] \arrow[d, "\partial"] & H_n(Y,B) \arrow[d, "\partial"] \\ H_{n-1}(A) \arrow[r, "f_*"] & H_{n-1}(B) \end{tikzcd}$$
3. (Long exact sequence) For any pair (X, A), there is a long exact sequence
$$\cdots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\partial} \cdots$$
where the maps are the obvious ones induced by the inclusion and quotient maps.
4. (Excision V.1) If $(X; A, B)$ is an excisive triad, i.e. X is the union of the interiors of A and B, then $\iota : (A, A \cap B) \to (X, B)$ induces an isomorphism $\iota_* H_n(A, A \cap B) \cong H_n(X, B)$.

   (Excision V.2) If $U \subseteq \overline{U} \subsetneq A \subset X$, with A open, then the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism $H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$.

   (Excision V.3) If $X = A \cup B$ where A and B are closed subsets, and so that $(A, A \cap B)$ and (X, B) are good pairs, i.e. NDR, then $\iota_* : H_n(A, A \cap B) \cong H_n(X, B)$.
5. (Dimension) The homology of a point has the form $H_n(pt) = 0$ for $n \neq 0$ and $H_0(pt) = A$, called the coefficient group.

6. A1. (Additivitiy) If $X = \coprod X_i$, then the map $\sum \iota_i : \oplus_i H_n(X_i) \to H_n(X)$ is an isomorphism for any n and any indexing set.

7. A2. (Weak equivalence) If $f : (X, A) \to (Y, B)$ is a weak equivalence, then $f_* : H_n(X, A) \cong H_n(Y, B)$.

Remark 0.2: Eilenberg-Steenrod definition differs from the above in the following way

1. the axioms are ordered differently, the category of spaces is different
2. V.2 of excision is used
3. there is no additivity axiom, and no weak equivalence axiom

Theorem 0.3: (Uniqueness of E-S homology theories) If H_* and K_* are E-S homology theories, not necessarily satisfying A1 and A2, then if the coefficient groups $\eta_0 : H_0(pt) \cong K_0(pt)$, then for any finite CW pair (X, A) we have natural transformations $\eta_n : H_n(X, A) \cong K_n(X, A)$ lifting η_0. In particular, they are isomorphic to singular homology with coefficients $H_0(pt)$.

If in addition, both H_* and K_* satisfy the additivity axiom, the result extends to CW pairs (not necessarily finite). Furthermore, if both homology theories satisfy the weak equivalence axiom, the result extends to pairs of <u>CGWH</u> spaces.

A similar statement holds for cohomology theories.

Definition 0.4: (Generalized reduced homology theories) A generalized reduced homology theory is a collection of functors h_n from $\underline{\text{CGWH}}_*$ to <u>Ab</u> along with natural transformations $e_n : h_n \to h_{n+1} \circ \Sigma$, which satisfy the following axioms:

A1. (Homotopy invariance) If $f \simeq g : X \to Y$, then $f_* = g_*$.

A2. (Exactness) If $\iota : A \rightarrowtail X$ is a cofibration, then the following sequence is exact

$$h_n(A) \xrightarrow{\iota_*} h_n(X) \xrightarrow{q_*} h_n(X/A).$$

A3. (Suspension) The homomorphism $e_n : h_n(X) \to h_{n+1}(\Sigma X)$ is an isomorphism.

A4. (Additivity) If $X = \vee_i X_i$, then the inclusions $\iota_i : X_i \to X$ induce an isomorphism $\sum \iota_{i*} : \sum h_n(X_i) \to h_n(X)$.

A5. (Weak equivalence) If $f : X \to Y$ is a weak equivalence, then f_* is an isomorphism.

Remark 0.5: One can go between generalized reduced theories and generalized unreduced theories easily by the following formulas:

1. For NDR pairs (X, A), also called good pairs, or $A \rightarrowtail X$ a cofibration,

$$H_n(X, A) = h_n(X_+/A_+) = \begin{cases} h_n(X/A) & if A \neq \emptyset \\ h_n(X_+) & if A = \emptyset \end{cases}$$

2. If (X, x_0) is a pointed space so that $\{x_0\}$ is a NDR, then $h_n(X) = H_n(X, x_0)$

Remark 0.6: The following construction is often useful. Let H_* be a generalized unreduced homology theory. If one only has a description of $H_*(X)$ for absolute spaces, one can recover a reduced theory by splitting off the coefficients using the following splitting: $pt \to X \to pt$ induces $H_*(pt) \to H_*(X) \to H_*(pt)$ says that $H_n(X) \cong H_n(pt) \oplus \widetilde{H}_n(X)$ for all n and X, where $\widetilde{H}_*$ is defined by this decomposition. The collection of functors $\widetilde{H}_*$ then make a reduced homology theory. The same construction works for cohomology as well.

Remark 0.7: Show how to get long exact sequence from the above axioms. Show how to get excision, Mayer-Vietoris.

The definitions for generalized reduced and unreduced cohomology theories are entirely analogous, just with some index shifting, and the additivity axiom changing to reflect the contravariance.

(I should add them for completeness)

1. Spectra

In what follows, we heavily use the existence of an adjunction $\Sigma \dashv \Omega$ on $\underline{\mathrm{Top}}_*$ which descends to an adjunction $\underline{\mathrm{hTop}}_*$, where Σ is the reduced suspension functor and Ω is the loop space functor. Explicitly, for any pointed spaces X and Y we have $\eta : [\Sigma X, Y] \cong [X, \Omega Y]$ a natural isomorphism of sets. First one shows $\eta : \underline{\mathrm{Top}}_*(\Sigma X, Y) \cong \underline{\mathrm{Top}}_*(X, \Omega Y)$ by sending a map $f(x,t) : \Sigma X \to Y$ to $f_t(x) = f(x,t)$. Over a point x in ΣX, there is a loop. This loop gets mapped to a loop in Y under f, and this describes the map. The inverse map is defined analogously. One can chase around the diagrams to see that the η as defined is natural, so it is an adjunction.

Whitehead's category of spectra

Definition 1.8: (Whitehead Spectrum) A Whitehead spectrum $\mathbb{E}$ is a sequence of based spaces E_n with maps $e_n : S^1 \wedge E_n \to E_{n+1}$. If $n \in \mathbb{Z}$ is allowed, we write $\underline{\mathrm{Sp}}^{\mathbb{Z}}$ for the category, and if we restrict to $n \geq 0$ we write $Sp^{\mathbb{N}}$. A morphism $\phi : \mathbb{E} \to \mathbb{F}$ of Whitehead spectra is a collection of maps $\phi_n : E_n \to F_n$ so that the following diagram commutes.

$$\begin{array}{ccc} S^1 \wedge E_n & \xrightarrow{e_n} & E_{n+1} \\ \downarrow{\scriptstyle \mathrm{id} \wedge \phi_n} & & \downarrow{\scriptstyle \phi_{n+1}} \\ S^1 \wedge F_n & \xrightarrow{f_n} & F_{n+1} \end{array}$$

Remark 1.9: In Whitehead's paper [**?**], his category of spaces is the full subcategory of $\underline{\mathrm{Top}}_*$ with objects that are homotopy equivalent to CW-complexes. Whitehead also allows positive and negative indices for spectra.

Definition 1.10: (Ω-spectrum) An Ω-spectrum is a Whitehead spectrum $\mathbb{E}$ so that the adjoint of the structure map $\widetilde{e}_n : E_n \to \Omega E_{n+1}$ is a homotopy equivalence. One could weaken the definition and just require the maps $\widetilde{e}_n$ be weak equivalences. (Perhaps denote the category of Ω spectra by $\underline{\mathrm{Sp}}^{\Omega}$.)

The Adams/Spanier category of CW-spectra

The category of symmetric simplicial set spectra

These definitions for spectra turn out to be inadequate for many reasons. In particular, the smash product of two spectra is very difficult to define, and only can be done in the homotopy category of spectra, i.e. the stable homotopy category. One way around this is to use symmetric spectra instead.

Definition 1.11: (Symmetric spectrum) A symmetric spectrum $\mathbb{X}$ is given by

1. a sequence of pointed (based) simplicial sets $X_0, X_1, \ldots$
2. a collection of maps $x_n : S^1 \wedge X_n \to X_{n+1}$,
3. a basepoint preserving left action of the symmetric group S_n on X_n such that the maps $x_n^p : S^p X_n \to X_{n+p}$ are $S_p \times S_n$-equivariant.

We now make the final condition more precise. Since $S^p \cong (S^1)^{\wedge p}$, the symmetric group S_p acts on S^p by permuting cells. We embed $\iota : S_p \times S_n \to S_{p+n}$ by having S_p permyte the first p elements and S_n permute the last n elements.

From the structure maps x_n, we construct the maps x_n^p in the obvious manner: $x_n^2 = x_{n+1} \circ \mathrm{id} \wedge x_n : S^1 \wedge S^1 \wedge X_n \to S^1 \wedge X_{n+1} \to X_{n+2}$ and so on. The equivariance condition then asserts the commutativity of the following square for all $\sigma \in S_p \times S_n$:

$$\begin{tikzcd} S^p \wedge X_n \arrow[r, "x_n^p"] \arrow[d, "\sigma\cdot"] & X_{n+p} \arrow[d, "\iota(\sigma)\cdot"] \\ S^p \wedge X_n \arrow[r, "x_n^p"] & X_{n+p} \end{tikzcd}$$

Example 1.12: For each $G \in \underline{\mathrm{Ab}}$, the Eilenberg-Maclane spaces $K(G,n)$ form an Ω-spectrum. We take $K_n = K(G,n)$, and write the spectrum as $\mathbb{K}$ or $\mathbb{K}(G)$. Since $[S^k, \Omega K(G,n)] \cong [S^{k+1}, K(G,n)]$ for all n and k, we see that $\Omega K(G,n)$ is a $K(G,n-1)$. Since up to homotopy equivalence there is only one $K(G,n)$, we obtain a map $\widetilde{k}_n : K(G,n) \to \Omega K(G,n+1)$ which is a homotopy equivalence. The adjoint maps $k_n : S^1 \wedge K(G,n) \to K(G,n+1)$ then make $\mathbb{K}$ a spectrum.

One can construct a simplicial set version of the spectrum $\mathbb{K}(G)$ using the Dold-Kan correspondence. See exercise 8.4.4 in Weibel's book. Also May's book on simplicial objects, section 23.

Example 1.13: The sphere spectrum and the Eilenberg-Maclane spectrum for $\mathbb{Z}$ are symmetric spectra. To any space X there is a suspension spectrum $\mathbb{X}$ given by $X_n = \Sigma^n X$ with $x_n = \mathrm{id}$.

1.1. The Brown Representability Theorem

Definition 1.14: To every Ω-spectrum $\mathbb{E}$, there is an associated generalized reduced cohomology theory $\mathbb{E}^*$ given by the formula $\mathbb{E}^n(X) = [X, E_n]$ where X is a based space. This satisfies the additivity axiom, but will not necessarily satisfy the weak equivalence axiom.

Definition 1.15: If $\mathbb{E}$ is just a Whitehead spectrum, one can still define a generalized reduced cohomology theory $\mathbb{E}^*$ by the formula $\widetilde{\mathbb{E}}^n(X) = \varinjlim [S^k \wedge X, E_{n+k}]$

where the direct limit is taken with respect to the evident maps. This cohomology theory will fail the additivity axiom unless it is an Ω-spectrum. Another notation for this is $\widetilde{H}^n(-;\mathbb{E})$.

To get unreduced theories, one artificially introduces basepoints into the construction, like in the discussion on generalized homology and cohomology theories above.

Definition 1.16: To every spectrum $\mathbb{E}$, there is an associated reduced homology theory as well. It is defined by the formula $\widetilde{\mathbb{E}}_n(X) = \varinjlim[S^{k+n}, X \wedge E_k]$, again with the evident maps. One can also write this as $\varinjlim \pi_{n+k}(X \wedge E_k)$. Another notation for this is $\widetilde{H}_n(-;\mathbb{E})$. For a non-reduced version, sneak in a disjoint basepoint: $\mathbb{E}_n(X) = \varinjlim[S^{k+n}, X_+ \wedge E_k]$.

$\mathbb{E}$ a spectrum	$\widetilde{\mathbb{E}}_n(X) = \varinjlim[S^{n+i}, X \wedge E_i]$
	$\mathbb{E}_n(X) = \varinjlim[S^{n+i}, X_+ \wedge E_i]$
	$\mathbb{E}_n(X/A) = \varinjlim[S^{n+i}, X/A \wedge E_i]$
	$\widetilde{\mathbb{E}}^n(X) = \varinjlim[S^i \wedge X, E_{n+i}]$
	$\mathbb{E}^n(X) = \varinjlim[S^i \wedge X_+, E_{n+i}]$
	$\mathbb{E}^n(X, A) = \varinjlim[S^i \wedge X/A, E_{n+i}]$
$\mathbb{E}$ an Ω-spectrum	$\widetilde{\mathbb{E}}^n(X) = [X, E_n]$
	$\mathbb{E}^n(X) = [X_+, E_n]$
	$\mathbb{E}^n(X/A) = [X/A, E_n]$

To see that the direct limit definition degenerates in the case of an Ω-spectrum, we need to use naturality of the adjunction. We need to show that the map $[X, E_n] \xrightarrow{\Sigma} [\Sigma X, \Sigma E_n] \xrightarrow{\epsilon_{n*}} [\Sigma X, E_{n+1}]$ is an isomorphism where $\epsilon_n = \eta^{-1}(e_n)$ the defining map of the adjunction. Observe that for any $f \in [X, E_n]$ and by naturality of the adjunction η, the following diagram is commutative.

$$\begin{tikzcd}
{[\Sigma E_n, E_{n+1}]} \arrow[r, "\eta"] \arrow[d, "\Sigma f^*"] & {[E_n, \Omega E_{n+1}]} \arrow[d, "f^*"] \\
{[\Sigma X, E_{n+1}]} \arrow[r, "\eta"] & {[X, \Omega E_{n+1}]}
\end{tikzcd}$$

Therefore, the following diagram is commutative

$$\begin{tikzcd}
{[X, E_n]} \arrow[r, "e_{n*}"] \arrow[d, "\Sigma"] & {[X, \Omega E_{n+1}]} \arrow[d, "\eta^{-1}"] \\
{[\Sigma X, \Sigma E_n]} \arrow[r, "\epsilon_{n*}"] & {[\Sigma X, E_{n+1}]}
\end{tikzcd}$$

and since $\eta^{-1} \circ e_{n-1_*}$ is an isomorphism, the of interest $\epsilon_{n*} \circ \Sigma$ is an isomorphism. We therefore see that all the maps in the direct limit are isomorphisms, from which we get the Ω-spectrum cohomology description. The map obtained by going around the diagram clockwise is the natural suspension isomorphism to the associated reduced cohomology theory. This argument shows that we can define the map the other way too, as long as we have an Ω-spectrum. As the map going counterclockise is more natural from the spectrum point of view, we should be using this description.

Example 1.17: In particular, if we take the spectrum $\mathbb{K}(G)$, the Eilenberg-Maclane spectrum for the group G, we get ordinary homology and cohomology with coefficient group G. The way one proves this is by using the uniqueness of Eilenberg-Maclane homology and cohomology theories with the dimension axiom. There is a map $T : [X, K(G, n)] \to H^n(X; G)$ which induces the isomorphism as well. Given $f \in [X, K(G, n)]$, there is an induced map $f^* : H^n(K(G, n); G) \to H^n(X; G)$ and a distinguished class $\alpha \in H^n(K(G; n); G)$. The map T is then defined by $T([f]) = f^*(\alpha)$. An explicit construction of α is possible, and can be found in Hatcher pg. 402.

These observations lead us to the question: Is every cohomology theory represented by a spectrum in this way? The answer is "yes" and is captured by the Brown Representability Theorem.

Theorem 1.18: Every reduced, additive cohomology theory on the category of basepointed CW complexes and basepoint-preserving maps has the form $\widetilde{H}^n(X) = [X, K_n]$ for some Ω-spectrum $\mathbb{K}$.

If additivity is dropped, the same conclusion holds but only on the category of finite CW complexes.

Every reduced, additive homology theory is represented by a spectrum $\mathbb{K}$ for all CW complexes. If additivity is dropped, then the statement holds only for finite CW complexes.

2. More on spectra

2.2. Homology and cohomology for spectra

In the above section on Brown representability, we saw that homology and cohomology could be phrased in terms of stable homotopy theory. This perspective is very useful, and coupled with the category of spectra, one can give fairly concrete map-based descriptions of homology and cohomology constructions. One can take this a step further, and define homology and cohomology not for just spaces, but for spectra as well.

To carry out this program, we need a suitable category $\mathcal{SH}$ which consists of spectra with maps being some sort of "stable map" that will agree with our notion if we are working with the spectra $\Sigma^\infty X$. If this all works out, the definitions for homotopy groups, homology, and cohomology will be the following.

$$\pi_n(\mathbb{A}) = [\Sigma^n \mathbb{S}, \mathbb{A}]_{\mathcal{SH}} \tag{1}$$
$$\mathbb{E}_n(\mathbb{A}) = [\Sigma^n \mathbb{S}, \mathbb{A} \wedge \mathbb{E}]_{\mathcal{SH}} \tag{2}$$
$$\mathbb{E}^n(\mathbb{A}) = [\mathbb{A}, \Sigma^n \mathbb{E}]_{\mathcal{SH}} \tag{3}$$

Spanier-Whitehead category, categories with suspension (Heller), etc.
Model structure on categories of spectra

Do the above but with symmetric simplicial set spectra

2.3. Homology and cohomology constructions with spectra

Ring spectra
cup product, cap product
$\mathbb{E}$-orientability
Thom isomorphism
Spanier-Whitehead duality
Poincaré-Lefschetz duality

2.4. Symmetric spectra and smash product

In the paper [**?**], the authors are able to define the smash product of symmetric spectra and work out many of its properties by realizing it as a tensor product in the category of $\mathbb{S}$-modules with respect to some symmetric monoidal category. What follows is a sketch of the construction with some additional comments on what is meant in the paper.

Definition 2.19: Let Σ be the subcategory of $\underline{\text{Set}}$ with objects $\overline{n} = \{1, 2, 3, ..., n\}$ for $n \geq 0$ (for $n = 0$ we take $\overline{n} = \emptyset$), and morphisms are required to be bijections. Therefore, whenever $n \neq m$ we have $\Sigma(\overline{n}, \overline{m}) = \emptyset$.

Definition 2.20: The category of symmetric sequences in $\mathfrak{C}$ is the functor category $\mathfrak{C}^{\Sigma}$. For the category of pointed simplicial sets $\underline{\text{sSet}}_*$, we write $\underline{\text{sSet}}_*^{\Sigma}$ or $\mathcal{S}_*^{\Sigma}$ for the category of symmetric sequences of simplicial sets.

An element $X \in \underline{\text{sSet}}_*^{\Sigma}$ is a sequence of pointed simplicial sets $X_0, X_1, \ldots$ with an action of the group Σ_n on X_n. A morphism $f : X \to Y \in \underline{\text{sSet}}_*^{\Sigma}$ is a sequence of Σ_n-equivariant maps $f_n : X_n \to Y_n$.

Definition 2.21: The tensor product in $\underline{\text{sSet}}_*^{\Sigma}$ of X and Y is the simplicial symmetric sequence given by the sequence of pointed simplicial sets

$$(X \otimes Y)_n = \vee_{p+q=n}(\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)$$

where Σ_n is the simplicial set with one non-degenerate 0-cell for every element of Σ_n and all other cells are degenerate.

Digression 2.22: We explain what $\wedge_{\Sigma_p \times \Sigma_q}$ is in a slightly more general context. Suppose A, B are two pointed simplicial sets for which A has a right G-action and B has a left G-action. Then $A \wedge_G B$ is the quotient of $A \wedge B$ where we declare for all a, b, g that $(ag, b) = (a, gb)$. More precisely, we have maps $A \times G \times B \rightrightarrows A \times B \longrightarrow A \wedge B$ given by $m_1(a, g, b) = (ag, b)$ and $m_2(a, g, b) = (a, gb)$. The coequalizer of $A \times G \times B \rightrightarrows A \wedge B$ is then $A \wedge_G B$. If instead of the smash product, we were using the cartesian product, the above construction is called the "balanced product" [**?**].

Now since $\Sigma_p \times \Sigma_q$ acts on $(\Sigma_n)_+$ by right-multiplication, we are able to make sense of the definition of $(X \otimes Y)_n$.

What is the action of Σ_n on $(X \otimes Y)_n$? It is given by left-multiplication with the factor $(\Sigma_n)_+$.

Proposition 2.23: Let $X, Y, Z \in \underline{\text{sSet}}_*^{\Sigma}$. Then there is a natural isomorphism

$$\underline{\text{sSet}}_*^{\Sigma}(X \otimes Y, Z) \cong \prod_{p,q} \underline{\text{sSet}}^{\Sigma_p \times \Sigma_q}(X_p \wedge Y_q, Z_{p+q}).$$

Proof. It is easy to work out explicitly with the following maps. Given $f : X \otimes Y \to Z$, for any n we have $f_n : \vee(\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \to Z_{p+q}$ which is Σ_n-equivariant. This is determined by universal property of wedge by the maps $f_{p,q} : (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \to Z_{p+q}$. Restricting this map to $(1) \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \to Z_{p+q}$ determines the map $X_p \wedge Y_q \to Z_{p+q}$ which is $\Sigma_p \times \Sigma_q$-equivariant because the action of $\Sigma_p \times \Sigma_q$ "moves over" to act on $X_p \wedge Y_q$.

Going the other way, with a family of such maps $\phi_{p,q}$, we define $f_{p,q}$ by $f_{p,q}(\sigma, x, y) = \sigma \cdot \phi_{p,q}(x, y)$. One needs to verify that this is well defined, but it does work. One then pieces these maps together to get a map $f \in \underline{\mathrm{sSet}}_*^{\Sigma}(X \otimes Y, Z)$. QED

With symmetric sequences, the tensor product is actually a symmetric monoidal product on the category. We need to define the twish map to be $\tau : X \otimes Y \to Y \otimes X$ by $\tau_{p,q}(\gamma, x, y) = (\gamma\rho_{q,p}, y, x)$ where $\rho_{q,p}$ is the q, p-shuffle, i.e. the permutation which moves the first q numbers to the end of $\overline{p+q}$, or equivalently, moves the last p numbers of $\overline{p+q}$ to the front. It is essential to have this shuffle in the twist map for the product to be symmetric.

It is easy enough to verify that the symmetric sphere sequence $\mathbb{S}$ given by $\mathbb{S}_n = (\mathbb{S}^1)^{\wedge n}$ with the evident action of Σ_n (permuting factors) is a commutative monoid in $\underline{\mathrm{sSet}}_*^{\Sigma}$. With this, we can then define the category of $\mathbb{S}$-modules. This category is equivalent to the category of symmetric spectra defined above.

To say that $\mathbb{S}$ is a commutative monoid in $\underline{\mathrm{sSet}}_*^{\Sigma}$ means that there are maps $u : 1 \to \mathbb{S}$ and $m : \mathbb{S} \otimes \mathbb{S} \to \mathbb{S}$. The symmetric sequence 1 is just $1_0 = S^0$ and $1_n = *$ for $n > 0$. The map u is just the inclusion. The map m is given by

$$m((\theta_1, ..., \theta_p), (\psi_1, ..., \psi_q)) = (\theta_1, ..., \theta_p, \psi_1, ..., \psi_q).$$

It is straightforward to verify the following diagrams commute. The commutativity of these diagrams is what it means for $(\mathbb{S}, u, m)$ to be a commutative monoid in $\underline{\mathrm{sSet}}_*^{\Sigma}$.

$$\begin{tikzcd} 1 \otimes \mathbb{S} \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & \mathbb{S} \otimes \mathbb{S} \arrow[d, "m"] \\ & \mathbb{S} \end{tikzcd} \quad \begin{tikzcd} \mathbb{S} \otimes \mathbb{S} \otimes \mathbb{S} \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"] & \mathbb{S} \otimes \mathbb{S} \arrow[d, "m"] \\ \mathbb{S} \otimes \mathbb{S} \arrow[r, "m"] & \mathbb{S} \end{tikzcd} \quad \begin{tikzcd} \mathbb{S} \otimes \mathbb{S} \arrow[r, "\tau"] \arrow[dr, "m"] & \mathbb{S} \otimes \mathbb{S} \arrow[d, "m"] \\ & \mathbb{S} \end{tikzcd}$$

A left $\mathbb{S}$-module is a symmetric sequence $X \in \underline{\mathrm{sSet}}_*^{\Sigma}$ with a pairing $\sigma : \mathbb{S} \otimes X \to X$, i.e. maps $\sigma_{p,q} : \mathbb{S}^p \wedge X_q \to X_{p+q}$ that must make the following diagrams commute.

$$\begin{tikzcd} 1 \otimes X \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & \mathbb{S} \otimes X \arrow[d, "\sigma"] \\ & \mathbb{S} \end{tikzcd} \quad \begin{tikzcd} \mathbb{S} \otimes \mathbb{S} \otimes X \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \sigma"] & \mathbb{S} \otimes X \arrow[d, "\sigma"] \\ \mathbb{S} \otimes X \arrow[r, "\sigma"] & X \end{tikzcd}$$

It is straightforward to verify that these commutativity conditions recover the requirements for a symmetric spectrum. Indeed a symmetric spectrum X also makes the underlying symmetric sequence into a left $\mathbb{S}$-module.

The fact that $\mathbb{S}$ is a commutative monoid allows us to construct a tensor product of $\mathbb{S}$-modules. This will be our smash product of symmetric spectra. Given a left $\mathbb{S}$-module X, we can also consider it as a right $\mathbb{S}$-module in a natural way via the twist morphism. That is, define $\overline{\sigma} = \sigma \circ \tau X \otimes \mathbb{S} \to \mathbb{S} \otimes X \to X$. This construction makes X into an $(\mathbb{S}, \mathbb{S})$-bimodule. With this structure in hand, we can then define the tensor product of left $\mathbb{S}$-modules.

Given X, Y left $\mathbb{S}$-modules, consider the diagram

$$X \otimes \mathbb{S} \otimes Y \underset{\mathrm{id} \otimes \sigma_Y}{\overset{\overline{\sigma_X} \otimes \mathrm{id}}{\rightrightarrows}} X \otimes Y .$$

The coequalizer of this diagram is what we call $X \otimes_{\mathbb{S}} Y$ or $X \wedge Y$. This is at first only a symmetric sequence. We can, however, equip it with the structure of a left $\mathbb{S}$-module by using the compatible left $\mathbb{S}$-module structure of X. So

$$(\sigma_{X \otimes Y})_{p,q} (\theta_1, ..., \theta_p, (x_1, ..., x_r, y_1, ..., y_s)) = ((\sigma_X)_{p,r}(\theta_1, ..., \theta_p, x_1, ..., x_r), y_1, ..., y_s) .$$

It is instructive to verify that $\Sigma^\infty A \wedge \Sigma^\infty B \cong \Sigma^\infty (A \wedge B)$ for $A, B \in \underline{\mathrm{sSet}}_*$.

Ring spectra (monoids in the category of the symm. monoidal cat. of spectra?)

3. Steenrod squares and cohomology operations

Definition 3.24: We briefly consider the *set valued* cohomology functors $H^n(-; A)$ with coefficients in an abelian group A. With this convention, a cohomology operation of type n, A, m, B is a natural transformation $\theta : H^n(-; A) \dot{\to} H^m(-; B)$.

Remark 3.25: The reason why we use the set valued functors is because of Yoneda's lemma. We are interested in $H^n(-; A) = [-, K(A, n)]$, which on the right hand side is a priori only a set. Recall $[X, K(A, n)] = \underline{\mathrm{hTop}}_*(X, K(A, n))$. With this in mind, a cohomology operation of type n, A, m, B is a natural transformation $\theta : \underline{\mathrm{hTop}}_*(-, K(A, n)) \dot{\to} \underline{\mathrm{hTop}}_*(-, K(B, m))$. The Yoneda lemma says the natural transformations are in 1-1 correspondence with $\underline{\mathrm{hTop}}_*(K(A, n), K(B, m))$.

Definition 3.26: The Steenrod squares $\mathrm{Sq}^i = \mathrm{Sq}^i_X : H^n(X; \mathbb{Z}_2) \to H^{n+i}(X; \mathbb{Z}_2)$ is a family of cohomology operations which satisfy the following properties:

1. $\mathrm{Sq}^i(f^*(\alpha)) = f^*(\mathrm{Sq}^i(\alpha))$ for $f : X \to Y$;
2. $\mathrm{Sq}^i(\alpha + \beta) = \mathrm{Sq}^i(\alpha) + \mathrm{Sq}^i(\beta)$;
3. $\mathrm{Sq}^i(\alpha \cup \beta) = \sum_j \mathrm{Sq}^j(\alpha) \cup \mathrm{Sq}^{i-j}(\beta)$;
4. $\mathrm{Sq}^i(\sigma\alpha) = \sigma(\mathrm{Sq}^i(\alpha))$ where $\sigma H^n(X; \mathbb{Z}_2) \cong H^{n+1}(X; \mathbb{Z}_2)$;
5. $\mathrm{Sq}^i(\alpha) = \alpha \cup \alpha$ if $i = |\alpha|$ and $\mathrm{Sq}^i(\alpha) = 0$ if $i > |\alpha|$;
6. $\mathrm{Sq}^0 = \mathrm{id}$;
7. Sq^1 is the $\mathbb{Z}_2$ Bockstein homomorphism β coming from the coefficient sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$.

Remark 3.27: There is a way to rearrange compositions of Sq^is by using the Adem relations.

$$\mathrm{Sq}^a \mathrm{Sq}^b = \sum_{j=0}^{[a/2]} \binom{b-j-1}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

if $a < 2b$

The Adem relations generate all the relations among the Steenrod squares. A composition of the Steenrod squares $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \mathrm{Sq}^{i_2} \cdots \mathrm{Sq}^{i_k}$ is called admissible if no Adem relation can be applied, i.e. $i_j \geq 2i_{j+1}$ for all j.

The Steenrod squares are elements of $\mathrm{Sq}^i \in [K(\mathbb{Z}_2, n), K(\mathbb{Z}_2, n+i)]$ for all n. Is the Steenrod algebra the algebra coming from stable limit of stable cohomology operations or something?

The construction of the Steenrod squares seems involved following Hatcher's approach. Some computations can be done using spectral sequences, see Davis and Kirk.

Steenrod squares show up in defining chern classes, and lots of cobordism theory computations.

4. Cobordism Theory–oriented cohomology theories

Definition 4.28: (Ring spectrum) A ring spectrum is a symmetric spectrum $\mathbb{E}$ with a map of spectra $\mathbb{E} \wedge \mathbb{E} \to \mathbb{E}$ which satisfies a few axioms. From Stong, we have maps $m_{p,q} : E_p \wedge E_q \to E_{p+q}$ and a map $u : \mathbb{S} \to \mathbb{E}$. These maps need to satisfy the following conditions:

1. The following diagram commutes up to homotopy and sign,

$$\begin{tikzcd}
\Sigma E_p \wedge E_q \arrow[r] & E_{p+1} \wedge E_q \arrow[dr] & \\
\Sigma(E_p \wedge E_q) \arrow[r] \arrow[u, "\lambda"] \arrow[d, "\mu"'] & \Sigma E_{p+q} \arrow[r] & E_{p+q+1} \\
E_p \wedge \Sigma E_q \arrow[r] & E_p \wedge E_{q+1} \arrow[ur] &
\end{tikzcd}$$

   i.e. $[m_{p+1,q} \circ (e_p \wedge 1) \circ \lambda] = [e_{p+q} \circ \Sigma m_{p+q}] = (-1)^p [m_{p,q+1} \circ (1 \wedge e_q) \circ \mu]$;

2. The following diagram commutes

$$\begin{tikzcd}
S^p \wedge E_q \arrow[r, "u_p \wedge 1"] \arrow[d, equal] & E_p \wedge E_q \arrow[d, "m_{p,q}"] & E_p \wedge S^q \arrow[l, "1 \wedge S^q"'] \arrow[d] \\
\Sigma^p E_q \arrow[r, "\widetilde{e}"] & E_{p+q} & S^q \wedge E_p \arrow[l, "(-1)^{pq}\widetilde{e}"']
\end{tikzcd}$$

What does it mean for a spectrum to be associative and commutative?

Definition 4.29: (Oriented cohomology theory) Let $\mathbb{E}$ be an Ω-spectrum and a ring spectrum. We say the cohomology theory $\widetilde{\mathbb{E}}^*$ is oriented if there is a cohomology class $\tau \in \widetilde{\mathbb{E}}^2(CP^\infty)$ which satisfies $\iota^*\tau = u_2 \in \widetilde{\mathbb{E}}^2(\mathbb{S}^2)$. This class τ then can be used to define an orientation class $e(\xi) \in \widetilde{\mathbb{E}}^2(B(\xi))$ for any $\mathbb{C}$ line bundle by pulling back τ along the classifying map $\xi : B(\xi) \to CP^\infty = BU(1)$.

We define an $\mathbb{E}$-orientation of ξ, a rank n real vector bundle to be an element $t \in \mathbb{E}^n(M\xi) = [M\xi, E_n]$, so that for all $x \in B(\xi)$ the pullback of t under the inclusion $\iota : S^n \cong \pi^{-1}(x) \to M\xi$ is to give the unit, i.e. $\iota^*(t) = u_n \in \widetilde{\mathbb{E}}^n(\mathbb{S}^n)$. The class t is called a Thom class for ξ. If we pullback the class t by the zero section $s_0 : B(\xi) \to M(\xi)$, we get a class $e(\xi) = s_0^*(t) \in \widetilde{\mathbb{E}}^2(B(\xi))$ called the Euler class of the bundle. These definitions parallel the usual definitions just with a generalized cohomology theory.

Example 4.30: MU is a ring spectrum, and indeed MU^* is an oriented cohomology theory. The characteristic classes one obtains are called connor-floyd classes $cf_n(\xi) \in MU^{2n}(B(\xi))$. This is the universal example of an oriented cohomology theory. That is, for any other oriented cohomology theory $\mathbb{E}$, there is a unique map $MU \to \mathbb{E}$.

5. Formulas in Homology and Cohomology theory

5.5. Tor and Ext

For finitely generated Abelian groups, the computation of Tor reduces to the following chart:

$\mathrm{Tor}(A,B)$	$B=\mathbb{Z}$	$B=\mathbb{Z}_m$
$A=\mathbb{Z}$	0	0
$A=\mathbb{Z}_n$	0	$\mathbb{Z}_{(m,n)}$

where $(m,n) = \gcd(m,n)$. We also have

$$\mathrm{Tor}(\oplus_i A_i, \oplus_j B_j) \cong \oplus_i, j\, \mathrm{Tor}(A_i, B_j)$$

and $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$.

For finitely generated Abelian groups, the computation of Ext reduces to the following chart:

$\mathrm{Ext}(A,B)$	$B=\mathbb{Z}$	$B=\mathbb{Z}_m$
$A=\mathbb{Z}$	0	0
$A=\mathbb{Z}_n$	$\mathbb{Z}_n$	$\mathbb{Z}_{(m,n)}$

We also have

$$\mathrm{Ext}(\oplus_i A_i, B) \cong \prod_i \mathrm{Ext}(A_i, B)$$

$$\mathrm{Ext}(A, \oplus B_i) \cong \oplus_i \mathrm{Ext}(A, B_i).$$

Another short way to remember this information is by the formulas

$$\mathrm{Tor}(A,B) \cong \mathrm{torsion}(A) \otimes_{\mathbb{Z}} \mathrm{torsion}(B)$$

and

$$\mathrm{Ext}(A, B)\mathrm{torsion}(A) \otimes_{\mathbb{Z}} B$$

Remark 5.31: To compute Tor and Ext for more general rings, it is useful to think of them as derived functors.

Here are the steps to compute $\mathrm{Tor}_n^R(A, B)$ with $A, B \in R - \underline{\mathrm{Mod}}$:

1. Take a projective, free, or flat resolution of A, written $P. \to A$.
2. Apply $- \otimes_R B$ to the resolution $P.$ to get $P. \otimes B$.
3. The resulting sequence $P. \otimes B$ is a chain complex, so taking homology, we define $\mathrm{Tor}_n^R = H_n(P. \otimes B)$.

Here are the steps to compute $\mathrm{Ext}_R^n(A, B)$ with $A, B \in R - \underline{\mathrm{Mod}}$:

1. Take a projective or free resolution $P. \to A$ of A.
2. Apply the contravariant functor $\hom(-, B)$ to $P.$.
3. The resulting object $\hom(P., B)$ is a cochain complex, so taking homology, we define $\mathrm{Ext}_R^n(A, B) = H^n(\hom(P., B))$.

Remark 5.32: For computations, it is often useful to use the (co)homological δ-functor properties of Tor and Ext.

Definition 5.33: A homological δ functor is a sequence of $a$dditive functors $h_n : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are abelian categories, with a collection of natural transformations δ. If $A \rightarrowtail B \twoheadrightarrow C$ is a short exact sequence in $\mathcal{A}$, we get $\delta : h_n(C) \to h_{n-1}(A)$. These are then required to fit into a long exact sequence for any short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{A}$:

$$\begin{tikzcd}
\cdots \arrow[r] & h_n(A) \arrow[r] & h_n(B) \arrow[r] & h_n(C) \arrow[dll, "\delta"] & \\
& h_{n-1}(A) \arrow[r] & h_{n-1}(B) \arrow[r] & h_{n-1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

A cohomological δ-functor consists of additive functors $h^n : \mathcal{A} \to \mathcal{B}$ with natural transformations $\delta : h^n(C) \to h^{n+1}(A)$ fitting into a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & h^n(A) \arrow[r] & h^n(B) \arrow[r] & h^n(C) \arrow[dll, "\delta"] & \\
& h^{n+1}(A) \arrow[r] & h^{n+1}(B) \arrow[r] & h^{n+1}(C) \arrow[r] & \cdots
\end{tikzcd}$$

Remark 5.34: Left derived functors of covariant functors yield homological delta functors.

Right derived functors of covariant functors yield cohomological delta functors.

Example 5.35: Explicitly, for Tor, to any short exact sequence $0 \to A \to B \to C \to 0$ in $R - \underline{\text{Mod}}$ and any $M \in R - \underline{\text{Mod}}$ there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Tor}_1^R(A,M) \arrow[r] & \mathrm{Tor}_1^R(B,M) \arrow[r] & \mathrm{Tor}_1^R(C,M) \arrow[dll, "\delta"] & \\
& A \otimes_R M \arrow[r] & B \otimes_R M \arrow[r] & C \otimes_R M \arrow[r] & 0.
\end{tikzcd}$$

With the same setup but for Ext, we get a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \hom_R(C,M) \arrow[r] & \hom_R(B,M) \arrow[r] & \hom(A,M) \arrow[dll, "\delta"] & \\
& \mathrm{Ext}_R^1(C,M) \arrow[r] & \mathrm{Ext}_R^1(B,M) \arrow[r] & \mathrm{Ext}_R^1(A,M) \arrow[r] & \cdots
\end{tikzcd}$$

Example 5.36: Add in dimension shifting for computations if time permits.

Theorem 5.37: If $R = \Lambda$ is a PID, then Ext^n and Tor_n vanish for $n > 1$.

5.6. Universal coefficient theorems

We first start off with the topological results, then review the algebraic results and try to obtain a unifying result.

Theorem 5.38: (Universal coefficient theorem for cohomology: cohomology in terms of homology) We take: (X, A) a pair of spaces, R a PID, $M \in R - \underline{\text{Mod}}$ and get the exact sequence (which splits)

$$0 \to \mathrm{Ext}_R(H_{q-1}(X, A; R), M) \to H^q(X, A; M) \to \hom_R(H_q(X, A; R), M) \to 0$$

and the absolute version which is exact (and splits)

$$0 \to \mathrm{Ext}_R(H_{q-1}(X; R), M) \to H^q(X; M) \to \hom_R(H_q(X; R), M) \to 0.$$

Note that taking $(X, *)$ gives a relative version of the theorem, getting rid of a superfluous term. One way to look at this theorem is that it is relating actual cohomology to its interpretation of being R-valued functions on homology classes. In general, the interpretation isn't exact and there is a correction term. To remember if the index in the Ext term is $q \pm 1$, you should think about the simple example:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "2\cdot"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$$H_2 = 0 \qquad H_1 = 0 \qquad H_0 = \mathbb{Z}_2 \qquad H_{-1} = 0$$

and upon dualizing

$$\begin{tikzcd}
0 & \hom(\mathbb{Z},\mathbb{Z}) \arrow[l] & \hom(\mathbb{Z},\mathbb{Z}) \arrow[l, "2\cdot"'] & 0 \arrow[l]
\end{tikzcd}$$

$$H^2 = 0 \qquad H^1 = \mathbb{Z}_2 \qquad H^0 = 0 \qquad H^{-1} = 0$$

The Ext term shows up because of the hom. The obvious map from $H^n(X;R) \to \hom(H_n(X), R)$ lets you know that the Ext term shows up on the left.

Theorem 5.39: (Universal coefficient theorem for cohomology: cohomology in terms of cohomology) If R is a PID, M is a $f$initely generated R-module, C_* is a free chain complex of R-modules, then the following sequence is split exact.

$$H^q(C_*) \otimes M \rightarrowtail H^q(C_*; M) \twoheadrightarrow \mathrm{Tor}^R(H^{q+1}(C_*), M)$$

The topological version is if X is a space, M is a finitely generated R-module, then

$$H^q(X) \otimes M \rightarrowtail H^q(X; M) \twoheadrightarrow \mathrm{Tor}^R(H^{q+1}(X), M)$$

Theorem 5.40: (Universal coefficient theorem for homology: homology in terms of homology) We take: (X, A) a pair of spaces, R a PID, M an $R - \underline{\mathrm{Mod}}$ and get the exact sequence (which splits)

$$0 \longrightarrow H_q(X, A; R) \otimes M \longrightarrow H_q(X, A; M) \longrightarrow \mathrm{Tor}^R(H_{q-1}(X, A; R), M) \longrightarrow 0$$

and the absolute version

$$0 \longrightarrow H_q(X; R) \otimes M \longrightarrow H_q(X; M) \longrightarrow \mathrm{Tor}^R(H_{q-1}(X; R), M) \longrightarrow 0$$

Theorem 5.41: (universal coefficient theorem for homology: homology in terms of cohomology) Let R be a PID, let C_* be a free chain complex of R-modules, suppose $H_q(C_*)$ is $f$initely generated for each q, let M be an R-module. Then the following sequence is exact, natural, and splits.

$$0 \longrightarrow \mathrm{Ext}_R(H^{q+1}(C_*), M) \longrightarrow H_q(C_*; M) \longrightarrow \hom(H^q(C_*), M) \longrightarrow 0$$

For a topological version, let X be a finite CW complex, M an R-module. Then the following sequence is exact, natural, and splits.

$$0 \longrightarrow \mathrm{Ext}_R(H^{q+1}(X), M) \longrightarrow H_q(X; M) \longrightarrow \hom(H^q(X), M) \longrightarrow 0$$

Theorem 5.42: (Algebraic Künneth theorem) Let C_* and D_* be chain complexes over a PID R. Suppose further that C_q is a free R-module. Then the following sequence is exact and splits (the splitting is not natural).

$$\bigoplus_{p+q=n} H_p(C_*) \otimes H_q(D_*) \rightarrowtail H_n(C_* \otimes D_*) \twoheadrightarrow \bigoplus_{p+q=n} \mathrm{Tor}^R\left(H_p(C_*), H_{q-1}(D_*)\right)$$

Theorem 5.43: (Künneth theorem) Let X and Y be topological spaces, let R be a PID. Then the following sequence is exact and splits (the splitting is not natural).

$$\bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \rightarrowtail H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n} \mathrm{Tor}^R\left(H_p(X), H_{q-1}(Y)\right)$$

Example 5.44: A simple example to see how the indices should be in the torsion term, we can consider $C_* = D_*$ given by $C_1 = C_0 = \mathbb{Z}$ and $\cdot 2 = d_1 : C_1 \to C_0$ as the chain map, all other terms and maps being 0. We can compute

$$H_*(C_*) \otimes H_*(D_*) = \begin{array}{|c|c} 0 & 0 \\ \hline \mathbb{Z}_2 & 0 \\ \hline \end{array}$$

and

$$C_* \otimes D_* = \begin{tikzcd} \mathbb{Z} \arrow[d, "\times 2"'] & \mathbb{Z} \arrow[l, "\times 2"'] \arrow[d, "\times(-2)"] \\ \mathbb{Z} & \mathbb{Z} \arrow[l, "\times 2"'] \end{tikzcd}$$

so that $T_* = \mathrm{Tot}^{\oplus}(C_* \otimes D_*)$,

$$T_* = \ 0 \longrightarrow \mathbb{Z} \xrightarrow{2 \oplus (-2)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{[2\ 2]} \mathbb{Z} \longrightarrow 0$$

and we compute $H_0(T_*) = \mathbb{Z}_2$, $H_1(T_*) = \mathbb{Z}_2$, $H_n(T_*) = 0$ for all other n. Since $H_0(C_*) = \mathbb{Z}_2$ is the only non-zero homology term, the Tor term which arises when trying to compute $H_1(C_* \otimes C_*)$ must have $H_0(C_*)$ in both slots. So we see the sum of the indices in the Tor term alwas is 1 less than the homology we are trying to compute. Remembering whether it shows up on C_* or D_* is only important if one of C_* and D_* is not a free chain complex.

Example 5.45: The UCT for homology in terms of homology can be obtained from the algebraic Künneth theorem by viewing the coefficient module as a chain complex concentrated in degree 0.

Theorem 5.46: (Algebraic Künneth theorem for cohomology.) This result is a bit more complicated, which explains all of the extra conditions in the above topological sequences. See Weibel pg. 90.

Theorem 5.47: Gysin sequence, Wang sequence are long exact sequences of (co)homology groups to fibrations $S^n \to E \to B$ respectively $F \to E \to \mathbb{S}^n$ with $n \geq 1$. They can be obtained by looking at the LSAH SS of the fibration and splicing together exact sequences. If $n = 0$, the result holds for fiber bundles, but the proof is more hands on, if one uses $\mathbb{Z}_2$ coefficients.

Theorem 5.48: Freudenthal suspension theorem says the suspension map $[\mathbb{S}^k, X] \to [\mathbb{S}^{k+1}, \Sigma X]$ for X a based, n-connected space is an isomorphism if $k < 2n+1$ and surjective if $k = 2n+1$.

Theorem 5.49: Poincaré duality says for M a compact, oriented manifold of dimension n, that there is an isomorphism $H^k(M) \cong H_{n-k}(M)$ for all k.

Theorem 5.50: Alexander duality is a formula describing the homology of a sphere minus a nice (compact, locally contractible, nonepty, proper) subspace K, and the subspace K. So $\widetilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \widetilde{H}^{n-i-1}(K; \mathbb{Z})$

Theorem 5.51: (Lefschetz duality) Suppose M is a compact R-orientable n-manifold with boundary ∂M. Then there is a class $[M] \in H_n(M, \partial M; R)$ giving isomorphisms $D_M : H^k(M;R) \to H_{n-k}(M, \partial M; R)$ and $D_M : H^k(M, \partial M; R) \to H_{n-k}(M, \partial M; R)$.

Definition 5.52: ($\varprojlim^1$) Suppose we have an inverse system of abelian groups

$$\cdots A_n \xrightarrow{\alpha_n} A_{n-1} \cdots A_1 \xrightarrow{\alpha_1} A_0$$

we can then take the inverse limit of this system, which is the subobject of $\prod A_i$ given by

$$\varprojlim A_i = \{a = (\ldots, a_2, a_1, a_0) \mid \forall k, \alpha_k(a_k) = a_{k-1}\}.$$

Define $\Delta : \prod A_i \to \prod A_i$ by

$$\Delta(\ldots, a_i, \ldots, a_1, a_0) = (\ldots, a_i - \alpha_{i+1}(a_{i+1}), \ldots, a_1 - \alpha_2(a_2), a_0 - \alpha_1(a_1))$$

or by $\Delta(a)_k = a_k - \alpha_{k+1}(a_{k+1})$. It is clear that $\varprojlim A_i = \ker \Delta$. Define $\varprojlim^1 A_i = \operatorname{coker} \Delta$. That is (unhelpful formula)

$$\varprojlim^1 A_i = \left\{ [a] \mid a \in \prod A_i, a \sim b \text{if} a - b \in \operatorname{im} \Delta \right\}.$$

Theorem 5.53: $\varprojlim^1$ is the first derived functor of inverse limit, all higher derived functors vanish. So $\varprojlim^1$ is a universal cohomological δ functor. So there is a long exact sequence to a short exact sequence of towers of abelian groups.

We have the Mittag-Leffler condition on an inverse system (tower of abelian groups) A_i. It says that if for any k there exists a $j \geq k$ such that the image $A_i \to A_k$ equals the image of $A_j \to A_k$ for all $i \geq j$. This is satisfied trivially if all maps are surjective.

The trivial Mittag-Leffler condition is if for each k there exists a $j > k$ such that $A_j \to A_k$ is zero.

Theorem 5.54: If the inverse system A_i satisfies the Mittag-Leffler condition, then $\varprojlim^1 A_i = 0$.

Theorem 5.55: Let $X = \varinjlim X_i$ be a CW complex (this is not necessarily the skeleton filtration!), and let h_* and h^* be generalized homology and cohomology theories. Then $h_k(X) = \varinjlim h_k(X_i)$ and there is an exact sequence

$$0 \longrightarrow \varprojlim^1 h^{n-1}(X_i) \longrightarrow h^n(\varinjlim X_i) \longrightarrow \varprojlim h^n(X_i) \longrightarrow 0$$

6. Cobordism Theory

Theorem 6.56: (The Thom isomorphism theorem) Let $\xi : E \to B$ be an orientable $\mathbb{R}$-vector bundle with rank n. To this bundle, there is a corresponding bundle $E_0 = E \setminus s_0(B)$, that is, all points in E corresponding to non-zero vectors in the fiber $\mathbb{R}^n$. The bundle E_0 is homotopy equivalent to an $\mathbb{S}^{n-1}$-bundle over B. Consider the natural inclusion $\iota : E_0 \to E$. There is a class $u \in H^n(E, E_0)$ so that the homomorphism $\Phi : H^k(B) \to H^{k+n}(E, E_0)$ given by $\phi(\omega) = \omega \cup u$ is an isomorphism for all k.

The pair $(E, E_0) \cong (D(\xi), S(\xi))$ that is, the disc bundle relative the sphere bundle, and from this perspective we see we are computing the cohomology of the Thom space of the bundle ξ, that is $MB = M(\xi) = D(\xi)/S(\xi)$. and $\widetilde{H}^*(M(\xi)) \cong H^*(E, E_0)$. Note: the Thom space is NOT an $\mathbb{S}^n$-bundle since the entire sphere bundle is crushed to a point! Not the sphere in each fiber, separately.

From the de Rham point of view, one can get away with talking about compactly supported cohomology in the vertical direction, and the isomorphism is then $H^{k+n}_{cv}(E) \cong H^k(B)$. One direction of the isomorphism is given by integration along the fiber, the other by wedging with the Thom class.

To this pair (E_0, E) there is a long exact cohomology sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(E, E_0) \arrow[r] & H^k(E) \arrow[r] & H^k(E_0) \arrow[dll, "\delta"] & \\
& H^{k+1}(E, E_0) \arrow[r] & H^{k+1}(E) \arrow[r] & H^{k+1}(E_0) \arrow[r] & \cdots .
\end{tikzcd}$$

equivalent to

$$\begin{tikzcd}
\cdots \arrow[r] & H^k(E, E_0) \arrow[r] & H^k(B) \arrow[r] & H^k(E_0) \arrow[dll, "\delta"] & \\
& H^{k+1}(E, E_0) \arrow[r] & H^{k+1}(B) \arrow[r] & H^{k+1}(E_0) \arrow[r] & \cdots .
\end{tikzcd}$$

A proof of the Thom isomorphism theorem can be obtained using a relative spectral sequence argument.

Definition 6.57: (Cobordism category) A cobordism category is a triple $(\mathscr{C}, \partial, i)$ where

1. $\mathscr{C}$ is a category with finite sums and an initial object $\emptyset$;
2. $\partial : \mathscr{C} \to \mathscr{C}$ is a sum-preserving (called additive in the literature) functor such that $\partial^2 = \emptyset$;
3. $i : \partial \to \mathrm{id}$ is a natural transformation
4. there is a small, (full subcategory) $\mathscr{C}_0$ such that every object of $\mathscr{C}$ is isomorphic to an object in $\mathscr{C}_0$.

Definition 6.58: If $\mathscr{C}$ is a cobordism category, we say $M, N \in \mathscr{C}$ are cobordant if there exists $U, V \in \mathscr{C}$ such that $M \oplus \partial U \cong N \oplus \partial V$.

Remark 6.59: In cobordism categories of manifolds with structure, the above cobordism relation is equivalent to the usual one. To a cobordism category $\mathscr{C}$ there is a cobordism semi-groups $\Omega(\mathscr{C})$ of cobordism classes of objects with the sum from the category.

Definition 6.60:

1. ((B,f)-structures on vector bundles) Let $f_r : B_r \to BO_r$ be a sequence of fibrations. Let $\xi : B(\xi) \to BO_r$ be a rank r vector bundle. A (B,f) structure on ξ is a lifting of ξ to B_r, i.e. $\tilde{\xi} : B(\xi) \to B_r$. The structures are defined up to vertical homotopy.

2. ((B,f,g)-structures on manifolds) (B,f) are as above. Now $g_r : B_r \to B_{r+1}$ so that

$$\begin{tikzcd} B_r \arrow[r, "g_r"] \arrow[d, "f_r"] & B_{r+1} \arrow[d, "f_{r+1}"] \\ BO_r \arrow[r, "\iota"] & BO_{r+1} \end{tikzcd}$$

commutes. A (B,f) structure on a manifold M is then a (B,f) structure on the stable normal bundle on M. That is, for $k >> 0$, embeddings $M \to \mathbb{R}^k$ are regularly isotopic, so the normal bundles are independent of the embedding. We then choose (B,f) structures on the normal bundles in the stable range so that the lifts are compatible with the commutative squares above.

Proposition 6.61: The cobordism semi-group for the cobordism category of manifolds with (B,f)-structures is denoted by MB_*. The semi-group is actually an abelian group in this case.

Give the standard examples: unoriented manifolds, oriented manifolds, framed cobordism,

Example 6.62: (MO_*) We have the computation $MO_*(pt) \cong \mathbb{Z}_2[x_2, x_4, x_5, x_6, x_8, \ldots]$ with generators x_i for $i \neq 2^n - 1$. The generators correspond to real projective spaces in even dimensions. The odd dimensional generators are given by Dold manifolds. For $n, m \geq 0$ consider the space $\mathbb{S}^n \times \mathbb{CP}^m$ and the involution $\iota(x,z) = (-x, \overline{z})$ where $-x$ is the point antipodal to x and $\overline{z}$ is obtained by conjugating each homogeneous coordinate for z. Dold then defines $P(n,m) = \mathbb{S}^n \times \mathbb{CP}^m/(x,z) \sim \iota(x,z)$. He then shows that these manifolds can be taken as generators of MO_*.

Stiefel-Whitney numbers are the means to distinguish cobordant manifolds. Stiefel-Whitney classes can be computed using their axioms in many cases. The definition of the Stiefel-Whitney classes is a bit more complicated.

Computation of $MO_*(X)$ can be done using cofibration sequence and the LSAH SS.

Example 6.63: (MSO_*) The description is not so simple. $MSO_* \otimes \mathbb{Q} \cong \mathbb{Q}[x_4, x_8, ...]$, generators given by CP^{2n}. MSO_* only has torsion of order 2, and only in dimensions not a multiple of 4. The groups are always finitely generated. Some explicit values $MSO_0 \cong \mathbb{Z}$, $MSO_1 = MSO_2 = MSO_3 = 0$, $MSO_4 \cong \mathbb{Z}4$, $MSO_5 \cong \mathbb{Z}_2$, $MSO_6 = MSO_7 = 0$.

The Dold manifolds from MO_* are in fact orientable in odd dimensions, so they determine non-trivial order two classes in MSO_*. Using the above calculations and the product structure, one can determine that $MSO_n \neq 0$ for all $n \geq 8$.

Classifying numbers are pontrjagin numbers and stiefel-whitney numbers. In particular, if M^{4k} is an oriented manifold, it bounds iff all pontrjagin numbers are zero.

Wall proved M^n oriented manifold bounds iff all pontrjagin numbers and all stiefel-whitney numbers vanish.

Pontrjagin classes are constructed by...

Example 6.64: (MU_*) Stably almost complex manifolds, i.e. (BU, f) structures. $MU_* \cong \mathbb{Z}[x_2, x_4, x_6, \cdots]$ with generators in all even dimensions given by (some complex algebraic variety [they are the Milnor hypersurfaces $H_{n,m}$]). The (B, f) structure is $B_{2r} = B_{2r+1} = BU_r$ with evident maps. Integer cohomology classes are used to distinguish elements, although it isn't clear exactly what they are. Related to chern classes somehow.

Description of stably almost complex structures is discussed briefly in Davis and Kirk. Add on trivial bundle to tangent bundle and equivalence classes of almost complex structures on those.

Description as Lazard ring with universal Formal Group Law.

Milnor computes MU_* formally, i.e. not using the geometric definition. One can compute $H_*(MU)$ and $H^*(MU)$ explicitly, see Adams's blue book p. 51. One has a Hurewicz map $MU_* = \pi_*(MU) \to H_*(MU)$ that Milnor uses to do the computation.

Definition 6.65: Stiefel-Whitney classes and Stiefel-Whitney numbers

Use axiomatic for stiefel whitney classes. Get them by pulling back well-chosen generators of cohomology of $BO(n)$. General definition uses thom iso and steenrod squares. $w_i(\xi) = \phi^{-1} \circ \mathrm{Sq}^i(\phi(1))$ where ϕ is the Thom isomorphism.

Definition 6.66: Euler class of an oriented vector bundle is the pull-back of the Thom class to the base by the zero section. The euler class is natural wrt oriented vector bundle maps.

The euler class with mod 2 coefficients gives the stiefel-whitney class.

Also there is a sum formula: $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$.

Definition 6.67: Chern classes. Top chern class is euler class, look at E_0 and make $2(n-1)$ dim'l vector bundle over that, pull back the euler class for that to get next lower chern class. repeat... Using a hermitian metric on the bundle.

Chern classes are natural wrt complex vector bundle maps.

Chern classes are stable vector bundle invariants.

Product theorem for total chern classes holds.

Definition 6.68: Pontrjagin classes and complexification of oriented vector bundles. Take an oriented bundle, complexify it by tensoring with $\mathbb{C}$, get a complex vector bundle. Look at the chern classes. The ones in dimension $4n + 2$ will be of order two (or zero, or something), the others are called pontrjagin classes. See this because the complexified bundle is isomorphic to its conjugate bundle.

Pontrjagin classes are natural and stable oriented vector bundle invariants.

There is almost a product formula, up to elements of order two.

Theorem 6.69: (Thom-Pontrjagin Theorem) The cobordism theory for (B, f) manifolds is related to stable homotopy theory by the equation

$$MB_n(pt) \cong \varinjlim \pi_{n+i}(MB_i).$$

Theorem 6.70: $\widetilde{MB}_*$ defines a reduced generalized homology theory. At very least, try to see that $\widetilde{MO}_*$ does.

Theorem 6.71: LSAH SS for generalized homology theories and computations.

7. Complex Cobordism

Complex cobordism is the cobordism theory of manifolds with a stable almost complex structure. In what follows, we develop the theory and present a sketch of the computation of MU_*.

Definition 7.72: (Normal bundle) Let X^n be a compact, smooth, n-dimensional manifold. If $\iota : X \to M$ is an embedding of smooth manifolds, we define the normal bundle of X via the embedding ι to be the bundle $\nu(\iota) = \iota^*(TM)/TX$. If we equip M with a Riemannian metric, $\nu(\iota)$ can be identified with the subbundle of TM given by $\nu(\iota)_x = \{v \in T_xM \,|\, \forall w \in \iota_*(TX), < v, w >= 0\}$, i.e. the orthogonal complement of $\iota_*(TX)$.

Every manifold X can be embedded into some $\mathbb{R}^{n+r}$ for $r >> 0$. Furthermore, if $\iota_1, \iota_2 : X \to \mathbb{R}^{n+r}$ are two embeddings with r sufficiently large, ι_1 and ι_2 are regularly homotopic, i.e. a homotopy $H : X \times I \to \mathbb{R}^{n+r}$ such that $H_t : X \to \mathbb{R}^{n+r}$ is an embedding for all $t \in I$, and $H_t* : TX \to T\mathbb{R}^{n+r}$ is continuous. This result is proven in [**?**][Theorem 8.4]. In the paper, Hirsch develops an obstruction to when two maps f, g are regularly homotopic $\Omega'(f, g)$. As we can embed $\iota_1, \iota_2 : X \to \mathbb{R}^{2n+1}$, the obstruction for these two embeddings to be regularly homotopic lies in $\pi_n(V_n(\mathbb{R}^{2n+1}) = 0$. Hence the result.

The above result is important, because it guarantees that we can talk about a well-defined stable normal bundle of X. Namely, take any embedding $\iota : X \to \mathbb{R}^{n+r}$, and consider the normal bundle $\nu(\iota)$. For any two different embeddings ι_1 or ι_2, a regular homotopy connecting them induces an isomorphism of the normal bundles. Furthermore, the bundle is stably defined. That is, if you have an embedding $\iota_r : X \to \mathbb{R}^{n+r}$ and $\iota_{r+1} : X \to \mathbb{R}^{n+r+1}$, the embedding ι_{r+1} is regularly homotopic to the embedding $j_{n+r} \circ \iota_r : X \to \mathbb{R}^{n+r} \subset \mathbb{R}^{n+r+1}$. It is clear that $\nu(j_{n+r} \circ \iota_r) \cong \nu(\iota_r) \oplus \epsilon^1$.[1]

Using the language of classifying spaces clarifies some of the above points. Let $G_r(\mathbb{R}^{n+r})$ denote the Grassmannian of r-planes in $\mathbb{R}^{n+r}$. From the map $j_{n+r} : \mathbb{R}^{n+r} \to \mathbb{R}^{n+r+1}$, we get an induced map $j_{n+r} : G_r(\mathbb{R}^{n+r} \to G_r(\mathbb{R}^{n+r+1})$ given by the image of a subspace under j_{n+r}. A classifying space for r-bundles is $BO_r := G_r = \varinjlim_n G_r(\mathbb{R}^{n+r})$. This is the classifying space for principal $O_r(\mathbb{R})$-bundles and r-dimensional real vector bundles.

The infinite Stiefel manifold of r-frames is defined in a similar way. Let $V_r(\mathbb{R}^{n+r}) = \{(v_1, \ldots, v_r) \,|\, \wedge_i v_i \neq 0\}$, i.e. it consists of ordered sets of r linearly independent vectors in $\mathbb{R}^{n+r}$, also called an r-frame. It is topologized as the subset of $(\mathbb{R}^{n+r})^r$. There is also an orthogonal Stiefel manifold $V_r^O(\mathbb{R}^{n+r})$ consisting of all orthonormal r-frames in $\mathbb{R}^{n+r}$. There is a natural map $p : V_r^O(\mathbb{R}^{n+r}) \to G_r(\mathbb{R}^{n+r})$ and likewise natural maps $j_{n+r} : V_r^O(\mathbb{R}^{n+r}) \to V_r^O(\mathbb{R}^{n+r+1})$. We define $V_r^O(\mathbb{R}) = \varinjlim_n V_r^O(\mathbb{R}^{n+r})$. The direct limits for the infinite Grassmannian and Stiefel manifolds are compatible

[1] Let ϵ^k denote the trivial bundle of rank k. Define $j_{n+r}(\sum_{i=1}^{n+r} x_i e_i) = \sum_{i=1}^{n+r} x_i e_i + 0e_{n+r+1}$

with the projection maps p, and so we have a map $p : V_r^O(\mathbb{R}) \to BO_r$. This map is a fibration with homotopy fiber the Lie group O_r.

A regular homotopy between embeddings $\iota_1, \iota_2 : X \to \mathbb{R}^{n+r}$ gives a homotopy $H : X \times I \to BO_r$, so that $H_0 = \nu(\iota_1)$ and $H_1 = \nu(\iota_2)$. Therefore, the normal bundles are indeed isomorphic as their structure maps are homotopic.[2]

Definition 7.73: (Classifying space for U_r) Let U_r be the unitary group of transformations of $\mathbb{C}^r$, i.e. those linear transformations $A : \mathbb{C}^r \to \mathbb{C}^r$ such that $AA^* = I$ where $(-)^*$ is the conjugate dual. It is straightforward to verify that $U_r \subseteq O_{2r}$ under the the canonical identification of $\mathbb{C}^r$ with $\mathbb{R}^{2r}$.

We therefore have an action of U_r on $V_{2r}^O(\mathbb{R})$ by restricting the action of O_{2r}. We define $BU_r = V_{2r}^O(\mathbb{R})/U_r$. As we may also identify BO_r with $V_r^O(\mathbb{R})/O_r$, we get a morphism $f_r : BU_r \to BO_{2r}$, which is actually a fibration. The homotopy fiber of this fibration is O_{2r}/U_r. It furthermore follows that BU_r is indeed the classifying space for unitary vector bundles by [M-1967][Theorem 7.4].

Definition 7.74: An almost complex structure on an r real dimensional vector bundle $\xi : X \to BO_r$ is a stable lift of ξ to $BU_{[r/2]}$. The lift must be strict, i.e. not just a lift up to homotopy! To lift ξ stably means that there exists some $r' >> r$ such that after composing ξ with the structure maps j_r for BO_r, we get a map $\xi : X \to BO_{r'}$ and this map has a lift $\tilde{\xi} : X \to BU_{[r'/2]}$.

$$\begin{tikzcd}
 & BU_{[r/2]} \arrow[r] \arrow[d, "f_{[r/2]}"] & BU_{[(r+1)/2]} \arrow[r] \arrow[d, "f_{[(r+1)/2]}"] & \cdots \arrow[r] & BU_{[r'/2]} \arrow[d, "f_{[r'/2]}"] \\
X \arrow[r, "\xi"] \arrow[urrrr, dashed, "\tilde{\xi}"] & BO_r \arrow[r, "j_r"] & BO_{r+1} \arrow[r, "j_{r+1}"] & \cdots \arrow[r] & BO_{r'}
\end{tikzcd}$$

Two almost complex structures $\tilde{\xi}_1$ and $\tilde{\xi}_2$, i.e. stable lifts of ξ, are equivalent if there exists a fiberwise homotopy $H : X \times I \to BU_{r'}$ for some $r' >> 0$ between $\tilde{\xi}_1$ and $\tilde{\xi}_2$, so that for all t, $f_{r'} \circ H_t = \xi$.

This definition is equivalent to the more intuitive definition of a continuous section of the bundle $J \in \Gamma(\mathrm{Hom}(\xi, \xi))$ such that for any $x \in X$, $J_x^2 = -\mathrm{id} : E_x \to E_x$.[3]

Definition 7.75: If X is a compact smooth manifold, a stably almost complex structure on X is an almost complex structure on the stable normal bundle of X.

Remark 7.76: Consider $X = pt$. There are two distinct stably almost complex structures on X. WLOG embed $\iota : X \to \mathbb{R}^r$. Then $\nu(\iota) = \epsilon^r$, and $\nu(\iota) : X \to BO_r$ is given by $\nu(\iota) : X \to G_r(\mathbb{R}^r)$. As the fiber of $BU_1 \to BO_2$ is $O(2)/U(1) \cong S^0$, there are two possible lifts of $\nu(\iota)$. As $O(2r)/U(r)$ always consists of two connected components and the inclusion $O(2)/U(1) \to O(2r)/U(r)$ never maps into just one component, there are stably two distinct lifts as well.

An easy way to think of the lift of $\nu(\iota) : X \to BU_1$ is by giving a 2-frame in $\mathbb{R}^2$ and looking at the class it determines in BU_1. There are two possible choices: (e_1, e_2) or $(e_1, -e_2)$.

The two possible stably almost complex structures on X are inverses to one another in MU_*. This can be seen by lifting $X \times I \to X \to BO_r$. The stably a.c.

[2] Can you give the construction?
[3] E is the total space of the bundle ξ.

structure on $X \times \{0\}$ is the given one, while the stably a.c. structure on $X \times \{1\}$ is the other structure. This is because the inward pointing normal at 0 and at 1 have different sign.

If we embed $X \to \mathbb{R}^2$, then $X \times I \to \mathbb{R}^3$, the inward pointing normal at $X \times \{0\}$ is e_3 while the inward pointing normal at $X \times \{1\}$ is $-e_3$. If the stably almost complex structure on X was given by (e_1, e_2), then the stably a.c. structure on $X \times \{0\}$ is given by (e_1, e_2, e_3) which is equivalent to the given one. But the stably almost complex structure on $X \times \{1\}$ is given by $(e_1, e_2, -e_3)$ which is easily seen to be inequivalent to (e_1, e_2, e_3).

With this, it is easy to reason that $MU_0 \cong \mathbb{Z}$. Similarly, one can also show with bare hands that $MU_1 \cong 0$.

The complete computation of MU_* presented by Stong, originally computed by Milnor and Quillen, we need to make use of some heavy machinery. The main insight is to recast the problem from geometric generators and relations into stable homotopy theory. The Pontrjagin-Thom construction does just that. To solve the stable homotopy theory problem, we make heavy use of the Steenrod algebra structure and

8. Model categories

9. Presheaf cohomology

Definition 9.77: Let $X \in \underline{\text{Top}}$, a presheaf $\mathcal{F}$ on X is a functor $\mathcal{F} : \text{Open}(X)^{op} \to \underline{\text{Ab}}$. A presheaf is called a sheaf if it satisfies the gluing axiom: if $x \in \mathcal{F}(U)$, $y \in \mathcal{F}(V)$ such that $x|_{U \cap V} = y|_{U \cap V}$, then there exists a unique element $z \in \mathcal{F}(U \cup V)$ which restricts to x and y on U and V respectively.

Example 9.78: Let $\pi : Y \to X$ be a map. For any $q \geq 0$ there is a cohomology presheaf $\mathscr{H}^q$ on X given by π which is defined as follows. For $U \in \text{Open}(X)$, define $\mathscr{H}^q(U) = H^q(\pi^{-1}(U))$, and for $\iota : U \to V$, define $\mathscr{H}^q(\iota) = j^* : \pi^{-1}(V) \to \pi^{-1}(U)$, where $j : \pi^{-1}(U) \to \pi^{-1}(V)$ is the inclusion.

Definition 9.79: Let $X \in \underline{\text{Top}}$, let $\mathcal{F}$ be a presheaf on X, and let $\mathfrak{U}$ be an open cover of X. The Čech complex of X wrt to $\mathfrak{U}$ and $\mathcal{F}$ is given by

$$\cdots \longrightarrow \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_1 \cdots \alpha_p}) \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_{p+1}} \mathcal{F}(U_{\alpha_1 \cdots \alpha_{p+1}}) \longrightarrow \cdots$$

with differential δ. The differential δ is given by first defining $\partial_i : \coprod U_{\alpha_0 \cdots \alpha_p} \to U_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_p}$ by $\partial_i|_{U_{\alpha_0 \cdots \alpha_p}} : U_{\alpha_0 \cdots \alpha_p} \subseteq U_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_p}$. Then define $\delta := \sum_i (-1)^i \partial_i$. A useful formula for δ is

$$(\delta\omega)_{\alpha_0 \cdots \alpha_{p+1}} = \sum_{i=0}^{p+1} (-1)^i \omega_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{p+1}}.$$

The cohomology of this chain complex is denoted by $H^*(\mathfrak{U}; \mathcal{F})$. One then defines $H^p(X; \mathcal{F}) := \varinjlim H^p(\mathfrak{U}; \mathcal{F})$.

Theorem 9.80: (Leray's construction) Let $\pi : Y \to X$ be a map of smooth manifolds, let $\mathfrak{U}$ be an open cover of X, let $\mathscr{H}^q$ be the cohomology presheaf given by the map π. There is a spectral sequence E with $E_2^{p,q} = H^p(\mathfrak{U}; \mathscr{H}^q)$ which converges to $H^*(X)$.

Question: Can we loosen the restrictions on the manifolds and maps in Leray's construction? Is there such a spectral sequence if we just have a CW complex and cellular maps?

Remark 9.81: Note that the cohomology with coefficients in a presheaf is very similar to Čech cohomology. Čech cohomology is obtained by computing the singular cohomology of the nerve of a cover $\mathfrak{U}$. For presheaf cohomology, the setup is exactly the same, except we apply the presheaf to the open sets and inclusion maps to get a different chain complex. Is there a way to make this description better? Is there some kind of thing definable on a singular set which behaves like the presheaf does in this specific case?

10. Homology with local coefficients

Homology with local coefficients arises in the LSAH SS if the base space of a fibration isn't simply connected.

Definition 10.82: Let X be a nice enough space so it has a universal cover $\widetilde{X}$ (locally path connected, semi-locally simply connected), let $G = \pi_1(X)$, and let A be a $\mathbb{Z}G = \mathbb{Z}[G]$-module. Then we define the singular chain complex for X with coefficients in A by $S_*(X; A) = S_*(\widetilde{X}) \otimes_{\mathbb{Z}G} A$. The homology of this complex is called the homology of X with local coefficients in A and is denoted by $H_*(X; A)$.

Definition 10.83: In a similar fashion, we also get cohomology with local coefficients in A by defining $S^*(X; A) = \hom_{\mathbb{Z}G}(S_*(\widetilde{X}), A)$. Cohomology with local coefficients in A is then the cohomology of this cochain complex.

Remark 10.84: If the action of $G = \pi_1(X)$ on A is trivial, i.e. $\rho : G \to \mathrm{Aut}(A)$ is the trivial homomorphism, then homology with local coefficients in A is just regular homology with coefficients in the abelian group A.

If $A = \mathbb{Z}G$, then $H_k(X; \mathbb{Z}G) \cong H_k(\widetilde{X}; \mathbb{Z})$.

Remark 10.85: If X is a CW structure, one can use the induced cell structure on $\widetilde{X}$ and the cellular chain complexes to define (co)homology with local coefficients.

Example 10.86: Let $F \to E \to B$ be a fibration. There is then an action of $\pi_1(B)$ on the homology or cohomology of the fiber. Let $[\alpha] \in \pi_1(B)$, then $[\alpha]$ lifts to a map $F \to F$ by using the homotopy lifting property in the following way.

$$\begin{tikzcd}
F \times \{0\} \arrow[rr, "\iota"] \arrow[d, "\iota_0"] & & E \arrow[d] \\
F \times [0,1] \arrow[r, "p"] \arrow[urr, dashed, "H"] & {[0,1]} \arrow[r, "\alpha"] & B
\end{tikzcd}$$

The map $H_1 : F \to F$ is then obtained by $H_1(x) = H(x, 1)$. It is a result that this map is homotopic to any other lift $\widetilde{H}$ of the map $F \times [0,1] \to B$. We then get an action $\pi_1(B) \to [F, F]$. With this action, we see that applying a generalized homology theory $\mathbb{E}_*$ or generalized cohomology theory $\mathbb{E}^*$ that $\pi_1(B)$ acts on $\mathbb{E}_*(F)$ resp. $\mathbb{E}^*(F)$. That is, there is a group homomorphism $\pi_1(B) \to \mathrm{Aut}(\mathbb{E}_*(F))$ resp. $\pi_1(B) \to \mathrm{Aut}(\mathbb{E}^*(F))$. Thus $\mathbb{E}_*(F)$ resp. $\mathbb{E}^*(F)$ is a $\mathbb{Z}\pi_1(B)$-module and so we can talk about (co)homology with local coefficients in the (co)homology of the fiber F. This construction is used in the LSAH SS.

11. Spectral sequences

LSAH SS

Grothendieck SS: need to figure out ex. with Leray SS and how it relates to the Leray SS for smooth manifolds and smooth maps. Need to compute right derived functors for the direct image functor of sheaves in this case. Somehow this turns out to be $\mathscr{H}^q$. Need to know what the global section functor is, etc. Should certainly know the statement of the base change spectral sequence for Tor.

Exact couples: Know what they are, how SS arises. ...

SS of a filtered cochain complex : Need to review. See weibel and Bott & Tu

Computing with SS: Path fibration, some homotopy groups of spheres, generalized homology groups, etc.

Hyperhomology: the hyperderived functors are pretty easy $\mathbb{L}_i$ and $\mathbb{R}^i$. Just need to know cartan-eilenberg resolutions of a chain complex, apply functor, take $\mathrm{Tot}^{\oplus}$ and compute homology. Functor needs to be right resp left exact, category needs enough projectives resp. injectives . Hyperderived functors are important as they show up in Grothendieck SS which is "an organizing principle of hom. alg."

12. Homological Algebra

12.7. Abelian categories and their descriptions

Theorem 12.87: Freyd-Mitchell embedding theorem

Example 12.88: Mapping cone, mapping cylinder

12.8. Derived Functors

Example 12.89: In $R - \underline{\mathrm{Mod}}$:

Given positive chain complexes $\mathcal{C}$ and $\mathcal{D}$ with $\mathcal{C}$ projective and $\mathcal{D}$ acyclic (i.e. $H_i(\mathcal{C}) = 0$ for $i \geq 0$) for any $f : H_0(\mathcal{C}) \to H_0(\mathcal{D})$ there exists $\phi : \mathcal{C} \to \mathcal{D}$ inducing f. All such ϕ are homotopic.

F is ...		apply F to a ...	then ...
Covariant Functor	Left derived L_n	projective resolution	take $H_n(-)$
	Right derived R_n	injective resolution	take $H^n(-)$
Contravariant Functor	Left derived L^n	injective resolution	take $H_n(-)$
	Right derived R^n	projective resolution	take $H^n(-)$

$$\mathrm{Tor}_n^{\Lambda}(A, B) := L_n(- \otimes B)(A) = L_n(A \otimes -)(B);$$
$$\mathrm{Ext}_{\Lambda}^n(A, B) := R^n(\mathrm{Hom}(-, B))(A) = R_n(\mathrm{Hom}(A, -))(B).$$

We can always view a contravariant functor $F : \mathfrak{C} \to \mathfrak{D}$ as a covariant functor $F^{\mathrm{op}} : \mathfrak{C}^{\mathrm{op}} \to \mathfrak{D}$. Hence when defining the left and right derived fuctors of a contravariant functor, we simply take the already defined left and right derived functors of F^{op}. The same goes with the definitions of left and right exact functors.

$- \otimes B$	covariant	right exact
$A \otimes -$	covariant	right exact
$\mathrm{Hom}(A, -)$	covariant	left exact
$\mathrm{Hom}(-, B)$	contravariant	left exact

For Λ-modules A and B, we have

$$E(A, B) \cong \mathrm{Ext}_\Lambda(A, B)$$

where an extension E of A by B is $B \to E \to A$, and the equivalence is natural.

13. Category Theory

13.9. Adjoint functors

Definition 13.90: Let $L : \mathfrak{C} \to \mathfrak{D}$ and $R : \mathfrak{D} \to \mathfrak{C}$ be functors. We say that L is left adjoint to R (or equivalently R is right adjoint to L) if there exists a natural equivalence $\eta : \mathfrak{D}(L-, -) \xrightarrow{\sim} \mathfrak{C}(-, R-)$ between the functors $\mathfrak{D}(F-, -), \mathfrak{C}(-, G-) : \mathfrak{C}^{op} \times \mathfrak{D} \to \underline{\mathrm{Set}}$. In this case, we write $\eta : F \dashv G$.

Definition 13.91: The unit of the adjunction is $\eta(\mathrm{id}_{LX}) : X \to RLX$, obtained from $\eta : \mathfrak{D}(LX, LX) \to \mathfrak{C}X, RLX$.

The counit of the adjunction is $\eta^{-1}(\mathrm{id}_{RY}) : LRY \to Y$ obtained from $\eta : \mathfrak{D}(LRY, Y) \to \mathfrak{C}(RY, RY)$.

Proposition 13.92: If $\varepsilon : \mathrm{id} \to GF$ and $\delta : FG \to \mathrm{id}$ are natural transformations and if the equation $\delta F \circ F\varepsilon = \mathrm{id}$ and $G\delta \circ \varepsilon G = \mathrm{id}$ hold, then $\eta : F \dashv G$, defined by $\eta(\phi) = G\phi \circ \varepsilon_X$, is a natural equivalence which shows F is left adjoint to G. Furthermore, ε and δ are the unit and counit of the adjunction η respectively.

Conversely, if $\eta : F \dashv G$ is a natural equivalence, then $\varepsilon_X := \eta(\mathrm{id}_{FX})$ and $\delta_Y := \eta^{-1}(\mathrm{id}_{GY})$ define natural transformations which satisfy the above equations.

Proposition 13.93: If $\eta : F \dashv G$ and $\eta' : F \dashv G'$, then there exists a natural equivalence between G and G'. We remark that for all $Y \in \mathfrak{D}$, we have $GY \cong G'Y$. Alternatively, G determines F up to natural equivalence.

Proposition 13.94: Let $F : \mathfrak{C} \to \mathfrak{D}$, $F' : \mathfrak{D} \to \mathfrak{C}$ be functors, and suppose there exist G and G' such that $\eta : F \dashv G$ and $\eta' : F' \dashv G'$. Then $\eta_{-,G'-} \circ \eta'_{F-,-} : F'F \dashv GG'$.

Proposition 13.95: Consider $F : \mathfrak{C} \to \mathfrak{D} \in \underline{\mathrm{Cat}}$. If $L \dashv F$ and a given UCC $(\mathfrak{i}, P^{\mathfrak{i}}_{-})$ exists in both $\mathfrak{C}$ and $\mathfrak{D}$, then the UCC commutes in $\mathfrak{D}$. That is, if $P^{\mathfrak{i}}_{\mathfrak{C}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$ and $P^{\mathfrak{i}}_{\mathfrak{D}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}$, then $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$. A similar result holds for all manner of permutations of left and right adjoints.

Proof. For the general result, the following diagram is helpful.

$$
\begin{tikzcd}[row sep=huge, column sep=huge]
\mathfrak{C} \arrow[r, "F"] \arrow[d, "P^{\mathfrak{i}}_{\mathfrak{C}}"] & \mathfrak{D} \arrow[l, bend right=40, "R"'] \arrow[l, bend left=40, "L"] \arrow[d, "P^{\mathfrak{i}}_{\mathfrak{D}}"'] \\
\mathfrak{i}\mathfrak{C} \arrow[r, "\mathfrak{i}F"] \arrow[u, bend left=40, "\mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}"] \arrow[u, bend right=40, "\mathcal{L}^{\mathfrak{i}}_{\mathfrak{C}}"'] & \mathfrak{i}\mathfrak{D} \arrow[l, bend right=40, "\mathfrak{i}L"'] \arrow[l, bend left=40, "\mathfrak{i}R"] \arrow[u, bend left=40, "\mathcal{L}^{\mathfrak{i}}_{\mathfrak{D}}"] \arrow[u, bend right=40, "\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}"']
\end{tikzcd}
$$

By proposition (composition of adjoints is adjoint), we have $\mathfrak{i}L \circ P^{\mathfrak{i}}_{\mathfrak{D}} \dashv \mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F$ and $P^{\mathfrak{i}}_{\mathfrak{C}} \circ L \dashv F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$. We see that $P^{\mathfrak{i}}_{\mathfrak{C}} \circ L = \mathfrak{i}L \circ P^{\mathfrak{i}}_{\mathfrak{D}}$ by the naturality of $P^{\mathfrak{i}}_{-} : \mathrm{id}_{\underline{\mathrm{Cat}}} \to \mathfrak{i}$, and therefore, by proposition (adjoints unique), there is a natural equivalence $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$. The proof of the remaining parts follow analogously.

$L \dashv F$	UUCs commute in $\mathfrak{C}$, i.e. $\mathcal{L}^{\mathfrak{i}}_{\mathfrak{C}} \circ \mathfrak{i}L \cong L \circ \mathcal{L}^{\mathfrak{i}}_{\mathfrak{D}}$	$L \dashv F$	UCCs commute in $\mathfrak{D}$, i.e. $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}}$
$F \dashv R$	UUCs commute in $\mathfrak{D}$, i.e. $\mathcal{L}^{\mathfrak{i}}_{\mathfrak{D}} \circ \mathfrak{i}F \cong F \circ \mathcal{L}^{\mathfrak{i}}_{\mathfrak{C}}$	$F \dashv R$	UCCs commute in $\mathfrak{C}$, i.e. $\mathcal{R}^{\mathfrak{i}}_{\mathfrak{C}} \circ \mathfrak{i}R \cong R \circ \mathcal{R}^{\mathfrak{i}}_{\mathfrak{D}}$

Let $\mathfrak{X} : I \to \mathfrak{D}$ and $\mathfrak{Y} : I \to \mathfrak{C}$ be diagrams. We then have the following table.

$L \dashv F$	colimits commute in $\mathfrak{C}$, $L(\mathrm{colim}(\mathfrak{X})) \cong \mathrm{colim}(L\mathfrak{X})$	$L \dashv F$	limits commute in $\mathfrak{D}$, $F\lim(\mathfrak{Y}) \cong \lim(F\mathfrak{Y})$
$F \dashv R$	colimits commute in $\mathfrak{D}$, $F(\mathrm{colim}(\mathfrak{Y})) \cong \mathrm{colim}(F\mathfrak{Y})$	$F \dashv R$	limits commute in $\mathfrak{C}$, $R(\lim(\mathfrak{X})) \cong \lim(R\mathfrak{X})$

QED

Definition 13.96: Let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor and consider $Y \in \mathfrak{D}$. A solution set for Y is a set $\{X_i \in \mathfrak{C} \,|\, i \in I\}$ and $\{f_i : Y \to FX_i\} \,|\, i \in I\}$ where I is a set (yes, a set!) if: for any $X \in \mathfrak{C}$ and any $\phi : Y \to FX$ there exists an i and $\overline{\phi} : X_i \to X$ such that the following diagram commutes

$$
\begin{tikzcd}
FX_i \arrow[d, "F\overline{\phi}"'] & Y \arrow[l, "f_i"'] \arrow[dl, "\phi"] \\
FX &
\end{tikzcd}
.
$$

Remark 13.97: It is easy to see that if $L \dashv F$ then $\{LY\}$ with $\{\varepsilon_Y : Y \to FLY\}$ is a solution set for Y by using the naturality of the adjunction. Thus for a left adjoint to F to exist, the functor F must satisfy the solution set condition.

Definition 13.98: Let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor and consider $Y \in \mathfrak{D}$. A cosolution set[4] for Y is a set $\{X_i \in \mathfrak{C} \,|\, i \in I\}$ and $\{f_i : FX_i \to Y\}$ where I is a set if: for any

[4] This terminology is not necessarily standard.

$X \in \mathfrak{C}$ and any $\phi : FX \to Y$ there exists an i and $\overline{\phi} : X \to X_i$ such that the following diagram commutes

$$\begin{tikzcd} FX_i \arrow[r, "f_i"] & Y \\ FX \arrow[u, "F\overline{\phi}"] \arrow[ur, "\phi"'] & \end{tikzcd} .$$

Remark 13.99: It is clear tht if $F \dashv R$ then $\{FRY\}$ with $\{\delta_Y : FRY \to Y\}$ is a cosolution set for Y. Thus for a right adjoint to F to exist, the functor F must satisfy the cosolution set condition.

The following theorem due to Freyd provides a partial converse to the above remarks. The conditions are slightly idealized, however, and it will not be of much use to us. We state the theorem without proof; see [ML-1971, § V.6] or [M-1967, § V.3] for a proof.

Theorem 13.100: (Freyd's Adjoint Functor Theorem) Suppose $\mathfrak{C}$ is a complete category (that is, all limits for I a small category exist) which has $\mathfrak{C}(X, Y)$ a set for all objects X, Y. Then a functor $F : \mathfrak{C} \to \mathfrak{D}$ has a left adjoint if and only if F preserves all (small) limits and there is a solution set for all $Y \in \mathfrak{D}$.

Theorem 13.101: (Special adjoint functor theorem)

13.10. Monoidal categories

References

[B&T-1982] Bott, R.; Tu, L.: *Differential Forms in Algebraic Topology*, New York (1982).

[A-2004] Adámek, A.; Herrlich, H.; Strecker, G.: *Abstract and Concrete Categories; The Joy of Cats*, on-line edition 2004.

[H-1970] Hilton, P.; Stammbach, U.: *A Course in Homological Algebra*, New York (1970).

[K-1958] Kan, Daniel: Adjoint Functors, *Transactions of the American Mathematics Society.* **87**, 294–329 (1958).

[ML-1971] Mac Lane, S.: *Categories for the Working Mathematician* New York (1971).

[M-1967] Mitchell, B.: *Theory of Categories*, New York (1967).

[S-1992] Sieradski, A.: *An Introduction to Topology and Homotopy*, Boston (1992).