Group Representations

Let $G$ be a group. We say that $G$ acts on a set $X$ (on the left) if there is a set map $G \times X \to X$, sending $(g, x)$ to $g \cdot x \in X$, such that $1 \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$.

Now fix a field $F$. A vector space $V$ over $F$ is called a $G$-module (or representation of $G$) if the group $G$ acts on the set $V$, and if for each $g \in G$ there is a linear transformation $\sigma(g) : V \to V$ such that $g \cdot x = \sigma(g)(x)$ for all $x \in V$. A trivial representation is a representation with $g \cdot x = x$ for all $g \in G$.

Let $GL(V)$ denote the group of linear automorphisms of $V$; if $V = F^d$ then $GL(V) = GL_d(F)$. If $V$ is a $G$-module then $\sigma : G \to GL(V)$ is a group homomorphism. Conversely, any group homomorphism $\sigma : G \to GL(V)$ makes $V$ into a $G$-module. Some authors take this as the definition of representation.

A representation $V$ of $G$ is also the same thing as a module over the ring $FG$. Here the group ring of $G$ is a vector space $FG$ with basis $G$, made into a ring with the product $(\sum \alpha_i g_i)(\sum \beta_j h_j) = \sum (\alpha_i \beta_j)(g_i h_j)$, $\alpha_i, \beta_j \in F$ and $g_i, h_j \in G$.

A $G$-map (= homomorphism of $G$-modules) is a linear transformation $f : V \to W$ commuting with the action of $G$ in the sense that $f(g \cdot v) = g \cdot f(v)$. Of course, this is the same thing as a homomorphism of modules over the ring $FG$.

Permutation representations. Let the set $X$ be a basis of a vector space $V$. Any action of $G$ on $X$ can be extended linearly into an action of $G$ on $V$; such a representation is called a permutation representation because $G$ permutes the basis. Each matrix $\sigma(g)$ consists of 0's and 1's. The regular representation is an example: $V$ is the group ring $FG$, $X = G$ and if $v = \sum a_i g_i$ then $g \cdot v = \sum a_i (g g_i)$.

1-dimensional representations. A 1-dimensional representation (of $G$ on $F$) is equivalent to a group map $G \xrightarrow{\sigma} F^*$. Since $F^*$ is an abelian group, the commutator subgroup $[G, G]$ must map to 1, so the representation factors through $G \xrightarrow{\sigma} G/[G, G]$. If $G$ has $n$ elements each $\sigma(g)$ must be an $n^{th}$ root of unity, because $g^n = 1$ in $G$. The absence of $n^{th}$ roots of unity in $F$ can affect the existence of 1-dimensional representations.

Let $C_n$ denote the cyclic group of order $n$, with generator $\theta$. It follows that the 1-dimensional representations of $C_n$ (over $F$) are in 1-1 correspondence with the set of $n^{th}$ roots of unity $\zeta$ in $F$ (take $\sigma(\theta) = \zeta$). The group $C_2$ has two 1-dimensional representations: the trivial representation and the sign representation ($\theta \cdot a = -a$). The cyclic group $C_3$ has three 1-dimensional representations if $F = \mathbb{C}$, but only one if $F = \mathbb{R}$.

Operations. Standard operations on vector spaces ($\oplus, \otimes, \Lambda^*$, etc.) also induce operations on $G$-modules. Let $V = F^m$ and $W = F^n$ be two representations. The direct sum $V \oplus W = F^{m+n}$ is a representation with $g \cdot (v + w) = (g \cdot v) + (g \cdot w)$, and the tensor product $V \otimes W = F^{mn}$ is a representation with $g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$.

Let $\Lambda^d V$ denote the $d^{th}$ exterior product of $V$, i.e., the vector space of dimension $\binom{m}{d}$ consisting of all alternating $d$-forms $v_1 \wedge \cdots \wedge v_d$ on $V$. The action of $G$ on $\Lambda^d V$ is given by the formula $g \cdot (v_1 \wedge \cdots \wedge v_d) = (g \cdot v_1) \wedge \cdots \wedge (g \cdot v_d)$. For example, if $d = m$ then under the usual identification of $\Lambda^m F^m$ with $F^d$ the action of $g$ on $F$ is multiplication by $\det(\sigma(g))$. 
Definition. A nonzero $G$-module $V$ is called irreducible (= a simple module) if no proper subspace is a $G$-submodule. $V$ is called completely reducible (= semisimple) if it is a direct sum of irreducible $G$-modules.

Clearly, every 1-dimensional representation is irreducible. If $\dim(V) = 2$, there is a simple test for irreducibility: $V$ is irreducible if no vector $v \neq 0$ in $V$ is an eigenvalue for all of the $2 \times 2$ matrices $\sigma(g)$, $g \in G$.

Here is a general test to see if $V$ is irreducible. For every $v \neq 0$, does the orbit of $v$ (the set $G \cdot v = \{g \cdot v, g \in G\}$, which includes $1 \cdot v = v$) span $V$? If so, $V$ is irreducible. If not, the span of $G \cdot v$ is a proper $G$-submodule.

Examples. 1) The regular representation of $C_2 = \{1, \theta\}$ on the plane is given by $\theta(x, y) = (y, x)$. The two vectors $(1, \pm 1)$ are eigenvectors so this representation is the direct sum $F(1, 1) \oplus F(1, -1)$.

2) The dihedral group $D_n$ ($n \geq 3$) is defined as the group of isometries in the plane fixing the regular $n$-gon; the 2-dimensional representation defining $D_n$ is irreducible (as the reflections have different eigenspaces, or because each $v$ and its rotate by $2\pi/n$ span the plane).

3) The quaternionic group $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ has an obvious 4-dimensional representation on the quaternions $\mathbb{H}$. (We take $F = \mathbb{R}$.) If $v \neq 0$, I claim that $\{v, iv, jv, kv\}$ is a basis of $\mathbb{H}$; this shows that $\mathbb{H}$ is an irreducible representation of $Q$. To show this, suppose given $a_i \in \mathbb{R}$ such that $a_1 v + a_2(iv) + a_3(jv) + a_4(kv) = 0$. Multiplying on the right by $v^{-1}$ yields $a_1 + a_2 i + a_3 j + a_4 k = 0$, so all the $a_i = 0$.

3) The symmetric group $S_4$ acts on the regular tetrahedron in $\mathbb{R}^3$ by permuting the 4 vertices. This extends by linearity to an action of $S_4$ on $\mathbb{R}^3$, which is irreducible (exercise!). More generally, $S_n$ acts on the regular $n$-simplex in $\mathbb{R}^{n-1}$, giving an irreducible $(n - 1)$-dimensional representation of $S_n$.

Schur’s Lemma. 1) $V$ is irreducible $\iff V \cong (FG)/I$ for some maximal left ideal.

2) If $V, W$ are irreducible, any nonzero $G$-map $f : V \to W$ is an isomorphism.

3) If $V$ is irreducible, the ring $\Delta = \text{End}_G(V)$ of all $G$-maps $V \to V$ is a division algebra. (A division algebra is an $F$-algebra in which every non-zero element is a unit.) If $F$ is algebraically closed then $\Delta = F$ (multiplication by scalars).

Proof. 1) Any choice of $v \neq 0$ in $V$ yields a nonzero $G$-map $FG \to V$ sending 1 to $v$. Its kernel $I$ is a left ideal, so its image is $FG/I$. This is a nonzero $G$-submodule of $V$, and every ideal $J$ containing $I$ yields a submodule $J/I$ of $V$. If $V$ is irreducible we must have $FG/I \cong V$ with no ideals $J$ containing $I$. 2) If $W$ is irreducible and $f \neq 0$, $f(V)$ must be $W$ and $\ker(f) \neq V$. When $V$ is irreducible this forces $\ker(f) = 0$, which means that $f$ is an isomorphism. 3) Any nonzero $G$-map $f : V \to V$ must be an isomorphism by 2), in which case $f^{-1}$ exists and is a $G$-map. This proves that every nonzero element $f$ is invertible in $\Delta$.

Examples with $F = \mathbb{R}$. The only (finite-dimensional) division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. (Over $F = \mathbb{C}$ the only f.d. division algebra is $\mathbb{C}$ itself.)

1) Consider the rotation representation of $C_3$ on the plane $\mathbb{R}^2$. The $2 \times 2$ matrices commuting with this action are products of scaling by $r$ and rotation by $\alpha$:

$$
\begin{pmatrix}
  r \cos(\alpha) & r \sin(\alpha) \\
  -r \sin(\alpha) & r \cos(\alpha)
\end{pmatrix} = r \cos(\alpha) + ir \sin(\alpha), \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$
These form a subring $\Delta$ of $M_2(\mathbb{R})$ isomorphic to $\mathbb{C}$.

2) For the canonical representation of the quaternion group $Q$ on $\mathbb{H}$, we have $\Delta = \mathbb{H}$. (Of course!)

**Corollary to Schur’s Lemma.** If $W \subset V$ is a submodule and $V$ is completely reducible, then $V = W \oplus W'$ for some complementary submodule $W'$.

**Proof.** Write $V = \oplus V_\alpha$ with $V_\alpha$ irreducible. By Zorn’s lemma, there is a largest family $\{\alpha_i\}$ so that $W \cap \oplus V_{\alpha_i} = 0$; set $W' = \oplus V_{\alpha_i}$. If $W \oplus W'$ isn’t $V$, it doesn’t contain some $V_\beta$; this would imply $(W \oplus W') \cap V_\beta = 0$, leading to a contradiction.

**Remark.** The regular representation is never irreducible (unless $G = 1$). To see this, recall that the norm element of $FG$ is the sum $N = \sum g$ of every element in $G$. Since $gN = N$ for all $g \in G$, $N$ generates a 1-dimensional submodule $F \cdot N$ of $FG$.

The next theorem states that $FG$ is completely reducible (when $1/|G|$ exists in $F$), and that it contains every irreducible representation at least once. Therefore $FG = F \cdot N \oplus W'$ for some $W'$. In fact, since $N^2 = |G| \cdot N$, the element $e = N/|G|$ is an idempotent of the ring $FG$ and $FG = W \oplus W'$ with $W' = FG(1 - e)$.

The hypothesis (that $1/|G|$ exists in $F$) fails only when $F$ has characteristic $p > 0$ and $p$ divides $|G|$. In this case, the regular representation is never completely reducible, because $F \cdot N \subset FG$ has no complement. (If $FG = F \cdot N \oplus W'$ then some nonzero multiple of $N$ must be idempotent, which is impossible because $N^2 = |G| \cdot N = 0$.)

**Maschke’s Theorem.** If $G$ is a finite group and $\frac{1}{|G|} \in F$, then:

1) Every representation of $G$ is completely reducible.

2) There are only a finite number $s$ of irreducible representations $V_i$ (up to isomorphism), with $V_i$ occurring $n_i \geq 1$ times in the regular representation of $G$ on $FG$.

We will write $\Delta_i$ for the division algebra $\text{End}_G(V_i)$, and set $d_i = \dim_F(\Delta_i)$.

3) The $i$th irreducible representation has dimension $d_in_i$. Hence

$$|G| = \sum_{i=1}^s d_in_i^2.$$ 

4) $FG$ is the product of the $s$ matrix rings $M_{n_i}(\Delta_i)$. The projection $FG \to M_{n_i}(\Delta_i)$ allows us to identify $V_i$ with the $M_{n_i}(\Delta_i)$-module $\Delta_i^{n_i}$.

Some explanation of parts 3) and 4) is in order. The matrix ring $M_{n_i}(\Delta_i)$ is the direct sum of its $n_i$ columns, each being the irreducible representation $V_i = \Delta_i^{n_i}$ of dimension $d_in_i$ over $F$. Summing over $i = 1, \ldots, s$ yields the decomposition of the $|G|$-dimensional representation $FG$ into its irreducible components.

**Corollary (Complex representations).** Suppose that $F = \mathbb{C}$. Then

$$|G| = \sum n_i^2,$$

where $n_i = \dim(V_i)$.

If $G$ is abelian then there are exactly $|G|$ irreducible representations, all of them $1$-dimensional. Every representation is a direct sum of $1$-dimensional representations.

Indeed, we must have $\Delta_i = \mathbb{C}$ and $FG = M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_s}(\mathbb{C})$. If $G$ is abelian, $FG$ is a commutative ring; this forces $n_1 = \cdots = n_s = 1$, or $FG = \Pi_{i=1}^s \mathbb{C}$.
Proposition. Let $c$ denote the number of conjugacy classes of elements of $G$. If $F = \mathbb{C}$ then there are $c$ irreducible representations of $G$. In general, the center $E_i$ of $\Delta_i$ is a finite field extension of $F$, and $c = \sum_{i=1}^s \dim_F(E_i) \geq s$.

The connection to Maschke’s theorem comes from the observation that the center of $FG$ is $\prod E_i$. Write $C_1, \ldots, C_c$ for the conjugacy classes of $G$. The $c$ elements $z_j = \sum \{g \in C_j\}$ are central elements of $FG$, and form a basis for the center of $FG$.

Examples. 1) If $F = \mathbb{C}$ then $C_3$ has three irreducible 1-dimensional representations. If $F = \mathbb{R}$ then $C_3$ has only two irreducible representations: the trivial representation $V_1 = \mathbb{R}$ and the rotation representation on the plane $V_2 = \mathbb{R}^2$.

2) The dihedral group $D_2 = C_2 \times C_2$ is abelian, so it has 4 one-dimensional representations—even over $\mathbb{R}$. The regular representation $FD_2$ is the sum of these 4 representations. Finding the irreducible representations of $D_3$ and $D_5$ is an exercise.

3) The dihedral group $D_4$ has 8 elements, and $D_4/[D_4, D_4]$ is $C_2 \times C_2$. Thus it has exactly 4 one-dimensional representations. We have already observed that $D_4$ has a 2-dimensional irreducible representation $V$ as its “birth certificate”. Since $8 = 4 \cdot 1 + 2^2$, this accounts for all the irreducible representations of $D_4$.

4) The quaternionic group $Q$ has 8 elements and 5 conjugacy classes. Since $Q/[Q, Q] = Q/\{\pm 1\} = C_2 \times C_2$, there are exactly 4 one-dimensional representations. Counting ($8 = 4 + 4$) shows there is exactly one other irreducible representation $V_5$, of dimension 2 or 4 depending on $F$. If $F = \mathbb{R}$, then $V_5$ is the 4-dimensional representation of $Q$ on $\mathbb{H}$; if $F = \mathbb{C}$ then $V_5$ is the 2-dimensional representation of $Q$ on $\mathbb{H} \cong \mathbb{C}^2$ (and $n_5 = 2$).

Exercises. 1) Consider the rotation representation of $C_3$ on the complex plane $\mathbb{C}^2$. Write this as the direct sum of two 1-dimensional representations over $F = \mathbb{C}$.

2) Provide details for the sketch given above that the 2-dimensional representation of $D_n$ is irreducible when $n \geq 3$.

3) Describe all irreducible representations of $D_3$ and $D_5$ over $\mathbb{R}$ and over $\mathbb{C}$. Hint: Find two actions of $D_5$ on the regular pentagon.

4) Prove that the 3-dimensional representation of $S_4$ arising from the action on the regular tetrahedron is irreducible.

5) Determine all irreducible complex representations of the alternating group $A_4$ (12 elements). Hint. Use the fact that $[A_4, A_4]$ has 4 elements to write down all group maps $A_4 \to \mathbb{C}^*$. Then let $G$ act on the set $X = \{(12)(34), (13)(24), (14)(23)\}$ of elements of $A_4$ by conjugation, and prove that $FX$ is irreducible.

6) If $V$ is irreducible and $W$ is any 1-dimensional representation of $G$, show that the tensor product $V \otimes W$ is also an irreducible representation of $G$.

Young Tableaux. Let $S_n$ denote the symmetric group on $n$ elements. The number of conjugacy classes of $S_n$ equals the number of unordered partitions of $n$; the unordered partition $\lambda = \{r_1, \ldots, r_h\}$ corresponds to the conjugacy class of $(1, \ldots, r_1) \cdots (n+1-r_h, \ldots, n)$. Since the order of the $r_i$ doesn’t matter, we always assume that $r_1 \geq r_2 \geq \ldots \geq r_h$. Each partition $\lambda$ determines an arrangement of $n$ empty boxes into $h$ rows, the $i^{th}$ row has $r_i$ boxes; such an arrangement is called a
Young Tableau of shape $\lambda$ and size $n$. The corresponding irreducible representation $S^\lambda$ of $S_n$ is sometimes called the Specht module of $\lambda$. We shall write $f^\lambda$ for $\dim(S^\lambda)$.

If we fill in the boxes of a Young tableau of shape $\lambda$ with the numbers $1, \ldots, n$ we get a Young diagram $D$. We call $D$ standard if a) the entries in every row are increasing, and b) the entries in every column are increasing. The number of standard Young diagrams of shape $\lambda$ equals $f^\lambda = \dim(S^\lambda)$.

There is a simple product formula for $f^\lambda$, called the hook formula. If $(i,j)$ is a box in a Young Tableau, the corresponding hook length $h_{ij}$ is the number of boxes in the “hook” $\{(i, k), k \geq j \} \cup \{(k, j), k \geq i \}$ with vertex $(i,j)$. The hook formula says that

$$\dim(S^\lambda) = f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h_{ij}}.$$ 

If $R$ (resp. $C$) denotes the subgroup of $S_n$ consisting of permutations which merely permute the entries in the rows (resp. in the columns) of $D$, then the Specht module may be described as $S^\lambda = (FS_n)f_D \subseteq FS_n$, where $f_D \in FS_n$ is the sum

$$f_D = \sum_{\tau \in C} (-1)^r \tau \sigma.$$

**Representations of $S_4$.** The only partitions of $n = 4$ are $\{1, 1, 1, 1\}$, $\{2, 1, 1\}$, $\{2, 2\}$, $\{3, 1\}$ and $\{4\}$, corresponding to the 5 Young tableau of size 4. Therefore there are exactly 5 irreducible representations of $S_4$. The only way to add up to 24 using five squares is $24 = 1+1+4+9+9$, so $S_4$ has two irreducible 3-dimensional representations (corresponding to two actions of $S_4$ on the regular tetrahedron), one irreducible 2-dimensional representation ($S_4$ acts on the triangle in the plane by $S_4 \to D_3 \subset GL_2(F)$) and two 1-dimensional representations (the trivial representation and the sign representation). Of course, the dimensions of these representations can also be found by the hook formula.

**Characters of finite groups.** For simplicity, we concentrate on representations of a finite group $G$ over $\mathbb{C}$. The character $\chi_V$ of a representation $V = \mathbb{C}^n$ is defined to be the set map $\chi_V : G \to \mathbb{C}$ sending $g$ to the trace of the matrix $\sigma(g)$. This map is independent of the choice of basis for $V$, since the trace is independent of this choice. This also shows that if $V$ and $W$ are isomorphic representations then $\chi_V = \chi_W$.

Note that $\chi_V$ determines the dimension of $V$, because $\chi_V(1) = \text{trace}(I) = \dim(V)$. We will see that in fact $\chi_V$ completely determines $V$ (over $F = \mathbb{C}$).

**Examples.** 1) Let $V$ be the 2-dimensional rotation representation of the cyclic group $C_n$. Then $\chi_V(\theta^k) = 2 \cos(2\pi k/n)$ for all $k$.

2) The character of the regular representation $V = \mathbb{C}G$ is easy to work out. The matrix $\sigma(g)$ consists of 0's and 1's, and the $(i, i)$ entry is 1 exactly when $g \cdot g_i = g_i$ in $G$. This never happens when $g \neq 1$, meaning that all diagonal entries are 0, and so the trace is 0. In conclusion, if $g \neq 1$ then $\chi_{CG}(g) = 0$.

3) The character of a 1-dimensional representation $V$ is $\chi_V(g) = \sigma(g)$, simply because the trace of a $1 \times 1$ matrix $(a)$ is $a$. These characters are not so interesting.

The characters $\chi_1, \ldots, \chi_s$ of the irreducible representations $V_1, \ldots, V_s$ are called the “irreducible” characters. Every character $\chi_V$ is a linear combination of the
irreducible characters in the vector space \( \mathbb{C}^G \) of all set maps \( G \to \mathbb{C} \). To see this, write \( V \) as the sum \( V = V_{i_1} \oplus \cdots \oplus V_{i_r} \) of irreducible representations. This puts the matrices \( \sigma(g) \) in block diagonal form, and we have \( \chi_V(g) = \chi_{i_1}(g) + \cdots + \chi_{i_r}(g) \) for all \( g \in G \). In particular, the character of the regular representation is \( \sum n_i \chi_{i_1} \), from which we deduce that for every \( g \neq 1 \) we have \( \sum n_i \chi_{i_1}(g) = 0 \). (And of course \( \sum n_i \chi_{i_1}(1) = \sum n_i^2 = |G| \).

A class function on \( G \) is a function \( \phi : G \to \mathbb{C} \) which is constant on conjugacy classes, i.e., if \( g' = hgh^{-1} \) then \( \phi(g') = \phi(g) \). The characters \( \chi_V \) are examples of class functions, because \( \chi_V(g) \) and \( \chi_V(g') \) are the traces of the matrices \( \sigma(g) \) and \( \sigma(g') = P \sigma(h) P^{-1}, P = \sigma(h) \). We choose representatives \( g_1, \ldots, g_c \) of the \( c \) conjugacy classes of \( G \); any class function (including \( \chi_V \)) is then completely determined by its values on these \( c \) elements. Thus the class functions form a \( c \)-dimensional vector subspace of \( \mathbb{C}^G \). We are going to show that the irreducible characters \( \chi_i \) of \( G \) form a basis of this vector space. Since they belong to this subspace, and there are \( s = c \) of them, it suffices to show that they are linearly independent. This follows from the following result, whose proof we omit.

**Orthogonality Relations.** The irreducible characters are orthogonal (with respect to the usual hermitian inner product) in the vector space \( \mathbb{C}^G \):

\[
\langle \chi_i | \chi_j \rangle = \sum_{g \in G} \chi_i(g)^* \chi_j(g) = \begin{cases} |G| & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}
\]

**Corollary.** The irreducible characters \( \{ \chi_1, \ldots, \chi_c \} \) form a basis for the vector space of class functions on \( G \).

Thus every class function \( \phi \) uniquely determines complex numbers \( a_1, \ldots, a_c \) such that \( \phi(g) = \sum a_i \chi_i(g) \) for all \( g \in G \). In fact, \( a_i = \frac{1}{|G|} \langle \phi | \chi_i \rangle \). In particular, the character \( \chi_V \) of any representation \( V \) uniquely determines integers \( m_i \) such that

\[
V \cong V_1^{m_1} \oplus \cdots \oplus V_1^{m_1} \oplus V_2^{m_2} \oplus \cdots \oplus V_c^{m_c} \oplus \cdots V_c^{m_c} .
\]

**Character tables.** The complex numbers \( \chi_i(g) \) assemble to form a \( c \times c \) matrix, called the character table of \( G \). The above results state that the character table tells us almost everything about all representations of \( G \). The Orthogonality Relations imply that the columns are linearly independent (being orthogonal). The character table is not quite a unitary matrix; it satisfies the relation \( A^* A = |G| \cdot I \) instead.

- The character tables of \( C_2 \) and \( C_3 \) are \( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \), \( \omega = e^{2\pi i/3} \).
- For \( S_3 \) there are 3 conjugacy classes, represented by: \( \{1, (12), (123)\} \). The character table for \( S_3 \) is

\[
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} .
\]