Σ Theory

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**Finitely Presented Groups**

**Definition:** A group is finitely presented if its group structure can be described with finitely many generators and finitely many relations.

A group $G$ is finitely generated if there is some finite number of elements $g_1, g_2, \ldots, g_n \in G$ for which all other elements of $G$ can be expressed as a finite product of these elements.

A group $G$ is finitely presented if it is finitely generated, say by $g_1, \ldots, g_n$, and $G \cong F(g_1, \ldots, g_n)/\mathcal{R}$ where $\mathcal{R}$ is finitely generated as a normal subgroup of the free group on $g_1, \ldots, g_n$.

**Example:** $\mathbb{Z} \times \mathbb{Z} \cong \langle a, b \mid ab = ba \rangle$
Questions:

- How can one determine if a given group is finitely presented?
- Is there something computable that tells us if a group is finitely presented or not?

Σ theory provides an answer to these questions for metabelian groups.
**Metabelian Groups**

**Definition:** A metabelian group is a group $G$ for which there exists $A \triangleleft G$ such that $A$ and $G/A$ are Abelian groups. That is, there is a short exact sequence

$$0 \to A \to G \to Q \to 1$$

where $A$ and $Q$ are Abelian.

**Example:**

- The class of metabelian groups contains more than just Abelian groups

  $$1 \to \mathbb{Z}_3 \to S_3 \to \mathbb{Z}_2 \to 1$$

- The alternating groups $A_n$ for $n \geq 5$ are non-Abelian simple groups; hence they are not metabelian groups.
**Q-Modules**

**Definition:** A $Q$-module is an Abelian group $A$ with an action of a group $Q$ on $A$, i.e.

- $\cdot : Q \times A \to A$
- $1 \cdot a = a$ for all $a \in A$
- $(qq') \cdot a = q \cdot (q' \cdot a)$

which satisfies:

- $q \cdot (a + a') = q \cdot a + q \cdot a'$ for all $a, a' \in A$.

This is equivalent to there being an “actual” $\mathbb{Z}[Q]$-module structure on $A$ where $\mathbb{Z}[Q]$ is the group ring.
\textbf{\(Q\)-module structure on \(A\)}

\[
\begin{array}{c}
0 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1
\end{array}
\]

- \(G\) acts on \(A\) by conjugation because \(A\) is a normal subgroup of \(G\).
- Since \(A\) is Abelian, it acts trivially on itself; hence there is a well defined \(G/A \cong Q\) action on \(A\).
  
We can use this extra structure of \(Q\) on \(A\) to determine when \(G\) is finitely presented.
- Given a \(Q\)-module \(A\), we define the semi-direct product \(A \rtimes Q\) where 
  \[(a, q) \rtimes (b, r) = (a + q \cdot b, qr).\]

\[
\begin{array}{c}
0 \rightarrow A \overset{\iota}{\rightarrow} A \rtimes Q \overset{\pi}{\rightarrow} Q \rightarrow 1
\end{array}
\]
OBSERVATIONS

We consider a general extension of $Q$ by $A$

$$0 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1.$$ 

- $Q$ not finitely generated $\Rightarrow$ $G$ not finitely presented
- $A$ not finitely generated $Q$-module $\Rightarrow$ $G$ not finitely presented
- $A$ and $Q$ finitely generated Abelian groups $\Rightarrow$ $G$ finitely presented

Thus the interesting case is where $Q$ is a finitely generated Abelian group and $A$ is infinitely generated as an Abelian group, but finitely generated as a $Q$-module.
Example

- \( A = \mathbb{Z}[1/2] \), i.e. the dyadic rationals, and take \( Q = \langle q \rangle \) the infinite cyclic group.
- \( Q \)-module structure on \( A \) given by
  \[
  q \cdot x = \frac{1}{2} x.
  \]
- \( \{1\} \) generates \( \mathbb{Z}[1/2] \) as a \( Q \)-module.
- \( \mathbb{Z}[1/2] \) is not a finitely generated Abelian group.
- We don’t need all powers of \( q \) to finitely generate \( A \) with this action; \( \{q^n \mid n \geq 0\} \) suffices.

**Question:** Is \( \mathbb{Z}[1/2] \rtimes \mathbb{Z} \) finitely presented?
**First Main Result of Σ Theory**

If $G$ is metabelian with

$$0 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$$

then $G$ is finitely presented if and only if $A \rtimes Q$ is.

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Extensions of $Q$ by $A$
**Special Case**

In the special case when $Q = \langle q \rangle$, define

$$Q_+ = \{q^n | n \geq 0\}, \ Q_- = \{q^n | n \leq 0\}.$$

Then the $\Sigma$-invariant may be defined as

$$\Sigma_A = \{Q_\varepsilon | A \text{ is finitely generated over } Q_\varepsilon\}$$

In this case, the second main result tells us that

- $G$ is finitely presented if and only if $\Sigma_A \neq \emptyset$,
- i.e. $A$ is finitely generated over $Q_+$ or $Q_-$. 
Example

Let $A = \mathbb{Z}[1/2]$, i.e. the dyadic rationals; let $Q = \langle q \rangle$.

- $Q$-module structures on $A$ are given by $q \cdot x = \pm 2^n x$ where $n \in \mathbb{Z}$. As long as $n \neq 0$, $A$ will be finitely generated over $Q$.

- Any $Q$-module structure on $A$ which makes $A$ finitely generated over $Q$ has $\Sigma_A \neq \emptyset$.

- Thus by the main result, every extension $G$ of $\mathbb{Z}$ by $\mathbb{Z}[1/2]$ which induces one of these module structures is finitely presented.

- In the case $q \cdot x = x$, the split extension extension is $\mathbb{Z}[1/2] \rtimes \mathbb{Z} \cong \mathbb{Z}[1/2] \times \mathbb{Z}$ is not even finitely generated.
Let \( A = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} a_i \) the free Abelian group on the generators \( \{ a_i \mid i \in \mathbb{Z} \} \), and let \( Q = \langle q \rangle \). A \( Q \)-module structure on \( A \) is given by \( q \cdot a_i = a_{i+1} \) and extended by linearity.

- \( A \) is not finitely generated as an Abelian group.

\[
\begin{array}{cccccc}
\text{a}_{-2} & \text{a}_{-1} & \text{a}_0 & \text{a}_1 & \text{a}_2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\text{q}^{-1} & \text{t} & \text{t} & \text{t} & \text{t} \\
\end{array}
\]

- \( A \) is finitely generated by \( \{ a_0 \} \) as a \( Q \)-module.
- \( A \) is not finitely generated over \( Q_+ = \{ q^n \mid n \geq 0 \} \) or \( Q_- = \{ q^n \mid n \leq 0 \} \).
- Therefore no extension which induces this \( Q \)-module structure on \( A \) is finitely presented. In particular \( \mathbb{Z} = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z} a_i \times \mathbb{Z} \) is not finitely presented.
**Interesting Computation**

- Let $A = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}a_i$ the free Abelian group on the generators $\{a_i \mid i \in \mathbb{Z}\}$, and let $Q = \langle q \rangle$.

- Every extension of $Q$ by $A$ is not finitely generated.

- Every $Q$-module structure on $A$ which finitely generates $A$ over $Q$ needs “both sides” of $Q$ to do it. That is, $A$ is never finitely generated over $Q_+$ or $Q_-$. 