

# BASS' $NK$ GROUPS AND $cdh$ -FIBRANT HOCHSCHILD HOMOLOGY.

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ABSTRACT. The  $K$ -theory of a polynomial ring  $R[t]$  contains the  $K$ -theory of  $R$  as a summand. For  $R$  commutative and containing  $\mathbb{Q}$ , we describe  $K_*(R[t])/K_*(R)$  in terms of Hochschild homology and the cohomology of Kähler differentials for the  $cdh$  topology.

We use this to address Bass' question, whether  $K_n(R) = K_n(R[t])$  implies  $K_n(R) = K_n(R[t_1, t_2])$ . The answer to this question is affirmative when  $R$  is essentially of finite type over the complex numbers, but negative in general.

In 1972, H. Bass posed the following question (see [2], question (VI)<sub>n</sub>):

Does  $K_n(R) = K_n(R[t])$  imply that  $K_n(R) = K_n(R[t_1, t_2])$ ?

One can rephrase the question in terms of Bass' groups  $NK_n$ , introduced in [1]:

Does  $NK_n(R) = 0$  imply that  $N^2K_n(R) = 0$ ?

More generally, for any functor  $F$  from rings to an abelian category, Bass defines  $NF(R)$  as the kernel of the map  $F(R[t]) \rightarrow F(R)$  induced by evaluation at  $t = 0$ , and  $N^2F = N(NF)$ . Bass' question was inspired by Traverso's theorem [24], from which it follows that  $N\text{Pic}(R) = 0$  implies  $N^2\text{Pic}(R) = 0$ .

In this paper, we give a new interpretation of the groups  $NK_n(R)$  in terms of Hochschild homology and the cohomology of Kähler differentials for the  $cdh$  topology, for commutative  $\mathbb{Q}$ -algebras. This allows us to give a counterexample to Bass' question in the companion paper [6] (see Theorem 0.2 below).

To state our main structural theorem, recall from [28] that each  $NK_n(R)$  has the structure of a module over the ring of big Witt vectors  $W(R)$ . It is convenient to use the countably infinite-dimensional  $\mathbb{Q}$ -vector spaces  $t\mathbb{Q}[t]$  and  $\Omega_{\mathbb{Q}[t]}^1$ . If  $M$  is any  $R$ -module, then  $M \otimes t\mathbb{Q}[t]$  and  $M \otimes \Omega_{\mathbb{Q}[t]}^1$  are naturally  $W(R)$ -modules by [10].

**Theorem 0.1.** *Let  $R$  be a commutative ring containing  $\mathbb{Q}$ . Then there is a  $W(R)$ -module isomorphism*

$$N^2K_n(R) \cong (NK_n(R) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(R) \otimes \Omega_{\mathbb{Q}[t]}^1).$$

*Thus  $K_n(R) = K_n(R[t_1, t_2])$  iff  $NK_n(R) = NK_{n-1}(R) = 0$  iff  $N^2K_n(R) = 0$ .*

*In addition, the following are equivalent for all  $p > 0$ :*

- a)  $K_n(R) = K_n(R[t_1, \dots, t_p])$
- b)  $NK_n(R) = 0$  and  $K_{n-1}(R) = K_{n-1}(R[t_1, \dots, t_{p-1}])$ .
- c)  $NK_q(R) = 0$  for all  $q$  such that  $n - p < q \leq n$ .

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The equivalence of (a), (b) and (c) is immediate by induction, using the formula for  $N^2K_n$ , and is included for its historical importance; see [25]. Theorem 0.1 also holds for the  $K$ -theory of schemes of finite type over a field; see 4.2 below.

Theorem 0.1 allows us to reformulate Bass' question as follows:

Does  $NK_n(R) = 0$  imply that  $NK_{n-1}(R) = 0$ ?

**Theorem 0.2.** *a) For any field  $F$  algebraic over  $\mathbb{Q}$ , the 2-dimensional normal algebra*

$$R = F[x, y, z]/(z^2 + y^3 + x^{10} + x^7y)$$

*has  $K_0(R) = K_0(R[t])$  but  $K_0(R) \neq K_0(R[t_1, t_2])$ .*

*b) Suppose  $R$  is essentially of finite type over a field of infinite transcendence degree over  $\mathbb{Q}$ . Then  $NK_n(R) = 0$  implies that  $R$  is  $K_n$ -regular and, in particular, that  $K_n(R) = K_n(R[t_1, t_2])$ .*

Part (a) is proven in the companion paper [6], using Theorem 0.1, while part (b) is proven below as Corollary 6.7.

The proof of Theorem 0.1 relies on methods developed in [5] and [7], which allow us to compute the groups  $NK_n$  and  $N^pK_n$  in terms of the Hochschild homology of  $R$ , and of the  $cdh$ -cohomology of the higher Kähler differentials  $\Omega^p$ , both relative to  $\mathbb{Q}$ . The groups  $NK_n(R)$  have a natural bigraded structure when  $\mathbb{Q} \subset R$ , and it is convenient to take advantage of this bigrading in stating our results. The bigrading comes from the eigenspaces  $NK_n^{(i)}(R)$  of the Adams operations  $\psi^k$  (arising from the  $\lambda$ -filtration) and the eigenspaces of the homothety operations  $[r]$  (i.e., base change for  $t \mapsto rt$ ). This bigrading will be explained in Sections 1 and 5; the general decomposition for Adams weight  $i$  has the form:

$$(0.3) \quad NK_n^{(i)}(R) \cong TK_n^{(i)}(R) \otimes_{\mathbb{Q}} t\mathbb{Q}[t].$$

Here  $TK_n^{(i)}$  denotes the *typical piece* of  $NK_n^{(i)}(R)$ , defined as the simultaneous eigenspace  $\{x \in NK_n^{(i)}(R) : [r]x = rx, r \in R\}$ . (See Example 1.6.) We provide a concrete description of the typical pieces in Theorem 5.1, reproduced here:

**Theorem 0.4.** *If  $R$  is a commutative  $\mathbb{Q}$ -algebra, then  $NK_n^{(i)}(R)$  is determined by its typical pieces  $TK_n^{(i)}(R)$  and (0.3). For  $i \neq n, n+1$  we have:*

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } i < n, \\ H_{cdh}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n+2. \end{cases}$$

*For  $i = n, n+1$ , we have an exact sequence:*

$$0 \rightarrow TK_{n+1}^{(n+1)}(R) \rightarrow \Omega_R^n \rightarrow H_{cdh}^0(R, \Omega^n) \rightarrow TK_n^{(n+1)}(R) \rightarrow 0.$$

The special case  $NK_0 = \oplus NK_0^{(i)}$  of Theorem 0.4 is that for  $R$  essentially of finite type over a field of characteristic zero, with  $d = \dim(R)$ ,

$$(0.5) \quad NK_0(R) \cong \left( (R^+/R_{\text{red}}) \oplus \bigoplus_{p=1}^{d-1} H_{cdh}^p(R, \Omega^p) \right) \otimes_{\mathbb{Q}} t\mathbb{Q}[t].$$

Here  $R^+$  is the seminormalization of  $R_{\text{red}}$ ; we show in Proposition 2.5 that  $R^+ = H_{cdh}^0(R, \mathcal{O})$ . The dimension zero case of Theorem 0.4 is also revealing:

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$
$TK_3^{(i)}(R)$	0	$HH_2^{(1)}(R)$	$\text{tors } \Omega_R^2$	$\Omega_{\text{cdh}}^3(R)/\Omega_R^3$	$H_{\text{cdh}}^1 \Omega^4$	0
$TK_2^{(i)}(R)$	0	$\text{tors } \Omega_R^1$	$\Omega_{\text{cdh}}^2(R)/\Omega_R^2$	$H_{\text{cdh}}^1 \Omega^3$	0	
$TK_1^{(i)}(R)$	$\text{nil}(R)$	$\Omega_{\text{cdh}}^1(R)/\Omega_R^1$	$H_{\text{cdh}}^1 \Omega^2$	0		
$TK_0^{(i)}(R)$	$R^+/R$	$H_{\text{cdh}}^1 \Omega^1$	0			
$TK_{-1}^{(i)}(R)$	$H_{\text{cdh}}^1 \mathcal{O}$	0				
$TK_{-2}^{(i)}(R)$	0					

Table 1. The groups  $TK_n^{(i)}(R)$  for  $n \leq 3$ ,  $\dim(R) = 2$ .

*Example 0.6.* If  $\dim(R) = 0$  then we get  $NK_n(R) \cong HH_{n-1}(R, I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$  for all  $n$ , where  $I$  is the nilradical of  $R$ . It is illuminating to compare this with Goodwillie's Theorem [12], which implies that  $NK_n(R) \cong NK_n(R, I) \cong NHC_{n-1}(R, I)$ . The identification comes from the standard observation (1.2) that the map  $HH_* \rightarrow HC_*$  induces  $NHC_*(R, I) \cong HH_*(R, I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ .

The calculations of Theorem 0.4 for small  $n$  are summarized in Table 1 when  $\dim(R) = 2$ . We will need the following cases of 0.4 in [6], to prove Theorem 0.2(a).

**Theorem 0.7.** *Let  $R$  be normal domain of dimension 2 which is essentially of finite type over  $\mathbb{Q}$ . Then*

- a)  $NK_0(R) = NK_0^{(2)}(R) \cong H_{\text{cdh}}^1(R, \Omega^1) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ , and
- b)  $NK_{-1}(R) = NK_{-1}^{(1)}(R) \cong H_{\text{cdh}}^1(R, \mathcal{O}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ .

Here is an overview of this paper: Section 1 reviews the bigrading on the Hochschild and cyclic homology of  $R[t]$  (and  $X \times \mathbb{A}^1$ ), and Section 2 reviews the  $cdh$ -fibrant analogue. Section 3 describes the sheaf cohomology of the fibers  $\mathcal{F}_{HH}(X)$ ,  $\mathcal{F}_{HC}(X)$ , etc. of  $HH(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, HH)$ , etc. In Section 4 we use these fibers to prove Theorem 0.1, by relating  $NK_{n+1}(X)$  to  $H^{-n}\mathcal{F}_{HH}(X)$ . We also show that Bass' question is negative for schemes in Lemma 4.5.

In Section 5, we give the detailed computations of the typical pieces  $TK_n^{(i)}(R)$  needed to establish (0.5) and Table 1; these computations employ the main result of [8]. In Section 6, we prove Theorem 0.2(b), that the answer to Bass' question is positive provided we are working over a sufficiently large base field. Finally, Section 7 describes how Theorem 0.7 changes if  $R$  is of finite type over an arbitrary field of characteristic 0: the map  $NK_0(R) \rightarrow H_{\text{cdh}}^1(R, \Omega_{/F}^1) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$  is onto, and an isomorphism if  $NK_{-1}(R) = 0$ .

**Notation.** All rings considered in this paper should be assumed to be commutative and noetherian, unless otherwise stated. Throughout this paper,  $k$  denotes a field of characteristic 0 and  $F$  is a field containing  $k$  as a subfield. We write  $\text{Sch}/k$  for the category of separated schemes essentially of finite type over  $k$ . If  $\mathcal{F}$  is a presheaf on  $\text{Sch}/k$ , we write  $\mathcal{F}_{\text{cdh}}$  for the associated  $cdh$  sheaf, and often simply write  $H_{\text{cdh}}^*(X, \mathcal{F})$  in place of the more formal  $H_{\text{cdh}}^*(X, \mathcal{F}_{\text{cdh}})$ .

If  $H$  is a functor on  $\text{Sch}/k$  and  $R$  is an algebra essentially of finite type, we occasionally write  $H(R)$  for  $H(\text{Spec } R)$ . For example,  $H_{\text{cdh}}^*(R, \Omega^i)$  is used for  $H_{\text{cdh}}^*(\text{Spec } R, \Omega^i)$ . Note that, because the  $cdh$  site is noetherian (every cover has

a finite subcovering)  $H_{\text{cdh}}^*$  sends inverse limits of schemes over diagrams with affine transition morphisms to direct limits.

If  $H$  is a contravariant functor from  $\text{Sch}/k$  to spectra, (co)chain complexes, or abelian groups that takes filtered inverse limits of schemes over diagrams with affine transition morphisms to colimits (as for example  $K$ ,  $HH$ ,  $\mathbb{H}_{\text{cdh}}(-, HH)$ , and  $\mathcal{F}_{HH}$ ), then for any  $k$ -algebra  $R$ , we abuse notation and write  $H(R)$  for the direct limit of the  $H(R_\alpha)$  taken over all subrings  $R_\alpha$  of  $R$  of finite type over  $k$ . (If  $R$  is essentially of finite type but not of finite type, the two definitions of  $H(R)$  agree up to canonical isomorphism.) In particular, we will use expressions like  $\mathbb{H}_{\text{cdh}}(R, HH)$  for general commutative  $\mathbb{Q}$ -algebras even though we do not define the *cdh*-topology for arbitrary  $\mathbb{Q}$ -schemes.

We use cohomological indexing for all chain complexes in this paper; for a complex  $C$ ,  $C[p]^q = C^{p+q}$ . For example, the Hochschild, cyclic, periodic, and negative cyclic homology of schemes over a field  $k$  can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [32] and [5, 2.7] for precise definitions. We shall write these presheaves as  $HH(/k)$ ,  $HC(/k)$ ,  $HP(/k)$  and  $HN(/k)$ , respectively, omitting  $k$  from the notation if it is clear from the context.

It is well known (see [31, 10.9.19]) that there is an Eilenberg-Mac Lane functor  $C \mapsto |C|$  from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies  $\pi_n(|C|) = H^{-n}(C)$ . For example, applying  $\pi_n$  to the Chern character  $K \rightarrow |HN|$  yields maps  $K_n(R) \rightarrow H^{-n}HN(R) = HN_n(R)$ . In this spirit, we will use descent terminology for presheaves of complexes.

## 1. THE BIGRADING ON $NHH$ AND $NHC$

Recall that  $k$  denotes a field of characteristic 0. In this section, we consider the Hochschild and cyclic homology of polynomial extensions of commutative  $k$ -algebras. No great originality is claimed. Throughout, we will use the chain level Hodge decompositions  $HH = \prod_{i \geq 0} HH^{(i)}$  and  $HC = \prod_{i \geq 0} HC^{(i)}$ .

The Künneth formula for Hochschild homology yields

$$(1.1) \quad NHH_n^{(i)}(R) \cong \left( HH_n^{(i)}(R) \otimes t\mathbb{Q}[t] \right) \oplus \left( HH_{n-1}^{(i-1)}(R) \otimes \Omega_{\mathbb{Q}[t]}^1 \right).$$

From the exact SBI sequence  $0 \rightarrow NHC_{n-1} \xrightarrow{B} NHH_n \xrightarrow{I} NHC_n \rightarrow 0$  (see [31, 9.9.1]), and induction on  $n$ , the map  $I$  induces canonical isomorphisms for each  $i$ :

$$(1.2) \quad NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes t\mathbb{Q}[t].$$

*Remark 1.3.* Both (1.1) and (1.2) generalize to non-affine quasi-compact schemes  $X$  over  $k$ . Indeed,  $NHH$  and  $NHC$  satisfy Zariski descent because  $HH$  and  $HC$  do and because, for any open cover  $\{U_i \rightarrow X\}$ , the collection  $\{U_i \times \mathbb{A}^1 \rightarrow X \times \mathbb{A}^1\}$  is also a cover. Thus we have

$$\begin{aligned} NHH^{(i)}(X) &\cong \mathbb{H}_{\text{Zar}}(X, NHH^{(i)}) \\ &\cong \mathbb{H}_{\text{Zar}}(X, HH^{(i)}) \otimes t\mathbb{Q}[t] \oplus \mathbb{H}_{\text{Zar}}(X, HH^{(i-1)})[1] \otimes \Omega_{\mathbb{Q}[t]}^1 \\ &\cong HH^{(i)}(X) \otimes t\mathbb{Q}[t] \oplus HH^{(i-1)}(X)[1] \otimes \Omega_{\mathbb{Q}[t]}^1, \end{aligned}$$

and  $NHC^{(i)}(X) = \mathbb{H}_{\text{Zar}}(X, NHC^{(i)}) \cong \mathbb{H}_{\text{Zar}}(X, HH^{(i)}) \otimes t\mathbb{Q}[t] \cong HH^{(i)}(X) \otimes t\mathbb{Q}[t]$ .

It is easy to iterate the construction  $F \mapsto NF$ . For example, we see from (1.1) and (1.2) that

$$(1.4) \quad N^2 HC_n^{(i)}(R) \cong \left( HH_n^{(i)}(R) \otimes t\mathbb{Q}[t] \otimes t\mathbb{Q}[t] \right) \oplus \left( HH_{n-1}^{(i-1)}(R) \otimes t\mathbb{Q}[t] \otimes \Omega_{\mathbb{Q}[t]}^1 \right).$$

By induction, we see that  $HH_{n-j}^{(i-j)}(R) \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}$  will occur  $\binom{p-1}{j}$  times as a summand of  $N^p HC_n^{(i)}(R)$  for all  $j \geq 0$ . We may write this as the formula:

$$(1.5) \quad N^p HC_n^{(i)}(R) \cong \bigoplus_{j=0}^{p-1} HH_{n-j}^{(i-j)}(R) \otimes_k \wedge^j k^{p-1} \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}.$$

**Cartier operations on  $NHH$  and  $NHC$ .** Let  $W(R)$  denote the ring of big Witt vectors over  $R$ ; it is well known that in characteristic 0 we have  $W(R) \cong \prod_1^\infty R$ . (See [28, p. 468] for example.) Cartier showed in [3] that the endomorphism ring  $\text{Carf}(R)$  of the additive functor underlying  $W$  consists of column-finite sums  $\sum V_m[r_{mn}]F_n$ , using the *homotheties*  $[r]$  (for  $r \in R$ ), and the Verschiebung and Frobenius operators  $V_m$  and  $F_m$ . Restricting the sum to  $m \geq m_0$  yields a descending sequence of ideals of  $\text{Carf}(R)$ , making it complete as a topological ring;  $W(R)$  is the complete topological subring of all sums  $\sum V_m[r_m]F_m$ ; see [3].

We will be interested in the intermediate (topological) subring  $\text{Carf}(R)$  of all row and column-finite sums  $\sum V_m[r_{mn}]F_n$ . As observed in [10, 2.14], there is an equivalence between the category of  $R$ -modules and the category of continuous  $\text{Carf}(R)$ -modules given by the constructions in the following example. (A left module  $M$  is *continuous* if the annihilator ideal of each element is an open left ideal.)

*Example 1.6.* If  $M$  is any  $R$ -module,  $N = M \otimes t\mathbb{Q}[t]$  is a continuous  $\text{Carf}(R)$ -module (and hence a  $W(R)$ -module) via the formulas:

$$[r]t^i = r^i t^i, \quad V_m(t^i) = t^{mi}, \quad F_m(t^i) = \begin{cases} m t^{i/m} & \text{if } m|i, \\ 0 & \text{else.} \end{cases}$$

The ring  $W(R) = \prod_1^\infty R$  acts on  $M \otimes t\mathbb{Q}[t]$  by  $(r_1, \dots, r_n, \dots) * \sum m_i t^i = \sum (r_i m_i) t^i$ . Conversely, every continuous  $\text{Carf}(R)$ -module  $N$  has a “typical piece”  $M$ , defined as the simultaneous eigenspace  $\{x \in N : [r]x = rx, r \in R\}$ , and  $N \cong M \otimes t\mathbb{Q}[t]$ .

Recall that we can define operators  $[r]$  on  $NHH_n(R)$  and  $NHC_n(R)$ , associated to the endomorphisms  $t \mapsto rt$  of  $R[t]$ . There are also operators  $V_m$  and  $F_m$ , defined via the ring inclusions  $R[t^m] \subset R[t]$  and their transfers. These operations commute with the Hodge decomposition. The following result follows immediately from [10, 4.11] using the observation that everything commutes with Adams operations.

**Proposition 1.7.** *The operators  $[r]$ ,  $V_m$  and  $F_m$  make each  $NHC_n^{(i)}(R)$  into a continuous  $\text{Carf}(R)$ -module, and hence a  $W(R)$ -module. The  $R$ -module  $HH_n^{(i)}(R)$  is its typical piece, and the canonical isomorphism  $NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes t\mathbb{Q}[t]$  of (1.2) is an isomorphism of  $\text{Carf}(R)$ -modules, the module structure on the right being given in Example 1.6.*

A similar structure theorem holds for  $NHH_n(R)$  and its Hodge components, using (1.1). However, it uses a non-standard  $R$ -module structure on the typical piece  $HH_n(R) \oplus HH_{n-1}(R)$ ; see [10, 3.3] for details.

*Remark 1.7.1.* The conclusions of Proposition 1.7 still hold for  $NHC_n^{(i)}(X)$  and  $HH_n^{(i)}(X)$  when  $X$  is any scheme, where  $W(R)$  and  $\text{Carf}(R)$  refer to the ring  $R = H^0(X, \mathcal{O})$ . That is,  $HH_n^{(i)}(X)$  is an  $R$ -module and  $NHC_n^{(i)}(X)$  is a continuous  $\text{Carf}(R)$ -module, isomorphic to  $HH_n^{(i)}(X) \otimes t\mathbb{Q}[t]$ .

This scheme version of 1.7 is not stated in [10], which was written before the cyclic homology of schemes was developed in [32]. However, the proof in [10] is easily adapted. Since the operators  $V_m$ ,  $F_m$  and  $[r]$  are defined on the underlying chain complexes in [10, 4.1], they extend to operations on the Hochschild and cyclic homology of schemes. The identities required to obtain continuous  $\text{Carf}(R)$ -module structures all come from the Künneth formula for the shuffle product on the chain complexes (see [10, 4.3]), so they also hold for the homology of schemes.

## 2. $cdh$ -FIBRANT $HH$ AND $NHC$

Now fix a field  $F$  containing  $k$ ; all schemes will lie in the category  $\text{Sch}/F$  (essentially of finite type over  $F$ ), in order to use the  $cdh$  topology on  $\text{Sch}/F$  of [22]. All rings will be commutative  $F$ -algebras; because they are filtered direct limits of finitely generated  $F$ -algebras, we can consider their  $cdh$ -cohomology.

If  $C$  is any (pre-)sheaf of cochain complexes on  $\text{Sch}/F$ , we can form the  $cdh$ -fibrant replacement  $X \mapsto \mathbb{H}_{cdh}(X, C)$  and write  $\mathbb{H}_{cdh}^n(X, C)$  for the  $n$ th cohomology of this complex. (The fibrant replacement is taken with respect to the local injective model structure, as in [5, 3.3].) For example, the  $cdh$ -fibrant replacement of a  $cdh$  sheaf  $C$  (concentrated in degree zero) is just an injective resolution, and  $\mathbb{H}_{cdh}^n(X, C)$  is the usual cohomology of the  $cdh$  sheaf associated to  $C$ .

Hochschild and cyclic homology, as well as differential forms, will be taken relative to  $k$ . For  $C = HH^{(i)}$ , it was shown in [7, Theorem 2.4] that

$$(2.1) \quad \mathbb{H}_{cdh}(X, HH^{(i)}) \cong \mathbb{H}_{cdh}(X, \Omega^i)[i].$$

This has the following consequence for  $C = NHH^{(i)}$  and  $NHC^{(i)}$ .

**Lemma 2.2.** *Let  $H^{(i)}$  denote either  $HH^{(i)}$  or  $HC^{(i)}$ , taken relative to a subfield  $k$  of  $F$ . Then  $\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) = \mathbb{H}_{cdh}(X, H^{(i)}) \oplus \mathbb{H}_{cdh}(X, NH^{(i)})$ , and:*

$$\begin{aligned} \mathbb{H}_{cdh}(X, NHH^{(i)}) &\cong (\mathbb{H}_{cdh}(X, \Omega^i)[i] \otimes t\mathbb{Q}[t]) \oplus \left( \mathbb{H}_{cdh}(X, \Omega^{i-1})[i] \otimes \Omega_{\mathbb{Q}[t]}^1 \right); \\ \mathbb{H}_{cdh}(X, NHC^{(i)}) &\cong \mathbb{H}_{cdh}(X, \Omega^i)[i] \otimes t\mathbb{Q}[t]. \end{aligned}$$

*Proof.* The displayed formulas follow from (1.1), (1.2) and (2.1), using the first assertion and the fact that  $-\otimes t\mathbb{Q}[t]$  commutes with  $\mathbb{H}_{cdh}$ . Thus it suffices to verify the first assertion. By resolution of singularities, we may assume that  $X$  is smooth.

Recall from [5, 3.2.2] that the restriction of the  $cdh$  topology to  $\text{Sm}/k$  is called the  $scdh$ -topology. The product of any  $scdh$  cover of  $X$  with  $\mathbb{A}^1$  is an  $scdh$  cover of  $X \times \mathbb{A}^1$ , and both  $HH^{(i)}$  and  $HC^{(i)}$  satisfy  $scdh$ -descent by [7, Thm. 2.4]. Now by Thomason's Cartan-Leray Theorem [23, 1.56] we have

$$\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) \cong \mathbb{H}_{cdh}(X, H^{(i)}(- \times \mathbb{A}^1)) \cong \mathbb{H}_{cdh}(X, H^{(i)}) \oplus \mathbb{H}_{cdh}(X, NH^{(i)}).$$

This gives the first assertion. Alternatively, we may prove the first assertion by induction on  $\dim(X)$ , using the definition of  $scdh$  descent to see that for smooth  $X$  we have  $H^{(i)}(X) = \mathbb{H}_{cdh}(X, H^{(i)})$  and

$$\mathbb{H}_{cdh}(X \times \mathbb{A}^1, H^{(i)}) = H^{(i)}(X \times \mathbb{A}^1) = H^{(i)}(X) \oplus NH^{(i)}(X).$$

In particular, the first assertion holds when  $\dim(X) = 0$ .  $\square$

*Remark 2.2.1.* If  $R$  is any commutative  $F$ -algebra, the formulas of Lemma 2.2 hold for  $X = \text{Spec}(R)$  by naturality. This is because we may write  $R = \varinjlim R_\alpha$ , where  $R_\alpha$  ranges over subrings of finite type over  $F$ , and  $\mathbb{H}_{cdh}(X, -) = \varinjlim \mathbb{H}_{cdh}(\text{Spec}(R_\alpha), -)$ .

**Corollary 2.3.** *If  $X = \text{Spec}(R)$  is in  $\text{Sch}/F$ , the modules  $\mathbb{H}_{cdh}^n(X, HH^{(i)})$  and  $\mathbb{H}_{cdh}^n(X, NHC^{(i)})$  are zero unless  $0 \leq n + i < \dim(X)$  and  $i \geq 0$ .*

*If  $n \geq \dim(X)$  and  $n > 0$  then  $\mathbb{H}_{cdh}^n(X, HH) = 0$ .*

*Proof.* Because  $\mathbb{H}_{cdh}^n(X, \Omega^i)[i] = H_{cdh}^{i+n}(X, \Omega^i)$ , this follows from (2.1), Lemma 2.2 and the fact that  $H_{cdh}^n(X, \Omega^i) = 0$  for  $n \geq \dim(X)$ ,  $n > 0$ . This bound is given in [5, 6.1] for  $i = 0$ , and in [7, 2.6] for general  $i$ .  $\square$

Here is a useful bound on the cohomology groups appearing in Lemma 2.2. Given  $X$ , let  $Q$  denote the total ring of fractions of  $X_{\text{red}}$ ; it is a finite product of fields  $Q_j$ , and we let  $e$  denote the maximum of the transcendence degrees  $\text{tr. deg}(Q_j/k)$ .

**Lemma 2.4.** *Let  $X$  be in  $\text{Sch}/F$ . If  $i > e$  then  $H_{cdh}^n(X, \Omega^i) = 0$  for all  $n$ .*

*Proof.* By [19, 12.24], we may assume  $X$  reduced. Since we may write  $X$  as an inverse limit of a sequence of affine morphisms with the same ring of total fractions  $Q$ , and  $cdh$ -cohomology sends such an inverse limit to a direct limit, we may also assume that  $X$  is of finite type over  $F$ . This implies that  $e = \dim(X) + \text{tr. deg}(F/k)$ .

The result is clear if  $\dim(X) = 0$ , since  $H_{cdh}^n(X, -) = H_{Zar}^n(X, -)$  in that case. Proceeding by induction on  $\dim(X)$ , choose a resolution of singularities  $X' \rightarrow X$  and observe that the singular locus  $Y$  and  $Y \times_X X'$  have smaller dimension. The hypothesis implies that  $\Omega^i = 0$  on  $X'_{Zar}$ , so  $H_{cdh}^n(X', \Omega^i) = 0$  by [7, 2.5]. The result now follows by induction from the Mayer-Vietoris sequence of [22, 12.1].  $\square$

If  $R$  is a commutative ring, we write  $R_{\text{red}}$  and  $R^+$  for the associated reduced ring and the seminormalization of  $R_{\text{red}}$ , respectively. These constructions are natural with respect to localization, so that we may form the seminormalization  $X^+$  of  $X_{\text{red}}$  for any scheme  $X$ . Because  $X^+ \rightarrow X$  is a universal homeomorphism, we have  $H_{cdh}^*(X, -) \cong H_{cdh}^*(X^+, -)$  for every  $X$  in  $\text{Sch}/k$ , for any field  $k$  of arbitrary characteristic. The case  $n = 0$  with coefficients  $\mathcal{O}_{cdh}$  is of special interest; recall our convention that  $H_{cdh}^0(X, \mathcal{O})$  denotes  $H_{cdh}^0(X, \mathcal{O}_{cdh})$ .

**Proposition 2.5.** *For any algebra  $R$ , we have  $H_{cdh}^0(\text{Spec } R, \mathcal{O}) = R^+$ . Moreover, for every  $X$  in  $\text{Sch}/F$  we have  $H_{cdh}^0(X, \mathcal{O}) = \mathcal{O}(X^+)$ .*

*Proof.* We may assume  $R$  and  $X$  are reduced. Writing  $R = \varinjlim R_\alpha$  as in Remark 2.2.1, we have  $R^+ = \varinjlim R_\alpha^+$  and  $H_{cdh}^0(R, \mathcal{O}) = \varinjlim H_{cdh}^0(R_\alpha, \mathcal{O})$ , so we may assume that  $R$  is of finite type. Thus the second assertion implies the first. Since  $H_{cdh}^0(-, \mathcal{O})$  and  $\mathcal{O}(-^+)$  are Zariski sheaves, it suffices to consider the case when  $X$  is affine.

Let  $X = \text{Spec } R$  be in  $\text{Sch}/F$ , with  $R$  reduced. There is an injection  $R \rightarrow Q$  with  $Q$  regular (for example,  $Q$  could be the total quotient ring of  $R$ ). By [5, 6.3],

$H_{\text{cdh}}^0(\text{Spec } Q, \mathcal{O}) = Q$ , so  $R$  injects into  $H_{\text{cdh}}^0(\text{Spec } R, \mathcal{O})$ . This implies that  $\mathcal{O}_{\text{red}}$  is a separated presheaf for the  $\text{cdh}$  topology on  $\text{Sch}/F$ . Thus, the ring  $H_{\text{cdh}}^0(X, \mathcal{O})$  is the direct limit over all  $\text{cdh}$ -covers  $p : U \rightarrow X$  of the Čech  $H^0$ . (See [SGA4, 3.2.3].)

Fix an element  $b \in H_{\text{cdh}}^0(\text{Spec } R, \mathcal{O})$  and represent it by  $b \in \mathcal{O}(U)$  for some  $\text{cdh}$  cover  $U \rightarrow X$ . Now recall from [19, 12.28] or [22, 5.9] that we may assume, by refining the  $\text{cdh}$  cover  $U \rightarrow X$ , that it factors as  $U \rightarrow X' \rightarrow X$  where  $X' \rightarrow X$  is proper birational  $\text{cdh}$  cover and  $U \rightarrow X'$  is a Nisnevich cover. If the images of  $b \in \mathcal{O}(U)$  agree in  $U \times_X U$ , i.e.,  $b$  is a Čech cycle for  $U/X$ , then its images agree in  $U \times_{X'} U$ , i.e., it is a Čech cycle for  $U/X'$ . But by faithfully flat descent,  $b$  descends to an element of  $\mathcal{O}(X')$ . Thus we can assume that  $U$  is proper and birational over  $X$ .

Next, we can assume that the Nisnevich cover  $p : U \rightarrow X$  is finite, surjective and birational. Indeed, since  $p$  is proper and birational we may consider the Stein factorization  $U \xrightarrow{q} Y \xrightarrow{r} X$ . By [EGAIII, 4.3] or [16, III.11.5 & proof],  $q_*(\mathcal{O}_U) = \mathcal{O}_Y$  and  $r$  is finite surjective and birational. By [22, 5.8],  $r$  is also a  $\text{cdh}$  cover. Because  $q_*(\mathcal{O}_U) = \mathcal{O}_Y$ , the canonical map  $\mathcal{O}_Y(Y) \rightarrow q_*(\mathcal{O}_U)(Y) = \mathcal{O}_U(U)$  is an isomorphism. Hence  $b$  descends to an element of  $\mathcal{O}(Y)$ . By Lemma 2.6,  $b$  lies in the seminormalization of  $R$ .  $\square$

**Lemma 2.6.** *Let  $A$  be a seminormal ring and  $B$  a ring between  $A$  and its normalization. Then the Čech complex  $A \rightarrow B \rightarrow B \otimes_A B$  is exact.*

*Proof.* We use Traverso's description of the seminormalization (see [24, p. 585]): the seminormalization of a ring  $A$  inside a ring  $B$  is

$$A^+ = \{b \in B \mid (\forall P \in \text{Spec } A) \ b \in A_P + \text{rad}(B_P)\}.$$

Let  $b \in B$  such that  $1 \otimes b = b \otimes 1$ . We have to show that  $b \in A_P + \text{rad}(B_P)$ , for all primes  $P$  of  $A$ . Let  $J = \text{rad}(B_P)$ ; since  $B_P/J$  is faithfully flat over the field  $A_P/P$ , the image of  $b$  in  $B_P/J$  lies in  $A_P/P$  by flat descent. That is,  $b \in A_P + J$ , as required.  $\square$

*Remark 2.7.* Even if  $X$  is affine seminormal, it can happen that  $H_{\text{cdh}}^i(X, \mathcal{O}) \neq 0$  for some  $i > 0$ . For example, if  $R$  denotes the subring  $F[x, g, yg]$  of  $F[x, y]$  for  $g = x^3 - y^2$  then it is easy to show that  $R$  is seminormal and that  $H_{\text{cdh}}^1(\text{Spec}(R), \mathcal{O}) = F$ , because the normalization of  $R$  is  $F[x, y]$  and the conductor ideal is  $gF[x, y]$ . For another example, the normal ring of Theorem 0.2 has  $H_{\text{cdh}}^1(X, \mathcal{O}) \neq 0$ , by Theorems 0.1 and 0.7(b).

### 3. THE FIBERS $\mathcal{F}_{HH}$ AND $\mathcal{F}_{HC}$

If  $C$  is a presheaf of complexes on  $\text{Sch}/F$ , we write  $\mathcal{F}_C$  for the shifted mapping cone of  $C \rightarrow \mathbb{H}_{\text{cdh}}(-, C)$ , so that we have a distinguished triangle:

$$(3.1) \quad \mathbb{H}_{\text{cdh}}(X, C)[-1] \rightarrow \mathcal{F}_C(X) \rightarrow C(X) \rightarrow \mathbb{H}_{\text{cdh}}(X, C)$$

*Example 3.1.1.* When  $C$  is concentrated in degree 0 we have  $H^n \mathcal{F}_C = 0$  for all  $n < 0$ . For  $C = \mathcal{O}$  and  $X = \text{Spec}(R)$ , we see from Proposition 2.5 that  $H^0 \mathcal{F}_{\mathcal{O}}(X) = \text{nil}(R)$ ,  $H^1 \mathcal{F}_{\mathcal{O}}(X) = R^+/R$ , and  $H^n \mathcal{F}_{\mathcal{O}}(X) = H_{\text{cdh}}^{n-1}(X, \mathcal{O})$  for  $n \geq 2$ . Note that, if  $X = \text{Spec } R \in \text{Sch}/F$ , then  $H^n \mathcal{F}_{\mathcal{O}}(X) = 0$  for  $n > \dim(X)$  by [5, 6.1].

We now consider the Hochschild and cyclic homology complexes, taken relative to a subfield  $k$  of  $F$ . For legibility, we write  $\mathcal{F}_{HH}^{(i)}$  for  $\mathcal{F}_{HH^{(i)}}$ , etc. By the usual homological yoga,  $\mathcal{F}_{HH}$  is the direct sum of the  $\mathcal{F}_{HH}^{(i)}$ ,  $i \geq 0$ , and similarly for  $\mathcal{F}_{HC}$ .



*Example 3.1.2.* If  $X$  is smooth over  $F$  then  $\mathcal{F}_{HH}(X) \simeq 0$  by [7, 2.4].

Lemma 2.2 and Remarks 2.2.1 and 1.3 imply the following analogue for  $N\mathcal{F}$ .

**Lemma 3.2.** *If  $X$  is in  $\text{Sch}/F$ , or if  $X = \text{Spec}(R)$  for an  $F$ -algebra  $R$ , we have quasi-isomorphisms:*

$$\begin{aligned} N\mathcal{F}_{HH}^{(i)}(X) &\cong \left( \mathcal{F}_{HH}^{(i)}(X) \otimes t\mathbb{Q}[t] \right) \oplus \left( \mathcal{F}_{HH}^{(i-1)}(X)[1] \otimes \Omega_{\mathbb{Q}[t]}^1 \right); \\ N\mathcal{F}_{HC}^{(i)}(X) &\cong \mathcal{F}_{HH}^{(i)}(X) \otimes t\mathbb{Q}[t]. \end{aligned}$$

Mimicking the argument that establishes (1.4) and (1.5) yields:

**Corollary 3.3.** *If  $X$  is in  $\text{Sch}/F$ , or if  $X = \text{Spec}(R)$  for an  $F$ -algebra  $R$ ,*

$$N^2\mathcal{F}_{HC}^{(i)}(X) \cong (\mathcal{F}_{HH}^{(i)}(X) \otimes t\mathbb{Q}[t] \otimes t\mathbb{Q}[t]) \oplus (\mathcal{F}_{HH}^{(i-1)}(X)[1] \otimes t\mathbb{Q}[t] \otimes \Omega_{\mathbb{Q}[t]}^1)$$

and

$$N^p\mathcal{F}_{HC}^{(i)}(X) \cong \bigoplus_{j=0}^{p-1} \mathcal{F}_{HH}^{(i-j)}(X)[j] \otimes_k \wedge^j k^{p-1} \otimes t\mathbb{Q}[t]^{\otimes p-j} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}.$$

The cohomology of the typical pieces  $\mathcal{F}_{HH}^{(i)}(R)$  is given as follows.

**Lemma 3.4.** *If  $R$  is an  $F$ -algebra and  $i \geq 0$ , then there is an exact sequence:*

$$0 \rightarrow H^{-i}\mathcal{F}_{HH}^{(i)}(R) \rightarrow \Omega_R^i \rightarrow H_{\text{cdh}}^0(R, \Omega^i) \rightarrow H^{1-i}\mathcal{F}_{HH}^{(i)}(R) \rightarrow 0.$$

For  $n \neq i, i-1$  we have:

$$H^{-n}\mathcal{F}_{HH}^{(i)}(R) \cong \begin{cases} HH_n^{(i)}(R) & \text{if } i < n, \\ H_{\text{cdh}}^{i-n-1}(R, \Omega^i) & \text{if } i \geq n+2. \end{cases}$$

*Proof.* As in Remark 2.2.1, we may assume  $R$  is of finite type. Since  $HH_i^{(i)}(R) = \Omega_R^i$  for all  $i \geq 0$ , and  $HH_n^{(i)}(R) = 0$  when  $i > n$  (see [31, 9.4.15] or [17, 4.5.10]), it suffices to use (2.1) and to observe that  $\mathbb{H}_{\text{cdh}}^{-n}(R, HH^{(i)}) = H_{\text{cdh}}^{i-n}(R, \Omega^i)$  vanishes when  $n > i$ .  $\square$

*Example 3.5.* Let  $X = \text{Spec}(R)$  be in  $\text{Sch}/F$ . Since  $HH^{(0)} = \mathcal{O}$ ,  $\mathcal{F}_{HH}^{(0)}(R)$  is described in Example 3.1.1. Applying Corollary 2.3 and Lemma 3.4 for  $i > 0$ , and using [7, 2.6] to bound the terms, we see that if  $d = \dim(R)$  then  $H^n\mathcal{F}_{HH}(X) = 0$  for  $n > d$ . If  $d = 1$ , then the only nonzero positive cohomology of  $\mathcal{F}_{HH}$  is  $H^1\mathcal{F}_{HH}(R) = R^+/R$ ; if  $d > 1$ , we have:

$$\begin{aligned} H^1\mathcal{F}_{HH}(R) &\cong (R^+/R) \oplus H_{\text{cdh}}^1(X, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(X, \Omega^{d-1}), \\ H^2\mathcal{F}_{HH}(R) &\cong H_{\text{cdh}}^1(X, \mathcal{O}) \oplus H_{\text{cdh}}^2(X, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(X, \Omega^{d-2}), \\ &\vdots \\ H^d\mathcal{F}_{HH}(R) &\cong H_{\text{cdh}}^{d-1}(X, \mathcal{O}). \end{aligned}$$

*Example 3.6.* When  $R$  is essentially of finite type over  $F$  and  $\text{tr. deg}(F/k) < \infty$ ,  $H^m\mathcal{F}_{HH}(R)$  is Hochschild homology for large negative  $m$ . To see this, observe that  $e = \text{tr. deg}(R/k)$ , the maximum transcendence degree of the residue fields of  $R$  at

its minimal primes, is finite. Using Lemmas 2.4 and 3.4, we get  $H^{-n}\mathcal{F}_{HH}^{(i)}(R) = 0$  and  $H^{-n}\mathcal{F}_{HH}^{(n)}(R) = \Omega_R^n$  for  $i > n > e$ , and hence

$$H^{-n}\mathcal{F}_{HH}(R) \cong HH_n(R) \text{ for all } n > e.$$

If  $R = k \oplus R_1 \oplus R_2 \oplus \dots$  is graded, and  $\widetilde{HC}_*(R) = HC_*(R)/HC_*(k)$ , it is well known that the map  $\widetilde{HC}_*(R) \xrightarrow{S} \widetilde{HC}_{*-2}(R)$  is zero. (See [31, 9.9.1] for example.) In Lemma 3.8 below, we prove a similar property for  $\mathcal{F}_{HH}$  and  $\mathcal{F}_{HC}$ , which we derive from Lemma 3.2 using the following trick.

**Standard Trick 3.7.** If  $R$  is a non-negatively graded algebra, there is an algebra map  $\nu : R \rightarrow R[t]$  sending  $r \in R_n$  to  $rt^n$ . The composition of  $\nu$  with evaluation at  $t = 0$  factors as  $R \rightarrow R_0 \rightarrow R$ , and so if  $H$  is a functor on algebras taking values in abelian groups, then the composition  $H(R) \xrightarrow{\nu} H(R[t]) \xrightarrow{t=0} H(R)$  is zero on the kernel  $\tilde{H}(R)$  of  $H(R) \rightarrow H(R_0)$ . Similarly, the composition of  $\nu$  with evaluation at  $t = 1$  is the identity. That is,  $\nu$  maps  $\tilde{H}(R)$  isomorphically onto a summand of  $NH(R)$ , and  $\tilde{H}(R)$  is in the image of  $(t = 1) : NH(R) \rightarrow H(R)$ .

**Lemma 3.8.** *If  $R = k \oplus R_1 \oplus \dots$  is a graded algebra then for each  $m$  the map  $\pi_m \mathcal{F}_{HC}(R) \xrightarrow{S} \pi_{m-2} \mathcal{F}_{HC}(R)$  is zero, and there is a split short exact sequence:*

$$0 \rightarrow \pi_{m-1} \mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{I} \pi_m \mathcal{F}_{HC}(R) \rightarrow 0.$$

*Similarly, there are split short exact sequences:*

$$0 \rightarrow \tilde{\mathbb{H}}_{\text{cdh}}^{m+1}(R, HC) \xrightarrow{B} \tilde{\mathbb{H}}_{\text{cdh}}^m(R, HH) \xrightarrow{I} \tilde{\mathbb{H}}_{\text{cdh}}^m(R, HC) \rightarrow 0.$$

*and*

$$0 \rightarrow \tilde{\mathbb{H}}_{\text{cdh}}^{m-1}(R, \Omega^{<i}) \xrightarrow{B} \tilde{H}_{\text{cdh}}^{m-i}(R, \Omega^i) \xrightarrow{I} \tilde{\mathbb{H}}_{\text{cdh}}^m(R, \Omega^{\leq i}) \rightarrow 0.$$

*Proof.* It suffices to show that  $I$  is onto and split. By [7, 2.4],  $\mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0$ , so  $\tilde{\mathcal{F}}_{HH} = \mathcal{F}_{HH}$  and  $\tilde{\mathcal{F}}_{HC} = \mathcal{F}_{HC}$ . By the standard trick 3.7, it suffices to show that the maps  $N\pi_m \mathcal{F}_{HH}(R) \rightarrow N\pi_m \mathcal{F}_{HC}(R)$  and  $N\tilde{\mathbb{H}}_{\text{cdh}}^m(R, HH) \rightarrow N\tilde{\mathbb{H}}_{\text{cdh}}^m(R, HC)$  are split surjections. But this is evident from the decompositions of  $N\mathcal{F}_{HC}^{(i)}(R)$  and  $\mathbb{H}_{\text{cdh}}(R, NHC^{(i)})$  in Lemmas 3.2 and 2.2.

The third sequence is obtained from the second one by taking the  $i^{\text{th}}$  component in the Hodge decomposition, described in Lemma 2.2.  $\square$

*Example 3.9.* Splicing the final sequences of Lemma 3.8 together, we see that the de Rham complexes are exact:

$$(3.9a) \quad 0 \rightarrow k \rightarrow R \xrightarrow{d} \tilde{H}_{\text{cdh}}^0(R, \Omega^1) \xrightarrow{d} \tilde{H}_{\text{cdh}}^0(R, \Omega^2) \rightarrow \dots$$

$$(3.9b) \quad 0 \rightarrow H_{\text{cdh}}^n(R, \mathcal{O}) \xrightarrow{d} H_{\text{cdh}}^n(R, \Omega^1) \xrightarrow{d} H_{\text{cdh}}^n(R, \Omega^2) \rightarrow \dots, \quad n > 0.$$

An analogous exact sequence

$$\dots \rightarrow \pi_{m-1} \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_m \mathcal{F}_{HH}(R) \xrightarrow{d} \pi_{m+1} \mathcal{F}_{HH}(R) \rightarrow \dots$$

is obtained by splicing the other sequences in Lemma 3.8. Using the interpretation of their Hodge components, described in Lemma 3.4, produces two more exact

sequences:

$$(3.9c) \quad 0 \rightarrow \text{nil}(R) \rightarrow \text{tors } \Omega_R^1 \rightarrow \text{tors } \Omega_R^2 \rightarrow \text{tors } \Omega_R^3 \rightarrow \cdots$$

$$(3.9d) \quad 0 \rightarrow (R^+/R) \rightarrow \Omega_{\text{cdh}}^1(R)/\Omega_R^1 \rightarrow \Omega_{\text{cdh}}^2(R)/\Omega_R^2 \rightarrow \cdots$$

Here we have written  $\Omega_{\text{cdh}}^i(R)$  for  $H_{\text{cdh}}^0(R, \Omega^i)$ , and  $\text{tors } \Omega_R^i$  is defined as the kernel of  $\Omega_R^i \rightarrow \Omega_{\text{cdh}}^i(R)$ ; the notation reflects the fact that if  $R$  is reduced then  $\text{tors } \Omega_R^i$  is the torsion submodule of  $\Omega_R^i$  (see Remark 5.6.1 below).

#### 4. BASS' GROUPS $NK_*(X)$

In this section, we relate algebraic  $K$ -theory to our Hochschild and cyclic homology calculations relative to the ground field  $k = \mathbb{Q}$ . Consider the trace map

$$NK_{n+1}(X) \rightarrow NHC_n(X) = NHC_n(X/\mathbb{Q})$$

induced by the Chern character. In the affine case, it is defined in [27]; for schemes it is defined using Zariski descent. As explained in [27], it arises from the Chern character from the spectrum  $NK(X)$  to the Eilenberg-Mac Lane spectrum  $|NHC(X)[1]|$  associated to the cochain complex  $NHC(X)[1]$ . Note that our indexing conventions are such that  $\pi_{n+1}|NHC(X)[1]| = H^{-n}NHC(X) = NHC_n(X)$ .

**Proposition 4.1.** *Suppose that  $R = \Gamma(X, \mathcal{O})$  for  $X$  in  $\text{Sch}/F$ , or that  $X = \text{Spec}(R)$  for an  $F$ -algebra  $R$ . Then for all  $n$ , the Chern character induces a natural isomorphism*

$$NK_{n+1}(X) \cong H^{-n}\mathcal{F}_{HH}(X) \otimes t\mathbb{Q}[t].$$

*This is an isomorphism of graded  $R$ -modules, and even  $\text{Carf}(R)$ -modules, identifying the operations  $[r]$ ,  $V_m$  and  $F_m$  on  $NK_*(X)$  with the operations on the right side described in Example 1.6.*

*Proof.* By Remark 2.2.1, we may suppose  $X \in \text{Sch}/F$ . By [7, 1.6], the Chern character  $K \rightarrow HN$  induces weak equivalences  $\mathcal{F}_K(X) \simeq |\mathcal{F}_{HC}(X)[1]|$  and  $\mathcal{F}_K(X \times \mathbb{A}^1) \simeq |\mathcal{F}_{HC}(X \times \mathbb{A}^1)[1]|$ . Since for any presheaf of spectra  $E$  we have a natural objectwise equivalence  $E(- \times \mathbb{A}^1) \simeq E \times NE$ , we obtain a natural weak equivalence from  $NK(X)$  to  $|N\mathcal{F}_{HC}(X)[1]|$ . Now take homotopy groups and apply Lemma 3.2.

As observed in [10, 4.12], the Chern character also commutes with the ring maps used to define the operators  $[r]$ ,  $V_m$ , and with the transfer for  $R[t^n] \rightarrow R[t]$  defining  $F_m$ . That is, it is a homomorphism of  $\text{Carf}(R)$ -modules. Since the transfer is defined via the ring map  $R[t] \rightarrow M_n(R[t^n])$ , followed by Morita invariance, there is no trouble in passing to schemes.  $\square$

We now come to one of our main results, which implies Corollary 0.1.

**Theorem 4.2.** *For all  $n$ ,  $N^2K_n(X) \cong (NK_n(X) \otimes t\mathbb{Q}[t]) \oplus (NK_{n-1}(X) \otimes \Omega_{\mathbb{Q}[t]}^1)$ , and*

$$N^{p+1}K_n(X) \cong \bigoplus_{j=0}^p NK_{n-j}(X) \otimes \wedge^j \mathbb{Q}^p \otimes (t\mathbb{Q}[t])^{\otimes(p-j)} \otimes (\Omega_{\mathbb{Q}[t]}^1)^{\otimes j}.$$

*This holds for every  $X$  in  $\text{Sch}/F$ , as well as for  $\text{Spec}(R)$  where  $R$  is an arbitrary commutative  $F$ -algebra.*

*Proof.* This is immediate from 4.1 and Corollary 3.3.  $\square$

*Remark 4.2.1.* Jim Davis has pointed out (see [9]) that a computation equivalent to 4.2 can also be derived — for arbitrary rings  $R$  — from the Farrell-Jones conjecture for the groups  $\mathbb{Z}^r$ . This particular case is covered by F. Quinn’s proof of hyperfinitary assembly for virtually abelian groups; see [20].

As an immediate consequence of 4.2 and [1, XII(7.3)], we deduce:

**Corollary 4.3.** *Suppose that  $X$  is in  $\text{Sch}/F$ , or that  $X = \text{Spec}(R)$  for an  $F$ -algebra  $R$ . Then:*

- a) *If  $NK_n(X) = NK_{n-1}(X) = 0$  then  $N^2K_n(X) = 0$ .*
- b) *If  $NK_n(X) = 0$  and  $K_{n-1}(X) = K_{n-1}(X \times \mathbb{A}^p)$  then  $K_n(X) = K_n(X \times \mathbb{A}^{p+1})$ .*
- c)  *$K_n(X) = K_n(X \times \mathbb{A}^p)$  if and only if  $NK_q(X) = 0$  for all  $q$  such that  $n-p < q \leq n$ .*

Recall that  $X$  is called  $K_n$ -regular if  $K_n(X) = K_n(X \times \mathbb{A}^p)$  for all  $p$ .

**Corollary 4.4.** *Suppose that  $X$  is in  $\text{Sch}/F$ , or that  $X = \text{Spec}(R)$  for an  $F$ -algebra  $R$ . Then the following conditions are equivalent:*

- a)  *$X$  is  $K_n$ -regular;*
- b)  *$NK_n(X) = 0$  and  $X$  is  $K_{n-1}$ -regular;*
- c)  *$NK_q(X) = 0$  for all  $q \leq n$ .*

*Remark 4.4.1.* This gives another proof of Vorst’s Theorem [25, 2.1] (in characteristic 0) that  $K_n$ -regularity implies  $K_{n-1}$ -regularity, and extends it to schemes.

The assumption that the scheme be affine is essential in Bass’ question — here is a non-affine example where the answer is negative.

**Negative answer to Bass’ question for nonaffine curves.** Let  $X$  be a smooth projective elliptic curve over a number field  $k$  and let  $L$  be a nontrivial degree zero line bundle with  $L^{\otimes 3}$  trivial. For example, if  $X$  is the Fermat cubic  $x^3 + y^3 = z^3$ , we may take the line bundle associated to the divisor  $P - Q$ , where  $P = (1 : 0 : 1)$  and  $Q = (0 : 1 : 1)$ .

**Lemma 4.5.** *Write  $Y$  for the nonreduced scheme with the same underlying space as  $X$  but with structure sheaf  $\mathcal{O}_Y = \mathcal{O}_X \oplus L = \text{Sym}(L)/(L^2)$ , that is,  $L$  is regarded as a square-zero ideal.*

*Then  $NK_7(Y) = 0$  but  $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega_{\mathbb{Q}[t]}^1$  is nonzero.*

*Proof.* In this setting, the relative Hochschild homology presheaf  $HH_n(Y, L)$  is the kernel of  $HH_n(Y) \rightarrow HH_n(X)$ ; sheafifying,  $\mathcal{HH}_n(Y, L)$  is the kernel of  $\mathcal{HH}_n(Y) \rightarrow \mathcal{HH}_n(X)$ . Since  $\Omega_X^1 \cong \mathcal{O}_X$  we see from Lemma 5.3 of [7] that  $\mathcal{HH}_n(Y, L)$  is:  $L^{\otimes 3} \oplus L^{\otimes 5}$  if  $n = 4$ ;  $L^{\otimes 5} \oplus L^{\otimes 5}$  if  $n = 5$ ; and  $L^{\otimes 5} \oplus L^{\otimes 7}$  if  $n = 6$ . By Serre duality,  $H^*(X, L^{\otimes i}) = 0$  if  $3 \nmid i$  (cf. [7, 5.1]). By Zariski descent, this implies that  $HH_5(Y, L) \cong H^1(X, \mathcal{HH}_4) \cong H^1(X, L^{\otimes 3}) \cong k$  and  $HH_6(Y, L) = 0$ . Since  $\mathcal{F}_{HH}(Y) \cong HH(Y, L)$ , it follows from 4.1 and 4.2 that  $NK_7(Y) = 0$  but  $NK_6(Y) \cong t\mathbb{Q}[t]$  and  $N^2K_7(Y) \cong NK_6(Y) \otimes \Omega_{\mathbb{Q}[t]}^1 \cong t\mathbb{Q}[t] \otimes \Omega_{\mathbb{Q}[t]}^1$ .  $\square$

We conclude this section by refining Proposition 4.1 and Corollary 4.3 to take account of the Adams/Hodge/ $\lambda$ -decompositions on K-theory and Hochschild homology, and by establishing the triviality of  $K_*^{(i)}(X)$  for  $i \leq 0$ .

Recall that by definition,  $K_n^{(i)}(X) = \{x \in K_n(X) \otimes \mathbb{Q} : \psi^k(x) = k^i x\}$ . For  $n < 0$ , the Adams operations cannot be defined integrally. However, it is possible to define the operations  $\psi^k$  on  $K_n(X) \otimes \mathbb{Q}$  for  $n < 0$  using descending induction on

$n$  and the formula  $\psi^k\{x, t\} = k\{\psi^k(x), t\}$  in  $K_{n+1}(X \times (\mathbb{A}^1 - 0))$  for  $x \in K_n(X)$  and  $\mathcal{O}(\mathbb{A}^1 - 0) = F[t, 1/t]$ . This definition was pointed out in [30, 8.4].

By [11, 2.3] or [8, 7.2], the Chern character  $NK_{n+1}(X) \rightarrow NHC_n(X)$  commutes with the Adams operations  $\psi^k$  in the sense that it sends  $NK_{n+1}^{(i+1)}(X)$  to  $NHC_n^{(i)}(X)$  for all  $i \leq n$  (and to 0 if  $i > n$ ). Here is the  $\lambda$ -decomposition of the isomorphism in Proposition 4.1:

**Proposition 4.6.** *Suppose that  $X \in \text{Sch}/F$ , or that  $X = \text{Spec}(R)$  for an  $F$ -algebra  $R$ . Then for all  $n$  and  $i$ , the Chern character induces a natural isomorphism:*

$$NK_n^{(i)}(X) \cong H^{1-n} \mathcal{F}_{HH}^{(i-1)}(X) \otimes t\mathbb{Q}[t].$$

*In particular, if  $i \leq 0$  then  $NK_n^{(i)}(X) = 0$  for all  $n$ .*

*Proof.* By [8], the Chern character  $K \rightarrow HN$  sends  $K^{(i)}(X)$  to  $HN^{(i)}(X)$ . The proof in [8] shows that the lift  $\mathcal{F}_K(X) \rightarrow \mathcal{F}_{HN}(X)$ , shown to be a weak equivalence in [7, 1.6], may be taken to send  $\mathcal{F}_K^{(i)}(X)$  to  $\mathcal{F}_{HN}^{(i)}(X)$ . Since  $HC \rightarrow HN$  sends  $HC^{(i-1)}$  to  $HN^{(i)}$ , the weak equivalence  $\mathcal{F}_{HC}[1] \simeq \mathcal{F}_{HN}$  identifies  $\mathcal{F}_{HC}^{(i-1)}[1]$  and  $\mathcal{F}_{HN}^{(i)}$ . Finally  $\mathcal{F}_{HH}^{(i-1)} = 0$  for  $i \leq 0$ .  $\square$

**Corollary 4.7.**  $K_n^{(i)}(X) \cong K_n^{(i)}(X \times \mathbb{A}^p)$  if and only if  $NK_{n-j}^{(i-j)}(X) = 0$  for all  $j = 0, \dots, p-1$ .

**Theorem 4.8.** *For  $X$  in  $\text{Sch}/F$  or  $X = \text{Spec}(R)$ , and all integers  $n$ , we have:*

- (1) *For  $i < 0$ ,  $K_n^{(i)}(X) = 0$ .*
- (2) *For  $i = 0$ ,  $K_n^{(0)}(X) \cong KH_n^{(0)}(X) \cong H_{\text{cdh}}^{-n}(X, \mathbb{Q})$ .*

Here  $KH$  denotes the homotopy  $K$ -theory of [29]. Theorem 4.8 answers Question 8.2 of [30].

*Proof.* We first show that  $K_n^{(i)}(X) \cong KH_n^{(i)}(X)$  when  $i \leq 0$ . Covering  $X$  with affine opens and using the Mayer-Vietoris sequences of [29, 5.1], it suffices to consider the case  $X = \text{Spec}(R)$ .

Since  $K(R)_{\mathbb{Q}}$  is the product of the eigen-components, the descent spectral sequence  $E_{p,q}^1 = N^p K_q(R)_{\mathbb{Q}} \Rightarrow KH_{p+q}(R)_{\mathbb{Q}}$  (see [29, 1.3]) breaks up into one for each eigen-component. If  $i \leq 0$ , the spectral sequence collapses by Proposition 4.6 to yield  $K_n^{(i)}(R) \cong KH_n^{(i)}(R)$  for all  $n$ .

To determine the groups  $KH_n^{(i)}(R)$  when  $i \leq 0$ , we use the  $cdh$  descent spectral sequence of [15, 1.1]. If  $i < 0$ , then the  $cdh$  sheaf  $K_{\text{cdh}}^{(i)}$  is trivial as  $X$  is locally smooth, so we have  $KH_n^{(i)}(R) = 0$  for all  $n$ . If  $i = 0$  then the  $cdh$  sheaf  $K_{\text{cdh}}^{(0)}$  is the sheaf  $\mathbb{Q}_{\text{cdh}}$ ; see [21, 2.8]. Hence we have  $K_n^{(0)}(R) = KH_n^{(0)}(R) = H_{\text{cdh}}^{-n}(X, \mathbb{Q})$ .  $\square$

## 5. THE TYPICAL PIECES $TK_n^{(i)}(R)$

In this section,  $R$  will be a commutative  $F$ -algebra. The default ground field  $k$  for Kähler differentials and Hochschild homology will be  $\mathbb{Q}$ .

As stated in (0.3), the Adams summands  $NK_n^{(i)}(R)$  of  $NK_n(R)$  decompose as  $NK_n^{(i)}(R) = TK_n^{(i)}(R) \otimes t\mathbb{Q}[t]$  for each  $n$  and  $i$ ; the decomposition is obtained from an action of finite Cartier operators precisely as the corresponding one for  $NHC$

and  $NHH$ , explained in Section 1. The typical pieces  $TK_n^{(i)}(R)$  are described by the following formulas.

**Theorem 5.1.** *Let  $R$  be a commutative  $F$ -algebra. For  $i \neq n, n+1$  we have:*

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } i < n, \\ H_{\text{cdh}}^{i-n-1}(R, \Omega^{i-1}) & \text{if } i \geq n+2. \end{cases}$$

*For  $i = n, n+1$ , the typical piece  $TK_n^{(i)}(R)$  is given by the exact sequence:*

$$0 \rightarrow TK_{n+1}^{(n+1)}(R) \rightarrow \Omega_R^n \rightarrow H_{\text{cdh}}^0(R, \Omega^n) \rightarrow TK_n^{(n+1)}(R) \rightarrow 0.$$

*Proof.* By Proposition 4.6,  $TK_n^{(i)} = H^{1-n}\mathcal{F}_{HH}^{(i-1)}$ . The rest is a restatement of Lemma 3.4.  $\square$

*Remark 5.1.1.* If  $R$  is essentially of finite type over a field  $F$  whose transcendence degree is finite over  $\mathbb{Q}$ , then the  $TK_n^{(i)}(R)$  are finitely generated  $R$ -modules. This fails if  $\text{tr.deg}(F/\mathbb{Q}) = \infty$  because then  $\Omega_{F/\mathbb{Q}}^i$  is infinite dimensional. For instance, Example 0.6 implies that, for  $R = F[x]/(x^2)$ , we have  $TK_2^{(2)}(R) = HH_1(R, x) = F \oplus \Omega_{F/\mathbb{Q}}^1$ .

**Corollary 5.2.** *Suppose that  $R$  is essentially of finite type over  $F$  and has dimension  $d$ . If  $n < 0$  then  $NK_n^{(i)}(R) = 0$  unless  $1 \leq i \leq d+n$ , in which case*

$$NK_n^{(i)}(R) = H_{\text{cdh}}^{i-n-1}(R, \Omega^{i-1}) \otimes t\mathbb{Q}[t].$$

*In particular,  $NK_n(R) = 0$  for all  $n \leq -d$ .*

*If  $d \geq 2$  then:*

$$\begin{aligned} NK_0(R) &\cong [(R^+/R) \oplus H_{\text{cdh}}^1(R, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(R, \Omega^{d-1})] \otimes t\mathbb{Q}[t], \\ NK_{-1}(R) &\cong [H_{\text{cdh}}^1(R, \mathcal{O}) \oplus H_{\text{cdh}}^2(R, \Omega^1) \oplus \cdots \oplus H_{\text{cdh}}^{d-1}(R, \Omega^{d-2})] \otimes t\mathbb{Q}[t], \\ &\vdots \\ NK_{1-d}(R) &\cong H_{\text{cdh}}^{d-1}(R, \mathcal{O}) \otimes t\mathbb{Q}[t]. \end{aligned}$$

*If  $d = 1$  then  $NK_0(R) = (R^+/R) \otimes t\mathbb{Q}[t]$  and  $NK_n(R) = 0$  for  $n < 0$ .*

*Remark 5.2.1.* The  $d = 1$  part of Theorem 5.2 holds for any 1-dimensional noetherian ring by [26, 2.8].

*Remark 5.2.2.* Observe that 5.2 and 4.4 imply that  $R$  is  $K_{-d}$ -regular. This recovers the affine case of one of the main results in [5].

Here is a special case of the calculations in Theorem 5.1, which proves Theorem 0.7. We will use it to construct the counterexample to Bass' question in the companion paper [6].

**Theorem 5.3.** *Let  $F$  be a field of characteristic 0 and  $R$  a normal domain of dimension 2, essentially of finite type over  $F$ . Then*

- a)  $H^1\mathcal{F}_{HH}(R/F) \cong H_{\text{cdh}}^1(R, \Omega_{F/F}^1)$ ,
- b)  $H^2\mathcal{F}_{HH}(R/F) \cong H_{\text{cdh}}^1(R, \mathcal{O})$ ,
- c)  $NK_0(R) \cong H_{\text{cdh}}^1(R, \Omega^1) \otimes t\mathbb{Q}[t]$ , and
- d)  $NK_{-1}(R) \cong H_{\text{cdh}}^1(R, \mathcal{O}) \otimes t\mathbb{Q}[t]$ .

*Proof.* Parts (a) and (b) are immediate from Example 3.5 and the fact that  $R$  is reduced and seminormal. Parts (c) and (d) follow from (a) and (b) using Proposition 4.1; cf. Corollary 5.2.  $\square$

We introduce some notation to make the statement of the next theorem more readable. The letter  $e$  denotes the maximum transcendence degree of the component fields in the total ring of fractions  $Q$  of  $R_{\text{red}}$ . For simplicity, we write  $\Omega_{\text{cdh}}^i(X)$  for  $H_{\text{cdh}}^0(X, \Omega^i)$ , and we have written  $\Omega_{\text{cdh}}^i(R)/\Omega_R^i$  for the cokernel of  $\Omega_R^i \rightarrow \Omega_{\text{cdh}}^i(R)$ .

**Definition 5.4.** For any commutative ring  $R$  containing  $\mathbb{Q}$ , we define:

$$E_n(R) = \Omega_{\text{cdh}}^n(R)/\Omega_R^n \oplus \bigoplus_{p=1}^{\infty} H_{\text{cdh}}^p(R, \Omega^{n+p});$$

$$\widetilde{HH}_n(R) = \ker(HH_n(R) \rightarrow \Omega_Q^n) = \ker(\Omega_R^n \rightarrow \Omega_Q^n) \oplus \bigoplus_{i=1}^{n-1} HH_n^{(i)}(R).$$

**Theorem 5.5.** *Let  $R$  be a commutative ring containing  $\mathbb{Q}$ . Then for all  $n$ :*

$$NK_n(R) \cong [\widetilde{HH}_{n-1}(R) \oplus E_n(R)] \otimes t\mathbb{Q}[t].$$

*If furthermore  $R$  is essentially of finite type over a field, and  $n \geq e + 2$ , then  $NK_n(R) \cong HH_{n-1}(R) \otimes t\mathbb{Q}[t]$ .*

*Proof.* Assembling the descriptions of the  $TK_n^{(i)}(R)$  in Theorem 5.1 yields the first assertion. The second part is immediate from this and 3.6.  $\square$

**Remark 5.5.1.** The Chern character  $NK_n(R) \rightarrow NHC_{n-1}(R) \cong HH_{n-1}(R) \otimes t\mathbb{Q}[t]$  is an isomorphism for  $n \geq e+2$ . If  $n \leq e+1$ , neither it nor the map  $H^{1-n}\mathcal{F}_{HH}(R) \rightarrow HH_{n-1}(R)$  of 4.1 need be a surjection.

In order to compare the torsion submodules  $\text{tors}\Omega_R^*$  with the typical pieces of  $NK_*(R)$ , we need the affine case of the following lemma. Following tradition, we write  $F(X)$  for the total ring of fractions of  $X_{\text{red}}$ . That is,  $F(X)$  is the product of the function fields of the irreducible components of  $X_{\text{red}}$ . When  $X = \text{Spec}(R)$  is affine, we write  $Q$  instead of  $F(X)$ .

**Lemma 5.6.** *Let  $X \in \text{Sch}/F$ ; for  $F(X)$  as above, the map  $\Omega_{\text{cdh}}^i(X) \rightarrow \Omega_{F(X)}^i$  is an injection.*

*Proof.* We may assume  $X$  reduced, and proceed by induction on  $d = \dim(X)$ , the case  $d = 0$  being trivial. Choose a resolution of singularities  $X' \rightarrow X$  and let  $Y$  be the singular locus of  $X$ , with  $Y' = Y \times_X X'$ . By [22, 12.1], there is a Mayer-Vietoris exact sequence

$$0 \rightarrow \Omega_{\text{cdh}}^i(X) \rightarrow \Omega_{\text{cdh}}^i(X') \oplus \Omega_{\text{cdh}}^i(Y) \rightarrow \Omega_{\text{cdh}}^i(Y') \xrightarrow{\partial} H_{\text{cdh}}^1(X, \Omega^i) \rightarrow \dots$$

Since  $F(Y) \subseteq F(Y')$ ,  $\Omega_{F(Y)}^i \subseteq \Omega_{F(Y')}^i$ . Because  $\dim(Y') < d$ , the inductive hypothesis implies that  $\Omega_{\text{cdh}}^i(Y) \rightarrow \Omega_{\text{cdh}}^i(Y')$  is an injection. Hence  $\Omega_{\text{cdh}}^i(X) \rightarrow \Omega_{\text{cdh}}^i(X')$  is an injection. But  $X'$  is smooth, so by  $scdh$  descent for  $\Omega^i$  (see [7, 2.5]) we have  $\Omega_{\text{cdh}}^i(X') \cong \Omega^i(X') \subset \Omega_{F(X')}^i = \Omega_{F(X)}^i$ .  $\square$

**Remark 5.6.1.** Lemma 5.6 remains true if, instead of  $\Omega^i$ , we use  $\Omega_{/k}^i$  for  $k \subseteq F$ . In particular, if  $X = \text{Spec}(R)$  is reduced affine, then  $\Omega_{\text{cdh}}^i(R/k) = H_{\text{cdh}}^0(R, \Omega_{/k}^i)$  injects into  $\Omega_{Q/k}^i$ . Thus  $\text{tors}(\Omega_{R/k}^i)$ , defined as the kernel of  $\Omega_{R/k}^i \rightarrow \Omega_{\text{cdh}}^i(R/k)$  in (3.9c), is the torsion submodule of  $\Omega_{R/k}^i$ .

**Corollary 5.7.** *For all  $n \geq 1$ ,  $TK_n^{(n)}(R) \cong \ker(\Omega_R^{n-1} \rightarrow \Omega_Q^{n-1})$ .*

*In particular if  $R$  is reduced, then  $TK_n^{(n)}(R)$  is the torsion submodule of  $\Omega_R^{n-1}$ .*

*Proof.* By 5.1,  $TK_n^{(n)}(R)$  is the kernel of  $\Omega_R^{n-1} \rightarrow \Omega_{\text{cdh}}^{n-1}(R)$ , so 5.6 applies.  $\square$

The typical pieces of  $NK_1^{(2)}(R)$  and  $NK_2^{(2)}(R)$  of 5.1 and 5.7 may be described as follows.

**Proposition 5.8.** *For all reduced  $F$ -algebras  $R$ , the typical pieces  $TK_1^{(2)}(R) = \Omega_{\text{cdh}}^1(R)/\Omega_R^1$  and  $TK_2^{(2)}(R) = \text{tors}(\Omega_R^1)$  fit into an exact sequence:*

$$0 \rightarrow \text{tors}(\Omega_R^1) \rightarrow \text{tors}(\Omega_{R/F}^1) \rightarrow \Omega_F^1 \otimes (R^+/R) \rightarrow \frac{\Omega_{\text{cdh}}^1(R)}{\Omega_R^1} \rightarrow \frac{\Omega_{\text{cdh}}^1(R/F)}{\Omega_{R/F}^1} \rightarrow 0.$$

*Proof.* We may assume  $\text{Spec } R \in \text{Sch}/F$ . Recall from [7, 4.2] that there is a bounded second quadrant homological spectral sequence for all  $p$  ( $0 \leq i < p, j \geq 0$ ):

$${}_pE_{-i,i+j}^1 = \Omega_{F/k}^i \otimes_F HH_{p-i+j}^{(p-i)}(R/F) \Rightarrow HH_{p+j}^{(p)}(R/k).$$

When  $p = 1$ , this spectral sequence degenerates to yield exactness of the bottom row in the following commutative diagram; the top row is the First Fundamental Exact Sequence for  $\Omega^1$  [31, 9.2.6].

$$\begin{array}{ccccccc} \Omega_F^1 \otimes R & \longrightarrow & \Omega_R^1 & \longrightarrow & \Omega_{R/F}^1 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 \longrightarrow & \Omega_F^1 \otimes R^+ & \longrightarrow & \Omega_{\text{cdh}}^1(R) & \longrightarrow & \Omega_{\text{cdh}}^1(R/F) & \longrightarrow 0. \end{array}$$

The upper left horizontal map is an injection because the left vertical map is an injection. Now apply the snake lemma, using Remark 5.6.1.  $\square$

## 6. BASS' QUESTION FOR ALGEBRAS OVER LARGE FIELDS.

We will now show that the answer to Bass' question is positive for algebras  $R$  essentially of finite type over a field  $F$  of infinite transcendence degree over  $\mathbb{Q}$ .

Recall from Proposition 4.1 that  $NK_{n+1}(R) \cong H^{-n}\mathcal{F}_{HH}(R/\mathbb{Q}) \otimes t\mathbb{Q}[t]$ . In light of this identification, the version of Bass' question stated before Theorem 0.2 becomes the case  $k = \mathbb{Q}$  of the following question:

$$(6.1) \quad \text{Does } H^m\mathcal{F}_{HH}(R/k) = 0 \text{ imply that } H^{m+1}\mathcal{F}_{HH}(R/k) = 0?$$

In Theorem 6.6, we show that the answer to question (6.1) is positive provided  $R$  is of finite type over a field  $F$  that has infinite transcendence degree over  $k$ . The proof is essentially a formal consequence of the Künneth formula in Lemma 6.3.

**Lemma 6.2.** *Let  $R$  be a commutative  $F$ -algebra, and suppose  $k$  is a subfield of  $F$ . Then  $H^{-*}\mathcal{F}_{HH}(R/k)$  and  $\mathbb{H}_{\text{cdh}}^{-*}(R, HH(/k))$  are graded modules over the graded ring  $\Omega_{F/k}^\bullet$ .*

*Proof.* As in 2.2.1, we may suppose that  $R$  is of finite type over  $F$ . Consider the functor on  $F$ -algebras that associates to an  $F$ -algebra  $A$  the Hochschild complex  $HH(A/k)$ . The shuffle product makes this into a functor to  $dg\text{-}HH(F/k)$ -modules. Since the  $cdh$ -site has a set of points (corresponding to valuations by [13, 2.1]), we can use a Godement resolution to find a model for the  $cdh$ -hypercohomology



$\mathbb{H}_{cdh}(-, HH(F/k))$  which is also a functor to  $dg\text{-}HH(F/k)$ -modules. It follows that there is a model for  $\mathcal{F}_{HH}(R/k)$  that is a  $dg\text{-}HH(F/k)$ -module, functorially in  $R$ . This implies the assertion, since  $\Omega_{F/k}^\bullet = H^{-\bullet} HH(F/k)$ .  $\square$

**Lemma 6.3.** (*Künneth Formula*) *Suppose that  $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$  are fields. Let  $R_0$  be an  $F_0$ -algebra, and set  $R = F \otimes_{F_0} R_0$ .*

- i) *Let  $T = \{t_i\}$  be transcendence basis of  $F/F_0$ ; writing  $F[dT]$  for the exterior algebra on the set  $\{dt_i\}$ , we have  $\Omega_{F/F_0}^\bullet = F[dT]$  and:*

$$\Omega_{F/k}^\bullet \cong F[dT] \otimes_{F_0} \Omega_{F_0/k}^\bullet$$

*In particular, the graded algebra homomorphism  $\Omega_{F_0/k}^\bullet \rightarrow \Omega_{F/k}^\bullet$  is flat.*

- ii)  $HH_*(R/k) \cong \Omega_{F/k}^\bullet \otimes_{\Omega_{F_0/k}^\bullet} HH_*(R_0/k) \cong F[dT] \otimes_{F_0} HH_*(R_0/k)$ .

*Proof.* It is classical that  $F[dT] = \Omega_{F/F_0}^\bullet$ . The tensor product decomposition of part i) follows from the fact that the fundamental sequence

$$0 \rightarrow F \otimes_{F_0} \Omega_{F_0}^1 \rightarrow \Omega_F^1 \rightarrow \Omega_{F/F_0}^1 \rightarrow 0$$

is split exact. This proves i). To prove ii), choose a free chain  $dg\text{-}F_0$ -algebra  $\Lambda$  and a surjective quasi-isomorphism of  $dg$ -algebras  $\Lambda \xrightarrow{\sim} R_0$ . Then  $\Lambda' = F \otimes_{F_0} \Lambda \rightarrow F \otimes_{F_0} R_0 = R$  is a free chain model of  $R$  as a  $k$ -algebra. Write  $\Omega_\Lambda^\bullet$  for differential forms; consider  $\Omega_\Lambda^\bullet$  as a chain  $dg$ -algebra with the differential  $\delta$  induced by that of  $\Lambda$ . Note  $\Lambda$  and  $\Lambda'$  are homologically regular in the sense of [4], so that Theorem 2.6 of [4] applies. Combining this with part (i), we obtain

$$\begin{aligned} HH_*(R) &= HH_*(\Lambda') = H_*(\Omega_{\Lambda'}^\bullet) \\ &= H_*(\Omega_F^\bullet \otimes_{\Omega_{F_0}^\bullet} \Omega_\Lambda^\bullet) = \Omega_F^\bullet \otimes_{\Omega_{F_0}^\bullet} H_*(\Omega_\Lambda^\bullet) \\ &= \Omega_F^\bullet \otimes_{\Omega_{F_0}^\bullet} HH_*(R_0). \end{aligned} \quad \square$$

Here is an easy consequence of Lemmas 6.2 and 6.3.

**Proposition 6.4.** *Suppose  $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$  are field extensions, that  $R_0$  is an  $F_0$ -algebra and  $R = F \otimes_{F_0} R_0$ . Then there is an isomorphism of graded  $\Omega_{F/k}^\bullet$ -modules*

$$F[dT] \otimes_{F_0} H^{-*} \mathcal{F}_{HH}(R_0/k) \cong H^{-*}(\mathcal{F}_{HH}(R/k)).$$

We also need the following lemma to prove the main result of this section.

**Lemma 6.5.** *Let  $R$  be essentially of finite type over  $F \supset \mathbb{Q}$ , and let  $H_n(R)$  denote either  $HH_n(R)$  or  $H^{-n} \mathcal{F}_{HH}(R)$ . Assume that  $H_{n_i}(R) = 0$  for some finite set  $\{n_1, \dots, n_r\}$  of positive integers. Then there exist an  $F$ -algebra of finite type  $R'$ , and a multiplicatively closed set  $S$  such that  $R \cong R'_S$  and  $H_{n_i}(R') = 0$  for  $1 \leq i \leq r$ .*

*Proof.* Because  $R$  is essentially of finite type, it is the localization  $R = R'_S$  of some finite type  $F$ -algebra  $R''$ . It is well known that  $HH_n(R'_S) \cong S^{-1} HH_n(R'')$  (see [31, 9.1.8]), and  $H^{-n} \mathcal{F}_{HH}(R'_S) \cong S^{-1} H^{-n} \mathcal{F}_{HH}(R'')$  by [7, 2.8–9].

Because  $R''$  is of finite type over  $F$ , we may write  $R'' = F \otimes_{F_0} R_0$  for some finitely generated field extension  $F_0$  of  $\mathbb{Q}$  and some finite type  $F_0$ -algebra  $R_0$ . Note  $R_0$  is essentially of finite type over  $\mathbb{Q}$ , whence  $H_p(R_0)$  is a finitely generated  $R_0$ -module ( $p \geq 0$ ). By 6.3 and/or 6.4,  $H_p(R'')$  is isomorphic, as an  $R''$ -module, to a direct sum of copies of  $R'' \otimes_{R_0} H_q(R_0)$  with  $q \leq p$ . In particular,  $M = \bigoplus_{i=1}^r H_{n_i}(R'')$  is

a finite sum of  $R''$ -modules, each of which is a — possibly infinite — direct sum of copies of one finitely generated module.

Given that  $M$  has this form, the hypothesis that  $S^{-1}M = 0$  implies that there exists a nonzero element  $s \in \text{Ann}(M) \cap S$ . Consider the finite type  $F$ -algebra  $R' = R''[1/s]$ . Then  $R \cong R'_S$  and we have  $\bigoplus_i H_{-n_i}(R') = M[1/s] = 0$ .  $\square$

**Theorem 6.6.** *Suppose  $k \subset F$  is an extension with  $\text{tr. deg}(F/k) = \infty$ , and  $R$  is essentially of finite type over  $F$ . If  $H^n(\mathcal{F}_{HH}(R/k)) = 0$ , then  $H^m(\mathcal{F}_{HH}(R/k)) = 0$  for all  $m \geq n$ .*

*Proof.* By Lemma 6.5, we may assume that  $R$  is of finite type over  $F$ . There is a finitely generated field extension  $F_0 \subset F$  of  $k$  and a finite type  $F_0$ -algebra  $R_0$  such that  $R = R_0 \otimes_{F_0} F$ . Note that  $\text{tr. deg}(F/F_0) = \infty$ . By Lemma 6.3 and Proposition 6.4,  $\Omega_{F/F_0}^i \otimes_{F_0} H^{n+i}(\mathcal{F}_{HH}(R_0/k))$  is a direct summand of  $H^n(\mathcal{F}_{HH}(R/k))$  for each  $i \geq 0$ . Since  $\Omega_{F/F_0}^i \neq 0$  for all  $i$ , all the  $H^{n+i}(\mathcal{F}_{HH}(R_0/k))$  vanish as well. Similarly,  $H^m(\mathcal{F}_{HH}(R/k))$  is a direct sum of copies of the groups  $\Omega_{F/F_0}^j \otimes_{F_0} H^{m+j}(\mathcal{F}_{HH}(R_0/k))$  for  $j \geq 0$ , all of which vanish when  $m \geq n$ , as we just observed.  $\square$

**Corollary 6.7.** *Let  $\mathbb{Q} \subset F$  be a field extension of infinite transcendence degree, and suppose  $R$  is essentially of finite type over  $F$ . Then  $NK_n(R) = 0$  implies that  $R$  is  $K_n$ -regular.*

*Proof.* Combine Theorem 6.6 with Proposition 4.1 and Corollary 4.4.  $\square$

Here is another proof of Corollary 6.7, which is essentially due to Murthy and Pedrini and given in their 1972 paper [18]; they stated the result only for  $n \leq 1$  because transfer maps for higher  $K$ -theory and the  $W(R)$ -module structure had not yet been discovered. We are grateful to Joseph Gubeladze [14] for pointing this out to the authors.

**Lemma 6.8.** *If  $R$  is an algebra over a field  $k$  of characteristic 0,  $N^p K_n(R[t]) \rightarrow N^p K_n(R \otimes_k k(t))$  is injective.*

*Proof.* The proof in [18, 1.3–1.6] goes through, taking into account that the norm map and localization sequences used there for  $K_0, K_1$  are now known for all  $K_n$ .  $\square$

**Lemma 6.9.** *Suppose that  $k$  is an algebraically closed field of infinite transcendence degree over  $\mathbb{Q}$ , and that  $R$  is a finitely generated  $k$ -algebra. If  $NK_n(R)$  is zero, then  $K_n(R) \xrightarrow{\cong} K_n(R[x_1, \dots, x_p])$  for all  $p > 0$ .*

*Proof.* Murthy and Pedrini prove this in [18, 2.1.]; although their result is only stated for  $i \leq 1$ , their proof works in general. Note that since  $NK_n(R)$  has the form  $TK_n(R) \otimes t\mathbb{Q}[t]$  by (0.3) (a result which was not known in 1972),  $NK_n(R)$  is torsionfree, and has finite rank if and only if it is zero.  $\square$

*Proof of Corollary 6.7.* Let  $\Phi$  denote the functor  $N^p K_n$ . If  $k \subset k_1$  is a finite algebraic field extension and  $R$  is a  $k$ -algebra, then  $\Phi(R) \rightarrow \Phi(R \otimes_k k_1)$  is an injection because its composition with the transfer  $\Phi(R \otimes_k k_1) \rightarrow \Phi(R)$  is multiplication by  $[k_1 : k]$ , and  $\Phi(R)$  is a torsionfree group. Since  $\Phi$  commutes with filtered colimits of rings,  $\Phi(R) \rightarrow \Phi(R \otimes_k \bar{k})$  is an injection. Thus Lemma 6.9 suffices to prove Corollary 6.7 when  $R$  is of finite type.  $\square$

7.  $NK_0$  OF SURFACES

We conclude with a general description for affine surfaces of the canonical map  $\Omega_F^1 \otimes_F NK_{-1} \rightarrow NK_0$ . This sheds light on the difference between the cases of small and large base fields, and also explains some results of [33].

If  $R$  is a 2-dimensional noetherian ring then  $NK_0(R)$  is the direct sum of  $NK_0^{(1)}(R) = N\text{Pic}(R)$  and  $NK_0^{(2)}(R)$

**Theorem 7.1.** *Let  $R$  be a 2-dimensional normal domain of finite type over a field  $F$  of characteristic 0. There is an exact sequence:*

$$\begin{aligned} 0 \rightarrow NK_1^{(2)}(R) &\rightarrow \left( H^0(R, \Omega_{R/F}^1) / \Omega_{R/F}^1 \right) \otimes t\mathbb{Q}[t] \\ &\rightarrow \Omega_F^1 \otimes_F NK_{-1}(R) \rightarrow NK_0(R) \rightarrow H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \otimes t\mathbb{Q}[t] \rightarrow 0. \end{aligned}$$

*Proof.* Consider the following short exact sequence of sheaves in  $(\text{Sch}/F)_{\text{cdh}}$ :

$$0 \rightarrow \Omega_F^1 \otimes_F \mathcal{O} \rightarrow \Omega^1 \rightarrow \Omega_{/F}^1 \rightarrow 0$$

Applying  $H_{\text{cdh}}$  yields

$$\begin{aligned} 0 \rightarrow \Omega_F^1 \otimes_F R &\xrightarrow{\iota} H^0(R, \Omega^1) \rightarrow H^0(R, \Omega_{R/F}^1) \xrightarrow{\partial} \\ &\Omega_F^1 \otimes_F H_{\text{cdh}}^1(R, \mathcal{O}) \rightarrow H_{\text{cdh}}^1(R, \Omega^1) \rightarrow H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \rightarrow 0 \end{aligned}$$

Note that, because  $\Omega_R^1 \rightarrow \Omega_{R/F}^1$  is onto, the map  $\partial$  kills the image of  $\Omega_{R/F}^1$ . Similarly, the image of  $\iota$  is contained in that of  $\Omega_R^1$ . Thus we obtain

$$\begin{aligned} 0 \rightarrow H^0(R, \Omega^1) / \Omega_R^1 &\rightarrow H^0(R, \Omega_{R/F}^1) / \Omega_{R/F}^1 \rightarrow \\ &\Omega_F^1 \otimes_F H_{\text{cdh}}^1(R, \mathcal{O}) \rightarrow H_{\text{cdh}}^1(R, \Omega^1) \rightarrow H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \rightarrow 0 \end{aligned}$$

Now apply  $\otimes t\mathbb{Q}[t]$  and use 5.1 and parts c) and d) of 5.3.  $\square$

**Corollary 7.2.** *Let  $R$  be a 2-dimensional normal domain of finite type over a field  $F$  of characteristic 0. If  $NK_{-1}(R) = 0$  then  $NK_0(R) \cong H_{\text{cdh}}^1(R, \Omega_{R/F}^1) \otimes t\mathbb{Q}[t]$ .*

*Example 7.3.* Let  $R$  be a 2-dimensional normal domain of finite type over  $\mathbb{Q}$ , and put  $R_F = R \otimes F$ . By 4.1 and 6.4,

$$(7.4) \quad NK_*(R_F) \cong NK_*(R) \otimes \Omega_{F/\mathbb{Q}}^*.$$

Keeping track of the  $\lambda$ -decomposition, as in 5.1, we see from Theorem 0.7 that

$$TK_1^{(2)}(R_F) \cong TK_1^{(2)}(R) \otimes F \cong H^0(R, \Omega^1) \otimes F / \Omega_R^1 \otimes F \cong H^0(R_F, \Omega_{R_F/F}^1) / \Omega_{R_F/F}^1.$$

From Theorem 7.1 we get an exact sequence

$$(7.5) \quad 0 \rightarrow \Omega_{F/\mathbb{Q}}^1 \otimes_F NK_{-1}(R_F) \rightarrow NK_0(R_F) \rightarrow H_{\text{cdh}}^1(R_F, \Omega_{R_F/F}^1) \otimes t\mathbb{Q}[t] \rightarrow 0$$

Using (7.4) and 0.7 again, we see that the sequence (7.5) is isomorphic to the sum

$$\begin{aligned} (0 \rightarrow \Omega_{F/\mathbb{Q}}^1 \otimes H_{\text{cdh}}^1(R, \mathcal{O}) \otimes t\mathbb{Q}[t] &\xrightarrow{\simeq} \Omega_{F/\mathbb{Q}}^1 \otimes H_{\text{cdh}}^1(R, \mathcal{O}) \otimes t\mathbb{Q}[t] \rightarrow 0 \rightarrow 0) \\ &\oplus \\ (0 \rightarrow 0 \rightarrow F \otimes H_{\text{cdh}}^1(R, \Omega^1) \otimes t\mathbb{Q}[t] &\xrightarrow{\simeq} F \otimes H_{\text{cdh}}^1(R, \Omega^1) \otimes t\mathbb{Q}[t] \rightarrow 0) \end{aligned}$$

For example, for  $R_F := F[x, y, z]/(z^2 + y^3 + x^{10} + x^7 y)$  the results of [6] show that:

$$NK_{-1}(R_F) = F \otimes t\mathbb{Q}[t]$$

$$NK_0(R_F) = \Omega_{F/\mathbb{Q}}^1 \otimes t\mathbb{Q}[t] \cong \bigoplus_{p=1}^{\text{tr. deg}(F)} F \otimes t\mathbb{Q}[t].$$

In other words, both typical pieces  $TK_{-1}(R_F)$  and  $TK_0(R_F)$  are  $F$ -vectorspaces, but while  $\dim_F TK_{-1}(R_F) = 1$  for all  $F$ , any cardinal number  $\kappa$  can be realized as  $\dim_F TK_0(R_F)$  for an appropriate  $F$ .

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