Localization for the K-Theory of Noncommutative Rings

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Abstract. If S is denominator set in a ring A, we describe the third term in the long exact sequence relating the K-theory of A and S\(^{-1}\)A. It is the Waldhausen K-theory of a category Perf(A, S). If A \(\rightarrow\) B is an analytic isomorphism along S, this third term satisfies excision, yielding a long exact Mayer-Vietoris sequence in K-theory.

The recent work [TT] of Thomason and Trobaugh establishes a localization theorem for the K-theory of commutative rings and quasi-compact quasi-separated schemes. This paper is partly an attempt to give a simple exposition of their proof in the important case A \(\rightarrow\) S\(^{-1}\)A, and partly an extension of their proof to the noncommutative case. When S consists of nonzeros-divisors, we recover the calculations of [GQ] and [Gr], since in that case our Perf(A, S) has the same K-theory as the exact category \(\mathcal{H}_S(A)\). We include an excision result which is new even in the commutative case.

To understand the statement of our localization theorem, we introduce some terms. Let A be a ring with unit. A strictly perfect complex \(P = P^*\) is a bounded chain complex of finitely generated projective left A-modules. A chain complex \(E = E^*\) of left A-modules is a perfect complex if there is a strictly perfect complex \(P^*\) and a quasi-isomorphism \(P \rightarrow E\). The category Perf(A) of perfect complexes forms a "Waldhausen" category, i.e., a category with cofibrations (degreewise split monics) and weak equivalences (quasi-isomorphisms). The K-theory of A is the same as the Waldhausen K-theory of Perf(A). (See 1.1 below.)

We are interested in localizing A at a multiplicatively closed subset S to form a left quotient ring S\(^{-1}\)A whose elements have the form s\(^{-1}\)a (s \(\in\) S, a \(\in\) A). This exists if S is a left denominator set, i.e., it satisfies the following conditions:

(i) ("Ore condition") (\(\forall s \in S, a \in A\))(\(\exists t \in S, b \in A\))ta = bs
(ii) (Annihilators) (\(\forall s \in S, a \in A\)) if as = 0 then (\(\exists t \in S\))ta = 0.

See [F], 16.9. This hypothesis is sufficient to make S\(^{-1}\)A flat as a right A-module, so that M \(\mapsto\) S\(^{-1}\)M = S\(^{-1}\)A \(\otimes\) A M is an exact functor from A-mod to S\(^{-1}\)A-mod. We remark that any central multiplicatively closed S will automatically be a left denominator set.

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Suppose that $S$ is a left denominator set in $A$. The category $\text{Perf}(A, S)$ of perfect $A$-module complexes $E$ such that $S^{-1}E$ is exact forms a Waldhausen subcategory of $\text{Perf}(A)$, so it makes sense to talk about its algebraic $K$-theory. We will see below that it has the same $K$-theory as the subcategories $\text{Perf}^{t}(A, S)$, $\text{Perf}_{b}(A, S)$ and $\text{Perf}^{-}(A, S)$ of strictly perfect, bounded, and bounded above complexes, respectively.

**Localization Theorem.** Let $S$ be a left denominator set in a ring $A$, for example a central multiplicatively closed set. Then there is a long exact sequence

$$
\cdots \to K_{n+1}(S^{-1}A) \xrightarrow{\partial} K_{n}\text{Perf}(A, S) \to K_{n}(A) \to K_{n}(S^{-1}A) \xrightarrow{\partial} \cdots
$$

valid for all integers.

**Formula.** Here is an explicit formula for the boundary map $\partial : K_{1}(S^{-1}A) \to K_{0}\text{Perf}(A, S)$. If $s \in S$, then $s$ is a unit of $S^{-1}A$ and $\partial(s)$ is represented by the complex

$$
0 \to A \xrightarrow{s} A \to 0
$$

concentrated in degrees 0 and 1. More generally, every matrix $\beta \in GL_{n}(S^{-1}A)$ is of the form $s^{-1}\alpha$ for some $s \in S$ and some $\alpha \in M_{n}(A)$. Then $\partial(\beta) = \partial(\alpha) - n\partial(s)$, where $\partial(\alpha)$ is represented by

$$
0 \to A^{n} \xrightarrow{\alpha} A^{n} \to 0.
$$

**Remark.** We will only construct a sequence ending in $K_{0}(A) \to K_{0}(S^{-1}A)$. However, the argument of [C] shows that we can define negative $K$-groups for $\text{Perf}(A, S)$ and continue the above sequence to negative values of $n$. On a spectrum level, if $K(A)$ denotes the nonconnective spectrum for the $K$-theory of $A$, the fiber $F$ of $K(A) \to K(S^{-1}A)$ is a nonconnective delooping of the usual connective $K$-theory spectrum for $\text{Perf}(A, S)$.

**Remark.** The proof of the Localization Theorem is much easier if we assume that $S$ is central. The difficulty with the general case is that clearing denominators in commutative diagrams is delicate. (See 3.1.)

**Remark.** It seems probable that the Localization Theorem remains valid if $A \to S^{-1}A$ is replaced by a flat epimorphism $A \to B$. However, the techniques of this paper do not immediately extend to that case. Several variants of the Localization Theorem may be found in the second author's thesis [Yao].

In order to compute with the Localization Theorem, we provide an excision result for analytic isomorphisms along $S$.

**Definition.** (Cf. [TT] and [We]) Let $S$ be a left denominator set in $A$. We say that a ring map $f : A \to B$ is a (left) analytic isomorphism along $S$ if

a) $f(S)$ is a left denominator set in $B$

b) $A/As \xrightarrow{\cong} B/Bs$ for all $s \in S$

c) $\text{Tor}^{A}_{p}(B, A/I) = 0$ for $p \neq 0$ and all left ideals $I$ of $A$ meeting $S$. 


Remark. Condition b) implies that $A/I \cong B/BI$ for every ideal $I$ meeting $S$. If $A$ is commutative, then c) is implied by the condition of [TT] that $B_P$ is flat over $A_P$ for all primes $P$ of $A$ meeting $S$. If $S$ consists of central nonzerodivisors in $A$, we will show in 5.5(b) below that c) is equivalent to the assertion that $f(S)$ consists of right (hence left) nonzerodivisors in $B$. Thus our notion includes the notion of analytic isomorphism used in [K] and [We]. The term “analytic isomorphism” comes from the fact that the $S$-adic completions $\hat{A} = \lim A/As$ and $\hat{B} = \lim B/Bs$ are isomorphic.

Excision Theorem. Let $A \to B$ be an analytic isomorphism along $S$. Then the total tensor product map

$$B \otimes_A^L : K(\text{Perf}^{-}(A, S)) \to K(\text{Perf}^{-}(B, S))$$

is a homotopy equivalence of spectra. Consequently, there is a long exact Mayer-Vietoris sequence (for all integers $n$):

$$\cdots \to K_{n+1}(S^{-1}B) \xrightarrow{\partial} K_n(A) \to K_n(B) \oplus K_n(S^{-1}A) \to K_n(S^{-1}B) \xrightarrow{\partial} \cdots$$

Remark. Our proof follows the proof of [TT, 3.19]. If $S$ is a central set of nonzerodivisors on $A$ and $B$, this result was proven by Karoubi [K] by showing that $\mathcal{H}_S(A) \approx \mathcal{H}_S(B)$. (See [We, 1.1]).

§1. The proof of the Localization Theorem.

The $K$-theory of $A$ is the $K$-theory of the category $\mathcal{P}(A)$ of fin. gen. projective left $A$-modules; either Quillen $K$-theory or Waldhausen $K$-theory may be used by [Wa, 1.9]. In order to compare the $K$-theory of $A$ to that of $\text{Perf}(A, S)$, we invoke the following result.

Lemma 1.1. (Waldhausen) The following subcategories of $\text{Perf}(A)$ have the same $K$-theory (the $K$-theory of $A$):

a) $\mathcal{P}(A)$, the fin. gen. projective $A$-modules
b) $\text{Perf}^r(A)$, the strictly perfect complexes
c) $\text{Perf}^b(A)$, the bounded perfect complexes
d) $\text{Perf}^{-}(A)$, the bounded above perfect complexes
e) $\text{Perf}(A)$, all perfect complexes.

In all cases, cofibrations are degreewise split monics, and the weak equivalences w are the quasi-isomorphisms.

Proof. By [Gi, 6.2], the categories a) and b) have the same $K$-theory. The inclusion of strictly perfect complexes in either bounded or bounded above perfect complexes satisfies the approximation property (App) of [Wa, 1.6.7] by a standard exercise (see [SGA 8, I.2.7.1] or [TT, 1.9.5]). The inclusion of category d) in category e) satisfies the dual approximation property (App$^*$) since we can truncate complexes. By the Approximation Theorem [Wa, 1.6.7], these categories have the same $K$-theory. □
Porism 1.2. It follows from the proof of 1.1 that the following subcategories of $\text{Perf}(A, S)$ have the same $K$-theory:

a) $\text{Perf}^a(A, S) = \text{Perf}^a(A) \cap \text{Perf}(A, S)$
b) $\text{Perf}^b(A, S) = \text{Perf}^b(A) \cap \text{Perf}(A, S)$
c) $\text{Perf}^c(A, S) = \text{Perf}^c(A) \cap \text{Perf}(A, S)$
d) $\text{Perf}(A, S)$

In all cases, cofibrations are degreewise split monics, and the weak equivalences $w$ are the quasi-isomorphisms.

In order to prove the Localization Theorem, we introduce a new notion of weak equivalence on the category with cofibrations $\text{Perf}(A)$. We let $v$ denote the class of all maps $E \to F$ such that $S^{-1}E \to S^{-1}F$ is a quasi-isomorphism. The subcategory of perfect complexes $v$-equivalent to zero is $\text{Perf}(A, S)$, so Waldhausen’s Fibration Theorem [W, 1.6.4] states that there is a homotopy fibration of spectra (yielding a long exact sequence on $K$-groups):

$$K(\text{Perf}(A, S)) \to K(\text{Perf}(A), w) \to K(\text{Perf}(A), v).$$

The middle term gives the $K$-theory of $A$, so it suffices to compare the right term to the $K$-theory of $S^{-1}A$.

Since $S^{-1}A$ is flat over $A$, localization provides an exact functor from $\text{Perf}(A)$ to $\text{Perf}(S^{-1}A)$. This functor not only factors through the change in weak equivalence (from $w$ to $v$), but it also factors through a category $B$, which we now define.

Definition 1.4. Let $B$ denote the full subcategory of $\text{Perf}(S^{-1}A)$ consisting of those perfect $S^{-1}A$-module complexes $E^*$ such that the class $[E^*]$ in $K_0(S^{-1}A)$ is in the image of the map $K_0(A) \to K_0(S^{-1}A)$. We make $B$ into a category with cofibrations (degreewise split monics) and weak equivalences (quasi-isomorphisms).

Thomason’s version of the Cofinality Theorem for Waldhausen $K$-theory [TT, 1.10.1] applies to the inclusion of $B$ in $\text{Perf}(S^{-1}A)$, proving that $K_n(B) \to K_n(S^{-1}A)$ is an isomorphism for $n \geq 1$, and that $K_0(B)$ is the image of $K_0(A) \to K_0(S^{-1}A)$. Therefore in order to prove the Localization Theorem it suffices to establish the following assertion:

$$K(\text{Perf}(A), v) \to K(B)$$

is a homotopy equivalence. This will be a consequence of the Thomason-Trobaugh Approximation Theorem [TT, 1.9.8], once we prove (in section 3 below) that the map of derived categories

$$T : v^{-1} \text{Perf}(A) \to w^{-1}B$$

is an equivalence. The first step is to show that every complex $E^*$ in $B$ is quasi-isomorphic to $S^{-1}P^*$ for some perfect $A$-module complex $P^*$, i.e., that every object of $w^{-1}B$ is isomorphic to $T(P^*)$ for some $P^*$ in $v^{-1} \text{Perf}(A)$. This is the subject of the next section.
§2. An Extension Criterion.

In this section we shall assume that $S$ is either central or a left denominator set in $A$. The following Exercise is trivial when $S$ is central. When $S$ is a left denominator set, it uses the fact that any finite subset $\{b_i\}$ of $S^{-1}A$ has a common denominator, i.e., is of the form $\{t^{-1}a_i\}$.

**Exercise 2.1.** Fix a left denominator set $S$ in $A$, and let $E^\bullet$ be a bounded chain complex of fin. gen. free left $S^{-1}A$-modules. Then there is a bounded complex $F^\bullet$ of fin. gen. free $A$-modules and an isomorphism $f : S^{-1}F^\bullet \to E^\bullet$ of $S^{-1}A$-module complexes. Moreover, if we use a choice of basis to represent the $f^i$ by matrices and assume that $E^i = 0$ for $i > n$, then $f^n = 1$ and every other $f^i$ is right multiplication by an element of $S$.

**Corollary 2.2.** If $P$ is a fin. gen. projective $A$-module and

$$0 \to E^m \to E^{m+1} \to \cdots \to E^{n-1} \to E^n \to 0$$

is an $S^{-1}A$-module complex with $E^i$ fin. gen. free for $i \neq n$ and $E^n \cong S^{-1}P$, then there is a bounded chain complex $P^\bullet$ of fin. gen. projective $A$-modules with $P^n = P$ and an isomorphism $f : S^{-1}P^\bullet \to E^\bullet$ of $S^{-1}A$-module complexes.

**Proof.** Choose $Q$ so that $P \oplus Q$ is fin. gen. free and apply 2.1 to $E^\bullet \oplus (S^{-1}Q(n))$ to get a free complex $F^\bullet$ with $F^n = P \oplus Q$ and an isomorphism $S^{-1}F^\bullet \cong E^\bullet \oplus (S^{-1}Q(n))$ in which

$$S^{-1}E^n \cong (S^{-1}P) \oplus (S^{-1}Q) \cong E^n \oplus (S^{-1}Q)$$

is the canonical map. Now set $P^\bullet = F^\bullet/Q(n)$.

**Corollary 2.3A.** If $E^\bullet$ is a strictly perfect $S^{-1}A$-module complex, then there is a bounded complex $F^\bullet$ of fin. gen. free $A$-modules, an $S^{-1}A$-module complex $D^\bullet$ and an isomorphism $S^{-1}F^\bullet \cong D^\bullet \oplus E^\bullet$ of $S^{-1}A$-module complexes.

**Proof.** Each $E^i$ is a fin. gen. projective $S^{-1}A$-module, so there are $S^{-1}A$-modules $D^i$ with $D^i \oplus E^i$ fin. gen. free. Assemble the $D^i$ into a complex (e.g., by 0 maps) and apply 2.1.

**Remark 2.3.1.** This is the elementary analogue of [TT, 5.5.1]. Thomason and Trobaugh need to work harder, invoking the derived category of $S^{-1}A$, because of their geometric context. In loc. cit., they state that "despite the flagrant triviality of the proof, this result is the key point in [TT]."

**Extension Criterion 2.4.** The following assertions are equivalent for every perfect $S^{-1}A$-module complex $E^\bullet$:

(i) $E^\bullet$ is quasi-isomorphic to $S^{-1}P^\bullet$ for some perfect $A$-module complex $P^\bullet$
(ii) The class $[E^\bullet] \in K_0(S^{-1}A)$ is in the image of $K_0(A) \to K_0(S^{-1}A)$.

**Proof.** That (i) implies (ii) is clear. For the converse, we may suppose that $E^\bullet$ is strictly perfect, so $[E^\bullet] = \sum (-1)^i [E^i]$. By adding short complexes of the form $0 \to D^i = D^{i+1} \to 0$, we may assume every $E^i$ is free except $E^n$, that $E^i = 0$ for $i > n$, and that $[E^n] = [S^{-1}P]$ for some projective $A$-module $P$. Hence $E^n$ and $S^{-1}P$ are stably isomorphic $S^{-1}A$-modules, i.e., $E^n \oplus (S^{-1}A)^r \cong S^{-1}(P \oplus A)^r$ for some $r$. Adding $(S^{-1}A)^r$ in dimensions $n - 1$ and $n$, we may assume that in fact $E^n \cong S^{-1}P$. Now apply 2.2 to obtain (i).
§3. Equivalence of Derived Categories.

If $w$ is a class of maps in a skeletally small additive category $C$, there is an additive category $w^{-1}C$ and a functor $Q : C \rightarrow w^{-1}C$ sending $w$ to isomorphisms which is universal in this respect. If $w$ is a multiplicative system $[H, 1.3]$, this is an especially nice construction, since $w^{-1}C$ has the same objects as $C$ and every morphism is represented by a diagram in $C$ of the form

$$E \xleftarrow{\alpha} E' \xrightarrow{\alpha} F.$$ 

This follows from the calculus of fractions $[V, 2.3.2] [H, 3.1]$

**Theorem 3.1.** Let $B \subseteq \text{Perf}(S^{-1}A)$ be as in (1.4), with $w$ being the quasi-isomorphisms. Let $v$ be the class of maps $v : E \rightarrow F$ in $\text{Perf}(A)$ such that $S^{-1}v : S^{-1}E \rightarrow S^{-1}F$ is a quasi-isomorphism. Then

$$T : v^{-1} \text{Perf}(A) \rightarrow w^{-1}B$$

is an equivalence of categories.

**Reduction.** The Extension Criterion 2.4 shows that every object of $B$, hence of $w^{-1}B$, comes from an object of $\text{Perf}(A)$. Therefore, it is enough to show that the functor $T$ is full and faithful. The following argument, copied from $[TT, 5.2.6]$, shows that it is enough to prove that $T$ is full, for this implies that $T$ is also faithful. Since $v$ is a multiplicative system, every map in $v^{-1} \text{Perf}(A)$ is represented as $E \xleftarrow{\nu'} E' \xrightarrow{\sigma} F$ with $\nu'$ in $v$. Suppose that $T$ sends this map (or equivalently, $\alpha$) to zero in $w^{-1}B \subseteq w^{-1} \text{Perf}(S^{-1}A) \subseteq D(S^{-1}A)$. Let $C$ be the mapping cone of $\alpha$, so that

$$C(1) \xrightarrow{x} E' \xrightarrow{\alpha} F \xrightarrow{\delta} C$$

forms a distinguished triangle of perfect $A$-module complexes. Since $\text{Hom}(E, -)$ is a cohomological functor $[V, 1.2] [H, 1.1]$ we have a diagram of abelian groups with exact rows:

$$\begin{array}{ccc}
\text{Hom}_v(E, C(1)) & \xrightarrow{x} & \text{Hom}_v(E, E') \xrightarrow{\alpha} \text{Hom}_v(E, F) \\
\downarrow T & & \downarrow T \\
\text{Hom}_w(S^{-1}E, S^{-1}C(1)) & \xrightarrow{x} & \text{Hom}_w(S^{-1}E, S^{-1}E') \xrightarrow{0} \text{Hom}_w(S^{-1}E, S^{-1}F)
\end{array}$$

For clarity, we have written $\text{Hom}_v$ (resp. $\text{Hom}_w$) for $\text{Hom}$ in the triangulated category $v^{-1} \text{Perf}(A)$ (resp. $w^{-1}B$). Since we have assumed that $T$ is full, the vertical maps are onto. Hence there is a map $E \xrightarrow{\nu} E' \xrightarrow{\sigma} C(1)$ in $v^{-1} \text{Perf}(A)$ such that $T(v^{-1}\nu) = 0$. This forces $\nu$ to be zero in $v^{-1} \text{Perf}(A)$, proving that $T$ is faithful.

**Proof that $T$ is full.** Note that $T$ is an additive functor between additive categories. As every strictly perfect complex is a direct summand of a bounded
f.g. free complex, we are reduced to showing that if $E$ and $F$ are bounded complexes of fin. gen. free $A$-modules, then

$$T : \text{Hom}_{w^{-1}\text{Perf}(A)}(E, F) \rightarrow \text{Hom}_{w^{-1}B}(S^{-1}E, S^{-1}F)$$

is onto. By [SGA6, I.2.7] and [V, I.2.4.2], every map in $w^{-1}B$ from $S^{-1}E$ to $S^{-1}F$ is represented by a chain map $\beta : S^{-1}E \rightarrow S^{-1}F$. Regarding the $\beta^a$ as matrices, we can clear denominators in the entries of the $\beta^a$ to find $A$-module maps $\alpha^n : E^n \rightarrow F^n$ and $s \in S$ so that $\beta^a = s^{-1}\alpha^a$ for all $n$. As a warmup we consider the easy case first.

**Easy case:** $S$ is central. Because over $S^{-1}A$ we have

$$(\alpha^{a^{-1}}d_F - d_E\alpha^a) = s(\beta^{a^{-1}}d_F - d_E\beta^a) = 0,$$

some $t \in S$ annihilates $\alpha^{a^{-1}}d_F - d_E\alpha^a$. Replacing $s$ by $ts$ and $\alpha^a$ by $t\alpha^a$, we have arranged that the $\{\alpha^a\}$ assemble to form a chain map $\alpha : E \rightarrow F$.

Multiplication by $s$ is a chain map $E \rightarrow E$ lying in $v$, and evidently the map

$$E \xrightarrow{s} E \xrightarrow{\alpha} F$$

in $v^{-1}\text{Perf}(A)$ maps to $\beta$. We are done in this case.

**General case.** When $S$ is not central, multiplication by $s$ is not even canonically defined, let alone a chain map. To define it, we fix a basis for the free modules $E^n$ and let $s : E \rightarrow E$ be the map represented by the diagonal matrix $sI$, i.e., multiply the basis elements by $s$ and extend linearly to $E$. Since $E^n = 0$ for $n > N$, we may use the following lemma, together with descending induction on $n$, to see that by changing our choice of the $\alpha^a$ (and $s_n \in S$ so that $\beta^a = s^{-1}_n\alpha^a$) we can find a new chain complex

$$E' : \cdots \rightarrow E^{n-1} \xrightarrow{e^{n-1}} E^n \xrightarrow{e^n} \cdots \rightarrow E^N \rightarrow 0$$

and a diagram of chain maps

$$E \xrightarrow{\{s_n\}} E' \xrightarrow{\{\alpha^a\}} F.$$

Since $\{s_n\}$ is in $v$, this represents a map in $v^{-1}\text{Perf}(A)$ lifting $\beta$. This will finish the proof.

**Lemma.** Suppose we are given $s_n \in S$ and $\alpha^a : E^n \rightarrow F$ so that $\beta^a = s^{-1}_n\alpha^a$, and a map $e^a : E^n \rightarrow E^{n+1}$ such that $s_n d_E = e^n s_{n+1}$. Then there is a map $e : E^{n-1} \rightarrow E^n$, an $s_{n-1} \in S$ and an $\alpha : E^{n-1} \rightarrow F^{n-1}$ so that $ee^a = 0$, $\beta^{n-1} = s^{-1}_{n-1}\alpha$ and such that the following diagram commutes:

$$
\begin{array}{ccc}
E^{n-1} & \xrightarrow{s_{n-1}} & E^{n-1} \\
\downarrow d_E & & \downarrow e \\
E^n & \xrightarrow{s_n} & E^n
\end{array}
\quad
\begin{array}{ccc}
E^{n-1} & \xrightarrow{\alpha} & F^{n-1} \\
\downarrow d_F & & \downarrow e \\
E^n & \xrightarrow{\alpha} & F^n
\end{array}
$$
Proof. Recall that $\beta^{n-1} = s^{-1}\alpha^{n-1}$ is given. Choose $e' \in S$ and $e' : E^{n-1} \to E^n$ so that $e's_n = s'd_E$. Then choose $t \in S$ and $a \in A$ so that $as' = ts$. Set $e'' = ae'$ and $\alpha'' = ta^n$, so that

$$(e''e^n)s_{n+1} = (ae')(s_n d_E) = a(s'd_E)d_E = 0$$

and over $S^{-1}A$ we have

$$(\alpha''d_F - e''\alpha^n) = t(s\beta^{n-1})d_F - (ae')(s_n \beta^n)$$

$$= as'(\beta^{n-1}d_F - d_E \beta^n)$$

$$= 0.$$

Therefore there is an $s'' \in S$ so that $s''e''e^n = 0$ and

$$s''(\alpha''d_F - e''\alpha^n) = 0.$$

Set $e = s''e''$, $s_{n-1} = s''as' = s''ts$ and $\alpha = s''\alpha''$. \[\square\]


The following two results are straightforward modifications of results in [TT, 2.4]. We need them for the excision result in the next section.

Recall from [SGA6, I.2] that an $A$-module chain complex $P^\bullet$ is said to be strictly pseudo-coherent if it is a bounded above complex of fin. gen. projective $A$-modules. A complex $E^\bullet$ is said to be pseudo-coherent if there is a quasi-isomorphism $P^\bullet \to E^\bullet$ with $P^\bullet$ strictly pseudo-coherent. Recall also that $\tau^n E$ is the good truncation

$$\cdots \to 0 \to d(E^{n-1}) \to E^n \to E^{n+1} \to \cdots.$$

**Theorem 4.1.** ([TT, 2.4.2]) Let $E$ be an $A$-module chain complex. The following are equivalent:

a) $E$ is pseudo-coherent

b) For all integers $n$ and $k$, and all directed systems $\{F_\alpha\}$ of $A$-module complexes, the canonical map (4.1.1) is an isomorphism.

(4.1.1) $$\lim_{\alpha} H^k(R \text{Hom}(E, \tau^n F_\alpha)) \cong H^k(R \text{Hom}(E, \lim_{\alpha} \tau^n F_\alpha))$$

c) Same as b) except we require the $F_\alpha$ to be strictly perfect

d) Same as c) except we require the $F_\alpha$ to be uniformly bounded above, and we require $E$ to be cohomologically bounded above.

e) For all integers $n$, and all directed systems $\{F_\alpha\}$ of $A$-module complexes, the canonical map (4.1.2) is an isomorphism.

(4.1.2) $$\lim_{\alpha} \text{Hom}_{D(A)}(E, \tau^n F_\alpha) \cong \text{Hom}_{D(A)}(E, \lim_{\alpha} \tau^n F_\alpha)$$

f) Same as e) except we require the $F_\alpha$ to be strictly perfect

g) Same as f) except we require the $F_\alpha$ to be uniformly bounded above, and we require $E$ to be cohomologically bounded above.
PROOF. We merely note the changes that are needed to modify the proof of [TT, 2.4.2]. Note that the meaning of "perfect" is slightly different in op. cit. In the proof that b) ⇒ e) we cite [H, I.6.4] instead of [TT, 2.4.1] to see that
\[ H^n R \text{Hom}(E, \tau^n F) = \text{Hom}_{D(A)}(E, \tau^n F). \]

In the proof that g) ⇒ a) we cite [SGA6, I.2.12 and I.2.7] instead of [TT, 2.2.13] to see that if \( E \oplus E' \) is \( n \)-pseudo-coherent then so is \( E \), and if \( E \) is \( n \)-pseudo-coherent for all \( n \) then \( E \) is pseudo-coherent. □

**Theorem 4.2.** ([TT, 2.4.3]) Let \( E \) be an \( A \)-module chain complex. The following are equivalent:

a) \( E \) is perfect

b) \( E \) is cohomologically bounded below, and for any directed system \( \{F_\alpha\} \) of \( A \)-module complexes, the canonical map (4.2.1) is an isomorphism.

\[
\lim_{\alpha} \text{Hom}_{D(A)}(E, F_\alpha) \xrightarrow{\cong} \text{Hom}_{D(A)}(E, \lim F_\alpha)
\]

(4.2.1)

c) \( E \) is cohomologically bounded, and (4.2.1) is an isomorphism for any directed system \( \{F_\alpha\} \) of strictly perfect complexes which is uniformly cohomologically bounded above.

**Proof.** We merely note the changes needed for the proof of [TT, 2.4.3] to go through. For a) ⇒ b) we cite [H, I.6.4] instead of [TT, 2.4.1], as above. For c) ⇒ a), the proof in [TT] shows that some \( E \oplus E' \) is isomorphic in \( D(A) \) to a strictly perfect complex. As in the proof of 4.1 above, this implies that \( E \) is pseudo-coherent. By [SGA6, I.5.8.1] \( E \) is perfect. □

§5. Excision.

In this section, we assume that \( f : A \to B \) is an analytic isomorphism along a left denominator set \( S \) in \( A \). In order to compare \( \text{Perf}(A, S) \) and \( \text{Perf}(B, S) \), we first compare the derived categories of \( A \) and \( B \).

Recall the construction of the total tensor product
\[
L_f^* = B \otimes^L_A - : D^- (A) \to D^- (B).
\]

Any bounded above \( A \)-module complex \( E \) has a quasi-isomorphism \( P \to E \) with a bounded above projective complex \( P \), and \( L_f^* (E) \) is \( B \otimes_A P \). The choice of \( P \) may be made functorial—use the total complex of the canonical free resolution—and therefore defines a functor from the category of bounded above \( A \)-module complexes to the category of bounded above \( B \)-module complexes. Restricting still further, but retaining the notation, we get functors
\[
L_f^* : \text{Perf}^- (A) \to \text{Perf}^- (B)
\]
\[
L_f^* : \text{Perf}^- (A, S) \to \text{Perf}^- (B, S).
\]

The former induces the map \( f^* : K(A) \to K(B) \). The latter map is the focal point of this section: we shall prove that it induces an isomorphism on \( K \)-theory.
PROPOSITION 5.1. Let $f : A \to B$ be an analytic isomorphism along $S$.

a) If $E$ is a bounded above complex of $A$-modules such that $S^{-1}E$ is exact, then the canonical map $E \to B \otimes_{A}^{L} E$ is an isomorphism in the derived category $D^{-}(A)$.

b) If $F$ is a bounded above complex of $B$-modules such that $S^{-1}F$ is exact, then the canonical map (obtained by thinking of $F$ as an $A$-module complex)

$$ B \otimes_{A}^{L} F \to F $$

is an isomorphism in the derived category $D^{-}(B)$.

PROOF. (Cf. [TT, 2.6.3 (a,b)]) For purposes of checking we may assume that $E$ and $F$ are also bounded below. The usual devissage argument (see op. cit.) now reduces to the case in which $E$ and $F$ are concentrated in one degree, i.e., $S$-torsion (left) modules. Since for every $A$-module $M$ we have

$$ H_{*}(B \otimes_{A}^{L} M) = \text{Tor}_{*}^{A}(B, M), $$

we are done by the following lemma. \[\]  

LEMMA 5.2. Let $A \to B$ be an analytic isomorphism along $S$. Then for every $S$-torsion left $A$-module $M$ we have $M \cong B \otimes_{A} M$ and $\text{Tor}_{p}^{A}(B, M) = 0$ for $p \neq 0$.

PROOF. If $M \cong A/I$ then $S$ meets $I$ and we are done by the definition of analytic isomorphism. An induction on the number of generators of $M$ proves this result if $M$ is finitely generated. As every $M$ is the union of its fin. gen. submodules, and Tor commutes with filtered colimits, the result holds for infinitely generated $M$ as well. \[\]

PROPOSITION 5.3. If $E$ is a perfect $B$-module complex with $S^{-1}E$ exact, then $E$ is also perfect as an $A$-module complex.

PROOF. (Cf. [TT, 2.6.3 (d)].) By truncating, we may assume that $E$ is bounded above. We appeal to criterion 4.2 (c) to see that $E$ is perfect, so let $\{F_{a}\}$ be a uniformly cohomologically bounded above directed system of strictly perfect $A$-module complexes. Given any bounded above complex $F$, let $\Gamma_{S}F$ denote the mapping cone of $F \to S^{-1}F$, translated by $+1$, so that $S^{-1}(\Gamma_{S}F)$ will be exact, and the natural map

$$ \text{Hom}_{D(A)}(E, \Gamma_{S}F) \to \text{Hom}_{D(A)}(E, F) $$

is an isomorphism. (To see this, use the long exact Hom sequence and note that $\text{Hom}(E, S^{-1}F)$ is trivial because $S^{-1}E \cong 0 \text{ in } D(A)$.) Using (5.1) and the adjointness property [V, 2.3.3], we see that if $E$ and $F$ are bounded above

$$ \text{Hom}_{D(A)}(E, \Gamma_{S}F) \cong \text{Hom}_{D(A)}(E, B \otimes_{A}^{L} \Gamma_{S}F) $$

$$ \cong \text{Hom}_{D(B)}(B \otimes_{A}^{L} E, B \otimes_{A}^{L} \Gamma_{S}F) $$

$$ \cong \text{Hom}_{D(B)}(E, B \otimes_{A}^{L} \Gamma_{S}F). $$
Now set $F = \lim_{\alpha} F_{\alpha}$, and note that $\lim_{\alpha} (\Gamma_S F_{\alpha}) \cong \Gamma_S F$.

Using criterion 4.2 (c) in $D(B)$ and the isomorphism above (first putting $F = F_{\alpha}$ and then $F = \lim_{\alpha} F_{\alpha}$), we therefore have

$$\lim_{\alpha} \text{Hom}_{D(A)}(E, \Gamma_S F_{\alpha}) \cong \lim_{\alpha} \text{Hom}_{D(B)}(E, B \otimes_A^L \Gamma_S F_{\alpha})$$
$$\cong \text{Hom}_{D(B)}(E, \lim_{\alpha} B \otimes_A^L \Gamma_S F_{\alpha})$$
$$\cong \text{Hom}_{D(B)}(E, B \otimes_A^L \Gamma_S F) \cong \text{Hom}_{D(A)}(E, \Gamma_S F).$$

Hence

$$\lim_{\alpha} \text{Hom}_{D(A)}(E, F_{\alpha}) \cong \text{Hom}_{D(A)}(E, F).$$

Using 4.2 (c), this proves that $E$ is perfect in $D(A)$.

**Corollary 5.4.** The forgetful functor from $B$-modules to $A$-modules induces a functor $u : \text{Perf}(B, S) \to \text{Perf}(A, S)$ and an equivalence of derived categories

$$\mathcal{w}^{-1} \text{Perf}(B, S) \approx \mathcal{w}^{-1} \text{Perf}(A, S),$$

whose inverse is the total tensor product $B \otimes_A^L \_$.  

**Proof of Excision Theorem.** By 5.1 and [Wa, 1.3.1] the compositions

$$\text{Perf}^{-}(A, S) \overset{L'}{\leftarrow} \text{Perf}^{-}(B, S) \overset{u}{\rightarrow} \text{Perf}^{-}(A, S)$$
$$\text{Perf}^{-}(B, S) \overset{u}{\rightarrow} \text{Perf}^{-}(A, S) \overset{L'}{\rightarrow} \text{Perf}^{-}(B, S)$$

induce maps on $K$-theory which are homotopy equivalent to the identity. The existence of the Mayer-Vietoris sequence is a formal consequence of the homotopy equivalence

$$K(\text{Perf}^{-}(A, S)) \sim K(\text{Perf}^{-}(B, S)),$$

given the Localization Theorem. (See, e.g., [We, 1.2]).

We conclude with the following promised result, that our notion of analytic isomorphism generalizes both the notion of “isomorphism infinitely near $Y$” of [TT] and the notion of analytic isomorphism used in [K] and [We].

**Lemma 5.5.** Let $S$ be central in $A$ and $f : A \to B$ a map such that $A/As \cong B/BS$ for all $s \in S$, and $f(S)$ is a left denominator set in $B$.

a) $f$ is an analytic isomorphism if and only if $\text{Tor}_p^A(B, A/As) = 0$ for $p \neq 0$ and all $s \in S$.

b) If $S$ consists of nonzerodivisors in $A$, then $f$ is an analytic isomorphism if and only if $S$ consists of right nonzerodivisors in $B$. 
Proof. If $s$ is a nonzerodivisor on $A$, then $\text{Tor}_p^A(B, A/As) = 0$ for $p > 1$ and $\text{Tor}_1^A(B, A/As) \cong \{b \in B : bs = 0\}$. Therefore a) implies b). To see a), let $I$ be a left ideal of $A$ containing $s \in S$ and set $J = As$. As $A/I \cong A/J \otimes_A A/I$ there is a spectral sequence

$$E^2_{pq} = \text{Tor}_p^A(\text{Tor}_q^A(B, A/J), A/I) \Rightarrow \text{Tor}_{p+q}^A(B, A/I).$$

If $\text{Tor}_q^A(B, A/As) = 0$ for $q \neq 0$ and $B/Bs \cong A/As$, the spectral sequence collapses to give

$$\text{Tor}_p^A(B, A/I) \cong \text{Tor}_p^A(A/J, A/I).$$

This vanishes for $p \neq 0$, proving (a).

Remark. The proof goes through if, instead of assuming $S$ central, we assume that $As$ is a 2-sided ideal of $A$ for all $s \in S$.

References


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