1. Bicategories.

The key idea in Waldhausen's approach to multiplicative structures is that one should consider bicategories (or double categories) as well as categories in K-theory. The best introduction to bicategories is [K-S]; [E] and [Mac, p. 44] also give formal definitions. We will recount and use the viewpoint of [S] of [Wa] in this section.

Associated to every small category \( S \) is a simplicial set \( S \), called its nerve: \( S_0 \) is the set of objects of \( S \), \( S_1 \) is the set of morphisms, \( S_2 \) is the set of pairs of composable morphisms (i.e., \( S_2 = S_1 \times S_1 \)), etc. The category \( S \) can be completely recovered from its nerve, and it is possible to write down axioms that describe which simplicial sets are nerves of categories. Identifying small categories and their nerves, we will think of a small category as a special kind of simplicial set.

Thinking of a simplicial set as a functor \( S_\cdot : \Delta^{op} \rightarrow (\text{Sets}) \), where \( \Delta \) is the category of finite ordinal numbers ([Mac, p. 171]), a bisimplicial set is a functor \( C_\cdot : \Delta^{op} \times \Delta^{op} \rightarrow (\text{Sets}) \). We may visualize \( S_\cdot \) as a lattice of sets:

\[
\begin{array}{cccc}
S_0 & S_1 & \cdots \\
\downarrow & \downarrow & \cdots \\
S_0 & S_1 & \cdots \\
\downarrow & \downarrow & \cdots \\
S_0 & S_1 & \cdots
\end{array}
\]

A (small) bicategory \( S_\cdot \) is then a special kind of bisimplicial set, namely one for which each of the simplicial sets \( S_n \), and

\[
\begin{array}{cccc}
S_0 & S_1 & \cdots \\
\downarrow & \downarrow & \cdots \\
S_0 & S_1 & \cdots \\
\downarrow & \downarrow & \cdots \\
S_0 & S_1 & \cdots
\end{array}
\]
$S_\infty$ are categories. The interchange law is automatic, as it merely states that $d_{i_4}^1 d_{i_3}^1 - d_{i_1}^1 d_{i_2}^1 : S_{i_2} = S_{i_1}$. We call the sets $S_{i_1}, S_{i_0}$ and $S_{i_0}, S_{i_0}$ the sets of bimorphisms, horizontal and vertical morphisms, and objects, respectively.

In the notation of [E-S], bimorphisms are called squares. The horizontal source and target maps $d_{i_1}^1 d_{i_0}^1: S_{i_2} = S_{i_1}$ strip off the left and right edges of the squares, while the vertical source and target maps $d_{i_0}^1 d_{i_1}^1: S_{i_2} = S_{i_0}$ strip off the top and bottom edges.

Two pertinent constructions of bicategories from categories $A, B, C$ are $\text{bi}(C)$, whose bimorphisms are commutative squares in $C$

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\Downarrow & & \Downarrow \\
B & \rightarrow & C
\end{array}
\]

and $A \otimes B$, whose objects are pairs $(A,B)$ in $\text{Obj}(A) \times \text{Obj}(B)$ and bimorphisms are pairs $(a:A \rightarrow A', b:B \rightarrow B')$ in $\text{Mor}(A) \times \text{Mor}(B)$.

We can represent $(a,b)$ as the square

\[
\begin{array}{ccc}
(A_0, B_0) & \xrightarrow{(a,1)} & (A_1, B_0) \\
(1,b) & \downarrow & (1,b) \\
(A_0, B_1) & \xrightarrow{(a,1)} & (A_1, B_1)
\end{array}
\]

The content of Lemma 3 on p. 170 of [Ma] is: if we think in terms of bisimplicial sets, then $A \otimes B$ is the product of $A$ and $B$ in the sense that $(A \otimes B)_n = A_n \times B_n$.

There is a geometric realization functor $B$ from bicategories (through bisimplicial sets) into topological spaces, described on [Wa, p. 164]. There is also a diagonalization functor 'diag' from bisimplicial sets into simplicial sets, and it is well known that $B \text{diag} = B$. Since $\text{diag}(A \otimes B)$ is the usual product of categories $A \otimes B$, we have $(B \text{diag})(A \otimes B) = B(A \otimes B).$ Since there is a map $\text{diag}(C) \rightarrow C$, $B$ is a retract of $B(\text{diag}(C))$.

From these remarks it is clear that for every functor $A \otimes B = C$ there is a commutative diagram

\[
\begin{array}{ccc}
RA \times RB & \rightarrow & BC \\
\downarrow & & \downarrow \\
B(A \otimes B) & \rightarrow & B(\text{diag}(C))
\end{array}
\]

2. Waldhausen's Product.

When $A$ is a small exact category, Waldhausen defines (on p. 194 of [Ma]) a bicategory $QA$ as follows. The bimorphisms are equivalence classes of commutative diagrams

\[
\begin{array}{ccc}
&& \\
& \Downarrow & \\
& \Downarrow & \\
& \Downarrow & \\
& \Downarrow & \\
& \Downarrow & \\
\end{array}
\]

in which the four little squares can be embedded in a $3 \times 3$ diagram with short exact rows and columns. Two diagrams are equivalent if they are isomorphic by an isomorphism which respects the identity on each corner object. Waldhausen proves (on p. 196) that the loop space $EQQA$ is homotopy equivalent to $BQA$ (the category QA is defined on $[Q, p.100]$). Thus by definition ([Q, p.103]) we have $K_pA = \tau_{p+1}BQA = \tau_{p+2}EQQA$. 
If $A$, $B$, $C$ are small exact categories, a functor $\mathbf{b}: A \times B \to C$ is called biexact if (i) each partial functor $\mathbf{b}(-): B \to C$, $(-)\mathbf{b}: A \to C$ is exact and if (ii) $A\mathbf{b} \circ \mathbf{b} = \mathbf{b} \circ B$ for distinguished zero objects $0$ of $A$, $B$, $C$. Note that we can assume $C$ skeletal if necessary to obtain the technical condition (ii). Given a biexact functor $\mathbf{b}$, there is an induced bicategory factorization

$$QA \otimes_{QB} QC \to h!(QC)$$

of the map of $Q$. The right-hand map is given on bimorphisms by "forgetting" the middle object in the diagram (2.1). The left-hand map is given on the bimorphism

$$(A_0, B_0) \mapsto (A_1, B_0) \mapsto (A_2, B_0)$$

$$(A_0, B_2) \mapsto (A_2, B_2)$$

$$(A_0, B_2) \mapsto (A_2, B_1)$$

of $QA \otimes_{QB} QC$ by adding the middle object $(A_2, B_2)$ as shown, and then applying $\mathbf{b}$. This factorization is pointed out in Proposition 9.2 of [W], where Waldhausen notes that the resulting map $BQA \otimes_{BQB} BQC$ of realizations vanishes on the subspace $BQA \otimes_{BQB}$ (because of the technical condition (ii)), and hence induces a map of topological spaces

(2.2)

$$BQA \otimes_{BQB} BQC \to BQC.$$

A brief account of the product map (2.2) is also given in [W].

If we take homotopy groups, we obtain (using, e.g., $[Bq(1.6)]$) a map

$$K_p(A) \otimes K_q(B) \to K_{p+q}(C).$$

(2.3) In the special case that $A = C$ and there is an object $b_0$ of $B$ so that $(-) \otimes b_0$ is the identity on $A$, there is a commutative diagram (Lemma 5.2.4 of [W]):

$$
\begin{array}{c}
\text{BQA} \otimes \text{BQC} \\
\downarrow \\
\text{BQA} \otimes \text{BQB} \to \text{BQC} \\
\end{array}
$$

The left vertical map comes from the inclusion of $\mathcal{S}^0$ into $\mathcal{Q}B$ by selection of the loop $[b_0]: \mathcal{S}^0 \to \mathcal{Q}B$. The fact that the top composite is the natural map is stated on p. 199, line 18 of [W].

When there is an associative pairing $B \times B \to B$, $K_q(B)$ becomes a graded ring; $K_q(B)$ has unit $[b_0]$ if $(-) \otimes b_0 = b_0(-) = \text{id}(b)$. The map $\otimes A \to A$ induces a right $K_q(B)$-module structure on $K_qA$ when the two evident functors $A \otimes B \to A$ agree up to natural isomorphism. These remarks apply notably to the case $B = \mathcal{Q}P$, the category of fin. gen. projective $k$-modules for a commutative ring $k$: tensor product makes $K_q(k)$ a graded commutative ring with unit, and for every $k$-algebra $A$ the group $K_q(A)$ is a $2$-sided $K_q(k)$-module.

3. Today's Product

Another approach to products in $K$-theory is to deal with symmetric monoidal categories, and invoke the ""*-qu"" theorem. This approach was first used by May in [L], using the category $\mathcal{F}(A) = \coprod_{B \in \mathcal{B}} CO(A)$, and later generalized by May in [May 1,2]. For simplicity, we first describe today's method, and then present the more complicated approach used by May.
The choice of an isomorphism $\Theta_{p,q}: A^p \otimes A^q \cong (A \otimes B)^{pq}$ for every $p$ and $q$ gives a pairing $\Theta(A) \otimes \Theta(B) \cong \Theta(A \otimes B)$. Since $\Theta(A) \otimes \Theta(B) = \Theta(A \otimes B) = 0$, it induces a map of topological spaces, which is the top row of the following diagram:

$$
\begin{array}{cccccc}
\text{BF}(A) & \otimes & \text{BF}(B) & \cong & \text{BF}(A \otimes B) \\
\downarrow \downarrow & & \downarrow \downarrow & & \downarrow \downarrow \\
\text{BG}_p(A) & \otimes & \text{BG}_q(B) & \cong & \text{BG}_{pq}(A \otimes B) \\
\downarrow & & \downarrow & & \downarrow \\
\text{BG}_p(A) & \otimes & \text{BG}_q(B) & \cong & \text{BG}_{pq}(A \otimes B) \\
\end{array}
$$

The convention is that $\text{BG}_p(A)$ denotes $\text{BG}_p(A)$ for $p \leq 2$; for $p > 2$ it denotes the result of the plus construction relative to $\text{BG}_p(A)$. The maps $\mathbb{f}_{pq}$ are the universal maps determined uniquely by the $\Theta_{pq}$. Today's idea is to define a map $\gamma$ making the diagram (3.1) commute up to homotopy. As before, taking homotopy groups yields a map

$$K_p(A) \otimes K_q(B) \to K_{pq}(A \otimes B).$$

If $k$ is a commutative ring, the map $k \otimes k \to k$ makes $K_p(A)$ into a graded ring; the maps $k \otimes A \to A$ make $K_p(A)$ into a $K_p(A)$-algebra for every $k$-algebra $A$.

Today first observes that $\mathbb{F}(A) = \bigoplus \mathbb{F}_p(A)$ is an $H$-space (see [M]) under direct sum, and that $x \circ (-), (-) \circ y$ are $H$-space maps for each $x \in \mathbb{F}(A), y \in \mathbb{F}(B)$. He next observes that $\bigoplus \mathbb{F}_p(A)$ is an

H-space (etc.) and that $f(x, y), f(y, y)$ are $H$-space maps; this is (2.1.2)(ii) of [L]. Now $\mathbb{F}(A)$ is the "group completion" of $\bigoplus \mathbb{F}_p(A)$ in the very strong sense that for every $x \in \mathbb{F}(A \otimes B)$ there is a positive integer $n$ such that $nx$ is in the image of $\bigoplus \mathbb{F}_p(A)$. Since $\mathbb{F}(A)$ is an $H$-group, there is a unique extension of $f$ to an $H$-space map

$$\gamma: [\mathbb{F}(A) \times \mathbb{F}(B)] \to \mathbb{F}(A \otimes B).$$

Specifically, if $x \in \mathbb{F}(A), y \in \mathbb{F}(B)$, we choose $x_0 \in \mathbb{F}(A), y_0 \in \mathbb{F}(B)$ such that $x = i_x(x_0), y = i_y(y_0)$, and define

$$\gamma(x, y) = (i_x(x_0), y_0) \circ (y_0, i_y(y_0)) = i_x(x_0) \circ y_0 = i_y(y_0) \circ x_0 = i_x(x_0) \circ i_y(y_0).$$

Here $x_0, y_0$ are the basepoints of $\mathbb{F}(A)$ and $\mathbb{F}(B)$, and we have used $i$ for $i_{A \otimes B}$. If we take $x$ and $y$ to be in the respective basepoint components, we recover the map

$$\text{colim} \gamma: \mathbb{F}(A) \times \mathbb{F}(B) \to \mathbb{F}(A \otimes B)$$

on the top of p. 332 of [L]. Since $\gamma$ is homotopically trivial on $[\mathbb{F}(A) \times \mathbb{F}(B)]$, $\gamma$ factors through the smash product to give the map $\gamma'$ of diagram (3.1). The choices used to define $\gamma$ mean that $\gamma'$ is only well-defined up to weak homotopy type.

In May's generalization, one considers "pairings" $A \otimes B = C$ of symmetric monoidal categories. This means that $A \otimes B = 0 \otimes B = 0$ and that there is a coherent natural distributivity axiom

$$(a \circ b \otimes c \circ b') = (a \circ b \circ c) \otimes (b \circ b').$$
Instead of making the technical notion of coherence precise, we refer the reader to §2 of [May 2] and content ourselves with the remark that $\text{B}_{\text{E}}(A) + \text{E}(B) = \text{E}(\text{A} \times \text{B})$ is such a pairing.

At this stage, we need to introduce the "group completion" map $\text{B}A \rightarrow \text{E}_{0}\text{BA}$, defined for every symmetric monoidal category $A$. For example, $\text{E}_{0}\text{BA}(A)$ is the space $\mathbb{Z} \times \text{SEG}^*(A)$. One way to construct the group completion is to use the $S^{-1}S$ construction of [QG]. Another way is to use an infinite loop space machine; for example, one can first obtain a $\Gamma$-space $\text{BA}$, use Segal's machine to obtain a spectrum $\text{E}_{0}\text{A}$, and take the zeroth space $\text{E}_{0}\text{BA}$. This latter approach has both the advantages and disadvantages inherent in infinite loop space machinery.

The point is that a pairing of symmetric monoidal categories functorially determines a pairing $\text{E}_{0}\text{BA} \times \text{E}_{0}\text{BB} \rightarrow \text{E}_{0}\text{BC}$ of infinite loop spaces. This follows, for example, from Theorems 1.6 and 2.1 of [May 2]. More is true: a pairing $\text{E}_{0}\text{BA} \times \text{E}_{0}\text{BB} \rightarrow \text{E}_{0}\text{BC}$ is determined in the stable category of infinite loop spectra, allowing spectrum level work to be performed. There is also a commutative diagram:

\[
\begin{array}{ccc}
\text{BA} \times \text{BB} & \rightarrow & \text{BA} = \text{BB}^* = \text{BB} \\
\downarrow & & \downarrow \\
\text{E}_{0}\text{BA} \times \text{E}_{0}\text{BB} & \rightarrow & \text{E}_{0}\text{BA} \times \text{E}_{0}\text{BB} \rightarrow \text{E}_{0}\text{BC}
\end{array}
\]  

in which the bottom composite is an infinite loop space map.

Commutativity of the diagram is Corollary 6.5 of [May 2]. As remarked in the introduction of [May 2], it is immediate from (3.1) and (3.2) that the product defined by May specializes to Loday’s product.

Here is an example of the usefulness of Loday’s product. Let $\text{En}_*\text{A}$ denote the skeletal category $\text{En}_*\text{A}$ of finite sets and their isomorphisms. For this symmetric monoidal category we have $\text{En}_*\text{En}_*\text{A} = \mathbb{Z} \times \text{SEG}^*(\text{En}_*\text{A})$, and a pairing $\text{En}_*\text{En}_*\text{A} \times \text{En}_*\text{En}_*\text{A} \rightarrow \text{En}_*\text{En}_*\text{A}$ induced by multiplication, which is discussed on [May 1, p. 161]. Consequently, $\text{En}_*\text{En}_*\text{A} = \mathbb{Z} \times \text{SEG}^*(\text{En}_*\text{A}) = \mathbb{Z}^\mathbb{A}$ is a graded commutative ring. The map $\text{En}_*\text{A} \rightarrow \mathbb{Z}(\mathbb{Z})$ embedding the symmetric group $\text{En}_*\text{A}$ into $\text{En}_*\text{A}$ induces a map of pairings in the senses of [May 1, p. 155] and [May 2, §2], hence a ring map $\mathbb{Z}^\mathbb{A} \rightarrow \mathbb{Z}^{\mathbb{A}}$.

4. Agreement of Product Structures

In order to directly compare Waldhausen’s product and Loday’s product, consider a pairing $A \times B = C$ of exact categories. The subcategories $\text{Is}(A)$, $\text{Is}(B)$, $\text{Is}(C)$ of isomorphisms are all symmetric monoidal categories, and the induced functor $\text{Is}(A) \times \text{Is}(B) = \text{Is}(C)$ is a pairing of symmetric monoidal categories. Waldhausen’s Lemma 9.2.6 in [Wa] states that the following diagram commutes up to basepoint preserving homotopy:

\[
\begin{array}{ccc}
\text{Bis}(A) \times \text{Bis}(B) & \rightarrow & \text{Bis}(C) \\
\downarrow & & \downarrow \\
\text{QR}_A \times \text{QR}_B & \rightarrow & \text{QR}_C \\
\downarrow & & \downarrow \\
\text{QR}_A \times \text{QR}_B & \rightarrow & \text{QR}_C
\end{array}
\]

The maps $\text{Bis}(A) \rightarrow \text{QR}_A$, etc., are described on p. 198 of [Wa]. The top arrow in (4.1) is induced from the composite $\text{Bis}(A) \times \text{Bis}(B) \rightarrow \text{Bis}(B) \otimes \text{Bis}(B) \rightarrow \text{Bis}(B) \times \text{Bis}(C) = \text{Bis}(C)$. 
which by (1.1) is the natural map $\mathbb{B}$. The bottom map is the double looping of Waldhausen's map (2.2), and we have already remarked on the lower right-hand homotopy equivalence (of $\mathcal{H}$-spaces!). There is a unique way, up to homotopy, to fill in the broken arrow so that the diagram remains homotopy commutative.

We point out that the broken arrow is induced from an $\mathcal{H}$-space map $(\mathcal{B}Q\mathcal{A})^* \to (\mathcal{Q}Q\mathcal{C})^* \to \mathcal{B}Q\mathcal{C}$. To see this, note that the functor $\mathcal{Q}\mathcal{A}Q\mathcal{B} \to \mathcal{Q}Q\mathcal{C}$ is a map of symmetric monoidal bicategories (the operation being sotwise direct sum), so that $\mathcal{B}Q\mathcal{A}Q\mathcal{B} \to \mathcal{H}(\mathcal{Q}\mathcal{A}Q\mathcal{B}) \to \mathcal{B}Q\mathcal{C}$ is an $\mathcal{H}$-space map (in fact it is an infinite loop space map).

Now suppose that all exact sequences in $A$ split. Then there is a basepoint preserving homotopy equivalence $\mathcal{B}Q\mathcal{A} \to \mathcal{B}Q\mathcal{A}$ so that

$$\begin{CD}
\mathcal{B}Q\mathcal{A} @>>> \mathcal{B}Q\mathcal{A} \\
\mathcal{B}Q\mathcal{A} \equiv \mathcal{B}Q\mathcal{A} \\
\mathcal{B}Q\mathcal{A} \equiv \mathcal{B}Q\mathcal{A}
\end{CD}$$

commutes (up to basepoint preserving homotopy). This is proven as (9.3.2) of [W], modulo the observation that (by (6.3) of [W]) the space $\mathcal{B}N\mathcal{A}(\mathcal{I}s(A))$ in (9.3.2) is just $\mathcal{E}_Q\mathcal{B}Q\mathcal{A}(A)$. Having said this, it follows that the top part of (4.1) induces the following homotopy commutative diagram of $\mathcal{H}$-spaces:

$$\begin{array}{c}
\mathcal{B}Q\mathcal{A} \to \mathcal{B}Q\mathcal{B} \\
\downarrow \\
\mathcal{B}Q\mathcal{B} \to \mathcal{B}Q\mathcal{C}
\end{array}$$

(4.2)

$$\begin{CD}
\mathcal{B}Q\mathcal{B} \to \mathcal{B}Q\mathcal{B} \\
\downarrow \\
\mathcal{E}_Q\mathcal{B}Q\mathcal{B} \to \mathcal{E}_Q\mathcal{B}Q\mathcal{B}
\end{CD}$$

The $\mathcal{H}$-space map $\gamma$ in (4.2) is uniquely determined, so it must be the same as the map $\gamma$ in today's construction, as generalized by May. Comparing (4.1) and (4.2), we see that the broken arrow $\tilde{\gamma}$ in (4.1) must be the same as the $\tilde{\gamma}$ in (3.1) and (3.2). We summarize this as follows:

Theorem 4.3 (Waldhausen). If $A \otimes B = C$ is a biequivalent pair of exact categories for which all exact sequences split, then the groups $K_A(A) = \pi_4(\mathcal{B}Q\mathcal{A})$ agree with the groups $K_A\mathcal{I}s(A) = \pi_4\mathcal{E}_Q\mathcal{B}Q\mathcal{B}(A)$. There are commutative diagrams

$$\begin{CD}
\mathcal{E}_Q\mathcal{B}Q\mathcal{B}(A) \to \mathcal{E}_Q\mathcal{B}Q\mathcal{B}(B) @>>> \mathcal{E}_Q\mathcal{B}Q\mathcal{B}(C) \\
\downarrow \\
\mathcal{B}Q\mathcal{A} \otimes \mathcal{B}Q\mathcal{B} @>>> \mathcal{B}Q\mathcal{C}
\end{CD}$$

$$\begin{CD}
K_A\mathcal{I}s(A) \otimes K_A\mathcal{I}s(B) @>>> K_A\mathcal{I}s(C) \\
\downarrow \\
K_A(A) \otimes K_A(B) @>>> K_A(C)
\end{CD}$$

in which the top maps are the Loday-May pairings, and the bottom maps are the Waldhausen pairings.

We can apply this result to the exact category $\mathcal{B}_E(A) = A$ of fin. gen. projective $A$-modules, etc. Since $\mathcal{E}_Q\mathcal{B}Q\mathcal{B}(A)$ is $K_A(A) = \mathcal{B}G\mathcal{A}(A)$, we obtain:

Corollary 4.4 (Waldhausen). Let $A, B$ be rings with unit. There is a homotopy commutative diagram

$$\begin{CD}
(K_A(A) = \mathcal{B}G\mathcal{A}(A)) \otimes (K_B(B) = \mathcal{B}G\mathcal{B}(B)) @>>> (K_A(AB) = \mathcal{B}G\mathcal{A}(AB)) \\
\downarrow \\
[\mathcal{B}Q\mathcal{A}(A)] \otimes [\mathcal{B}Q\mathcal{B}(B)] @>>> [\mathcal{E}_Q\mathcal{B}Q\mathcal{B}(AB)].
\end{CD}$$
where the top arrow is today's pairing and the bottom arrow is Waldhausen's product. Thus the two pairings agree on homotopy to give the same graded map

$$K_{n}(A) \otimes K_{n}(B) \rightarrow K_{n}(A \otimes B).$$

**Remark.** Waldhausen gave the argument for Theorem (4.3) on p.255 of [Wa] for the special case $A = Z, B = ZG$, in order to show that the map $\pi_{n}(B;Z) \rightarrow K_{n}(ZG)$ constructed on [Wa, p. 227] agrees with today's map on [L, p. 226].

If we had used the skeletal subcategories $F(A) \subseteq \mathcal{P}(A)$ of free modules, we would get $F(A) = 
\mathcal{F}(A)$ when $A$ is commutative, or more generally when $A^{\otimes n}$ for $m \geq n$. In this case we would write (4.4) as the commutative diagram

$$\begin{array}{c}
\mathcal{K}(A) \otimes \mathcal{K}(B) \\
\mathcal{K}(A \otimes B)
\end{array}$$

5. **Relative K-theory.**

When $I$ is an ideal in a ring $A$, we will construct a pairing $K_{n}(A, I) \otimes K_{n}(B, I) \rightarrow K_{n}(A \otimes B, I)$ so that:

If $B$ is commutative and $A$ is a $B$-algebra, the map

$$K_{n}(A, I) \otimes K_{n}(B, I) \rightarrow K_{n}(A \otimes B, I)$$

makes $K_{n}(A, I)$ a graded $K_{n}(B, I)$-module.

There are two approaches to this problem, corresponding to the two types of product. We will first describe May's approach, which is more conceptually straightforward, and then describe Waldhausen's more subtle method.

In May's approach, the basic object is the category $\mathcal{F}(B)$ of free $B$-modules. One has a "morphism of pairings of symmetric monoidal categories" (q.v. 12 of [May 2]):

$$\begin{array}{c}
\mathcal{F}(A) \otimes \mathcal{F}(B) \\
\mathcal{F}(A \otimes B)
\end{array}$$

The machinery described in Theorem 1.6 of [May 2] produces a "morphism of pairings in the stable category" which is adequately represented by commutativity of the right-hand portion of the following diagram:

$$\begin{array}{c}
\text{Fiber}(A, I) \rightarrow B_{F} \\
\text{Fiber}(A \otimes B, I)
\end{array}$$

In (5.2) we have written $\text{Fiber}(A, I)$ for the homotopy fiber of the map $B_{F}(A) \rightarrow B_{F}(A, I)$, and similarly for fiber $(A \otimes B, I)$. This being said, it is standard that there is a broken arrow making (5.2) a map of (infinite loop space) fibrations. Since $K_{n}(A, I) = K_{n}(\text{Fiber}(A, I))$, the homotopy groups of (5.2) yield a map of long exact sequences:

$$\begin{array}{c}
\cdots \rightarrow K_{p}(A, I) \otimes K_{q}(B) \\
\rightarrow K_{p}(A \otimes B)
\end{array}$$

The problem with this approach is that the broken arrow in (5.2) is not unique and, unless care is taken, will not make (5.1) hold.
Happily, there is enough structure in the categories involved to save the day. The functoriality of May's approach makes the right-hand square in (5.2) commute, and the details of associativity can be checked directly. This is done in detail in [May 3].

The second approach to relative pairings is due to Waldhausen and is implicit in 17 of [Wa]. The idea is to use simplicial exact categories (SEC's) to produce a model for $D^2$-fiber$(A,i)$, and do all work at the category-theoretic level.

To an exact category $A$, Waldhausen associates an SEC $S(A)$, and proves in [Wa, (7.1)] that $\text{GROF}_A = BQA$. An object of the exact category $S(A)$ is a sequence

$$A : A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n$$

of admissible monics in $A$, together with choices of objects $A_{i+1}(i\rightarrow j)$ and isomorphisms $A_{i+1}/A_i$. The $i$th face map $d_i : S(A) \rightarrow S(A)$ is induced by dropping the index $i$. For example, the sequence for $d_0(A)$ is

$$A_0 \rightarrow A_1 \rightarrow \ldots \rightarrow A_n.$$

Next, we suppose given an exact functor $f : A \rightarrow A'$. Waldhausen constructs on [Wa, p. 182] an SEC denoted $F_i(f)$ fitting into a sequence of SEC's,

$$(5.3) \quad A \rightarrow A' \rightarrow F_i(f) - S(A) - S(A'),$$

Of course, in (5.3) we consider $A$ and $A'$ to be constant simplicial exact categories.

Briefly, an object of the exact category $F_i(f)$ is a triple $(A',A,A)$; an object $A'$ of $S(A')$, an object $A$ of $S(A)$, and an isomorphism $f(A) \cong d_0(A')$ in $S(A')$. The map $A' \rightarrow F_i(f)$ sends $A'$ to the object $(A'\leftarrow A'\leftarrow \ldots \leftarrow A,0,0)$ in $F_i(f)$, and the map $F_i(f) \rightarrow S(A)$ sends $(A',A,A)$ to $A$.

The content of Propositions (7.1) and (7.2) of [Wa] is that

$$BQA \rightarrow BQA' \rightarrow BQF_i(f) \rightarrow BQS(A) \rightarrow BQS(A'),$$

is a fibration sequence up to homotopy. Thus $GROF_i(f)$ is a model for the fiber of $BQA \rightarrow BQA'$; if we set $X_0(f) = \pi_{p+1}BQF_i(f)$, there is a long exact sequence

$$(5.4) \quad \ldots \rightarrow X_{n+1}(A) \rightarrow X_n(A') \rightarrow X_n(f) \rightarrow X_n(A) \rightarrow X_n(A').$$

In particular: if $A = \mathbb{E}(A), A' = \mathbb{E}(A')$, then $X_0(A,A') = \pi_{p+1}BQF_i(f)$.

Given this category-theoretic encoding of the relative term in $K$-theory, we can construct relative pairings with ease. One starts with a commutative diagram

$$(5.5) \quad \begin{array}{ccc} A \times E & \rightarrow & \mathbb{C} \\ f \times E & \downarrow & \mathbb{C}' \\ A' \times E & \rightarrow & \mathbb{C}' \end{array}$$

in which the horizontal arrows are biexact. The functors $\otimes$ induce simplicial biexact functors $S(A \times E) \rightarrow S(C)$ and $F_i(f \otimes E) \rightarrow F_i(f')$ in an obvious manner. The result is the commutative diagram of SEC's.
analogous to (5.3):

\[ A \otimes B \rightarrow A' \otimes B \rightarrow F_*(f) \otimes B \rightarrow S_! A \otimes B \rightarrow S_! A' \otimes B \]

\[ C \rightarrow C' \rightarrow F_*(f) \rightarrow S_! C \rightarrow S_! C' \]

Following (5.2) above, there is a commutative diagram of simplicial bicategories, the middle of which is

\[ QA' \otimes QS \rightarrow QF_*(f) \otimes QS \rightarrow QS, A \otimes QS \]

\[ QQC' \rightarrow QF_*(f') \rightarrow QQC_*(C) \]

(5.6)

Geometric realization yields a map of fibrations sequences, the middle of which is

\[ BQA' \otimes BQS \rightarrow BQF_*(f) \otimes BQS \rightarrow BQS, A \otimes BQS \]

\[ BQC' \rightarrow BQF_*(f') \rightarrow BQC_*(C) \]

Taking homotopy groups yields the map of long exact sequences

\[ \cdots \rightarrow \pi_{\geq 1} A' \otimes K_2(B) \rightarrow \pi_{\geq 1} (f) \otimes K_2(B) \rightarrow K_2(A) \otimes K_2(B) \cdots \]

\[ \cdots \rightarrow \pi_{\geq 1} A \otimes K_2(C') \rightarrow \pi_{\geq 1} (f') \otimes K_2(C') \rightarrow K_2(A) \otimes K_2(C') \cdots \]

the middle vertical arrow being the desired pairing.

Now suppose that we are in the situation of (2.3) above, i.e., that \( A = C \) and \( A' = C' \). We assume that there is an associative pairing:

\[ B \otimes B = B, (-) \otimes B_0 \]

is the identity functor on \( A, A' \) and \( B \), and that (5.5) fits into an "associativity axiom" cube (going from \( A \otimes B \) to \( A' \)). Then the two evident functors \( F_*(f) \otimes B \rightarrow F_*(f) \) agree (up to natural isomorphism), so that \( K_*(f) \) is a graded

\[ K_*(B) - \text{module in such a way that (5.4) is a sequence of } K_*(B) \text{-modules.} \]

The above paragraph applies to the situation of (5.1). We take (5.5) to be the diagram

\[ Z(A) \otimes Z(B) \rightarrow Z(ABB) \]

\[ Z(A/B) \otimes Z(B) \rightarrow Z(A/BB) \]

All hypotheses are met, and \( K_*(A, I) \) is a graded \( K_*(B) \text{-module.} \)

6. Products in \( K \)-theory

There is another type of \( K \)-theory with product: the Karoubi-Villamayor groups \( K_*(A) \). This theory makes sense for any ring (with or without "one"), and is uniquely determined by the axioms given in [K-V]. One way to define them is to set \( K_0(A) = K_0(A) \), and for \( p > 0 \) to define \( K_*(A) \) by exactness of

\[ 0 \rightarrow K_*(A) \otimes K_0(B) \rightarrow K_0(B) \rightarrow \]

\[ K_0(B^{p-1}) \rightarrow \]

Here we have used the notation \( A = (t^2) A[t] e A[t] \otimes A[t] \rightarrow A \) for any ring \( A \), and defined \( \partial A \) by iteration: \( \partial^p A = \mathcal{E} (t^2, t^2 A[t_1, \ldots, t_p]) \).

In [K] Karoubi constructs a pairing for \( K \)-theory from the following "usual" pairing for \( K_0 \): When \( A \) is a ring without "one" we can define \( K_*(A) = K_0(B^A, A) \) for any ring \( R \) with "one" with an \( R \)-algebra structure on \( A \); this definition is independent of the choice of \( R \). Since \( (Z(A) \otimes Z(B)) = K_0(ABB) \) for \( A = Z \otimes A \otimes B \),

we can define the bilinear map \( \gamma_0 \) as the composite:

\[ K_0(A) \otimes K_0(B) \rightarrow K_0(Z(A) \otimes Z(B)) \rightarrow K_0(ABB) \rightarrow K_0(ABB). \]
One interpretation of Karoubi's construction of pairings is this:

It is easy to show that $K_p(E(MB)) + K_q(E(MB))$ and $K_0(E(MB)) + K_0(E(MB))$ are injections for every $A,B$. We then have the following commutative square with exact rows (for $p,q \geq 0$):

\[
\begin{array}{c}
0 \rightarrow K_p(A) \rightarrow K_q(B) \rightarrow K_0(\mathbb{Z}^p) \rightarrow K_0(\mathbb{Z}^q) \rightarrow K_0(E(MB)) \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \rightarrow K_p(B) \rightarrow K_q(A) \rightarrow K_0(\mathbb{Z}^p) \rightarrow K_0(\mathbb{Z}^q) \rightarrow K_0(E(MB))
\end{array}
\]

(6.1)

It follows that the broken arrow $\delta_{p,q}$ is defined when $p, q \geq 1$.

When $p = 0, q \geq 1$ and $p \geq 1, q = 0$ it is also easy to induce maps $\delta_{p,q}$. Karoubi then proves the following theorem in [K, p. 78]:

**Theorem (6.2)** The maps $\delta_{p,q} : K_p(A) \otimes K_q(B) \rightarrow K_{p+q}(M)$ are the unique natural bilinear maps satisfying the following axioms:

(i) $\delta_{0,0}$ is the "usual" product $K_0(A) \otimes K_0(B) \rightarrow K_0(M)$

(ii) Every map of $f$-fibrations (see [K-V])

\[
\begin{array}{cc}
0 & \rightarrow A \otimes B \rightarrow A \rightarrow A' \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \rightarrow C' \rightarrow C \rightarrow C'' \rightarrow 0
\end{array}
\]

(6.2a)

gives rise to a commutative diagram

\[
\begin{array}{c}
K_p(A'') \otimes K_q(B) \rightarrow K_{p+q}(C'') \rightarrow K_p(B) \otimes K_q(A') \\
\downarrow & \downarrow & \downarrow \\
K_p(A') \otimes K_q(B) \rightarrow K_{p+q}(C') \rightarrow K_p(B) \otimes K_q(A').
\end{array}
\]

For a ring $A$ with "one" we can define a simplicial ring $A_+$, which in degree $n$ is the coordinate ring $A[x_0, \ldots, x_n]/(x_1=1)$ of the "standard $n$-simplex," the face and degeneracy maps being dictated by the geometry. Applying $BG\xi$ gives the simplicial topological space $p \rightarrow BG\xi(A)$, and we have for $s \geq 1$ that

\[K_* (A) \rightarrow \pi_s BG\xi(A),\]

This is proven in [A, p. 65]. The map of simplicial spaces $BG\xi(A) \times BG\xi(A)$ induces the map $K_* (A) \rightarrow K_* (A)$ of homotopy groups.

Today's pairing now induces a pairing in $KV$-theory in a completely canonical way: the choice of isomorphisms $A^p \otimes A^q \rightarrow \mathbb{Z}(M)$ completely determines a simplicial pairing $K(A) \times K(B) \rightarrow K(M)$, and this in turn yields a map of simplicial topological spaces

\[\gamma : \mathbb{Z}[BG\xi(A)] \rightarrow \mathbb{Z}[BG\xi(B)] \rightarrow \mathbb{Z}[BG\xi(M)].\]

Applying geometric realization yields a map

\[|\gamma| : \mathbb{Z}[BG\xi(A)] \rightarrow \mathbb{Z}[BG\xi(B)] \rightarrow \mathbb{Z}[BG\xi(M)].\]

Another way to proceed is to use Waldhausen's pairing and a Q-version of the map $K_* \rightarrow KV$. Define the simplicial subcategory $\mathcal{E}(A)$ of $E(A)$ by letting $\mathcal{E}(A)$ denote the full subcategory of $E(A)$ of projective $A_+$-modules extended from $A$ (i.e., isomorphic to some $\mathbb{Z}[BG\xi(A)]$). We then have

\[K_0(A) \rightarrow \mathbb{Z}[BG\xi(A)],\]

(6.3)

\[K_* (A) \rightarrow \pi_* (\mathbb{Z}[BG\xi(A)]).
\]
This is Theorem 2.1 of [We]; the technical reason for using $E_2(A)$ instead of $E_2(A)$ is that $G|BQ(A)| = [K_2(A)|^*|BGZ(A)|$, and the space $[K_0(A)|$ need not be $K_0(A)$. We now induce a pairing from Waldhausen's product:

External $\otimes$ gives a biexact functor $\pi^*(A)*\pi^*(B) = \pi^*(A\otimes B)$, and so a morphism of simplicial bicateories

$$\pi^*(A)*\pi^*(B) \rightarrow \pi^*(A\otimes B),$$

which realizes to a map of topological spaces:

$$[BQ(A]^*-[BQ(B] \rightarrow [BQ(A\otimes B].$$

Since each $BQ(P)$ is connected, we have that

$$G[BQ(A] = [BQ(A] = [BQ(B],$$

and therefore we have an induced map of homotopy groups

$$\pi^*(A)*\pi^*(B) \rightarrow \pi^*(A\otimes B),$$

defined for $p, q \geq 0$. In view of (6.5) and (4.4), it is clear that the map (6.4) agrees with today's $\otimes$.

In view of Theorem (6.2), we can show that the pairing (6.5) agrees with Waldhausen's pairing (6.1) by checking the two axioms. Since Waldhausen's map $K_0(A) \otimes K_0(B) = K_0(A\otimes B)$ agrees with the classical external product used by Waldhausen, we only have to check axiom (ii). We can assume that $A, A^\prime, C, C^\prime$, and $B$ have a "one".

so that the commutative diagram (6.2a) gives rise to a commutative diagram of bisimplicial bicateories analogous to (5.6):

$$\pi^*(A^\prime) \otimes \pi^*(B) \rightarrow \pi^*(A) \otimes \pi^*(B) \rightarrow \pi^*(A \otimes B) \rightarrow \pi^*(A \otimes B) \rightarrow \pi^*(A \otimes B).$$

Applying geometric realization gives a map of fibrations at each level, and the assumption that the exact sequences of rings were $GL$-fibrations implies that we have global fibrations, i.e., that the rows in the following diagram are fibrations of topological spaces:

$$[BQ(A^\prime)]-[BQ(B)] \rightarrow [BQ(A)\otimes BQ(B)] \rightarrow [BQ(A)\otimes BQ(B)] \rightarrow [BQ(A)\otimes BQ(B)].$$

The fact that $\pi^*(A) \otimes \pi^*(B) = \pi^*(A \otimes B)$ follows from consideration of the long exact homotopy sequences of the rows (and $L$), and the commutative diagram

$$\pi^*(A) \otimes \pi^*(B) \rightarrow \pi^*(A \otimes B),$$

translates into the diagram of axiom (ii). We summarize this:

Proposition (6.6) The pairing (6.5) on $K_0$-theory induced from the pairing on $K_0$-theory satisfies the axioms of Theorem (6.2), and so agrees with Waldhausen's product.
References


