BASS' NK GROUPS AND *cdh*-FIBRANT HOCHSCHILD HOMOLOGY.

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Abstract. The K-theory of a polynomial ring $R[t]$ contains the K-theory of R as a summand. For R commutative and containing $\mathbb{Q}$, we describe $K_*(R[t])/K_*(R)$ in terms of Hochschild homology and the cohomology of Kähler differentials for the *cdh* topology.

We use this to address Bass' question, on whether $K_n(R) = K_n(R[t])$ implies $K_n(R) = K_n(R[t_1, t_2])$. We provide an example of a ring R for which this fails; on the other hand, we show that the answer to this question is affirmative when R is essentially of finite type over the complex numbers.

In 1972, H. Bass posed the following question (see [2], question $(\mathrm{VI})_n$):

Does $K_n(R) = K_n(R[t])$ imply that $K_n(R) = K_n(R[t_1, t_2])$?

One can phrase the question in terms of Bass' Nil groups as follows: "Does $NK_n(R) = 0$ imply that $N^2K_n(R) = 0$?"

The notation NK_n, introduced in [1], is defined more generally for any functor F from rings to an abelian category; $NF(R)$ is the kernel of the map $F(R[t]) \to F(R)$ induced by evaluation at $t = 0$, and $N^2F = N(NF)$. Bass' question was inspired by Traverso's theorem [31], from which it follows that $N\,\mathrm{Pic}(R) = 0$ implies $N^2\,\mathrm{Pic}(R) = 0$.

In this paper, we give a fairly complete answer to Bass' question for rings containing $\mathbb{Q}$. The answer breaks up into cases:

Theorem 0.1. *a) For any number field K, there exists a normal surface singularity R over K such that $K_0(R) = K_0(R[t])$ but $K_0(R) \neq K_0(R[t_1, t_2])$.*

b) Suppose R is essentially of finite type over a field of infinite transcendence degree over $\mathbb{Q}$. Then $NK_n(R) = 0$ implies that R is K_n-regular and, in particular, that $K_n(R) = K_n(R[t_1, t_2])$.

The proof of both these results relies on methods developed in [5] and [8], which allow us to compute the groups N^pK_n in terms of certain homological functors derived from cyclic and Hochschild homology (see Theorem 5.5). One consequence of our computation is the following structural result, implying that it suffices to compute the groups NK_n for various n to determine the groups N^pK_n.

Theorem 0.2. *Let R be a commutative ring containing $\mathbb{Q}$.*

a) $K_n(R) = K_n(R[t_1, t_2])$ *iff* $NK_n(R) = NK_{n-1}(R) = 0$ *iff* $N^2K_n(R) = 0$.

b) $K_n(R) = K_n(R[t_1, ..., t_p])$ *iff* $NK_q(R) = 0$ *for all q such that* $n - p < q \leq n$.

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c) $K_n(R) = K_n(R[t_1, ..., t_p])$ *iff*

$$NK_n(R) = 0 \text{ and } K_{n-1}(R) = K_{n-1}(R[t_1, ..., t_{p-1}]).$$

This result allows us to reformulate Bass' question as follows:

$$\text{Does } NK_n(R) = 0 \text{ imply that } NK_{n-1}(R) = 0? \tag{0.3}$$

We obtain the promised computation of the groups $NK_n(R)$ in terms of the Hochschild homology of R, and of the *cdh*-cohomology of the higher Kähler differentials Ω^p, both relative to $\mathbb{Q}$. A special case is that for R essentially of finite type over a field of characteristic zero,

$$NK_0(R) \cong \left((R^+/R_{\text{red}}) \oplus \bigoplus_{p=1}^{\dim(R)-1} H^p_{\text{cdh}}(R, \Omega^p) \right) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]. \tag{0.4}$$

Here R^+ is the seminormalization of R_{red}; we show in Proposition 2.5 that $R^+ = H^0_{\text{cdh}}(R, \mathcal{O})$. The length of $H^q_{\text{cdh}}(R, \Omega^p)$ for $q > 0$ is the du Bois invariant $b^{p,q}$ of [29] when both are defined; see (7.8).

Here are two more important examples of our general calculations: the dimension zero case (0.5) and the normal surface case (0.6), which will be used to construct the counterexample (with $b^{0,1} = 1$ and $b^{1,1} = 0$) in Theorem 0.1.

Example 0.5. If $\dim(R) = 0$ then we get $NK_n(R) \cong HH_{n-1}(R, I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$ for all n, where I is the nilradical of R. It is illuminating to compare this with Goodwillie's Theorem [15], which implies that $NK_n(R) \cong NK_n(R, I) \cong NHC_{n-1}(R, I)$. The identification comes from the standard observation (1.2) that the map $HH_* \to HC_*$ induces $NHC_*(R, I) \cong HH_*(R, I) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$.

Theorem 0.6. *Let R be normal domain of dimension 2 and essentially of finite type over $\mathbb{Q}$. Then*

a) $NK_0(R) \cong H^1_{\text{cdh}}(R, \Omega^1) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$, *and*
b) $NK_{-1}(R) \cong H^1_{\text{cdh}}(R, \mathcal{O}) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]$.

As (0.4) illustrates, the groups $NK_n(R)$ have a natural bigraded structure when $\mathbb{Q} \subset R$, and it is convenient to take advantage of this bigrading in stating our results. The bigrading comes from the λ-filtration and the homothety operations, and will be explained in Sections 1 and 5; the general decomposition has the form:

$$NK_n^{(i)}(R) \cong TK_n^{(i)}(R) \otimes_{\mathbb{Q}} t\mathbb{Q}[t]. \tag{0.7}$$

Here i is the Adams weight, and $TK_n^{(i)}$ denotes the typical piece of $NK_n^{(i)}(R)$ under the decomposition by the action of homothety operators.

When $\dim(R) = 2$, our calculations for small n are summarized in Table 1 below; the general result is stated in Theorem 5.1 and Corollary 5.2.

Here is an overview of this paper: Section 1 reviews the well known bigrading on the Hochschild and cyclic homology of $R[t]$ (and $X \times \mathbb{A}^1$), and Section 2 reviews the *cdh*-fibrant analogue. Section 3 describes the sheaf cohomology of the fibers $\mathcal{F}_{HH}(X)$, $\mathcal{F}_{HC}(X)$, etc. of $HH(X) \to \mathbb{H}_{\text{cdh}}(X, HH)$, etc. In Section 4 we use these fibers to prove Theorem 0.2, by relating $NK_{n+1}(X)$ to $H^{-n}\mathcal{F}_{HH}(X)$. We also show (easily) that Bass' question is negative for schemes in Lemma 4.5.

In Section 5, we give the detailed computations of the typical pieces $TK_n^{(i)}(R)$ needed to establish (0.4), Theorem 0.6 and Table 1; these computations employ the

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$
$TK_3^{(i)}(R)$	0	$HH_2^{(1)}(R)$	$\mathrm{tors}\,\Omega^2_R$	$\Omega^3_{\mathrm{cdh}}(R)/\Omega^3_R$	$H^1_{\mathrm{cdh}}\Omega^4$	0
$TK_2^{(i)}(R)$	0	$\mathrm{tors}\,\Omega^1_R$	$\Omega^2_{\mathrm{cdh}}(R)/\Omega^2_R$	$H^1_{\mathrm{cdh}}\Omega^3$	0	
$TK_1^{(i)}(R)$	$\mathrm{nil}(R)$	$\Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R$	$H^1_{\mathrm{cdh}}\Omega^2$	0		
$TK_0^{(i)}(R)$	R^+/R	$H^1_{\mathrm{cdh}}\Omega^1$	0			
$TK_{-1}^{(i)}(R)$	$H^1_{\mathrm{cdh}}\mathcal{O}$	0				
$TK_{-2}(R)$	0					

Table 1. The groups $TK_n^{(i)}(R)$ for $n \le 3$, $\dim(R) = 2$.

main result of [9]. In Section 6, we prove Theorem 0.1(b), that the answer to Bass' question is positive provided we are working over a sufficiently large base field.

The final three sections (7–9) concern the construction of the counterexample mentioned in Theorem 0.1(a). The necessary calculations of the du Bois invariants (that is, *cdh*-cohomology groups of differentials) are provided in Sections 7 and 8, drawing on Wahl's paper [33]; it is then an easy step to obtain the actual example, which we do in Theorem 9.1. Section 9 also contains a closer look at the Bass Nil groups of normal surfaces over general fields of characteristic zero.

Notation. All rings considered in this paper should be assumed to be commutative and noetherian, unless otherwise stated. Throughout this paper, k denotes a field of characteristic 0 and F is a field containing k as a subfield. We write Sch/k for the category of separated schemes essentially of finite type over k. If H is a functor on Sch/k and R is an algebra essentially of finite type, we occasionally write $H(R)$ for $H(\mathrm{Spec}\,R)$. For example, $H^*_{\mathrm{cdh}}(R,\Omega^i)$ is used for $H^*_{\mathrm{cdh}}(\mathrm{Spec}\,R,\Omega^i)$.

If H is a functor from Sch/k to spectra, (co)chain complexes, or abelian groups that commutes (up to natural equivalence) with inverse limits of schemes over diagrams with affine transition morphisms whenever it is defined (as for example K, HH, $\mathbb{H}_{\mathrm{cdh}}(-,HH)$, and $\mathcal{F}_{HH}$), then for any k-algebra R, we abuse notation and write $H(R)$ for the direct limit of the $H(R_\alpha)$ taken over all subrings R_α of R of finite type over k. (If R is essentially of finite type but not of finite type, the two definitions of $H(R)$ agree up to canonical isomorphism.) In particular, we will use expressions like $\mathbb{H}_{\mathrm{cdh}}(R,HH)$ for general commutative $\mathbb{Q}$-algebras even though we do not define the *cdh*-topology for arbitrary $\mathbb{Q}$-schemes.

We use cohomological indexing for all chain complexes in this paper; for a complex C, $C[p]^q = C^{p+q}$. For example, the Hochschild, cyclic, periodic, and negative cyclic homology of schemes over a field k can be defined using the Zariski hypercohomology of certain presheaves of complexes; see [40] and [5, 2.7] for precise definitions. We shall write these presheaves as $HH(/k)$, $HC(/k)$, $HP(/k)$ and $HN(/k)$, respectively, omitting k from the notation if it is clear from the context.

It is well known that there is an Eilenberg-Mac Lane functor $C \mapsto |C|$ from chain complexes of abelian groups to spectra, and from presheaves of chain complexes of abelian groups to presheaves of spectra. This functor sends quasi-isomorphisms of complexes to weak homotopy equivalences of spectra, and satisfies $\pi_n(|C|) = H^{-n}(C)$. For example, applying π_n to the Chern character $K \to |HN|$ yields maps

$K_n(R) \to H^{-n}HN(R) = HN_n(R)$. In this spirit, we will use descent terminology for presheaves of complexes.

Finally, we set $V = t\mathbb{Q}[t]$, the vector space used to decompose functors into typical pieces, which was introduced in (0.4) and (0.7). We also set $dV = \Omega^1_{\mathbb{Q}[t]/\mathbb{Q}}$. Observe that $d : V \to dV$ is an isomorphism.

1. The bigrading on NHH and NHC

Recall that k denotes a field of characteristic 0. In this section, we consider the Hochschild and cyclic homology of polynomial extensions of commutative k-algebras. No great originality is claimed. Throughout, we will use the chain level Hodge decompositions $HH = \prod_{i\ge 0} HH^{(i)}$ and $HC = \prod_{i\ge 0} HC^{(i)}$.

Recall the notation $V = t\mathbb{Q}[t]$ and $dV = \Omega^1_{\mathbb{Q}[t]}$. As $d : V \to dV$ is an isomorphism, the Künneth formula for Hochschild homology yields

$$NHH_n^{(i)}(R) \cong \left(HH_n^{(i)}(R) \otimes V\right) \oplus \left(HH_{n-1}^{(i-1)}(R) \otimes dV\right). \tag{1.1}$$

From the exact SBI sequence $0 \to NHC_{n-1} \xrightarrow{B} NHH_n \xrightarrow{I} NHC_n \to 0$ (see [39, 9.9.1]), and induction on n, the map I induces canonical isomorphisms for each i:

$$NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes V. \tag{1.2}$$

Remark 1.3. Both (1.1) and (1.2) generalize to non-affine quasi-compact schemes X over k. Indeed, NHH and NHC satisfy Zariski descent because HH and HC do and because, for any open cover $\{U_i \to X\}$, the collection $\{U_i \times \mathbb{A}^1 \to X \times \mathbb{A}^1\}$ is also a cover. Thus we have

$$\begin{aligned} NHH^{(i)}(X) &\cong \mathbb{H}_{Zar}(X, NHH^{(i)}) \\ &\cong \mathbb{H}_{Zar}(X, HH^{(i)}) \otimes V \quad \oplus \quad \mathbb{H}_{Zar}(X, HH^{(i-1)})[1] \otimes dV \\ &\cong HH^{(i)}(X) \otimes V \quad \oplus \quad HH^{(i-1)}(X)[1] \otimes dV, \end{aligned}$$

and $NHC^{(i)}(X) = \mathbb{H}_{Zar}(X, NHC^{(i)}) \cong \mathbb{H}_{Zar}(X, HH^{(i)}) \otimes V \cong HH^{(i)}(X) \otimes V$.

It is easy to iterate the construction $F \mapsto NF$. For example, we see from (1.1) and (1.2) that

$$N^2HC_n^{(i)}(R) \cong \left(HH_n^{(i)}(R) \otimes V \otimes V\right) \oplus \left(HH_{n-1}^{(i-1)}(R) \otimes V \otimes dV\right). \tag{1.4}$$

By induction, we see that $HH_{n-j}^{(i-j)}(R) \otimes V^{\otimes(p-j)} \otimes (dV)^{\otimes j}$ will occur $\binom{p-1}{j}$ times as a summand of $N^pHC_n^{(i)}(R)$ for all $j \ge 0$. We may write this as the formula:

$$N^pHC_n^{(i)}(R) \cong \bigoplus_{j=0}^{p-1} HH_{n-j}^{(i-j)}(R) \otimes_k \wedge^j k^{p-1} \otimes V^{\otimes(p-j)} \otimes (dV)^{\otimes j}. \tag{1.5}$$

Cartier operations on NHH and NHC. Let $W(R)$ denote the ring of big Witt vectors over R; it is well known that in characteristic 0 we have $W(R) \cong \prod_1^\infty R$. Cartier showed in [3] that the endomorphism ring $\mathrm{Cart}(R)$ of the additive functor underlying W consists of column-finite sums $\sum V_m[r_{mn}]F_n$, using the *homotheties* $[r]$ (for $r \in R$), and the Verschiebung V_m and Frobenius F_m operators.

We will be interested in the intermediate subring $\mathrm{Carf}(R)$ of all row and column-finite sums $\sum V_m[r_{mn}]F_n$. As observed in [11, 2.14], there is an equivalence between the category of R-modules and the category of continuous $\mathrm{Carf}(R)$-modules given by the constructions in the following example:

Example 1.6. Set $V = t\mathbb{Q}[t]$. If M is any R-module, $N = M \otimes V$ is a continuous $\mathrm{Carf}(R)$-module via the formulas:

$$[r]t^i = r^i t^i, \quad V_m(t^i) = t^{mi}, \quad F_m(t^i) = \begin{cases} mt^{i/m} & \text{if } m|i, \\ 0 & \text{else.} \end{cases}$$

The subring $W(R) = \prod_1^\infty R$ acts on $M \otimes V$ by $(r_1, ..., r_n, ...) * \sum m_i t^i = \sum (r_i m_i) t^i$. Conversely, every continuous $\mathrm{Carf}(R)$-module N has a "typical piece" M, defined as the simultaneous eigenspace $\{x \in N : [r]x = rx,\ r \in R\}$, and $N \cong M \otimes V$.

Recall that we can define operators $[r]$ on $NHH_n(R)$ and $NHC_n(R)$, associated to the endomorphisms $t \mapsto rt$ of $R[t]$. There are also operators V_m and F_m, defined via the ring inclusions $R[t^m] \subset R[t]$ and their transfers. These operations commute with the Hodge decomposition. The following result follows immediately from [11, 4.11] using the observation that everything commutes with Adams operations.

Proposition 1.7. *The operators* $[r]$, V_m *and* F_m *make each* $NHC_n^{(i)}(R)$ *into a continuous* $\mathrm{Carf}(R)$*-module, and hence a* $W(R)$*-module. The* R*-module* $HH_n^{(i)}(R)$ *is its typical piece, and the canonical isomorphism* $NHC_n^{(i)}(R) \cong HH_n^{(i)}(R) \otimes V$ *of* (1.2) *is an isomorphism of* $\mathrm{Carf}(R)$*-modules, the module structure on the right being given in Example 1.6.*

A similar structure theorem holds for $NHH_n(R)$ and its Hodge components, using (1.1). However, it uses a non-standard R-module structure on the typical piece $HH_n(R) \oplus HH_{n-1}(R)$; see [11, 3.3] for details.

Remark 1.7.1. If X is a scheme and $R = H^0(X, \mathcal{O})$, then the conclusions of Proposition 1.7 still hold. This is because $NHC_n^{(i)}(X)$ is a continuous $\mathrm{Carf}(R)$-module, isomorphic to $HH_n^{(i)}(X) \otimes V$.

This scheme version of 1.7 is not stated in [11], which was written before the cyclic homology of schemes was developed in [40]. However, the proof in [11] is easily adapted. Since the operators V_m, F_m and $[r]$ are defined on the underlying chain complexes in [11, 4.1], they extend to operations on the Hochschild and cyclic homology of schemes. The identities required to obtain continuous $\mathrm{Carf}(R)$-module structures all come from the Künneth formula for the shuffle product on the chain complexes (see [11, 4.3]), so they also hold for the homology of schemes.

2. *cdh*-FIBRANT HH AND NHC

Now fix a field F containing k; all schemes will lie in the category Sch/F (essentially of finite type over F), in order to use the *cdh* topology on Sch/F of [30]. All rings will be commutative F-algebras; because they are filtered direct limits of affine schemes in Sch/F, we can consider their *cdh*-cohomology.

If C is any (pre-)sheaf of cochain complexes on Sch/F, we can form the *cdh*-fibrant replacement $X \mapsto \mathbb{H}_{\mathrm{cdh}}(X, C)$ and write $\mathbb{H}^n_{\mathrm{cdh}}(X, C)$ for the nth cohomology of this complex. For example, the *cdh*-fibrant replacement of a *cdh* sheaf C (concentrated in degree zero) is just an injective resolution, and $\mathbb{H}^n_{\mathrm{cdh}}(X, C)$ is the usual cohomology of the *cdh* sheaf associated to C.

Hochschild and cyclic homology will be taken relative to k. For $C = HH^{(i)}$, it was shown in [8, Theorem 2.4] that

$$\mathbb{H}_{\mathrm{cdh}}(X, HH^{(i)}) \cong \mathbb{H}_{\mathrm{cdh}}(X, \Omega^i)[i]. \tag{2.1}$$

This has the following consequence for $C = NHH^{(i)}$ and $NHC^{(i)}$.

Lemma 2.2. *Let $H^{(i)}$ denote either $HH^{(i)}$ or $HC^{(i)}$, taken relative to a subfield k of F. Then $\mathbb{H}_{\mathrm{cdh}}(X \times \mathbb{A}^1, H^{(i)}) = \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)}) \oplus \mathbb{H}_{\mathrm{cdh}}(X, NH^{(i)})$, and:*

$$\mathbb{H}_{\mathrm{cdh}}(X, NHH^{(i)}) \cong \big(\mathbb{H}_{\mathrm{cdh}}(X, \Omega^i)[i] \otimes V\big) \oplus \big(\mathbb{H}_{\mathrm{cdh}}(X, \Omega^{i-1})[i] \otimes dV\big);$$

$$\mathbb{H}_{\mathrm{cdh}}(X, NHC^{(i)}) \cong \mathbb{H}_{\mathrm{cdh}}(X, \Omega^i)[i] \otimes V.$$

Proof. The product of any smooth *cdh* cover of X with $\mathbb{A}^1$ is a smooth *cdh* cover of $X \times \mathbb{A}^1$, and both $HH^{(i)}$ and $HC^{(i)}$ satisfy *scdh*-descent by [8, Thm. 2.4], so:

$$\mathbb{H}_{\mathrm{cdh}}(X \times \mathbb{A}^1, H^{(i)}) \cong \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)}(- \times \mathbb{A}^1)) \cong \mathbb{H}_{\mathrm{cdh}}(X, H^{(i)}) \oplus \mathbb{H}_{\mathrm{cdh}}(X, NH^{(i)}).$$

The displayed formulas follow from (1.1), (1.2) and (2.1), using the fact that $- \otimes V$ commutes with $\mathbb{H}_{\mathrm{cdh}}$. □

Remark 2.2.1. If R is any commutative F-algebra, the formulas of Lemma 2.2 hold for $X = \mathrm{Spec}(R)$ by naturality. This is because we may write $R = \varinjlim R_\alpha$, where R_α ranges over subrings of finite type over F, and $\mathbb{H}_{\mathrm{cdh}}(X, -) = \varinjlim \mathbb{H}_{\mathrm{cdh}}(\mathrm{Spec}(R_\alpha), -)$.

Corollary 2.3. *If $X = \mathrm{Spec}(R)$ is in Sch/F, the modules $\mathbb{H}^n_{\mathrm{cdh}}(X, HH^{(i)})$ and $\mathbb{H}^n_{\mathrm{cdh}}(X, NHC^{(i)})$ are zero unless $0 \le n + i < \dim(X)$ and $i \ge 0$.*

If $n \ge \dim(X)$ and $n > 0$ then $\mathbb{H}^n_{\mathrm{cdh}}(X, HH) = 0$.

Proof. Because $\mathbb{H}^n_{\mathrm{cdh}}(X, \Omega^i)[i] = H^{i+n}_{\mathrm{cdh}}(X, \Omega^i)$, this follows from (2.1), Lemma 2.2 and the fact that $H^n_{\mathrm{cdh}}(X, \Omega^i) = 0$ for $n \ge \dim(X)$, $n > 0$. This bound is given in [5, 6.1] for $i = 0$, and in [8, 2.6] for general i. □

Here is a useful bound on the cohomology groups appearing in Lemma 2.2. Given X, let Q denote the total ring of fractions of X_{red}; it is a finite product of fields Q_j, and we let e denote the maximum of the transcendence degrees $\mathrm{tr.deg}(Q_j/k)$.

Lemma 2.4. *Let X be in Sch/F. If $i > e$ then $H^n_{\mathrm{cdh}}(X, \Omega^i) = 0$ for all n.*

Proof. By [23, 12.24], we may assume X reduced. Since we may write X as an inverse limit of a sequence of affine morphisms with the same ring of total factions Q, and *cdh*-cohomology commutes with such an inverse limit, we may also assume that X is of finite type over F. This implies that $e = \dim(X) + \mathrm{tr.deg}(F/k)$.

The result is clear if $\dim(X) = 0$, since $H^n_{\mathrm{cdh}}(X, -) = H^n_{Zar}(X, -)$ in that case. Proceeding by induction on $\dim(X)$, choose a resolution of singularities $X' \to X$ and observe that the singular locus Y and $Y \times_X X'$ have smaller dimension. The hypothesis implies that $\Omega^i = 0$ on X'_{Zar}, so $H^n_{\mathrm{cdh}}(X', \Omega^i) = 0$ by [8, 2.5]. The result now follows by induction from the Mayer-Vietoris sequence of [30, 12.1]. □

If R is a commutative ring, we write R_{red} and R^+ for the associated reduced ring and the seminormalization of R_{red}, respectively. These constructions are natural with respect to localization, so that we may form the seminormalization X^+ of X_{red} for any scheme X. Because $X^+ \to X$ is a universal homeomorphism, we have $H^*_{\mathrm{cdh}}(X, -) \cong H^*_{\mathrm{cdh}}(X^+, -)$ for every X in Sch/k, for any field k of arbitrary characteristic. In particular, $H^n_{\mathrm{cdh}}(X, \mathcal{O}) \cong H^n_{\mathrm{cdh}}(X^+, \mathcal{O})$ for all n. The case $n = 0$ is of special interest:

Proposition 2.5. *For any algebra R, we have $H^0_{\rm cdh}(\operatorname{Spec} R, \mathcal{O}) = R^+$. Moreover, for every X in Sch/F we have $H^0_{\rm cdh}(X, \mathcal{O}) = \mathcal{O}(X^+)$.*

Proof. We may assume R and X are reduced. Writing $R = \varinjlim R_\alpha$ as in Remark 2.2.1, we have $R^+ = \varinjlim R^+_\alpha$ and $H^0_{\rm cdh}(R, \mathcal{O}) = \varinjlim H^0_{\rm cdh}(R_\alpha, \mathcal{O})$, so we may assume that R is of finite type. Thus the second assertion implies the first. Since $H^0_{\rm cdh}(-, \mathcal{O})$ and $\mathcal{O}(-^+)$ are Zariski sheaves, it suffices to consider the case when X is affine.

Let $X = \operatorname{Spec} R$ be in Sch/F, with R reduced. There is an injection $R \to Q$ with Q regular (for example, Q could be the total quotient ring of R). By [5, 6.3], $H^0_{\rm cdh}(\operatorname{Spec} Q, \mathcal{O}) = Q$, so R injects into $H^0_{\rm cdh}(\operatorname{Spec} R, \mathcal{O})$. This implies that $\mathcal{O}_{\rm red}$ is a separated presheaf for the *cdh* topology on Sch/F. Thus, the ring $H^0_{\rm cdh}(X, \mathcal{O})$ is the direct limit over all cdh-covers $p : U \to X$ of the Čech H^0. (See [SGA4, 3.2.3].)

Fix an element $b \in H^0_{\rm cdh}(\operatorname{Spec} R, \mathcal{O})$ and represent it by $b \in \mathcal{O}(U)$ for some *cdh* cover $U \to X$. Now recall from [23, 12.28] or [30, 5.9] that we may assume, by refining the *cdh* cover $U \to X$, that it factors as $U \to X' \to X$ where $X' \to X$ is proper birational *cdh* cover and $U \to X'$ is a Nisnevich cover. If the images of $b \in \mathcal{O}(U)$ agree in $U \times_X U$, *i.e.,* b is a Čech cycle for U/X, then its images agree in $U \times_{X'} U$, *i.e.,* it is a Čech cycle for U/X'. But by faithfully flat descent, b descends to an element of $\mathcal{O}(X')$. Thus we can assume that U is proper and birational over X.

Next, we can assume that the Nisnevich cover $p : U \to X$ is finite, surjective and birational. Indeed, since p is proper and birational we may consider the Stein factorization $U \xrightarrow{q} Y \xrightarrow{r} X$. By [EGAIII, 4.3] or [20, III.11.5 & proof], $q_*(\mathcal{O}_U) = \mathcal{O}_Y$ and r is finite surjective and birational. By [30, 5.8], r is also a *cdh* cover. Because $q_*(O_U) = O_Y$, the canonical map $\mathcal{O}_Y(Y) \to q_*(\mathcal{O}_U)(Y) = \mathcal{O}_U(U)$ is an isomorphism. Hence b descends to an element of $\mathcal{O}(Y)$. By Lemma 2.6, b lies in the seminormalization of R. ☐

Lemma 2.6. *Let A be a seminormal ring and B a ring between A and its normalization. Then the Čech complex $A \to B \to B \otimes_A B$ is exact.*

Proof. We use Traverso's description of the seminormalization (see [31, p. 585]): the seminormalization of a ring A inside a ring B is

$$A^+ = \{b \in B \mid (\forall P \in \operatorname{Spec} A)\ b \in A_P + \operatorname{rad}(B_P)\}.$$

Let $b \in B$ such that $1 \otimes b = b \otimes 1$. We have to show that $b \in A_P + \operatorname{rad}(B_P)$, for all primes P of A. Let $J = \operatorname{rad}(B_P)$; since B_P/J is faithfully flat over the field A_P/P, the image of b in B_P/J lies in A_P/P by flat descent. That is, $b \in A_P + J$, as required. ☐

Remark 2.7. Even if X is affine seminormal, it can still happen that $H^i_{\rm cdh}(X, \mathcal{O}) \neq 0$ for some $i > 0$. For example, if R denotes the subring $F[x, g, yg]$ of $F[x, y]$ for $g = x^3 - y^2$ then it is easy to show that R is seminormal and that $H^1_{\rm cdh}(\operatorname{Spec}(R), \mathcal{O}) = F$, because the normalization of R is $F[x, y]$ and the conductor ideal is $gF[x, y]$. For another example, the normal ring of Theorem 0.1 has $H^1_{\rm cdh}(X, \mathcal{O}) \neq 0$, by Theorems 0.2(a) and 0.6(b).

3. The fibers $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$

If C is a presheaf of complexes on Sch/F, we write $\mathcal{F}_C$ for the shifted mapping cone of $C \to \mathbb{H}_{\mathrm{cdh}}(-, C)$, so that we have a distinguished triangle:

$$\mathbb{H}_{\mathrm{cdh}}(X, C)[-1] \to \mathcal{F}_C(X) \to C(X) \to \mathbb{H}_{\mathrm{cdh}}(X, C) \tag{3.1}$$

Example 3.1.1. When C is concentrated in degree 0 we have $H^n\mathcal{F}_C = 0$ for all $n < 0$. For $C = \mathcal{O}$ and $X = \mathrm{Spec}(R)$, we see from Proposition 2.5 that $H^0\mathcal{F}_{\mathcal{O}}(X) = \mathrm{nil}(R)$, $H^1\mathcal{F}_{\mathcal{O}}(X) = R^+/R$, and $H^n\mathcal{F}_{\mathcal{O}}(X) = H^{n-1}_{\mathrm{cdh}}(X, \mathcal{O})$ for $n \geq 2$. Note that, if $X = \mathrm{Spec}\, R \in \mathrm{Sch}/F$, then $H^n\mathcal{F}_{\mathcal{O}}(X) = 0$ for $n > \dim(X)$ by [5, 6.1].

We now consider the Hochschild and cyclic homology complexes, taken relative to a subfield k of F. For legibility, we write $\mathcal{F}^{(i)}_{HH}$ for $\mathcal{F}_{HH^{(i)}}$, etc. By the usual homological yoga, $\mathcal{F}_{HH}$ is the direct sum of the $\mathcal{F}^{(i)}_{HH}$, $i \geq 0$, and similarly for $\mathcal{F}_{HC}$.

Example 3.1.2. If X is smooth over F then $\mathcal{F}_{HH}(X) \simeq 0$ by [8, 2.4].

Lemma 2.2 and Remarks 2.2.1 and 1.3 imply the following analogue for $N\mathcal{F}$.

Lemma 3.2. *If X is in Sch/F, or if $X = \mathrm{Spec}(R)$ for an F-algebra R, we have quasi-isomorphisms:*

$$N\mathcal{F}^{(i)}_{HH}(X) \cong \left(\mathcal{F}^{(i)}_{HH}(X) \otimes V\right) \oplus \left(\mathcal{F}^{(i-1)}_{HH}(X)[1] \otimes dV\right);$$
$$N\mathcal{F}^{(i)}_{HC}(X) \cong \mathcal{F}^{(i)}_{HH}(X) \otimes V.$$

Mimicking the argument that establishes (1.4) and (1.5) yields:

Corollary 3.3. *If X is in Sch/F, or if $X = \mathrm{Spec}(R)$ for an F-algebra R,*

$$N^2\mathcal{F}^{(i)}_{HC}(X) \cong \left(\mathcal{F}^{(i)}_{HH}(X) \otimes V \otimes V\right) \oplus \left(\mathcal{F}^{(i-1)}_{HH}(X)[1] \otimes V \otimes dV\right)$$

and

$$N^p\mathcal{F}^{(i)}_{HC}(X) \cong \bigoplus_{j=0}^{p-1} \mathcal{F}^{(i-j)}_{HH}(X)[j] \otimes_k \wedge^j k^{p-1} \otimes V^{\otimes(p-j)} \otimes (dV)^{\otimes j}.$$

The cohomology of the typical pieces $\mathcal{F}^{(i)}_{HH}(R)$ is given as follows.

Lemma 3.4. *If R is an F-algebra and $i \geq 0$, then there is an exact sequence:*

$$0 \to H^{-i}\mathcal{F}^{(i)}_{HH}(R) \to \Omega^i_R \to H^0_{\mathrm{cdh}}(R, \Omega^i) \to H^{1-i}\mathcal{F}^{(i)}_{HH}(R) \to 0.$$

For $n \neq i, i-1$ we have:

$$H^{-n}\mathcal{F}^{(i)}_{HH}(R) \cong \begin{cases} HH^{(i)}_n(R) & \text{if } i < n, \\ H^{i-n-1}_{\mathrm{cdh}}(R, \Omega^i) & \text{if } i \geq n+2. \end{cases}$$

Proof. As in Remark 2.2.1, we may assume R is of finite type. Since $HH^{(i)}_i(R) = \Omega^i_R$ for all $i \geq 0$, and $HH^{(i)}_n(R) = 0$ when $i > n$ (see [39, 9.4.15] or [22, 4.5.10]), it suffices to use (2.1) and to observe that $\mathbb{H}^{-n}_{\mathrm{cdh}}(R, HH^{(i)}) = H^{i-n}_{\mathrm{cdh}}(R, \Omega^i)$ vanishes when $n > i$. □

Example 3.5. Let $X = \operatorname{Spec}(R)$ be in Sch$/F$. Since $HH^{(0)} = \mathcal{O}$, $\mathcal{F}^{(0)}_{HH}(R)$ is described in Example 3.1.1. Applying Corollary 2.3 and Lemma 3.4 for $i > 0$, and using [8, 2.6] to bound the terms, we see that if $d = \dim(R)$ then $H^n\mathcal{F}_{HH}(X) = 0$ for $n > d$. If $d = 1$, then the only nonzero positive cohomology of $\mathcal{F}_{HH}$ is $H^1\mathcal{F}_{HH}(R) = R^+/R$; if $d > 1$, we have:

$$\begin{aligned} H^1\mathcal{F}_{HH}(R) &\cong \quad (R^+/R) \quad \oplus H^1_{\mathrm{cdh}}(X,\Omega^1) \oplus \cdots \oplus H^{d-1}_{\mathrm{cdh}}(X,\Omega^{d-1}),\\ H^2\mathcal{F}_{HH}(R) &\cong H^1_{\mathrm{cdh}}(X,\mathcal{O}) \oplus H^2_{\mathrm{cdh}}(X,\Omega^1) \oplus \cdots \oplus H^{d-1}_{\mathrm{cdh}}(X,\Omega^{d-2}),\\ &\vdots \qquad\quad \vdots\\ H^d\mathcal{F}_{HH}(R) &\cong H^{d-1}_{\mathrm{cdh}}(X,\mathcal{O}). \end{aligned}$$

Example 3.6. When R is essentially of finite type over F and $\operatorname{tr.deg}(F/k) < \infty$, $H^m\mathcal{F}_{HH}(R)$ is Hochschild homology for large negative m. To see this, observe that $e = \operatorname{tr.deg}(R/k)$, the maximum transcendence degree of the residue fields of R at its minimal primes, is finite. Using Lemmas 2.4 and 3.4, we get $H^{-n}\mathcal{F}^{(i)}_{HH}(R) = 0$ and $H^{-n}\mathcal{F}^{(n)}_{HH}(R) = \Omega^n_R$ for $i > n > e$, and hence

$$H^{-n}\mathcal{F}_{HH}(R) \cong HH_n(R) \text{ for all } n > e.$$

If $R = k \oplus R_1 \oplus R_2 \oplus \dots$ is graded, and $\widetilde{HC}_*(R) = HC_*(R)/HC_*(k)$, it is well known that the map $\widetilde{HC}_*(R) \xrightarrow{S} \widetilde{HC}_{*-2}(R)$ is zero. In Lemma 3.8 below, we prove a similar property for $\mathcal{F}_{HH}$ and $\mathcal{F}_{HC}$, which we derive from Lemma 3.2 using the following trick.

Standard Trick 3.7. If R is a non-negatively graded algebra, there is an algebra map $\nu : R \to R[t]$ sending $r \in R_n$ to rt^n. The composition of ν with evaluation at $t = 0$ factors as $R \to R_0 \to R$, and so if H is a functor on algebras taking values in abelian groups, then the composition $H(R) \xrightarrow{\nu} H(R[t]) \xrightarrow{t=0} H(R)$ is zero on the kernel $\widetilde{H}(R)$ of $H(R) \to H(R_0)$. Similarly, the composition of ν with evaluation at $t = 1$ is the identity. That is, ν maps $\widetilde{H}(R)$ isomorphically onto a summand of $NH(R)$, and $\widetilde{H}(R)$ is in the image of $(t = 1) : NH(R) \to H(R)$.

Lemma 3.8. *If $R = k \oplus R_1 \oplus \cdots$ is a graded algebra then for each m the map $\pi_m\mathcal{F}_{HC}(R) \xrightarrow{S} \pi_{m-2}\mathcal{F}_{HC}(R)$ is zero, and there is a split short exact sequence:*

$$0 \to \pi_{m-1}\mathcal{F}_{HC}(R) \xrightarrow{B} \pi_m\mathcal{F}_{HH}(R) \xrightarrow{I} \pi_m\mathcal{F}_{HC}(R) \to 0.$$

Similarly, there are split short exact sequences:

$$0 \to \tilde{\mathbb{H}}^{m+1}_{\mathrm{cdh}}(R, HC) \xrightarrow{B} \tilde{\mathbb{H}}^m_{\mathrm{cdh}}(R, HH) \xrightarrow{I} \tilde{\mathbb{H}}^m_{\mathrm{cdh}}(R, HC) \to 0.$$

and

$$0 \to \tilde{\mathbb{H}}^{m-1}_{\mathrm{cdh}}(R, \Omega^{<i}) \xrightarrow{B} \tilde{H}^{m-i}_{\mathrm{cdh}}(R, \Omega^i) \xrightarrow{I} \tilde{\mathbb{H}}^m_{\mathrm{cdh}}(R, \Omega^{\le i}) \to 0.$$

Proof. It suffices to show that I is onto and split. By [8, 2.2], $\mathcal{F}_{HH}(k) = \mathcal{F}_{HC}(k) = 0$, so $\tilde{\mathcal{F}}_{HH} = \mathcal{F}_{HH}$ and $\tilde{\mathcal{F}}_{HC} = \mathcal{F}_{HC}$. By the standard trick 3.7, it suffices to show that the maps $N\pi_m\mathcal{F}_{HH}(R) \to N\pi_m\mathcal{F}_{HC}(R)$ and $N\mathbb{H}^m_{\mathrm{cdh}}(R, HH) \to$

$N\mathbb{H}^m_{\mathrm{cdh}}(R,HC)$ are split surjections. But this is evident from the respective decompositions of their terms in Lemmas 3.2 and 2.2, such as

$$\mathbb{H}_{\mathrm{cdh}}(R,NHC^{(i)}) \simeq H_{\mathrm{cdh}}(R,\Omega^i)[i]\otimes_{\mathbb{Q}} t\mathbb{Q}[t]$$
$$N\mathcal{F}^{(i)}_{HC}(R) \simeq \mathcal{F}^{(i)}_{HH}(R)\otimes_{\mathbb{Q}} t\mathbb{Q}[t]$$
$$N\mathcal{F}_{HC} \simeq \mathcal{F}_{HH}\otimes_{\mathbb{Q}} t\mathbb{Q}[t].$$

The third sequence is obtained from the second one by taking the i^{th} component in the Hodge decomposition, described in Lemma 2.2. □

Example 3.9. Splicing the final sequences of Lemma 3.8 together, we see that the de Rham complexes are exact:

$$0\to k\to R\xrightarrow{d}\tilde{H}^0_{\mathrm{cdh}}(R,\Omega^1)\xrightarrow{d}\tilde{H}^0_{\mathrm{cdh}}(R,\Omega^2)\to\cdots \tag{3.9a}$$

$$0\to H^n_{\mathrm{cdh}}(R,\mathcal{O})\xrightarrow{d}H^n_{\mathrm{cdh}}(R,\Omega^1)\xrightarrow{d}H^n_{\mathrm{cdh}}(R,\Omega^2)\to\cdots, \qquad n>0. \tag{3.9b}$$

An analogous exact sequence

$$\cdots\to\pi_{m-1}\mathcal{F}_{HH}(R)\xrightarrow{d}\pi_m\mathcal{F}_{HH}(R)\xrightarrow{d}\pi_{m+1}\mathcal{F}_{HH}(R)\to\cdots$$

is obtained by splicing the other sequences in Lemma 3.8. Using the interpretation of their Hodge components, described in Lemma 3.4, produces two more exact sequences:

$$0\to\mathrm{nil}(R)\to\mathrm{tors}\,\Omega^1_R\to\mathrm{tors}\,\Omega^2_R\to\mathrm{tors}\,\Omega^3_R\to\cdots \tag{3.9c}$$

$$0\to(R^+/R)\to\Omega^1_{\mathrm{cdh}}(R)/\Omega^1_R\to\Omega^2_{\mathrm{cdh}}(R)/\Omega^2_R\to\cdots. \tag{3.9d}$$

Here we have written $\Omega^i_{\mathrm{cdh}}(R)$ for $H^0_{\mathrm{cdh}}(R,\Omega^i)$, and if R is not reduced then $\mathrm{tors}\,\Omega^i_R$ should be interpreted as the kernel of $\Omega^i_R\to\Omega^i_{\mathrm{cdh}}(R)$.

4. Bass' groups $NK_*(X)$

In this section, we relate algebraic K-theory to our Hochschild and cyclic homology calculations relative to the ground field $k=\mathbb{Q}$. Consider the trace map

$$NK_{n+1}(X)\to NHC_n(X)=NHC_n(X/\mathbb{Q})$$

induced by the Chern character. In the affine case, it is defined in [35]; for schemes it is defined using Zariski descent. As explained in [35], it arises from the Chern character from the spectrum $NK(X)$ to the Eilenberg-Mac Lane spectrum $|NHC(X)[1]|$ associated to the cochain complex $NHC(X)[1]$. Note that our indexing conventions are such that $\pi_{n+1}|NHC(X)[1]|=H^{-n}NHC(X)=NHC_n(X)$.

Proposition 4.1. *Suppose that X is in* Sch/F*, or that $X=\mathrm{Spec}(R)$ for an F-algebra R. Then for all n, the Chern character induces a natural isomorphism*

$$NK_{n+1}(X)\cong H^{-n}\mathcal{F}_{HH}(X)\otimes V.$$

This is an isomorphism of graded R-modules, and even Carf(R)*-modules, identifying the operations $[r]$, V_m and F_m on $NK_*(X)$ with the operations on the right side described in Example 1.6.*

Proof. By Remark 2.2.1, we may suppose $X \in \mathrm{Sch}/F$. By [8, 1.7], the Chern character $K \to HN$ induces weak equivalences $\mathcal{F}_K(X) \simeq |\mathcal{F}_{HC}(X)[1]|$ and $\mathcal{F}_K(X \times \mathbb{A}^1) \simeq |\mathcal{F}_{HC}(X \times \mathbb{A}^1)[1]|$. Since for any presheaf of spectra E we have a natural objectwise equivalence $E(- \times \mathbb{A}^1) \simeq E \times NE$, we obtain a natural weak equivalence from $NK(X)$ to $|N\mathcal{F}_{HC}(X)[1]|$. Now take homotopy groups and apply Lemma 3.2.

As observed in [11, 4.12], the Chern character also commutes with the ring maps used to define the operators $[r]$, V_m, and with the transfer for $R[t^n] \to R[t]$ defining F_m. That is, it is a homomorphism of $\mathrm{Carf}(R)$-modules. Since the transfer is defined via the ring map $R[t] \to M_n(R[t^n])$, followed by Morita invariance, there is no trouble in passing to schemes. □

Corollary 4.2. *For all n, $N^2K_n(X) \cong \big(NK_n(X) \otimes V\big) \oplus \big(NK_{n-1}(X) \otimes dV\big)$, and*

$$N^{p+1}K_n(X) \cong \bigoplus_{j=0}^{p} NK_{n-j}(X) \otimes \wedge^j \mathbb{Q}^p \otimes V^{\otimes(p-j)} \otimes (dV)^{\otimes j}.$$

This holds for every X in Sch/F, as well as for $\mathrm{Spec}(R)$ where R is an arbitrary commutative F-algebra.

Proof. This is immediate from 4.1 and Corollary 3.3. □

Remark 4.2.1. Jim Davis has pointed out (see [10]) that a computation equivalent to 4.2 can also be derived — for arbitrary rings R — from the Farrell-Jones conjecture for the groups $\mathbb{Z}^r$. This particular case is covered by F. Quinn's proof of hyperelementary assembly for virtually abelian groups; see [27].

We now come to one of our main results, which implies Theorem 0.2, and which is immediate from 4.2 and [1, XII(7.3)].

Theorem 4.3. *Suppose that X is in Sch/F, or that $X = \mathrm{Spec}(R)$ for an F-algebra R. Then:*

a) *If $NK_n(X) = NK_{n-1}(X) = 0$ then $N^2K_n(X) = 0$.*
b) *If $NK_n(X) = 0$ and $K_{n-1}(X) = K_{n-1}(X \times \mathbb{A}^p)$ then $K_n(X) = K_n(X \times \mathbb{A}^{p+1})$.*
c) *$K_n(X) = K_n(X \times \mathbb{A}^p)$ if and only if $NK_q(X) = 0$ for all q such that $n - p < q \leq n$.*

Recall that X is called *K_n-regular* if $K_n(X) = K_n(X \times \mathbb{A}^p)$ for all p.

Corollary 4.4. *Suppose that X is in Sch/F, or that $X = \mathrm{Spec}(R)$ for an F-algebra R. Then the following conditions are equivalent:*

a) *X is K_n-regular;*
b) *$NK_n(X) = 0$ and X is K_{n-1}-regular;*
c) *$NK_q(X) = 0$ for all $q \leq n$.*

Remark 4.4.1. This gives another proof of Vorst's Theorem [32, 2.1] (in characteristic 0) that K_n-regularity implies K_{n-1}-regularity, and extends it to schemes.

The assumption that the scheme be affine is essential in Bass' question — here is a non-affine example where the answer is negative.

Negative answer to Bass' question for nonaffine curves. Let X be a smooth projective elliptic curve over a number field k and let L be a nontrivial degree zero line bundle with $L^{\otimes 3}$ trivial. For example, if X is the Fermat cubic $x^3 + y^3 = z^3$, we may take the line bundle associated to the divisor $P - Q$, where $P = (1 : 0 : 1)$ and $Q = (0 : 1 : 1)$.

Lemma 4.5. *Write Y for the nonreduced scheme with the same underlying space as X but with structure sheaf $\mathcal{O}_Y = \mathcal{O}_X \oplus L = \mathrm{Sym}(L)/(L^2)$, that is, L is regarded as a square-zero ideal.*

Then $NK_7(Y) = 0$ but $N^2K_7(Y) \cong NK_6(Y) \otimes dV$ is nonzero.

Proof. Since $\Omega^1_X \cong \mathcal{O}_X$ we see from Lemma 5.3 of [8] that the relative Hochschild homology sheaf $\mathcal{HH}_n(Y, L)$ is: $L^{\otimes 3} \oplus L^{\otimes 5}$ if $n = 4$; $L^{\otimes 5} \oplus L^{\otimes 5}$ if $n = 5$; and $L^{\otimes 5} \oplus L^{\otimes 7}$ if $n = 6$. By Serre duality, $H^*(X, L^{\otimes i}) = 0$ if $3 \nmid i$ (cf. [8, 5.1]). By Zariski descent, this implies that $HH_5(Y, L) \cong H^1(X, \mathcal{HH}_4) \cong H^1(X, L^{\otimes 3}) \cong k$ and $HH_6(Y, L) = 0$. Since $\mathcal{F}_{HH}(Y) \cong HH(Y, L)$, it follows from 4.1 and 4.2 that $NK_7(Y) = 0$ but $NK_6(Y) \cong V$ and $N^2K_7(Y) \cong NK_6(Y) \otimes dV \cong V \otimes dV$. □

We conclude this section by refining Proposition 4.1 and Theorem 4.3 to take account of the Adams/Hodge/λ-decompositions on K-theory and Hochschild homology, and by establishing the triviality of $K^{(i)}_*(X)$ for $i \le 0$.

Recall that by definition, $K^{(i)}_n(X) = \{x \in K_n(X) \otimes \mathbb{Q} : \psi^k(x) = k^i x\}$. For $n < 0$, the Adams operations cannot be defined integrally. However, it is possible to define the operations ψ^k on $K_n(X) \otimes \mathbb{Q}$ for $n < 0$ using descending induction on n and the formula $\psi^k\{x, t\} = k\{\psi^k(x), t\}$ in $K_{n+1}(X \times (\mathbb{A}^1 - 0))$ for $x \in K_n(X)$. This definition was pointed out in [38, 8.4].

By [14, 2.3] or [9, 7.2], the Chern character $NK_{n+1}(X) \to NHC_n(X)$ commutes with the Adams operations ψ^k in the sense that it sends $NK^{(i+1)}_{n+1}(X)$ to $NHC^{(i)}_n(X)$ for all $i \le n$ (and to 0 if $i > n$). Here is the λ-decomposition of the isomorphism in Proposition 4.1:

Proposition 4.6. *Suppose that $X \in \mathrm{Sch}/F$, or that $X = \mathrm{Spec}(R)$ for an F-algebra R. Then for all n and i, the Chern character induces a natural isomorphism:*

$$NK^{(i)}_n(X) \cong H^{1-n}\mathcal{F}^{(i-1)}_{HH}(X) \otimes V.$$

In particular, if $i \le 0$ then $NK^{(i)}_n(X) = 0$ for all n.

Proof. By [9], the Chern character $K \to HN$ sends $K^{(i)}(X)$ to $HN^{(i)}(X)$. The proof in [9] shows that the lift $\mathcal{F}_K(X) \to \mathcal{F}_{HN}(X)$, shown to be a weak equivalence in [8, 1.6], may be taken to send $\mathcal{F}^{(i)}_K(X)$ to $\mathcal{F}^{(i)}_{HN}(X)$. Since $HC \to HN$ sends $HC^{(i-1)}$ to $HN^{(i)}$, the weak equivalence $\mathcal{F}_{HC}[1] \simeq \mathcal{F}_{HN}$ identifies $\mathcal{F}^{(i-1)}_{HC}[1]$ and $\mathcal{F}^{(i)}_{HN}$. Finally $\mathcal{F}^{(i-1)}_{HH} = 0$ for $i \le 0$. □

Corollary 4.7. *$K^{(i)}_n(X) \cong K^{(i)}_n(X \times \mathbb{A}^p)$ if and only if $NK^{(i-j)}_{n-j}(X) = 0$ for all $j = 0, ..., p - 1$.*

Theorem 4.8. *For X in Sch/F or $X = \mathrm{Spec}(R)$, and all integers n, we have:*

(1) *For $i < 0$, $K^{(i)}_n(X) = 0$.*
(2) *For $i = 0$, $K^{(0)}_n(X) \cong KH^{(0)}_n(X) \cong H^{-n}_{\mathrm{cdh}}(X, \mathbb{Q})$.*

This answers Question 8.2 of [38].

Proof. We first show that $K_n^{(i)}(X) \cong KH_n^{(i)}(X)$ when $i \le 0$. Covering X with affine opens and using the Mayer-Vietoris sequences of [37, 5.1], it suffices to consider the case $X = \mathrm{Spec}(R)$.

Since $K(R)$ is the product of the eigen-components, the descent spectral sequence $E^1_{p,q} = N^pK_q(R) \Rightarrow KH_{p+q}(R)$ (see [37, 1.3]) breaks up into one for each eigen-component. If $i \le 0$, the spectral sequence collapses by Proposition 4.6 to yield $K_n^{(i)}(R) \cong KH_n^{(i)}(R)$ for all n.

To determine the groups $KH_n^{(i)}(R)$ when $i \le 0$, we use the cdh descent spectral sequence of [19, 1.1]. If $i < 0$, then the cdh sheaf $K^{(i)}_{\mathrm{cdh}}$ is trivial as X is locally smooth, so we have $KH_n^{(i)}(R) = 0$ for all n. If $i = 0$ then the cdh sheaf $K^{(0)}_{\mathrm{cdh}}$ is the sheaf $\mathbb{Q}_{\mathrm{cdh}}$; see [28, 2.8]. Hence we have $K_n^{(0)}(R) = KH_n^{(0)}(R) = H^{-n}_{\mathrm{cdh}}(X, \mathbb{Q})$. □

5. The typical pieces $TK_n^{(i)}(R)$

In this section, R will be a commutative F-algebra. The default ground field k for Kähler differentials and Hochschild homology will be $\mathbb{Q}$.

As stated in (0.7), the Adams summands $NK_n^{(i)}(R)$ of $NK_n(R)$ decompose as $NK_n^{(i)}(R) = TK_n^{(i)}(R) \otimes V$ for each n and i; the decomposition is obtained from an action of finite Cartier operators precisely as the corresponding one for NHC and NHH, explained in Section 1. The typical pieces $TK_n^{(i)}(R)$ are described by the following formulas.

Theorem 5.1. *Let R be a commutative F-algebra. For $i = n, n+1$, the typical piece $TK_n^{(i)}(R)$ is given by the exact sequence:*

$$0 \to TK_{n+1}^{(n+1)}(R) \to \Omega^n_R \to H^0_{\mathrm{cdh}}(R, \Omega^n) \to TK_n^{(n+1)}(R) \to 0.$$

For $i \ne n, n+1$ we have:

$$TK_n^{(i)}(R) \cong \begin{cases} HH_{n-1}^{(i-1)}(R), & \text{if } i < n, \\ H^{i-n-1}_{\mathrm{cdh}}(R, \Omega^{i-1}) & \text{if } i \ge n+2. \end{cases}$$

Proof. By Proposition 4.6, $TK_n^{(i)} = H^{1-n}\mathcal{F}_{HH}^{(i-1)}$. The rest is a restatement of Lemma 3.4. □

Remark 5.1.1. If R is essentially of finite type over a field F whose transcendence degree is finite over $\mathbb{Q}$, then the $TK_n^{(i)}(R)$ are finitely generated R-modules. This fails if $\mathrm{tr.deg}(F/\mathbb{Q}) = \infty$ because then $\Omega^i_{F/\mathbb{Q}}$ is infinite dimensional. For instance, Example 0.5 implies that, for $R = F[x]/(x^2)$, we have $TK_2^{(2)}(R) = HH_1(R, x) = F \oplus \Omega^1_{F/\mathbb{Q}}$.

Corollary 5.2. *Suppose that R is essentially of finite type over F and has dimension d. If $n < 0$ then $NK_n^{(i)}(R) = 0$ unless $1 \le i \le d + n$, in which case*

$$NK_n^{(i)}(R) = H^{i-n-1}_{\mathrm{cdh}}(R, \Omega^{i-1}) \otimes V.$$

In particular, $NK_n(R) = 0$ for all $n \le -d$.

If $d \geq 2$ then:

$$\begin{aligned} NK_0(R) &\cong \big[\, (R^+/R) \;\oplus H^1_{\mathrm{cdh}}(R,\Omega^1) \oplus \cdots \oplus H^{d-1}_{\mathrm{cdh}}(R,\Omega^{d-1})\big] \otimes V, \\ NK_{-1}(R) &\cong \big[H^1_{\mathrm{cdh}}(R,\mathcal{O}) \oplus H^2_{\mathrm{cdh}}(R,\Omega^1) \oplus \cdots \oplus H^{d-1}_{\mathrm{cdh}}(R,\Omega^{d-2})\big] \otimes V, \\ &\vdots \qquad\qquad \vdots \\ NK_{1-d}(R) &\cong H^{d-1}_{\mathrm{cdh}}(R,\mathcal{O}) \otimes V. \end{aligned}$$

If $d = 1$ then $NK_0(R) = (R^+/R) \otimes V$ and $NK_n(R) = 0$ for $n < 0$.

Remark 5.2.1. The $d = 1$ part of Theorem 5.2 holds for any 1-dimensional noetherian ring by [34, 2.8].

Remark 5.2.2. Observe that 5.2 and 4.4 imply that R is K_{-d}-regular. This recovers the affine case of one of the main results in [5].

Here is a special case of the calculations in Theorem 5.1, which proves Theorem 0.6. We will use it to construct the counterexample to Bass' question in Section 9.

Theorem 5.3. *Let F be a field of characteristic 0 and R a normal domain of dimension 2, essentially of finite type over F. Then*

a) $H^1\mathcal{F}_{HH}(R/F) \cong H^1_{\mathrm{cdh}}(R,\Omega^1_{/F})$,
b) $H^2\mathcal{F}_{HH}(R/F) \cong H^1_{\mathrm{cdh}}(R,\mathcal{O})$,
c) $NK_0(R) \cong H^1_{\mathrm{cdh}}(R,\Omega^1) \otimes V$, *and*
d) $NK_{-1}(R) \cong H^1_{\mathrm{cdh}}(R,\mathcal{O}) \otimes V$.

Proof. Parts (a) and (b) are immediate from Example 3.5 and the fact that R is reduced and seminormal. Parts (c) and (d) follow from (a) and (b) using Corollary 5.2. □

We introduce some notation to make the statement of the next theorem more readable. The letter e denotes the maximum transcendence degree of the component fields in the total ring of fractions Q of R_{red}. For simplicity, we write $\Omega^i_{\mathrm{cdh}}(X)$ for $H^0_{\mathrm{cdh}}(X,\Omega^i)$, and we have written $\Omega^i_{\mathrm{cdh}}(R)/\Omega^i_R$ for the cokernel of $\Omega^i_R \to \Omega^i_{\mathrm{cdh}}(R)$.

Definition 5.4. For any commutative ring R containing $\mathbb{Q}$, we define:

$$E_n(R) = \Omega^n_{\mathrm{cdh}}(R)/\Omega^n_R \oplus \bigoplus_{p=1}^{\infty} H^p_{\mathrm{cdh}}(R,\Omega^{n+p});$$

$$\widetilde{HH}_n(R) = \ker\big(HH_n(R) \to \Omega^n_Q\big) = \ker(\Omega^n_R \to \Omega^n_Q) \oplus \bigoplus_{i=1}^{n-1} HH^{(i)}_n(R).$$

Theorem 5.5. *Let R be a commutative ring containing $\mathbb{Q}$. Then for all n:*

$$NK_n(R) \cong \big[\widetilde{HH}_{n-1}(R) \oplus E_n(R)\big] \otimes V.$$

If furthermore R is essentially of finite type over a field, and $n \geq e+2$, then $NK_n(R) \cong HH_{n-1}(R) \otimes V$.

Proof. Assembling the descriptions of the $TK^{(i)}_n(R)$ in Theorem 5.1 yields the first assertion. The second part is immediate from this and 3.6. □

Remark 5.5.1. The isomorphism for $n \geq e+2$ is given by the Chern character $NK_n(R) \to NHC_{n-1}(R) \cong HH_{n-1}(R) \otimes V$. This part of Theorem 5.5 fails for $n \leq e+1$, by 4.1, because $H^{1-n}\mathcal{F}_{HH}(R) \to HH_{n-1}(R)$ need not be a surjection.

In order to compare the torsion submodules $\operatorname{tors}\Omega^*_R$ with the typical pieces of $NK_*(R)$, we need the affine case of the following lemma. Following tradition, we write $F(X)$ for the total ring of fractions of X_{red}. That is, $F(X)$ is the product of the function fields of the components of X_{red}. When $X = \operatorname{Spec}(R)$ is affine, we write Q instead of $F(X)$.

Lemma 5.6. *Let $X \in \mathrm{Sch}/F$; for $F(X)$ as above, the map $\Omega^i_{\text{cdh}}(X) \to \Omega^i_{F(X)}$ is an injection.*

Proof. We may assume X reduced, and proceed by induction on $d = \dim(X)$, the case $d = 0$ being trivial. Choose a resolution of singularities $X' \to X$ and let Y be the singular locus of X, with $Y' = Y \times_X X'$. By [30, 12.1], there is a Mayer-Vietoris exact sequence

$$0 \to H^0_{\text{cdh}}(X, \Omega^i) \to H^0_{\text{cdh}}(X', \Omega^i) \oplus H^0_{\text{cdh}}(Y, \Omega^i) \to H^0_{\text{cdh}}(Y', \Omega^i) \to \cdots .$$

Since $F(Y) \subseteq F(Y')$, $\Omega^i_{F(Y)} \subseteq \Omega^i_{F(Y')}$. Because $\dim(Y') < d$, the inductive hypothesis implies that $\Omega^i_{\text{cdh}}(Y) \to \Omega^i_{\text{cdh}}(Y')$ is an injection. Hence $\Omega^i_{\text{cdh}}(X) \to \Omega^i_{\text{cdh}}(X')$ is an injection. But X' is smooth, so by *scdh* descent for Ω^i (see [8, 2.5]) we have $\Omega^i_{\text{cdh}}(X') \cong \Omega^i(X') \subset \Omega^i_{F(X')} = \Omega^i_{F(X)}$. □

Remark 5.6.1. Lemma 5.6 remains true if, instead of Ω^i, we use $\Omega^i_{/k}$ for $k \subseteq F$. In particular, if $X = \operatorname{Spec}(R)$ is reduced affine, then $\Omega^i_{\text{cdh}}(R/k) = H^0_{\text{cdh}}(R, \Omega^i_{/k})$ injects into $\Omega^i_{Q/k}$; hence $\operatorname{tors}(\Omega^i_{R/k})$ is the kernel of $\Omega^i_{R/k} \to \Omega^i_{\text{cdh}}(R/k)$.

Corollary 5.7. *For all $n \geq 1$, $TK^{(n)}_n(R) \cong \ker(\Omega^{n-1}_{R/\mathbb{Q}} \to \Omega^{n-1}_{Q/\mathbb{Q}})$.*
In particular if R is reduced, then $TK^{(n)}_n(R) = \operatorname{tors}(\Omega^{n-1}_{R/\mathbb{Q}})$.

Proof. By 5.1, $TK^{(n)}_n(R)$ is the kernel of $\Omega^{n-1}_{R/\mathbb{Q}} \to \Omega^{n-1}_{\text{cdh}}(R)$, so 5.6 applies. □

The typical pieces of $NK^{(2)}_1(R)$ and $NK^{(2)}_2(R)$ of 5.1 and 5.7 may be described as follows.

Proposition 5.8. *For all reduced F-algebras R, the typical pieces $TK^{(2)}_1(R) = \Omega^1_{\text{cdh}}(R)/\Omega^1_R$ and $TK^{(2)}_2(R) = \operatorname{tors}(\Omega^1_R)$ fit into an exact sequence:*

$$0 \to \operatorname{tors}(\Omega^1_R) \to \operatorname{tors}(\Omega^1_{R/F}) \to \Omega^1_F \otimes (R^+/R) \to \frac{\Omega^1_{\text{cdh}}(R)}{\Omega^1_R} \to \frac{\Omega^1_{\text{cdh}}(R/F)}{\Omega^1_{R/F}} \to 0.$$

Proof. We may assume $\operatorname{Spec} R \in \mathrm{Sch}/F$. The spectral sequence of [8, 4.2] degenerates for $p = 1$ to yield exactness of the bottom row in the following commutative diagram; the top row is the First Fundamental Exact Sequence for Ω^1 [39, 9.2.6].

$$\begin{array}{ccccccccc}
 & & \Omega^1_F \otimes R & \longrightarrow & \Omega^1_R & \longrightarrow & \Omega^1_{R/F} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_F \otimes R^+ & \longrightarrow & \Omega^1_{\text{cdh}}(R) & \longrightarrow & \Omega^1_{\text{cdh}}(R/F) & \longrightarrow & 0.
\end{array}$$

The upper left horizontal map is an injection because the left vertical map is an injection. Now apply the snake lemma, using Remark 5.6.1. □

6. Bass' question for algebras over large fields.

We will now show that the answer to Bass' question is positive for algebras R essentially of finite type over a field F of infinite transcendence degree over $\mathbb{Q}$.

Recall from Proposition 4.1 that $NK_{n+1}(R) \cong H^{-n}\mathcal{F}_{HH}(R/\mathbb{Q}) \otimes V$. In light of this identification, version 0.3 of Bass' question becomes the case $k = \mathbb{Q}$ of the following question:

(6.1) Does $H^m\mathcal{F}_{HH}(R/k) = 0$ imply that $H^{m+1}\mathcal{F}_{HH}(R/k) = 0$?

We also show that the answer to question (6.1) is positive provided R is of finite type over a field F that has infinite transcendence degree over k. The proof is essentially a formal consequence of the Künneth formula in Lemma 6.3.

Lemma 6.2. *Let R be a commutative F-algebra, and suppose k is a subfield of F. Then $H^{-*}\mathcal{F}_{HH}(R/k)$ and $\mathbb{H}^{-*}_{\rm cdh}(R, HH(/k))$ are graded modules over the graded ring $\Omega^\bullet_{F/k}$.*

Proof. As in 2.2.1, we may suppose that R is of finite type over F. Consider the functor on F-algebras that associates to an F-algebra A the Hochschild complex $HH(A/k)$. The shuffle product makes this into a functor to *dg*-$HH(F/k)$-modules. Since the *cdh*-site has a set of points (corresponding to valuations), we can use a Godement resolution to find a model for the *cdh*-hypercohomology $\mathbb{H}_{\rm cdh}(-, HH(/k))$ which is also a functor to *dg*-$HH(F/k)$-modules. It follows that there is a model for $\mathcal{F}_{HH}(R/k)$ that is a *dg*-$HH(F/k)$-module, functorially in R. This implies the assertion, since $\Omega^\bullet_{F/k} = H^{-\bullet}HH(F/k)$. □

Lemma 6.3. *(Künneth Formula) Suppose that $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$ are fields. Let R_0 be an F_0-algebra, and set $R = F \otimes_{F_0} R_0$.*

i) *Let $T = \{t_i\}$ be transcendence basis of F/F_0; write $dT = \{dt_i\}$. Then*
$$\Omega^\bullet_{F/k} \cong F[dT] \otimes_{F_0} \Omega^\bullet_{F_0/k}$$
In particular, the graded algebra homomorphism $\Omega^\bullet_{F_0/k} \to \Omega^\bullet_{F/k}$ is flat.

ii) $HH_*(R/k) \cong \Omega^\bullet_{F/k} \otimes_{\Omega^\bullet_{F_0/k}} HH_*(R_0/k)$.

Proof. Note $F[dT] = \Omega^\bullet_{F/F_0}$; the tensor product decomposition of part i) follows from the fact that the fundamental sequence
$$0 \to F \otimes_{F_0} \Omega^1_{F_0} \to \Omega^1_F \to \Omega^1_{F/F_0} \to 0$$
is split exact. This proves i). To prove ii), choose a free chain *dg*-F_0-algebra Λ and a surjective quasi-isomorphism of *dg*-algebras $\Lambda \overset{\sim}{\twoheadrightarrow} R_0$. Then $\Lambda' = F \otimes_{F_0} \Lambda \to F \otimes_{F_0} R_0 = R$ is a free chain model of R as a k-algebra. Write $\Omega^\bullet_\Lambda$ for differential forms; consider $\Omega^\bullet_\Lambda$ as a chain *dg*-algebra with the differential δ induced by that of Λ. Note Λ and Λ' are homologically regular in the sense of [4], so that Theorem 2.6 of [4] applies. Combining this with part (i), we obtain
$$\begin{aligned} HH_*(R) =& HH_*(\Lambda') = H_*(\Omega^\bullet_{\Lambda'}) \\ =& H_*(\Omega^\bullet_F \otimes_{\Omega^\bullet_{F_0}} \Omega^\bullet_\Lambda) = \Omega^\bullet_F \otimes_{\Omega^\bullet_{F_0}} H_*(\Omega^\bullet_\Lambda) \\ =& \ \Omega^\bullet_F \otimes_{\Omega^\bullet_{F_0}} HH_*(R_0). \end{aligned}$$
□

Here is an easy consequence of Lemmas 6.2 and 6.3.

Proposition 6.4. *Suppose $\mathbb{Q} \subseteq k \subseteq F_0 \subseteq F$ are field extensions, that R_0 is an F_0-algebra and $R = F \otimes_{F_0} R_0$. Then there is an isomorphism of graded $\Omega^\bullet_{F/k}$-modules*

$$\Omega^\bullet_{F/k} \otimes_{\Omega^\bullet_{F_0/k}} H^{-*}\mathcal{F}_{HH}(R_0/k) \cong H^{-*}(\mathcal{F}_{HH}(R/k)).$$

We also need the following lemma to prove the main result of this section.

Lemma 6.5. *Let R be essentially of finite type over $F \supset \mathbb{Q}$, and let $H_n(R)$ denote either $HH_n(R)$ or $H^{-n}\mathcal{F}_{HH}(R)$. Assume that $H_{n_i}(R) = 0$ for some finite set $\{n_1, \dots, n_r\}$ of positive integers. Then there exist an F-algebra of finite type R', and a prime ideal $P \in \operatorname{Spec} R'$, such that $R \cong R'_P$ and $H_{n_i}(R') = 0$ for $1 \le i \le r$.*

Proof. Because R is essentially of finite type, we have $R = R''_P$ for some finite type F-algebra R''. Because R'' is of finite type over F, we may write $R'' = F \otimes_{F_0} R_0$ for some finitely generated field extension F_0 of $\mathbb{Q}$ and some finite type F_0-algebra R_0. Note R_0 is essentially of finite type over $\mathbb{Q}$, whence $H_p(R_0)$ is a finitely generated R_0-module ($p \ge 0$). By 6.3 and/or 6.4, $H_p(R'')$ is isomorphic, as an R''-module, to a direct sum of copies of $R'' \otimes_{R_0} H_q(R_0)$ with $q \le p$. In particular, $M = \bigoplus_{i=1}^r H_{n_i}(R'')$ is a finite sum of R''-modules, each of which is a — possibly infinite — direct sum of copies of one finitely generated module. Given that M has this form, the hypothesis that $M_P = 0$ implies that there exists a nonzero element $f \in Ann(M)$. Consider the finite type F-algebra $R' = R''[1/f]$. Then $P \in \operatorname{Spec}(R')$, $R \cong R'_P$ and we have $\bigoplus_i H_{-n_i}(R') = M[1/f] = 0$. □

Theorem 6.6. *Suppose $k \subset F$ is an extension with $\operatorname{tr.deg}(F/k) = \infty$, and R is essentially of finite type over F. If $H^n(\mathcal{F}_{HH}(R/k)) = 0$, then $H^m(\mathcal{F}_{HH}(R/k)) = 0$ for all $m \ge n$.*

Proof. By Lemma 6.5, we may assume that R is of finite type over F. There is a finitely generated field extension $F_0 \subset F$ of k and a finite type F_0-algebra R_0 such that $R = R_0 \otimes_{F_0} F$. Note that $\operatorname{tr.deg}(F/F_0) = \infty$. By Proposition 6.4, $\Omega^i_{F/k} \otimes_{F_0} H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ is a direct summand of $H^n(\mathcal{F}_{HH}(R/k))$ for each $i \ge 0$. Since $\Omega^i_{F/k} \ne 0$ for all i, all the $H^{n+i}(\mathcal{F}_{HH}(R_0/k))$ vanish as well. Similarly, $H^m(\mathcal{F}_{HH}(R/k))$ is a direct sum of copies of the groups $\Omega^j_{F/k} \otimes_{F_0} H^{m+j}(\mathcal{F}_{HH}(R_0/k))$ for $j \ge 0$, all of which vanish when $m \ge n$, as we just observed. □

Corollary 6.7. *Let $\mathbb{Q} \subset F$ be a field extension of infinite transcendence degree, and suppose R is essentially of finite type over F. Then $NK_n(R) = 0$ implies that R is K_n-regular.*

Proof. Combine Theorem 6.6 with Proposition 4.1 and Corollary 4.4. □

7. du Bois invariants, χ^p and deformations

We now prepare for our counterexample by constructing generalized du Bois invariants $b^{p,q}$ and their Euler characteristic χ^p; for $q > 0$ they coincide with the du Bois invariants of [29]. We then show (Theorem 7.11) that the χ^p are invariant under appropriate deformations; this result may be of independent interest. All differential forms, Hochschild homology, etc, will be taken over F for the rest of this paper. We remark that several of the results we quote from here on — in particular anything involving du Bois complexes — have been proved under the assumption that the base field is $\mathbb{C}$; however, flat base change implies that they all remain valid over arbitrary fields F of characteristic 0.

Fix a scheme $X \in \mathrm{Sch}/F$ and choose a proper simplicial hyperresolution $\pi : Y_\bullet \to X$. Following [12] we fix p and we consider the p-th *du Bois complex*

$$\underline{\Omega}^p_X = \mathbb{R}\pi_*\Omega^p_{Y_\bullet}.$$

Du Bois shows in [12] that the assignment $X \mapsto \underline{\Omega}^p_X$ is natural in X up to unique isomorphism in the derived category. The relevance for us lies in the fact that the Zariski hypercohomology of the complex $\underline{\Omega}^p_X$ computes $H^*_{\mathrm{cdh}}(X, \Omega^p)$:

Lemma 7.1. *Let $X \in \mathrm{Sch}/F$ and $p \geq 0$. Then there is a natural isomorphism*

$$\mathbb{H}^*_{\mathrm{zar}}(X, \underline{\Omega}^p_X) \cong H^*_{\mathrm{cdh}}(X, \Omega^p).$$

Proof. (Cf. the proof of [6, 4.1].) Recall that $H^*_{\mathrm{cdh}}(X, \Omega^p)$ is the Zariski hypercohomology of the complex $\mathbb{R}a_*a^*\Omega^p|_X$, where $a : (\mathrm{Sch}/F)_{\mathrm{cdh}} \to (\mathrm{Sch}/F)_{\mathrm{zar}}$ is the morphism of sites and $|_X$ denotes the restriction from the big Zariski site $(\mathrm{Sch}/F)_{\mathrm{zar}}$ to X_{zar}. Let $Y_\bullet \to X$ be a simplicial hyperresolution. By [8, 2.5], we have a quasi-isomorphism on X_{zar}

$$\Omega^p_{Y_n} \xrightarrow{\simeq} \mathbb{R}a_*a^*\Omega^p|_{Y_n}$$

since each Y_n is smooth. Using also [6, 4.3], we have a diagram of equivalences

$$\mathbb{R}a_*a^*\Omega^p|_X \xrightarrow{\simeq} \mathbb{R}\pi_*(\mathbb{R}a_*a^*\Omega^p|_{Y_\bullet}) \xleftarrow{\simeq} \mathbb{R}\pi_*\Omega^p_{Y_\bullet} = \underline{\Omega}^p_X.$$

Applying $\mathbb{H}^*_{\mathrm{zar}}(X, -)$ yields $H^*_{\mathrm{cdh}}(X, \Omega^p) \cong \mathbb{H}^*_{\mathrm{zar}}(X, \underline{\Omega}^p)$. ∎

Isolated singularities. Suppose that $\mathrm{Sing}(X)$ is an isolated point x. Choose a good resolution $\pi : Y \to X$, meaning that Y is smooth, π is proper and $E = \pi^{-1}(x)_{\mathrm{red}}$ is a normal crossings divisor with smooth components. Then by [12, 4.8, 4.11] we have a distinguished triangle

$$0 \to \underline{\Omega}^p_X \to \mathbb{R}\pi_*\Omega^p_Y \oplus \Omega^p_x \to \mathbb{R}\pi_*\underline{\Omega}^p_E \to 0.$$

To rewrite this, let $E_1, \dots, E_t$ denote the (smooth) components of E, and define

$$Y_n = \begin{cases} Y \amalg x_0 & n = 0 \\ \coprod_{i_1 < \cdots < i_n} E_{i_1} \times_Y \cdots \times_Y E_{i_n} & n > 0. \end{cases} \tag{7.2}$$

By [12, 4.10], the complex $\underline{\Omega}^p_E$ is quasi-isomorphic to (the total complex of)

$$\underline{\Omega}^p_{Y_1} \to \underline{\Omega}^p_{Y_2} \to \cdots$$

The maps in this complex are given by the usual alternating sum of restriction maps, since it arises from a coskeletal hyperresolution of E. Generically writing $\pi : Y_n \to X$ for the canonical map from Y_n to X, we have

$$\underline{\Omega}^p_X \simeq \mathrm{Tot}\Big(\mathbb{R}\pi_*\underline{\Omega}^p_{Y_0} \to \mathbb{R}\pi_*\underline{\Omega}^p_{Y_1} \to \mathbb{R}\pi_*\underline{\Omega}^p_{Y_2} \to \cdots\Big). \tag{7.3}$$

Now suppose that $\dim(X) = 2$. Because $\Omega^p_{E_{i,j}} = 0$ for $i \neq j$ and $p > 0$, (7.3) reduces to: $\underline{\Omega}^p_X \simeq \mathrm{Tot}\left(\mathbb{R}\pi_*\Omega^p_Y \to \bigoplus_i \mathbb{R}\pi_*\Omega^p_{E_i}\right)$ for $p > 0$, and

$$\underline{\Omega}^0_X \simeq \mathrm{Tot}\Big(\mathbb{R}\pi_*\mathcal{O}_Y \oplus \mathcal{O}_x \to \bigoplus_i \mathbb{R}\pi_*\mathcal{O}_{E_i} \to \bigoplus_{i<j} \mathbb{R}\pi_*\mathcal{O}_{E_i \cap E_j}\Big).$$

In other words, in the notation of [33],

$$\underline{\Omega}^0_X \simeq \mathbb{R}\pi_*\mathcal{O}_Y(-E) \oplus \mathcal{O}_x \quad \text{and} \quad \underline{\Omega}^p_X \simeq \mathbb{R}\pi_*\left(\Omega^p_Y(\log E)(-E)\right), \quad p > 0. \tag{7.4}$$

Du Bois invariants. Suppose for simplicity that $X = \operatorname{Spec} R$, where R is a domain, essentially of finite type over F. For any $p \geq 0$, there is a map from the p-th higher cotangent complex $\mathcal{L}^p_X$ to the p-th du Bois complex $\underline{\Omega}^p_X$, obtained by composing the isomorphism $H_0(\mathcal{L}^p_X) \cong \Omega^p_X$ and the natural map $\Omega^p_X \to H^0(\underline{\Omega}^p_X)$.

Definition 7.5. We define the cochain complex C^p_X of quasi-coherent $\mathcal{O}_X$-modules by

$$C^p_X := \operatorname{cone}(\mathcal{L}^p_X \to \underline{\Omega}^p_X).$$

That is, we have a triangle $\mathcal{L}^p_X \to \underline{\Omega}^p_X \to C^p_X \to \mathcal{L}^p_X[1]$.

By Lemma 7.1, (2.1) and [22, 4.5.13],

$$\mathbb{H}^i(C^p_X) = H^{i+1-p}(\mathcal{F}^{(p)}_{HH}(X)). \tag{7.6}$$

Note that because our Kähler differentials are taken over F, the cohomology sheaves of C^p_X are coherent. Comparing with Lemma 3.4, we conclude that:

$$\mathbb{H}^q(C^p_X) = \begin{cases} \mathbb{H}^q(X, \underline{\Omega}^p) & \text{for } q \geq 1 \\ \operatorname{coker}\left(\Omega^p_X \to \mathbb{H}^0(X, \underline{\Omega}^p)\right) & \text{for } q = 0 \\ \ker\left(\Omega^p_X \to \mathbb{H}^0(X, \underline{\Omega}^p)\right) & \text{for } q = -1 \\ D^{(p)}_{-1-q}(X) & \text{for } q \leq -2, \end{cases} \tag{7.7}$$

where $D^{(p)}_n$ denotes *André-Quillen homology.* Recall that $D^{(p)}_n(R) \cong HH^{(p)}_{p+n}(R)$.

In particular, if X has isolated singularities, then each of the homology modules $H^n(C^p_X)$ is of finite length. In this case we may define, following and expanding on Steenbrink's definition [29], the *generalized du Bois invariants* to be the numbers

$$b^{p,q} = b^{p,q}_X = \operatorname{length} \mathbb{H}^q(C^p_X), \quad \text{for } p \geq 0 \text{ and } q \in \mathbb{Z}. \tag{7.8}$$

Example 7.8.1. Since R is a domain, $b^{0,0} = \operatorname{length}_R(R^+/R)$ and $b^{0,q} = 0$ for $q < 0$. And if $q > 0$ then (7.4) yields $b^{0,q} = h^q(\mathcal{O}_Y(-E))$.

If, moreover, X is locally a complete intersection, then $HH^{(p)}_n(R) = 0$ for $n \gg 0$ (see [13]); hence it follows from (7.7) that C^p_X is homologically bounded. In this case, we define $\chi^p(X)$ to be the Euler characteristic of C^p_X:

$$\chi^p(X) := \sum\nolimits_q (-1)^q b^{p,q}. \tag{7.9}$$

Lemma 7.10. *If $X = \operatorname{Spec}(R)$ for a ring R that admits a non-negative grading with $R_0 = k$, then $\sum (-1)^p b^{p,q} = 0$ for all q.*

Proof. The cases $q = -1$, $q = 0$, $q > 0$ follow from (7.7) using (3.9c), (3.9d) and (3.9b), respectively. For $q < -1$ it follows from Goodwillie's Theorem [39, 9.9.1]. □

The invariants χ^p are invariant under deformations of the sort described in the following theorem, where we write X_s for the fiber of X over a closed point $s \in S$.

Theorem 7.11. *Suppose $X \to S$ is a flat local complete intersection map of affine varieties with S smooth and such that the singular locus X_{sing} of X is finite and étale over S. Suppose, in addition, that one can find a projective map $\pi : Y \to X$ such that Y is smooth and such that the reduced, irreducible components $E_1, \dots, E_m$ of $Y \times_X X_{\text{sing}}$ are smooth over S and satisfy the property that each*

$$E_{i_1,\dots,i_t} := E_{i_1} \times_Y E_{i_2} \times_Y \cdots \times_Y E_{i_t} \to S$$

is smooth ($1 \leq i_1, \dots, i_t \leq m$). Then $\chi^p(X_s)$ is independent of the closed point s.

Suppose in addition that a finite group G acts on both X and Y and that π and $X \to S$ are equivariant, where we declare the action of G on S to be trivial. Assume that $X/G \to S$ is a flat local complete intersection such that $(X/G)_{\mathrm{sing}}$ is finite and étale over S. Then $\chi^p(X_s/G)$ is independent of the closed point $s \in S$.

Proof. In analogy with Definition 7.5, we use (7.3) to define a relative version of C^p:

$$C^p_{X/S} := \mathrm{Tot}\left(\mathcal{L}^p_{X/S} \to \mathbb{R}\pi_*\underline{\Omega}^p_{Y_0/S} \to \mathbb{R}\pi_*\underline{\Omega}^p_{Y_1/S} \to \mathbb{R}\pi_*\underline{\Omega}^p_{Y_2/S} \to \cdots\right),$$

where, as in (7.2),

$$Y_n = \begin{cases} Y \amalg X_{\mathrm{sing}} & n = 0 \\ \coprod_{i_1 < \cdots < i_n} E_{i_1,\dots,i_n} & n > 0 \end{cases} \tag{7.11a}$$

and $\mathcal{L}^p_{X/S}$ is the p-th cotangent complex for $X \to S$; the map $\mathcal{L}^p_{X/S} \to \mathbb{R}\pi_*\Omega^p_{Y_0/S}$ is induced by the composite of the natural maps $\mathcal{L}^p_{X/S} \to \Omega^p_{X/S} \to \pi_*\Omega^p_{Y_0/S}$.

The complex $C^p_{X/S}$ is a complex of quasi-coherent $\mathcal{O}_X$-modules with only finitely many non-zero homology sheaves, each of which is coherent. Moreover, each such homology sheaf is supported on the singular locus of X, which maps finitely to S. By restriction of scalars along the affine map $X \to S$, we may therefore regard $C^p_{X/S}$ as a complex of quasi-coherent $\mathcal{O}_S$-modules whose homology is coherent. As such, this complex determines a class $[C^p_{X/S}]$ in $G_0(S) = K_0(S)$. Explicitly, this class is the alternating sum of these homology modules.

For any point $s \in S$, let $j_s : s \to S$ be the induced map of schemes and let $j_s^* : K_0(S) \to K_0(s) \cong \mathbb{Z}$ be the pull-back map in K-theory. Note that for any s, the map j_s^* sends the class of a locally free $\mathcal{O}_S$-module to its rank. Consequently, the map $j_s^* : K_0(S) \to \mathbb{Z}$ does not depend on the choice of $s \in S$. We now prove that for any closed point $s \in S$:

$$j_s^*[C^p_{X/S}] = [C^p_{X_s/s}]. \tag{7.11b}$$

Since the class $[C^p_{X_s/s}]$ in $K_0(s) = \mathbb{Z}$ is $\chi^p(X_s)$ when $s \in S$ is a closed point, this will prove the first assertion of the Theorem.

Note first of all that if $\mathcal{F}^\bullet$ is any complex of quasi-coherent $\mathcal{O}_S$-modules with bounded, coherent homology, then $j_s^*[\mathcal{F}^\bullet] = [\mathbb{L}j_s^*\mathcal{F}^\bullet]$, where $\mathbb{L}j_s^*$ denotes the left derived functor associated to j_s^*. For any n, let $\tilde{\pi} : Y_n \to S$ be the structure map, which we are supposing to be smooth and hence flat. Thus $\tilde{\pi}$ and j_s are Tor-independent over S. Consider the pullback diagram

$$\begin{array}{ccc} (Y_n)_s & \xrightarrow{\alpha_s} & Y_n \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ s & \xrightarrow{j_s} & S \end{array}$$

By [SGA6, IV.3.1], we have

$$\mathbb{L}j_s^*\mathbb{R}\tilde{\pi}_*\Omega^p_{Y_n/S} \simeq \mathbb{R}q_*\mathbb{L}\alpha_s^*\Omega^p_{Y_n/S}$$

Since Y_n/S is smooth, $\Omega^p_{Y_n/S}$ is locally free and we have

$$\mathbb{L}\alpha_s^*\Omega^p_{Y_n/S} = \alpha_s^*\Omega^p_{Y_n/S} \cong \Omega^p_{(Y_n)_s/s}.$$

Hence

$$\mathbb{L}j_s^*\mathbb{R}\tilde{\pi}_*\Omega^p_{Y_n/S} \simeq \mathbb{R}q_*\Omega^p_{(Y_n)_s/s}. \tag{7.11c}$$

Similarly, it is a standard property of the cotangent complex that

$$j_s^*\mathcal{L}^p_{X/S} \simeq \mathbb{L}j_s^*\mathcal{L}^p_{X/S} \simeq \mathcal{L}^p_{X_s/s}.$$

Combining these, we get the formula

$$j_s^*[C^p_{X/S}] = \Big[\cdots \to \mathcal{L}^p_{X_s/s} \to \mathbb{R}q_*\Omega^p_{(Y_0)_s/s} \to \mathbb{R}q_*\Omega^p_{(Y_1)_s/s} \to \cdots\Big].$$

Finally, if s is a closed point then by (7.3) we have

$$\underline{\Omega}^p_{X_s} \simeq \Big(\mathbb{R}q_*\Omega^p_{(Y_0)_s/s} \to \mathbb{R}q_*\Omega^p_{(Y_1)_s/s} \to \cdots\Big)$$

and hence the formula $j_s^*[C^p_{X/S}] = [C^p_{X_s/s}]$ of (7.11b), proving the first assertion.

Suppose now that a finite group G acts on X and Y as in the statement of the Theorem. Let Y_n be as in (7.11a) above; then G acts on $Y_n \to S$ and hence on $\underline{\Omega}^p_{Y_n/S}$ and $\mathbb{R}\tilde{\pi}_*\underline{\Omega}^p_{Y_n/S}$ for all n. For each $s \in S$, the group G acts also on $\underline{\Omega}^p_{(Y_n)_s}$.

Since G is a finite group and we are in characteristic 0, taking G-invariants is exact. This implies the key property we will need, proven in [12, 5.12], namely that

$$\underline{\Omega}^p_{(Y_n)_s/G} \simeq (\underline{\Omega}^p_{(Y_n)_s})^G \simeq (\Omega^p_{(Y_n)_s})^G.$$

Since taking G-invariants also commutes with $\mathbb{R}q_*$, this property implies that

$$\mathbb{R}q_*\big(\underline{\Omega}^p_{(Y_n)_s/G}\big) \simeq \mathbb{R}q_*\big((\Omega^p_{(Y_n)_s})^G\big) \simeq (\mathbb{R}q_*\Omega^p_{(Y_n)_s})^G. \tag{7.11d}$$

Define the analogue $D^p_{X/S}$ of $C^p_{X/S}$ by

$$D^p_{X/S} = \Big(\mathcal{L}^p_{(X/G)/S} \to (\mathbb{R}\pi_*\Omega^p_{Y_0/S})^G \to (\mathbb{R}\pi_*\Omega^p_{Y_1/S})^G \to \cdots\Big).$$

Now taking G-invariants commutes with $\mathbb{L}j_s^*$. Using (7.11c) and (7.11d), we have

$$\mathbb{L}j_s^*((\mathbb{R}\tilde{\pi}_*\Omega^p_{Y_i/S})^G) \simeq (\mathbb{L}j_s^*(\mathbb{R}\tilde{\pi}_*\Omega^p_{Y_i/S}))^G \simeq (\mathbb{R}q_*\Omega^p_{(Y_i)_s})^G \simeq \mathbb{R}q_*(\underline{\Omega}^p_{(Y_i)_s/G)}).$$

Finally, observe that a similar argument as that used to prove (7.3) shows that

$$\underline{\Omega}^p_{X_s/G} \simeq \Big(\mathbb{R}q_*\underline{\Omega}^p_{((Y_0)_s/G)} \to \mathbb{R}q_*\underline{\Omega}^p_{((Y_1)_s/G)} \to \cdots\Big)$$

Indeed, $X_s/G, Y_s/G$, and the $(E_i)_s/G$ satisfy the same hypotheses as do X, Y, and the E_i, except for smoothness, so that the results in [12, 4.8, 4.10, 4.11] apply. It follows that

$$j_s^*[D^p_{X/S}] \simeq [C^p_{(X_s/G)}].$$

Since the class of $[C^p_{X_s/G}]$ in $K_0(s) = \mathbb{Z}$ is $\chi^p(X_s/G)$, it is independent of s. □

8. Isolated (hyper)surface singularities.

In this section we consider the du Bois invariants of a two-dimensional isolated hypersurface singularity X. That is, $X = \operatorname{Spec} R$ where $R = F[x,y,z]/(f(x,y,z))$ and $\Omega^3_{R/F} \cong R/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is supported at the origin (*i.e.,* the point x_0 defined by the maximal ideal (x,y,z)). The analytic analogues of some of our results are due to Steenbrink and may be found in Wahl's paper [33].

We will need the following well known calculation of $\Omega^p_R = \Omega^p_{R/F}$. Recall that the *Tjurina number* τ is:

$$\tau = \mathrm{length}_R \left(R/\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z} \right) \right).$$

Lemma 8.1. *Let $X = \operatorname{Spec} R$ be a 2-dimensional isolated hypersurface singularity. Then each of the following R-modules has length equal to τ:*

$$\Omega^3_R \cong \mathrm{Ext}^1_R(\Omega^1_R, R) \cong \mathrm{Ext}^2_R(\Omega^2_R, R), \quad \mathrm{tors}(\Omega^2_R) \cong \mathrm{Ext}^1_R(\Omega^2_R, R).$$

Proof. Write $R = P/f$, where $P = F[x,y,z]$, and consider the complex $\mathcal{K}$ of free R-modules, whose boundary maps are induced by exterior multiplication with df, indexed with R in degree 0:

$$\mathcal{K}: \quad 0 \to R \overset{\wedge df}{\to} \Omega^1_P \otimes_P R \overset{\wedge df}{\to} \Omega^2_P \otimes_P R \overset{\wedge df}{\to} \Omega^3_P \otimes_P R \to 0.$$

By [24, p. 326], the n-th cohomology of the complex $\mathcal{K}$ is the torsion submodule of Ω^n_R. In the isolated singularity case considered here, it follows from Lebelt's results [21] (see also [25, Prop. 1]) that Ω^n_R is a torsionfree module for $n \le 1$. In particular, we have free resolutions:

$$0 \to R \overset{\wedge df}{\to} \Omega^1_P \otimes_P R \to \Omega^1_R \to 0$$

$$0 \to R \overset{\wedge df}{\to} \Omega^1_P \otimes_P R \overset{\wedge df}{\to} \Omega^2_P \otimes_P R \to \Omega^2_R \to 0$$

Moreover the perfect pairing $\Omega^p_P \otimes_P \Omega^{3-p}_P \to \Omega^3_P \cong P$ induces a perfect pairing $\mathcal{K}^p \otimes_R \mathcal{K}^{3-p} \to \mathcal{K}^3 \cong R$. From this we get an isomorphism of complexes $\mathrm{Hom}_R(\mathcal{K}, R)[-3] \cong \mathcal{K}$. It follows that

$$\mathrm{Ext}^1_R(\Omega^1_R, R) = H^3(\mathcal{K}) = \Omega^3_R \cong R/\left(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z} \right),$$

$$\mathrm{Ext}^1_R(\Omega^2_R, R) = H^2(\mathcal{K}) = \mathrm{tors}\,\Omega^2_R.$$

By definition, the length of the first of these modules is τ; by [25, Thm. 3], the second module also has length τ. □

Recall the definition of the (generalized) du Bois invariants $b^{p,q}$ from (7.8).

Proposition 8.2. *Let $X = \operatorname{Spec} R$ be a 2-dimensional isolated hypersurface singularity. Then the following hold:*

a) $b^{p,q} = 0$ *unless* $p + q \in \{1, 2\}$.
b) $b^{1-q,q} = b^{2-q,q} = \tau$ *for all* $q < 0$.
c) $b^{0,2} = 0,$ *and* $b^{0,1} = -\chi^0$.

Proof. a) For $q > 0$, (a) is a particular case of a general statement for isolated singularities proved by Steenbrink in [29, Thm. 1], since $b^{p,q}$ is the length of $\mathbb{H}^q(X, \underline{\Omega}^p_X)$ by (7.7). In our case Steenbrink's result is immediate from Grauert-Riemenschneider vanishing [16, Satz 2.3] and from the fact, proved in [8, Prop. 2.6], that for an affine surface X,

(8.2a) $$H^2_{\mathrm{cdh}}(X, \Omega^p) = 0 \qquad (p \ge 0).$$

If $q = 0$ and $p > 2$, (a) holds since then $a^*\Omega^p = 0$. If $q = p = 0$, it holds since R is normal, hence seminormal. For $q = -1$, (a) holds because $\Omega^p_R = 0$ for $p > 3$, that R and Ω^1_R are torsionfree, and Remark 5.6.1. For $q \le -2$, we have

(8.2b) $$H_q(C^p_X) = D^{(p)}_{-1-q}(R) = HH^{(p)}_{p-q-1}(R) = \mathrm{tors}(\Omega^{p+q+1}_R)$$

which is zero unless $p+q \in \{1,2\}$, by a result of Michler [24].

b) This follows from (8.2b) and the fact that the kernel of $\Omega^n_R \to H^0_{\mathrm{cdh}}(X,\Omega^n)$ is $\mathrm{tors}(\Omega^n_R)$ (see Remark 5.6.1), using [25, Thm. 3] (see Lemma 8.1).

c) That $b^{0,2}=0$ is a particular case of (8.2a). The rest follows from parts (a) and the definition (7.9) of χ^0. □

Proposition 8.3. *Let $X = \operatorname{Spec} R$ be a 2-dimensional isolated hypersurface singularity. Further let $\pi : Y \to X$ be a good resolution, E the exceptional divisor, $E_1,\dots,E_r$ its reduced irreducible components, g_i the genus of E_i, and l the number of loops in the incidence graph. Put $g = \sum_i g_i$ and $p_g = \mathrm{length}_R\, H^1(Y,\mathcal{O}_Y)$.*

a) *The map $H^n_{\mathrm{cdh}}(X,\mathcal{O}) \to H^n_{\mathrm{cdh}}(Y,\mathcal{O}) = H^n(Y,\mathcal{O})$ is an isomorphism for $n \neq 1$, and an injection for $n=1$. We have*
$$b^{0,1} = p_g - g - l.$$
In particular $H^1_{\mathrm{cdh}}(X,\mathcal{O}) \to H^1(Y,\mathcal{O})$ is an isomorphism $\iff$ $g=l=0$.

b) *$H^n_{\mathrm{cdh}}(X,\Omega^2) \cong H^n(Y,\Omega^2)$ for $n \geq 0$. In particular, $H^n_{\mathrm{cdh}}(X,\Omega^2)=0$ for $n\geq 1$.*

c) *$\mathrm{Ext}^i_R(H^0(Y,\Omega^2),R) \cong H^i(Y,\mathcal{O}_Y)$. In particular, $Ext^2_R(H^0(Y,\Omega^2),R)=0$.*

d) *$b^{1,0} \leq \tau$.*

e) *$b^{2,0} = \tau - p_g$, and $\chi^2 = -p_g$.*

Proof. To prove (a), observe that R is normal and $Y \to X$ is projective, so that $R = H^0_{\mathrm{cdh}}(X,\mathcal{O}) = H^0(Y,\mathcal{O})$ by Zariski's Main Theorem (and Proposition 2.5). Since $Y \to X$ has fibers of dimension at most 1, and X is affine,

(8.3a)
$$H^2(Y,\mathcal{F}) = H^0(X, R^2\pi_*\mathcal{F}) = 0$$

for all coherent sheaves $\mathcal{F}$. In particular, $H^2(Y,\mathcal{O})=0$. Similarly, $H^2_{\mathrm{cdh}}(X,\mathcal{O})=0$ by [5, Theorem 6.1]. Since $\operatorname{Sing} X = \{x_0\}$, we have a blowup square

(8.3b)
$$\begin{array}{ccc} E & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \\ x_0 & \longrightarrow & X \end{array}$$

From the Mayer-Vietoris sequence associated to this square, we extract the short exact sequence
$$0 \to H^1_{\mathrm{cdh}}(X,\mathcal{O}) \to H^1(Y,\mathcal{O}) \to H^1_{\mathrm{cdh}}(E,\mathcal{O}) \to 0.$$
Hence $b^{0,1} = \mathrm{length}_R\, H^1(Y,\mathcal{O}) - \mathrm{length}_R\, H^1_{\mathrm{cdh}}(E,\mathcal{O})$. Applying descent to the cover $\coprod_i E_i \to E$, we obtain $\mathrm{length}_R\, H^1_{\mathrm{cdh}}(E,\mathcal{O}) = l+g$.

For (b), the isomorphisms $H^n_{\mathrm{cdh}}(X,\Omega^2) \cong H^n(Y,\Omega^2)$ follow from the Mayer-Vietoris sequence associated to the square (8.3b). By Grauert-Riemenschneider vanishing [16, Satz 2.3], $\mathbb{R}\pi_*\Omega^2_Y \simeq \pi_*\Omega^2_Y$, so $H^n(Y,\Omega^2) = H^0(X, R^n\pi_*\Omega^2_Y)$ vanishes for $n>0$ because X is affine.

To prove (c), recall that $\omega_X \cong \mathcal{O}_X[2]$ because X is an affine hypersurface. For any bounded complex of quasi-coherent sheaves $\mathcal{F}^\bullet$ on Y, Grothendieck-Serre duality gives a quasi-isomorphism:
$$\mathbb{R}\pi_*\mathbb{R}\operatorname{Hom}_Y(\mathcal{F}^\bullet,\Omega^2_Y) \simeq \mathbb{R}\operatorname{Hom}_X(\mathbb{R}\pi_*\mathcal{F}^\bullet,\mathcal{O}_X)$$
Taking $\mathcal{F}^\bullet = \Omega^p_Y$ and using the duality pairing on Y,
$$\mathbb{R}\operatorname{Hom}_Y(\Omega^p_Y,\Omega^2_Y) \simeq \operatorname{Hom}_Y(\Omega^p_Y,\Omega^2_Y) \cong \Omega^{2-p}_Y,$$

we get a spectral sequence

$$\mathrm{Ext}^i_R(H^j(Y,\Omega^p),R) \Rightarrow H^{i-j}(Y,\Omega^{2-p}). \tag{8.3c}$$

Taking $p=2$ and using Grauert-Riemenschneider vanishing [16, Satz 2.3], which gives $H^j(Y,\Omega^2)=0$ for $j>0$, we obtain the conclusion of (c):

$$\mathrm{Ext}^i_R(H^0(Y,\Omega^2),R) \cong H^i(Y,\mathcal{O}_Y).$$

In particular, by (8.3a), $\mathrm{Ext}^2_R(H^0(Y,\Omega^2),R)=0$.

To prove (d), recall that $b^{1,0}$ is the length of the R-module $L=\mathbb{H}^0(C^1_X)$. Since $b^{1,-1}=0$ by Proposition 8.2, it follows from (7.7) that we have an exact sequence

$$0\to\Omega^1_R\to H^0_{\mathrm{cdh}}(X,\Omega^1)\to L\to 0. \tag{8.3d}$$

From (8.3d) we get the exact sequence

$$\mathrm{Ext}^1_R(\Omega^1_R,R)\to\mathrm{Ext}^2_R(L,R)\to\mathrm{Ext}^2_R(H^0_{\mathrm{cdh}}(X,\Omega^1),R). \tag{8.3e}$$

From the spectral sequence (8.3c) with $p=1$, we have an exact sequence

$$\mathrm{Hom}_R(H^1(Y,\Omega^1),R)\xrightarrow{d_2}\mathrm{Ext}^2_R(H^0(Y,\Omega^1),R)\to H^2(Y,\Omega^1).$$

Since the R-module $H^1(Y,\Omega^1)$ is supported at x_0, $\mathrm{Hom}_R(H^1(Y,\Omega^1),R)=0$. The right side also vanishes, by (8.3a), so we get $\mathrm{Ext}^2_R(H^0(Y,\Omega^1),R)=0$.

By part (a), the map $H^0_{\mathrm{cdh}}(X,\Omega^1)\to H^0(Y,\Omega^1)$ is injective, so the map

$$\mathrm{Ext}^2_R(H^0(Y,\Omega^1),R)\to\mathrm{Ext}^2_R(H^0_{\mathrm{cdh}}(X,\Omega^1),R)$$

is surjective and hence

$$\mathrm{Ext}^2_R(H^0_{\mathrm{cdh}}(X,\Omega^1),R)=0.$$

From (8.3e) we get that $\mathrm{Ext}^1_R(\Omega^1_R,R)\to\mathrm{Ext}^2_R(L,R)$ is surjective and hence

$$\begin{aligned} b^{1,0}=\mathrm{length}_R(L) &= \mathrm{length}_R(\mathrm{Ext}^2_R(L,R)) \\ &\le \mathrm{length}_R(\mathrm{Ext}^1_R(\Omega^1_R,R)) \\ &= \tau,\ \text{ by Lemma 8.1.} \end{aligned}$$

To prove (e), define finite length R-modules N and M so that

$$0\to N\to\Omega^2_R\to H^0(Y,\Omega^2)\to M\to 0 \tag{8.3f}$$

is exact. By definition (7.8) and the fact that R is Gorenstein, we get

$$b^{2,0}=\mathrm{length}_R(M)=\mathrm{length}_R(\mathrm{Ext}^2_R(M,R)). \tag{8.3g}$$

Because N has finite length, $\mathrm{Ext}^i(N,R)=0$ for $i<2$ and hence there are isomorphisms

$$\mathrm{Ext}^i(\Omega^2_R/N,R)\xrightarrow{\cong}\mathrm{Ext}^i(\Omega^2_R,R)\qquad(i<2).$$

Using this together with part (c) and (8.3f), we get an exact sequence

$$0\to H^1(Y,\mathcal{O}_Y)\to\mathrm{Ext}^1_R(\Omega^2_R,R)\to\mathrm{Ext}^2_R(M,R)\to 0.$$

Using this sequence, and taking into account Lemma 8.1 and (8.3g), we get

$$b^{2,0}=\tau-\mathrm{length}_R H^1(Y,\mathcal{O})=\tau-p_g.$$

By 8.2(a,b), this yields $\chi^2=b^{2,0}-\tau=-p_g$. □

9. Wahl's example.

Using the general results of the preceding sections, we can now prove:

Theorem 9.1. *Let F be a field of characteristic 0 and*

$$R = F[x, y, z]/(z^2 + y^3 + x^{10} + ax^7y),$$

for any nonzero $a \in F$. Then $b^{0,1} = 1$ and $b^{1,1} = 0$. That is,

a) $H^1_{\mathrm{cdh}}(R, \mathcal{O}) \cong F$ *and*
b) $H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) = 0$.

Therefore, $H^1\mathcal{F}_{HH}(R/F) = 0$ but $H^2\mathcal{F}_{HH}(R/F) \neq 0$. In particular, if F is a number field, then R gives a negative answer to Bass' question:

$$K_0(R) = K_0(R[t]) \text{ but } K_0(R[t_1, t_2]) \cong K_0(R) \oplus stF[s, t].$$

Remark 9.1.1. The cdh cohomology groups in question may computed using an explicit description of a resolution of singularities, together with the self-intersection numbers of the exceptional components. For the surface in Theorem 9.1 for all values of a (including 0), the resolution data was checked for us by Liz Sell, and is displayed in Figure 1.

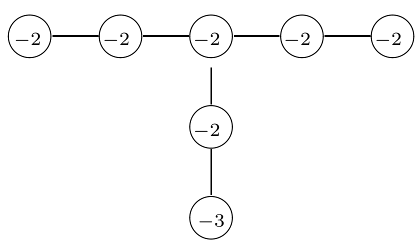

Figure 1. The Resolution graph for $z^2 + y^3 + x^{10} + ax^7y$

The proof we shall give here will be a straightforward application of the invariance of χ^p (Theorem 7.11), applied to the specific example:

$$X = \operatorname{Spec} F[x, y, z, t]/(z^2 + y^3 + x^{10} + tx^7y). \tag{9.2}$$

Consider the map $X \to S = \operatorname{Spec} F[t]$ induced by the obvious inclusion, and write X_s for the fiber over $s \in S$. When s is the point $t = a$ we have $X_s = \operatorname{Spec}(R)$ for the ring R in Theorem 9.1.

Proposition 9.3. *Let X be the affine variety of (9.2). Then the integer $\chi^p(X_s)$ is independent of the choice of closed point $s \in S$.*

Proof. Since the value of χ^p does not change upon passing to a finite extension, we may assume that F contains a primitive 30-th root of unity. Put

$$\tilde{X} = \operatorname{Spec} F[u, v, w, t]/(u^{30} + v^{30} + w^{30} + tu^{21}v^{10})$$

Let $G = \mu_3 \times \mu_{10} \times \mu_{15}$ act on $\tilde{X}$ by scalar multiplication on the variables x, y, z so that the assignment $x = u^3$, $y = v^{10}$ and $z = w^{15}$ identifies X with $\tilde{X}/G$.

One may readily verify that $X \to S$ is a flat local complete intersection whose singular locus is defined by $x = y = z = 0$ and hence maps isomorphically onto

S. The singular locus of $\tilde{X}$ is defined by $u = v = w = 0$ and hence also maps isomorphically onto S. Let $\tilde{Y}$ be the blowup of $\tilde{X}$ along its singular locus. One may readily verify that

$$\tilde{Y} = \mathrm{Proj}\left(\frac{F[t,u,v,w,A,B,C]}{(A^{30}+B^{30}+C^{30}+tuB^{10}A^{20}, uB-vA, uC-wA, vC-wB)}\right),$$

where t,u,v,w have degree 0 and A,B,C have degree 1. It is also easy to verify that $\tilde{Y} \to S$ is smooth. The fiber of $\tilde{Y} \to \tilde{X}$ over $\tilde{X}_{\mathrm{sing}}$ is

$$\tilde{E} = \mathrm{Proj}\, F[t,A,B,C]/(A^{30}+B^{30}+C^{30}) \cong S \times E_0$$

where E_0 is a smooth curve. We see that all the hypotheses of Theorem 7.11 are satsified. □

Now the proof of Theorem 9.1 is straightforward. The idea is that X_0 is quasi-homogeneous, so its du Bois invariants are easy to compute. This gives χ^p and we can then apply Propositions 8.2 and 8.3 along with a calculation of the Tjurina number to compute $b^{p,q}(X_s)$ for $s \neq 0$.

Example 9.4. The surface $X_0 = \mathrm{Spec}\, F[x,y,z]/(z^2+y^3+x^{10})$ is discussed by Wahl in [33, 4.4]. By elementary calculations, described in [33, 4.3], it is easy to calculate the invariants $\tau = 1 \cdot 2 \cdot 9 = 18$, $g = 0$ and

$$p_g = \dim\left(F[x,y,z]/(\tfrac{\partial f}{\partial x}, \tfrac{\partial f}{\partial y}, \tfrac{\partial f}{\partial z})\right)_{\leq 2} = 1$$

where $f = z^2+y^3+x^{10}$. Moreover, as with any isolated normal surface singularity defined by a non-negatively graded ring, we have $l = 0$ by [26, Theorem 2.3.1]. (Or, one may see $l = 0$ from the graph of Figure 1.) Using Lemma 7.10 and Proposition 8.3(a,e), this yields

$$b^{1,1} = b^{0,1} = p_g - g - l = 1, \quad b^{1,0} = b^{2,0} = \tau - p_g = 17.$$

By Proposition 8.2(a), $\chi^0 = -b^{0,1} = -1$, $\chi^1 = b^{1,0} - b^{1,1} = 16$, $\chi^2 = -1$.

Proof of Theorem 9.1. By Theorem 7.11, $\chi^p(X_s)$ does not depend on s and we write $\chi^p = \chi^p_s$. By Proposition 8.2(c), $b^{0,1} = -\chi^0$ is also independent of s. For the choice $s = 0$, we have $b^{0,1}_0 = 1$ by [33, 4.4] (see Example 9.4). This proves assertion (a).

To compute $b^{1,1}$ when $a \neq 0$, we use the calculation of $\tau(X_a)$ given in [33, 4.4]:

$$\tau(X_a) = \begin{cases} 18 & a = 0 \\ 16 & a \neq 0. \end{cases} \tag{9.5}$$

By Proposition 8.3(d)

$$b^{1,0}(X_a) \leq \tau(X_a) = 16 \quad \text{for all } a \neq 0. \tag{9.6}$$

By the invariance of χ^1 (see Proposition 9.3), Example 9.4 and (9.6), we have

$$\begin{aligned} 16 = \chi^1 &= b^{1,0}(X_a) - b^{1,1}(X_a) \\ &\leq 16 - b^{1,1}(X_a) \end{aligned}$$

for any $a \neq 0$, and hence $0 = b^{1,1}(X_a) = \dim_F H^1_{\mathrm{cdh}}(X_a, \Omega^1_{/F})$. □

Remark 9.7. a) In (9.5) of the proof above, we refer to the calculation of the Tjurina numbers τ stated by Wahl in [33, 4.4]. These can be checked directly using the Tjurina function of the SINGULAR library `sing.lib` ([17], [18]).

b) Steenbrink uses analytic methods to define an invariant α and proves that $b^{1,1} = p_g - g - l - \alpha$; see [33, (1.9.1)]. Comparing with Proposition 8.2(a), and using GAGA, we see that $\alpha = b^{0,1} - b^{1,1}$. It is this invariant that is computed by Wahl in [33, 4.4].

We finish this section with a general description of the relationship between NK_{-1} and NK_0 of surfaces. This sheds light on the difference between the cases of small and large base fields, and also explains some results of [41].

Theorem 9.8. *Let R be a 2-dimensional normal domain of finite type over a field F of characteristic 0. There is an exact sequence:*

$$0 \to NK_1^{(2)}(R) \to \Big(H^0(R, \Omega^1_{/F})/\Omega^1_{R/F}\Big) \otimes V \\ \to \Omega^1_F \otimes_F NK_{-1}(R) \to NK_0(R) \to H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) \otimes V \to 0.$$

Proof. Consider the following short exact sequence of sheaves in $(\mathrm{Sch}/F)_{\mathrm{cdh}}$:

$$0 \to \Omega^1_F \otimes_F \mathcal{O} \to \Omega^1 \to \Omega^1_{/F} \to 0$$

Applying H_{cdh} yields

$$0 \to \Omega^1_F \otimes_F R \xrightarrow{\iota} H^0(R, \Omega^1) \to H^0(R, \Omega^1_{/F}) \xrightarrow{\partial} \\ \Omega^1_F \otimes_F H^1_{\mathrm{cdh}}(R, \mathcal{O}) \to H^1_{\mathrm{cdh}}(R, \Omega^1) \to H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) \to 0$$

Note that, because $\Omega^1_R \to \Omega^1_{R/F}$ is onto, the map ∂ kills the image of $\Omega^1_{R/F}$. Similarly, the image of ι is contained in that of Ω^1_R. Thus we obtain

$$0 \to H^0(R, \Omega^1)/\Omega^1_R \to H^0(R, \Omega^1_{/F})/\Omega^1_{R/F} \to \\ \Omega^1_F \otimes_F H^1_{\mathrm{cdh}}(R, \mathcal{O}) \to H^1_{\mathrm{cdh}}(R, \Omega^1) \to H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) \to 0$$

Now apply $\otimes V$ and use 5.1 and parts c) and d) of 5.3. ☐

Corollary 9.9. *Let R be a 2-dimensional normal domain of finite type over a field F of characteristic 0. If $NK_{-1}(R) = 0$ then $NK_0(R) \cong H^1_{\mathrm{cdh}}(R, \Omega^1_{/F}) \otimes V$.*

Example 9.10. Let R be a 2-dimensional normal domain of finite type over $\mathbb{Q}$, and put $R_F = R \otimes F$. By 4.1 and 6.4,

$$NK_*(R_F) \cong NK_*(R) \otimes \Omega^*_{F/\mathbb{Q}}. \tag{9.11}$$

Keeping track of the λ-decomposition, as in 5.1, we see from Theorem 0.6 that

$$TK_1^{(2)}(R_F) \cong TK_1^{(2)}(R) \otimes F \cong H^0(R, \Omega^1) \otimes F/\Omega^1_R \otimes F \cong H^0(R_F, \Omega^1_{/F})/\Omega^1_{R_F/F}.$$

From Theorem 9.8 we get an exact sequence

$$0 \to \Omega^1_{F/\mathbb{Q}} \otimes_F NK_{-1}(R_F) \to NK_0(R_F) \to H^1_{\mathrm{cdh}}(R_F, \Omega^1_{/F}) \otimes V \to 0 \tag{9.12}$$

Using (9.11) and 0.6 again, we see that the sequence (9.12) is isomorphic to the sum

$$(0 \to \Omega^1_{F/\mathbb{Q}} \otimes H^1_{\mathrm{cdh}}(R,\mathcal{O}) \otimes V \xrightarrow{\simeq} \Omega^1_{F/\mathbb{Q}} \otimes H^1_{\mathrm{cdh}}(R,\mathcal{O}) \otimes V \to 0 \to 0)$$
$$\oplus$$
$$(0 \to 0 \to F \otimes H^1_{\mathrm{cdh}}(R,\Omega^1) \otimes V \xrightarrow{\simeq} F \otimes H^1_{\mathrm{cdh}}(R,\Omega^1) \otimes V \to 0)$$

For example, for $R_F := F[x,y,z]/z^2 + y^3 + x^{10} + x^7 y$ we have

$$NK_{-1}(R_F) = F \otimes V$$
$$NK_0(R_F) = \Omega^1_{F/\mathbb{Q}} \otimes V \cong \bigoplus_{p=1}^{\mathrm{tr.\,deg}(F)} F \otimes V.$$

In other words, both typical pieces $TK_{-1}(R_F)$ and $TK_0(R_F)$ are F-vectorspaces, but while $\dim_F TK_{-1}(R_F) = 1$ for all F, any cardinal number κ can be realized as $\dim_F TK_0(R_F)$ for an appropriate F.

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