The Path Fibration

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In this paper, we will investigate the notion of cohomology with respect to a presheaf and how it relates to the cohomology of topological objects such as fiber bundles and fibrations. The goal being to compute $H^q(\Omega(\mathbb{S}^n);\mathbb{Z})$, that is, the cohomology groups of the loop space of an n-sphere.

In order to make the paper readable, we will assume working knowledge with the basics of: manifolds; fiber bundles; de Rham cohomology (H^*_{dR}), Čech-de Rham cohomology (H^*_D), Čech cohomology ($H^*(B,\mathcal{F})$) and singular cohomology with coefficients in A (denoted by $H^*(B;A)$). Some results will be cited without proof, but the statements will be given in the appendix. Our discussion is broken up into three sections: in section 1, we introduce presheaves and correct a few mistakes in [Bo] concerning these definitions; in section 2, we investigate the path fibration of a topological space; and in section 3, we compute $H^*(\Omega(\mathbb{S}^n);\mathbb{Z})$ (see Notation 1 in section 4). Throughout this paper, all maps are continuous unless otherwise noted. As another disclaimer, when we work with fiber bundles or manifolds, we will be assuming they are smooth, and the involved maps are smooth. Also in this situation, contractible will mean diffeomorphic to some $\mathbb{R}^k$, whereas in the general case, we will take contractible to mean that there exists a deformation retraction of the space to a point.[1]

1 Presheaves and Cohomology

Definition 1. Let $(B,\mathcal{T})$ be a topological space. We define a category Open(B) by setting its objects to be $\mathcal{O}\,\mathrm{Open}(B) := \mathcal{T}$ and the morphisms to be

$$\mathcal{M}\,\mathrm{Open}(B) := \left\{ i_U^V : V \subseteq U \mid V, U \in \mathcal{T} \right\}.$$

Definition 2. A presheaf on a space B is a contravariant functor $\mathcal{F} : \mathrm{Open}(B) \to \mathfrak{C}$ where $\mathfrak{C}$ is typically $\underline{\mathrm{Ab}}$, $\underline{\mathrm{Gp}}$ or $\underline{\mathrm{Mod}_{\mathbb{R}}}$.

Warning: The following definitions differ from those in [Bo, pg. 109] which we will show to be flawed.

Example 1. The trivial presheaf with group G on B is given by: $\mathcal{F}(U) = G$ for all U, and $\mathcal{F}(i_U^V) = \mathrm{id}_G$ for all morphisms i_U^V. This trivially is a functor $\mathcal{F} : \mathrm{Open}(B) \to \underline{\mathrm{Ab}}$, hence is a presheaf.

Definition 3. A constant presheaf with group G on a space B is a presheaf $\mathcal{F}$ with the property that for any contractible open set $U \in \mathrm{Open}(B)$, $\mathcal{F}(U) = G$ and for any contractible open sets $V \subseteq U$, $\mathcal{F}(i_U^V) = \mathrm{id}_G$.

[1] This definition of a contractible space is not universally accepted. Some authors instead say a space is contractible if the space is homotopic to a point.

Remark 1. The trivial presheaf is a constant presheaf, but it is in general not the only one. There is a constant presheaf which sends coproducts to products, i.e. $\mathcal{F}(U \coprod V) = G \prod G$ for U, V contractible.

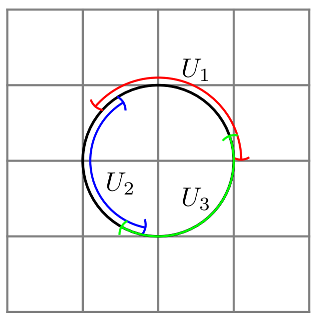
Definition 4. A locally constant presheaf with group G on B is a presheaf $\mathcal{F}$ which is locally naturally equivalent to a constant presheaf. That is, for every point $x \in B$, there is a neighborhood $N(x)$ for which $\mathcal{F}|_{N(x)}$ is a constant presheaf.

Example 2. For a fiber bundle $\pi : E \to B$ with fiber F, the presheaf $\mathcal{H}^q : \mathrm{Open}(B) \to \underline{\mathrm{Mod}}_{\mathbb{R}}$ given by $\mathcal{H}^q(U) := H^q_{dR}(\pi^{-1}(U)$ for all $U \in \mathrm{Open}(B)$ with the morphisms $\mathcal{H}^q(i) := i^*$ for all i.

This is a locally constant presheaf because for any contractible open set U, we have $\mathcal{H}^q(U) = H^q_{dR}(\pi^{-1}(U)) = H^q_{dR}(F)$ by the Poincaré lemma.

If it is not clear from context what we mean by $\mathcal{H}^q$, we will write $\mathcal{H}^q(\pi : E \to B)$. In [Bo], the ambiguous notation $\mathcal{H}^q(F)$ where F is the fiber of $\pi : E \to B$ is used. This does indeed have its merits, so we will use this notation as well if it is clear what is intended.

Example 3. Not every locally constant presheaf is constant. If $\pi : M \to \mathbb{S}^1$ is the Möbius band, one will see that the presheaf $\mathcal{H}^0$ is not constant. Consider the following good cover of $\mathbb{S}^1$:



and suppose the Möbius band M is chosen such that the twist occurs in $U_1 \cap U_2$. Then, one computes $\mathcal{H}^0(U_i) = \mathcal{H}^0(U_i \cap U_j) = \mathbb{R}$ for all i and j. However, exactly one of the maps $\mathcal{H}^0(i_1^{12})$ or $\mathcal{H}^0(i_2^{12})$ must be the isomorphism which sends $1 \longmapsto -1$. Therefore, the presheaf $\mathcal{H}^0$ cannot be constant.

Remark 2 (Discrepancy with [Bo]). The above definitions, as already mentioned, differ from those found in [Bo, pg. 109]. The definitions from [Bo] are contained in the following excerpt:

> The *trivial presheaf with group* G is the presheaf $\mathcal{F}$ which associates to every *connected* open set the group G and to every inclusion $V \subset U$ the identity map: $\mathcal{F}(U) \to \mathcal{F}(V)$. We say that a presheaf is a *constant presheaf* if it is isomorphic to the trivial presheaf, and that it is a *locally constant presheaf* if it is locally isomorphic to the trivial presheaf, i.e., every point has a neighborhood U so that $\mathcal{F}|_U$ is a constant presheaf.

First off, the structure imposed on $\mathcal{F}$ in the definition of a trivial presheaf is not enough to make it unique. See remark 1.

Secondly, for a fiber bundle $\pi : E \to B$ with fiber F, the presheaf $\mathcal{H}^q(\pi : E \to B)$ is *not* locally constant according to this definition. Consider an $\mathbb{S}^1$-bundle over B which has dimension 2. For any U contractible about a point x, i.e. (in this context) $U \cong \mathbb{D}^2$, we have $\mathcal{H}^1(U) \cong \mathbb{R}$. Contained in $\mathbb{D}^2$, there is an open annulus which we dennote by A. As A has the cohomology of $\mathbb{S}^1$, we have $\mathcal{H}^1(A) = H^1_{dR}(A \times \mathbb{S}^1) \cong \mathbb{R} \times \mathbb{R}$. Thus we cannot have $\mathcal{H}^1(i_U^A) : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ the identity map. Therefore, following the definitions from [Bo], the presheaves $\mathcal{H}^q(\pi : E \to B)$ are *not* locally constant.

This error propagates further in the section, however. It is claimed that the presheaf $\mathcal{H}^q$ with respect to the trivial bundle $M \times F$ provides an example of a locally constant presheaf which is not constant in general. There are two ways one could attempt to remedy the situation: one can either use the definition for locally constant by [Bo] and not consider $\mathcal{H}^q$ as locally constant, or change the definition to the one which has been given in definitions 3 and 4. Then this example makes sense only if M is contractible, and then $\mathcal{H}^q(M) = H^q(F)$ by the Poincaré lemma. Furthermore, it can be easily seen that indeed $\mathcal{H}^q$ must be a constant presheaf.

A nice property about fiber bundles that we would like to have more generally, is that there is a spectral sequence which converges to the cohomology of the fiber bundle, and we have in many cases a good understanding of the E_2 term, e.g. Leray's Theorem. These two facts help us compute the cohomology of many spaces, and thus compute Čech cohomology of spaces with respect to the locally constant presheaf $\mathcal{H}^q$. We hope to generalize these properties to arbitrary locally constant presheaves, but how far can one take it? What properties and results still hold?

One important property about a constant presheaf $\mathcal{F}$, and thus of $\mathcal{H}^q$ of a fiber bundle with simply connected base and fiber with finite dimensional cohomology, is that

$$H^p(B, \mathcal{F}) \cong \mathcal{F} \otimes_{\mathbb{Z}} H^p(B, \mathbb{Z}) \tag{1}$$

or when working with manifolds over $\mathbb{R}$, $H^p(B, \mathcal{F}) \cong \mathcal{F} \otimes_{\mathbb{R}} H^p(B, \mathbb{R}) \cong \mathcal{F} \otimes_{\mathbb{R}} H^p_{dR}(B)$.

Our first abstraction is to consider the construction of $C^p(\mathfrak{U}, \mathcal{H}^q)$[2] for a general surjective map $\pi : E \to B$. In general, $\mathcal{H}^q$ is not a loacally constant presheaf *for a general surjection*, and so, we do not get the isomorphism in equation 1 as evinced by the next example.

Example 4. Consider the projection $\pi : \mathbb{S}^1 \to [-1, 1]$ which vertically projects the unit circle onto the horizontal axis. If we consider the good cover

$$\mathfrak{U} = \{[-1, -1/4)\,, (-1/2, 1/2)\,, (1/4, 1]\}$$

of $I := [-1, 1]$. The presheaf $\mathcal{H}^0$ on I is not locally constant. This is easily seen by computing $H^0(\pi^{-1}(B(0;\epsilon))) \cong \mathbb{R}^2$ and $H^0(\pi^{-1}(B(1;\epsilon))) \cong \mathbb{R}$ for ϵ sufficiently small. Since these computations hold for all sufficiently small ϵ, no neighborhoods $N(0)$ or $N(1)$ can be chosen such that $\mathcal{H}^0|_{N(0)}(U_0) \cong \mathcal{H}^0|_{N(1)}(U_1)$ for $U_0 \subseteq N(0)$ and $U_1 \subseteq N(1)$ contractible.

By the Poincaré lemma, $\mathcal{H}^q(U) = 0$ for all $q \neq 0$, and for dimensional reasons $E_2 = E_\infty = H^*_{dR}(\mathbb{S}^1)$. Pictorially, we have

$$E_2 = \begin{array}{|c|c|c} & & \\ \hline \mathbb{R} & \mathbb{R} & \\ \hline \end{array}$$

[2] see the appendix for more on the Čech complex

Thus we cannot rely on our knowledge of $H^p_{dR}(I)$ to tell us if $H^p(\mathfrak{U}, \mathcal{H}^0)$ vanishes or not as we can for constant presheaves.

2 The Path Fibration

One way to get some of these properties is to consider the concept of a fibration or a fibering. This construction will give us some of our sought out properties. The main property we get is the following:

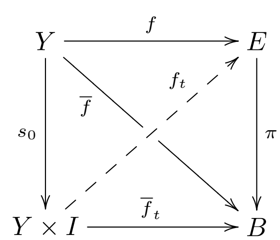
$$\text{for } \pi : E \to B, \exists \text{ a space } F \ni \forall \text{ contractible } U \subseteq B, \mathcal{H}^q(U) \cong H^q(F), \tag{2}$$

or, in other words, $\mathcal{H}^q$ are locally constant presheaves with respective groups $H^q(F)$ for some fixed space. With this extra structure, we can consider the spectral sequence with $E_2^{p,q} = H^p(B, \mathcal{H}^q)$ which then converges to $H^*_D(E)$. The more general nature of $\pi : E \to B$ lets us consider more spaces, but the property 2 permits some techniques to simplify matters. In this section, we are considering general spaces, continuous maps, and integer singular cohomology. We denote the functor which associates a space with its singular integer cochain abelian group as S^*.

Definition 5. Let B be a space with basepoint $*$. The path space of B is the set $P(B) = \{\gamma : [0,1] \to B \,|\, \gamma(0) = *\}$ given the compact-open topology. It is equipped with the canonical projection $\pi : P(B) \to B$ given by $\pi(\gamma) = \gamma(1)$. We denote $\Omega(B) := \pi^{-1}(*)$

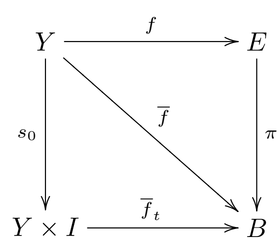
Definition 6. A fibration or fibering of B is a map $\pi : E \to B$ which satisfies the following homotopy covering property:

> Given any map $f : Y \to E$ and any homotopy $\overline{f}_t : Y \times I \to B$ of $\overline{f} := \pi \circ f$ such that $s_0 \circ \overline{f} : t = \overline{f}$ (where $s_0(y) = (y, 0)$), there exists a homotopy f_t of f which covers $\overline{f}_t$. That is, there is a commutative diagram (3)



It is clear that covering spaces are fibrations over their base space. We can also see that the path space $P(B)$ is a fibration over the base space B.

Example 5. We show that $\pi : P(B) \to B$ is a fibration. We thus begin with a diagram



and seek to lift $\overline{f}_t$ to a homotopy $f_t : Y \times I \to E$. By definition, $f(y)$ is a path in B which has terminal point $\overline{f}_0(y) = \pi(f(y))$. For $t_0 \in I$ define $f_{t_0}(y) = f(y) * \overline{f}_t(y)|_{[0,t_0]}$ where the $*$ denotes the path product of first traversing $f(y)$, and then traversing $\overline{f}_t(y)|_{[0,t_0]}$. This is evidently a proper lifting as $f_0(y) = f(y)$ and $\pi(f_{t_0}(y)) = \pi(f(y) * \overline{f}_t(y)|_{[0,t_0]}) = f_{t_0}(y)$. Hence $\pi : P(B) \to B$ is a fibration.

Proposition 1. a.) For any contractible subset $U \subseteq B$, we have $\pi^{-1}(U)$ homotopic to $\pi^{-1}(p)$ for $p \in U$.

b.) If $p, q \in B$ are in the same path component, then $\pi^{-1}(p)$ is homotopy equivalent to $\pi^{-1}(q)$. If B is path connected, then $\pi^{-1}(p) \simeq \pi^{-1}(*) = \Omega B$.

Proof. a.) Since U is contractible, there is a homotopy $F_t : U \times I \to U$ such that $F_1 = \mathrm{id}_U$ and $F_0(x) = p$ for all $x \in U$ and $p \in U$. We thus have a path $F_t(x) : I \to U$ with initial point x and terminal point p.

We claim that the maps $\iota : \pi^{-1}(p) \to \pi^{-1}(U)$ and $\phi : \pi^{-1}(U) \to \pi^{-1}(p)$ where

$$\phi(f : I \to B) := f \cdot F_t(f(1)) = \begin{cases} f(2x) & 0 \le x \le 1/2 \\ F_{2x-1}(f(1)) & 1/2 \le x \le 1 \end{cases}$$

are homotopically inverse to one another. The non-trivial homotopy is showing $\iota\phi \simeq \mathrm{id}_{\pi^{-1}(U)}$. The desired homotopy is given by

$$G_t(f) = \begin{cases} f((1-t/2)^{-1} \cdot x) & 0 \le x \le 1 - t/2 \\ F_{\frac{2x}{t} - \frac{2}{t} + 1} & 1 - t/2 \le x \le 1 \end{cases}$$

This homotopy continuously stretches the path f along the path $F_t(f(1))$ until it reaches $\phi(f)$. It is a routine matter of working with the compact-open topology that the map G_t is continuous, and therefore, a homotopy.

b.) By assumption, there is a path γ with initial point p and terminal point q; also γ^{-1} has initial point q and terminal point p. We define $\phi_p : \pi^{-1}(p) \to \pi^{-1}(q)$ and $\phi_q : \pi^{-1}(q) \to \pi^{-1}(p)$ by: $\phi_p(f) := f \cdot \gamma$ (the product path), and $\phi_q(f) := f \cdot \gamma^{-1}$. The homotopy equivalences $\phi_q \circ \phi_p \simeq \mathrm{id}_{\pi^{-1}(p)}$ and $\phi_p \circ \phi_q \simeq \mathrm{id}_{\pi^{-1}(q)}$ are given by a similar construction as was seen in part a.) of the proof of this proposition.

□

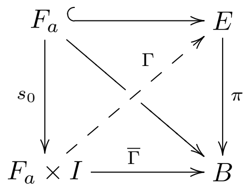
Proposition 2. a.) Any two fibers of a fibering over an arcwise-connected space have the same homotopy type.

i. A path $\gamma : [0,1] \to B$ with $\gamma(0) = a$ and $\gamma(1) = b$ induces a (not unique) map $\Gamma_1 : F_a \to F_b$. We also denote this induced map by $\mathcal{I}(\gamma)$.

ii. If $\gamma \simeq \mu$ are homotopic paths, then they induce homotopic maps $\Gamma_1 \simeq \mathrm{M}_1$.

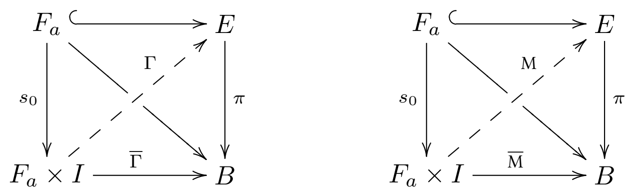
b.) For every contractible open set U, the inverse image $\pi^{-1}U$ has the homotopy type of the fiber F_a, where $a \in U$.

Proof. a.) i. From the path γ, we define $\overline{\Gamma} : F_a \times I \to B$ by $\overline{\Gamma}_t(y) = \gamma(t)$ or $\overline{\Gamma} = \gamma \circ \pi_2$. We thus have the diagram:

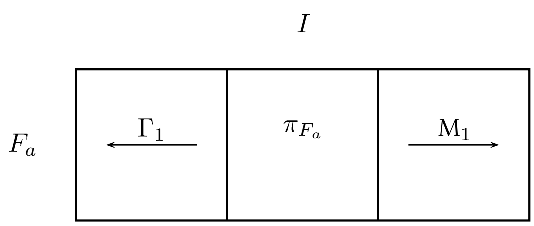


As the above diagram is commutative, $\operatorname{im} \Gamma_1 \subseteq F_b$, and we identify the map $\Gamma_1 : F_a \times \{1\} \to F_b \hookrightarrow E$ with $\Gamma_1 : F_a \to F_b$.

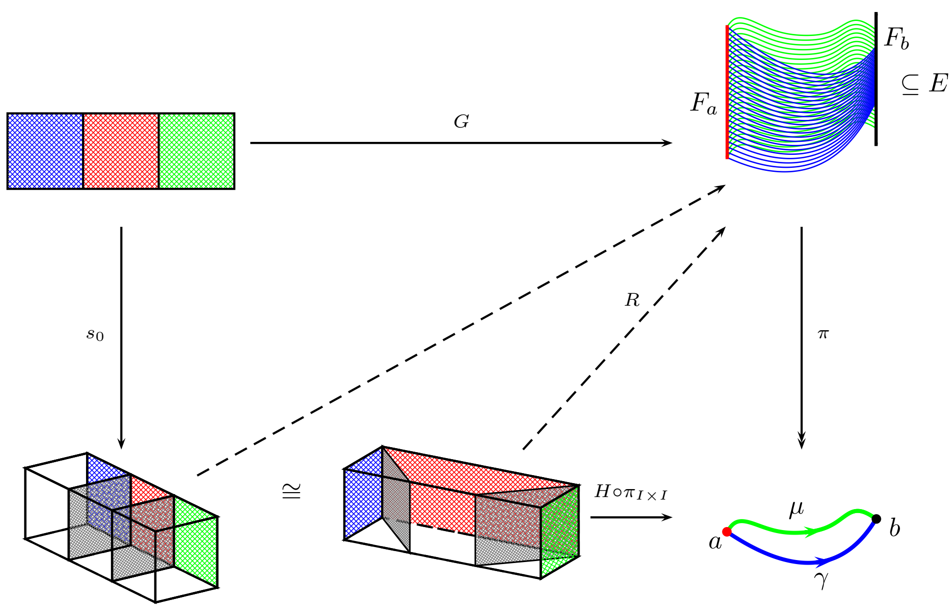
ii. Let $H : \gamma \xrightarrow{\simeq} \mu$. As we have



we see that $\mathrm{M}_0 = \Gamma_0$. Thus we may construct $G : F_a \times I \to E$ by



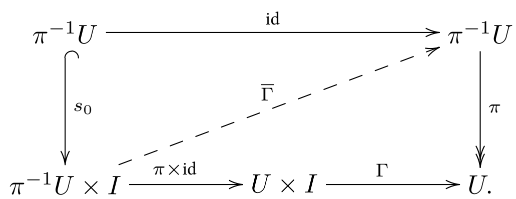
which is evidently continuous by the continuity of Γ, M and the fact that the definitions agree on the subspaces where they intersect, i.e. $t = 1/3, 2/3$. As $\gamma \simeq \mu$, we have a homotopy $H : I \times I \to B$ such that $H_0 = \gamma$, $H_1 = \mu$. The maps G and H then induce the following diagram by the covering homotopy property of E.



we thus see that R covers the homotopy H, so that $\mathrm{M}_1 = R_1(y,1) : F_a \times \{1\} \times \{1\} \to F_b, \Gamma_1 = R_0(y,1)$ and thus $R_t(y,1)$ is a homotopy of the maps $\mathrm{M}_1 \simeq \Gamma_1$.

We now prove the result. By the assumption that B is arcwise connected, given any two points $a, b \in B$, there is a path γ connecting them. We take $\gamma(0) = a$ and $\gamma(1) = b$. The paths γ and γ^{-1} induce maps of the fibers $\mathcal{I}(\gamma) : F_a \to F_b$ and $\mathcal{I}(\gamma^{-1}) : F_b \to F_a$. By the previous result, we have $\mathcal{I}(\gamma^{-1}) \circ \mathcal{I}(\gamma) \simeq \mathcal{I}(\gamma^{-1} \cdot \gamma) \simeq \mathcal{I}(a) = \mathrm{id}_{F_a}$ and similarly, $\mathcal{I}(\gamma) \circ \mathcal{I}(\gamma^{-1}) \simeq \mathrm{id}_{F_b}$ which establishes the claim.

b.) By assumption that $U \subseteq B$ is contractible, there is a deformation retraction of U to a point p which we denote by $\Gamma : U \times I \to U$. With this deformation retraction, we create the following diagram with the help of the homotopy covering property:

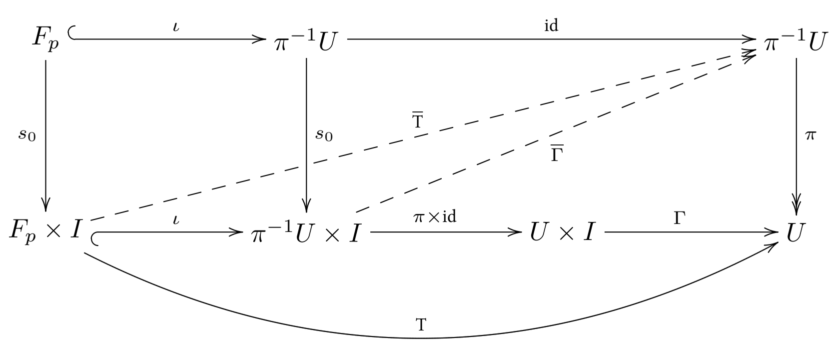


Since $\mathrm{im}\,\overline{\Gamma}_1 \subseteq \pi^{-1}(p) = F_p$, we can factor $\overline{\Gamma}_1$ as

$$\overline{\Gamma}_1 = \iota \circ \phi : \pi^{-1}U \xrightarrow{\phi} F_p \xhookrightarrow{\iota} \pi^{-1}U\ .$$

Thus $\overline{\Gamma}$ gives us a homotopy between $\mathrm{id}_{\pi^{-1}U}$ and $\iota \circ \phi$.

We now show that $\phi \circ \iota \simeq \mathrm{id}_{F_p}$. By the covering homotopy property, we have the following diagram:



Observe that we may take $\overline{\mathrm{T}} = \overline{\Gamma} \circ \iota$ as the covering homotopy of the map T. As Γ is a deformation retraction, we have $\mathrm{im}\,\overline{\mathrm{T}} \subseteq F_p$, and thus we may factor this map as

$$\overline{\mathrm{T}}_t = \iota \circ \Phi_t : F_p \times I \xrightarrow{\Phi} F_p \xhookrightarrow{\iota} \pi^{-1}U\ .$$

With this, we verify that $\Phi_0 = \mathrm{id}_{F_p}$ and $\Phi_1 = \phi \circ \iota$. Thus Φ establishes that $\mathrm{id}_{F_p} \simeq \phi \circ \iota$, with which we conclude that $\pi^{-1}U \simeq F_p$ $\square$

Remark 3. I am curious to know if the same proposition holds true if the other definition of contractibility is used.

Proposition 3. If $\pi : E \to B$ is a fibering where B is simply connected and E is path connected, then the fiber is path connected.

Proof. As the $E_2^{0,0}$ term trivially survives to E_∞, we have $E_2^{0,0} = E_\infty^{0,0} = H^0(E) = \mathbb{Z}$. By equation 1, we have $E_2^{0,0} = H^0(B, H^0(F)) = H^0(F)$ from which the proposition follows. $\square$

3 The Cohomology of $\Omega(\mathbb{S}^n)$

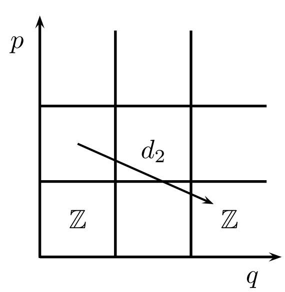
We begin our computation by illustrating the technique with the special case with $n = 2$. We thus have the following situation

$$\begin{array}{ccc} \Omega\mathbb{S}^2 & \longrightarrow & P\mathbb{S}^2 \\ & & \downarrow{\scriptstyle \pi} \\ & & \mathbb{S}^2 \end{array}$$

that is, $\mathbb{S}^2$ is the base space of the path fibration $P\mathbb{S}^2$, and the associated fiber is $\Omega\mathbb{S}^2$. As $\mathbb{S}^2$ is simply connected, we conclude by the results 4 and 5 that the locally constant presheaves $\mathcal{H}^q(\pi : P\mathbb{S}^2 \to \mathbb{S}^2)$ on $\mathbb{S}^2$ are indeed constant. We write $\mathcal{H}^q(\Omega\mathbb{S}^2) = H^q(\Omega\mathbb{S}^2)$ as it is the constant presheaf with group $H^q(\Omega\mathbb{S}^2)$. We therefore have $E_2^{p,q} = H^p(\mathbb{S}^2, \mathcal{H}^q(\Omega\mathbb{S}^2))$. Utilizing equation 1, we then have the zeroth column given by $E_2^{0,q} = H^0(\mathbb{S}^2, H^q(\Omega\mathbb{S}^2)) = H^q(\Omega\mathbb{S}^2)$. By proposition 3, we can also compute that the bottom row is given by $E_2^{p,0} = H^p(\mathbb{S}^2, H^0(\Omega\mathbb{S}^2)) = H^p(\mathbb{S}^2, \mathbb{Z})$. As

$$H^p(\mathbb{S}^2, \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

we conclude that all colums other than $E_2^{0,q}$ and $E_2^{2,q}$ are zero by equation 1. From the description of the domain and codomain of the differentials d_i, we see that $d_i = 0$ for all $i \geq 3$. We thus have the following diagram of E_2:



We prove by induction that $H^q(\Omega\mathbb{S}^2) = \mathbb{Z}$ for all q. It has already been explained that the base case with $q = 0$ is satisfied. For our induction step, suppose $H^q(\Omega\mathbb{S}^2) = E_2^{0,q} = \mathbb{Z}$. By equation 1, we conclude that $E_2^{2,q} = \mathbb{Z}$. Consider the map $d_2^{0,q+1} : E_2^{0,q+1} \to E_2^{2,q}$. Since

$P\mathbb{S}^2$ is contractible, $E_3 = E_\infty = H^*_D(P\mathbb{S}^2)$ only has one nonzero term $E_3^{0,0} = \mathbb{Z}$, from which we conclude the differential $d_2^{0,q+1} : E_2^{0,q+1} \to E_2^{2,q}$ must yield trivial cohomology. That is, $\ker d_2^{0,q+1} = 0$, and $\operatorname{im} d_2^{0,q+1} = \mathbb{R}$, or d_2 is an isomorphism. Therefore, $E_2^{0,q+1} = H^{q+1}(\Omega\mathbb{S}^2)$ and the proof is complete.

An entirely analogous proof works for the general case of $\Omega\mathbb{S}^n$ for $n \geq 2$ to give

$$H^q(\Omega\mathbb{S}^n) = \begin{cases} \mathbb{Z} & q = k(n-1) \text{ for } k \in \mathbb{N}_0 \\ 0 & \text{otherwise} \end{cases}$$

4 Appendix

Notation 1. The notation $H^*(B; G)$ represents singular cohomology of the topological space B with coefficients in G where G is most often a group or vector space. We will later see $H^*(B, \mathcal{F})$ which arises from a different concept!

Definition 7. We now delve into computing cohomology of a space with respect to a presheaf $\mathcal{F}$. To do so, one first defines the Čech complex of an open cover $\mathfrak{G} = \{G_\alpha\}$ of a space B with respect to $\mathcal{F}$. This is given by $C^p(\mathfrak{G}, \mathcal{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(G_{\alpha_1 \cdots \alpha_p})$ with differential δ. The differential δ is given by first defining $\partial_i : \coprod U_{\alpha_0 \cdots \alpha_p} \to U_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_p}$ by $\partial_i|_{U_{\alpha_0 \cdots \alpha_p}} : U_{\alpha_0 \cdots \alpha_p} \subseteq U_{\alpha_0 \cdots \hat{\alpha_i} \cdots \alpha_p}$. Then define $\delta := \sum_i (-1)^i \partial_i$. One then defines $H^p(B, \mathcal{F}) := \varinjlim H^p(\mathfrak{G}, \mathcal{F})$.

Lemma 1. If M is a manifold, then $H^*_{dR}(M \times \mathbb{R}) \cong H^*_{dR}(M)$. We obtain as a corollary $H^*(M \times \mathbb{R}^k) = H^*(M)$.

Proposition 4. Let $\mathfrak{U}$ be an open cover of a connected topological space B and $N(\mathfrak{U})$ is the nerve of the cover. If $\pi_1(N(\mathfrak{U})) = 0$, then every locally constant presheaf on $\mathfrak{U}$ is constant.

Proposition 5. If the space B has a good cover $\mathfrak{U}$, then $\pi_1(B) \cong \pi_1(N(\mathfrak{U}))$.

References

[Bo] Bott, R.; Tu, L.: *Differential Forms in Algebraic Topology*, Springer-Verlag, New York 1982.